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Subhrajit Roy, Animesh Roy and Amar P. Misra

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Subhrajit Roy,[a] Animesh Roy,[b] and Amar P. Misra[c]

AFFILIATIONS

Department of Mathematics, Siksha Bhavana, Visva-Bharati University, Santiniketan 731 235, West Bengal, India

[a] **Electronic mail:** suvo.math88@gmail.com
[b] **Electronic mail:** aroyiitd@gmail.com
[c] **Author to whom correspondence should be addressed:** apmisra@visva-bharati.ac.in

ABSTRACT

The nonlinear dynamics of circularly polarized dispersive Alfvén wave (AW) envelopes coupled to the driven ion-sound waves of plasma slow response is studied in a uniform magnetoplasma. By restricting the wave dynamics to a few number of harmonic modes, a low-dimensional dynamical model is proposed to describe the nonlinear wave–wave interactions. It is found that two subintervals of the wave number of modulation k of AW envelope exist, namely, $(3/4)k_c < k < k_c$ and $0 < k < (3/4)k_c$, where k_c is the critical value of k below which the modulational instability (MI) occurs. In the former, where the MI growth rate is low, the periodic and/or quasi-periodic states are shown to occur, whereas the latter, where the MI growth is high, brings about the chaotic states. The existence of these states is established by the analyses of Lyapunov exponent spectra together with the bifurcation diagram and phase-space portraits of dynamical variables. Furthermore, the complexities of chaotic phase spaces in the nonlinear motion are measured by the estimations of the correlation dimension as well as the approximate entropy and compared with those for the known Hénon map and the Lorenz system in which a good qualitative agreement is noted. The chaotic motion, thus, predicted in a low-dimensional model can be a prerequisite for the onset of Alfvénic wave turbulence to be observed in a higher dimensional model that is relevant in the Earth's ionosphere and magnetosphere.

The generation of envelope solitons in the nonlinear interactions of high-frequency wave electric field and low-frequency plasma density perturbations has been recognized as one of the most important features in the context of plasma heating, transport of plasma particles, as well as wave turbulence in modern physics. One particular class of such solitons is the Alfvén solitons that are circularly polarized high-frequency dispersive waves trapped by the plasma density troughs of low-frequency perturbations. This work proposes a new low-dimensional dynamical model to govern the nonlinear interactions of these dispersive Alfvén waves (DAWs) with low-frequency plasma density fluctuations and shows how the nonlinear dynamics can transit from periodic to chaotic states. The complexity of such chaotic states is also measured by means of the correlation dimension (CD) and the approximate entropy and compared with those for the known Hénon map and the Lorenz system. The existence of chaos in the evolution of Alfvénic wave envelopes can be a good indication for the onset of Alfvénic wave turbulence that is relevant in the Earth's ionosphere and magnetosphere.

I. INTRODUCTION

Alfvén waves are typical magnetohydrodynamic (MHD) waves that travel along the magnetic field lines and can be excited in any electrically conducting fluid permeated by a magnetic field. Such waves can be dispersive in warm electron–ion magnetoplasmas due to the effects of finite ion Larmor radius and the electron pressure gradient force. However, in cold plasmas, they may become dispersive due to finite values of the wave frequency (in comparison with the ion-cyclotron frequency) and the electron inertial force.[1] Since the theoretical description of their existence by Alfvén in 1942[2] and experimental verification by Lundquist in 1949,[3] the Alfvén waves (especially with large amplitude) have been known to play significant roles in transporting energy and momentum in many geophysical and astrophysical MHD flows including the solar corona and the solar wind. They have also been observed in the Earth's magnetosphere,[4] in interplanetary plasmas[5] and in the solar photosphere[6] and proposed as the origin of geomagnetic jerks.[7] Furthermore, the dispersive Alfvén waves (DAWs) can have a wide range of applications in laboratory and space plasmas.[8,9]

Large amplitude Alfvén waves interacting with plasmas can give rise to different nonlinear effects, including the parametric decay of three-wave interactions,[10] stimulated Raman and Brillouin scattering,[11] modulational instability (MI) of wave envelopes,[12] plasma background density modification due to the Alfvén wave ponderomotive force, the Alfvén solitons,[13] as well as the formation of Alfvén vortices and related phenomena[14,15] that have been observed in the Earth's ionosphere and magnetosphere. For some other important nonlinear effects involving Alfvén waves in plasmas, readers are referred to the review work of Shukla and Stenflo.[16] Furthermore, the formation of envelope solitons associated with the modulational instability due to the nonlinear interaction of high-frequency wave electric field and low-frequency ion density perturbations has been known to be one of the most important features in the context of chaos and wave turbulence in plasmas.[17,18] When the electric field intensity is so high that the wave number of modulation exceeds its threshold value, the envelopes are essentially trapped by the density cavities of plasma slow response, and the interactions result in chaos. As this chaotic process develops in a low-dimensional dynamical system, the rate of transfer (or redistribution) of energy from lower to higher harmonic modes (from large to small spatial length scales) becomes faster, leading to strong wave turbulence. Such scenarios have been reported in different contexts by means of Zakharov-like equations in plasmas.[17–20]

The nonlinear coupling of circularly polarized dispersive Alfvén waves and ion density perturbations associated with plasma slow motion has been studied by Shukla *et al.*[8] in a uniform magnetoplasma. They proposed a set of coupled nonlinear equations for the wave electric field and the plasma density perturbation, which admits a localized DAW envelope accompanied by a plasma density depression. However, the theory of nonlinear wave–wave interactions associated with the DAWs has not been studied yet. The purpose of the present work is to reconsider this model's equations and to study the dynamical features of nonlinear three-wave interactions numerically in a low-dimensional dynamical model. We show that the transition from order to chaos is indeed possible when the wave number of modulation is within the domain of the excitation of three-wave modes. The existence of periodic, quasiperiodic, and chaotic states is confirmed by inspecting the Lyapunov exponent spectra, the bifurcation diagram, and phase-space portraits of dynamical variables. The complexities of chaotic phase spaces are also examined by the estimations of correlation dimension (CD) and the approximate entropy (ApEn), and the obtained results are compared with those for the Hénon map and the Lorenz system. Good qualitative agreements of the results are noticed.

This manuscript is organized as follows: In Sec. II, the modulational instability of AW envelopes is studied and the construction of a low-dimensional dynamical model from a higher dimensional system is shown. The basic dynamical properties of the low-dimensional system is studied and the existence of periodic, quasi-periodic, or chaotic states are shown in Sec. III. In Sec. IV, the complexities of chaotic phase spaces are measured and compared with those for the Lorenz system and Hénon map. Finally, the results are concluded in Sec. V.

II. LOW-DIMENSIONAL MODEL

The nonlinear interactions of circularly polarized dispersive Alfvén wave envelopes propagating along the constant magnetic field $\mathbf{B}_0 = B_0\hat{z}$ and the slowly varying electron/ion density fluctuations that are driven by the Alfvén wave ponderomotive force can be described by the following set of coupled equations:[8,21]

$$\left(\frac{\partial}{\partial t} + V_A\frac{\partial}{\partial z}\right)E_\perp - \frac{V_A^2}{2n_0}\frac{\partial}{\partial z}(n_1E_\perp) \pm i\frac{V_A^2}{2\omega_{ci}}\frac{\partial^2}{\partial z^2}E_\perp = 0, \tag{1}$$

$$\left(\frac{\partial^2}{\partial t^2} - C_s^2\frac{\partial^2}{\partial z^2}\right)n_1 = -\frac{n_0V_A^2}{B_0^2}\frac{\partial^2}{\partial z^2}|E_\perp^2|, \tag{2}$$

where $E_\perp$ is the perpendicular (to $\hat{z}$) component of the wave electric field, n_1 is the plasma number density perturbation (with n_0 denoting the equilibrium value), $V_A = B_0/\sqrt{4\pi n_0 m_i}$ is the Alfvén velocity, and $\omega_{ci} = eB_0/cm_i$ is the ion cyclotron frequency with e denoting the elementary charge, c is the speed of light in vacuum, and m_i is the ion mass. Also, $C_s = \sqrt{T_e/m_i}$ is the ion-sound speed, with T_e denoting the electron thermal energy. For the description of the linear theory of circularly polarized dispersive Alfvén waves and the derivations of the nonlinear coupled Eqs. (1) and (2), readers are referred to the work of Shukla *et al.*[8]

By defining the dimensionless quantities according to $t \to t\omega_{ci}$, $z \to z\omega_{ci}/C_s$, $n \to n_1/n_0$, and $E \to cE_\perp/C_sB_0$, Eqs. (1) and (2) can be reproduced as

$$\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial z^2} = -\alpha^2\frac{\partial^2}{\partial z^2}|E|^2, \tag{3}$$

$$\frac{\partial E}{\partial t} + \beta\frac{\partial E}{\partial z} - \frac{\beta}{2}\frac{\partial}{\partial z}(nE) + i\gamma\frac{\partial^2 E}{\partial z^2} = 0, \tag{4}$$

where $\alpha = V_A/c$, $\beta = V_A/C_s$, and $\gamma = \pm\beta^2/2$. Here, the $\pm$ sign in γ corresponds to the right- and left-circularly polarized AWs.

Looking for the modulation of the AW amplitude and thereby making the ansatz,

$$\begin{aligned} E(z,t) &= E_1(z,t)\exp[i\theta(z,t)],\\ n(z,t) &= \tilde{n}(z,t)\exp\left(ikz - i\omega t\right) + \text{c.c.},\\ E_1(z,t) &= E_0 + \tilde{E}\exp\left(ikz - i\omega t\right) + \text{c.c.},\\ \theta(z,t) &= \theta_0 + \tilde{\theta}\exp\left(ikz - i\omega t\right) + \text{c.c.}, \end{aligned} \tag{5}$$

where E_1 and θ are slowly varying functions of z and t, and $\tilde{E} \ll E_0$, $\tilde{\theta} \ll \theta_0$, we obtain from Eqs. (3) and (4) the following linear dispersion relation for the modulated DAW envelope (for details, see Appendix A),

$$(\omega^2 - k^2)\left[(\omega - \beta k)^2 - \gamma^2k^4\right] + \alpha^2\beta\gamma k^5|E_0|^2 = 0. \tag{6}$$

For the modulational instability, we assume $\omega \approx \beta k + i\Gamma$ with $\beta k \gg \Gamma$. The instability growth rate is then obtained as

$$\Gamma = \sqrt{\gamma k^3\left(\frac{\beta\alpha^2|E_0|^2}{\beta^2 - 1} - \gamma k\right)}. \tag{7}$$

Thus, the modulational instability sets in for $0 < k < k_c$, where $k_c \equiv 2\alpha^2|E_0|^2/\beta|\left(\beta^2-1\right)|$ is the critical wave number with $\beta > 1$, and the maximum growth rate is attained at $k = (3/4)k_c$. From the expression of Γ we find that the growth rate increases in the interval $0 \lesssim k \lesssim (3/4)k_c$, reaches a maximum at $k = (3/4)k_c$, and then decreases with k with a cutoff at $k = k_c$. However, if the electric field intensity is so high that the MI threshold exceeds the decay instability threshold, the DAWs may be trapped by the ion-sound density perturbations. In this case, the interaction between the circularly polarized DAWs and the ion-sound waves may result in a turbulence in which the transfer or redistribution of wave energy among different modes can take place.[17,20,22] On the other hand, in the adiabatic limit, i.e., the quasi-stationary response of density fluctuations, the second order time derivative in Eq. (3) can be disregarded. The resulting equation is then the derivative NLS (DNLS) equation given by

$$\left(\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial z}\right)E-\frac{1}{2}\beta\alpha^2\frac{\partial}{\partial z}\left(|E|^2E\right)+i\gamma\frac{\partial^2 E}{\partial z^2}=0. \tag{8}$$

Equation (8) is clearly integrable[23,24] and, hence, nonchaotic. So, it can have a localized solution for the wave electric field envelopes.

Equations (3) and (4) are, in general, multidimensional and can describe the evolution of an infinite number of wave modes. However, a few number modes may be assumed to participate actively in the nonlinear wave–wave interactions. Such cases are not only common in the Alfvénic wave turbulence but also occur in the parametric instabilities of high- and low-frequency wave interactions close to the instability threshold. In this situation, a low-dimensional model with a few truncated modes is well applicable to study the basic features of the full wave dynamics of Eqs. (3) and (4). Here, one must note that the specific details of the low-dimensional model strongly depend on the range of the wave number of modulation k. So, considering the nonlinear dynamics among a few number of wave modes, we expand the electric field envelope $E(z,t)$ and the density perturbation $n(z,t)$ as

$$\begin{aligned}E(z,t)&=\sum_{m=-M/2}^{+M/2}E_m(t)e^{imkz}=\sum_{m=-M/2}^{+M/2}\rho_m(t)e^{\theta_m(t)}e^{imkz}\\&=E_0(t)+E_{-1}(t)e^{-ikz}+E_1(t)e^{ikz},\end{aligned} \tag{9}$$

$$n(z,t)=\sum_{m=-M/2}^{+M/2}n_m(t)e^{imkz}=n_0(t)+n_1(t)e^{ikz}+n_{-1}(t)e^{-ikz}, \tag{10}$$

where $M=[k^{-1}]$ denotes the number of modes to be selected in the interactions, $E_{-m}=E_m$, and $n_{-m}=n_m$. For three-wave interactions, we choose $M=2$ and following the same approach as in Refs. 18 and 22, we obtain from Eqs. (3) and (4) the following set of reduced equations:

$$\ddot{n}_1-k^2n_1=\alpha^2k^2n_0\sin\psi\cos\phi, \tag{11}$$

$$\dot{\psi}=\beta k n_1\sin\phi, \tag{12}$$

$$\dot{\phi}=k(\beta-\gamma k)-\frac{1}{2}\beta k n_1\tan\frac{\psi}{2}\cos\phi, \tag{13}$$

where the dot denotes differentiation with respect to t, $\phi=\theta_0-\theta_1$, $\psi=2w$, and $n_0=|E_{-1}|^2+|E_1|^2+|E_0|^2$ is the conserved plasmon number. The detailed derivations of Eqs. (11)–(13) are given in Appendix B. The system of Eqs. (11) to (13) can be recast as an autonomous system,

$$\begin{aligned}\dot{x}_1&=\beta_0x_2\sin x_4,\\\dot{x}_2&=x_3,\\\dot{x}_3&=-k^2x_2+\alpha_0^2\sin x_1\cos x_4,\\\dot{x}_4&=\gamma_0-\frac{1}{2}\beta_0x_2\tan\frac{x_1}{2}\cos x_4,\end{aligned} \tag{14}$$

where $\beta_0=\beta k$, $\alpha_0=\alpha k\sqrt{n_0}$, and $\gamma_0=k(\beta-\gamma k)$ of which the key parameters are α, β, and k. Also, for the sake of convenience, we have redefined the variables as $\psi=x_1$, $n_1=x_2$, $\dot{n}_1=x_3$, and $\phi=x_4$.

III. DYNAMICAL PROPERTIES

In this section, we numerically study the linear stability analysis of Eq. (14) and look for different parameter regimes for the existence of periodic, quasiperiodic, and chaotic states on the basis of Lyapunov exponent spectra, bifurcation diagram, and phase-space portraits.

A. Equilibrium points and eigenvalues

As a starting point, we calculate the equilibrium points by equating the right-hand sides of Eq. (14) to zero and finding solutions for x_1, x_2, x_3, x_4 as $(x_{10}, x_{20}, 0, n\pi)$, where $x_{10}=4n\pi \pm 2\sin^{-1}\left(\pm\sqrt{\gamma_0k^2/\alpha_0^2\beta_0}\right)$ and $x_{20}=\pm(-1)^n\left(2k/\beta_0\right)\sqrt{\left(\beta-\gamma k\right)\left(\alpha^2n_0\beta-\beta+\gamma k\right)}$ with n being zero or an integer. Thus, there are primarily four types of equilibrium points, namely, $P_1\equiv\left(x_{10}^+, x_{20}^+, 0, n\pi\right)$, $P_2\equiv\left(x_{10}^-, x_{20}^-, 0, n\pi\right)$, $P_3\equiv\left(x_{10}^+, x_{20}^-, 0, n\pi\right)$, and $P_4\equiv\left(x_{10}^-, x_{20}^+, 0, n\pi\right)$, where $x_{10}^{\pm}$ $(x_{20}^{\pm})$ are the values corresponding to the $\pm$ signs in x_{10} (x_{20}) (for details, see Appendix C). [Here, we are not considering any sign convention in γ applicable to right- or left-circularly polarized AWs.] We note that $(0,0,0,0)$ is not an equilibrium point since for $(0,0,0,0)$ to be an equilibrium point, one must have $k\sim 2/\beta$, which may not satisfy the restriction $k<k_c$ for some typical parameter regimes with $\alpha \ll 1$, $E_0>1$, and $\beta>1$. Furthermore, for real values of x_{10} and x_{20}, one must have $\alpha^2n_0\lesssim 1$ and $(2/\beta)\left(1-\alpha^2n_0\right)\lesssim k\lesssim(2/\beta)$. Next, applying the transformation around the equilibrium point, i.e., $x_1'=x_1-x_{10}$, $x_2'=x_2-x_{20}$, $x_3'=x_3-x_{30}$, and $x_4'=x_4-x_{40}$, we obtain a linearized system of the form: $dX'/dt=JX'$, where J is the Jacobian matrix and $X'=(x_1', x_2', x_3', x_4')$. The eigenvalues ($\lambda$) corresponding to each of these equilibrium points can be obtained from the relation $JX'=\lambda X'$ and then the stability of the system (14) can be studied by the nature of these eigenvalues. The Jacobian matrix J is given by

$$J = \begin{bmatrix} 0 & \beta_0 \sin x_4 & 0 & \beta_0 x_2 \cos x_4 \\ 0 & 0 & 1 & 0 \\ \alpha_0^2 \cos x_1 \cos x_4 & -k^2 & 0 & -\alpha_0^2 \sin x_1 \sin x_4 \\ -\frac{1}{4}\beta_0 x_2 \sec^2\left(\frac{x_1}{2}\right)\cos x_4 & -\frac{1}{2}\beta_0 \tan\left(\frac{x_1}{2}\right)\cos x_4 & 0 & \frac{1}{2}\beta_0 x_2 \tan\left(\frac{x_1}{2}\right)\sin x_4 \end{bmatrix}, \tag{15}$$

which at the equilibrium point $(x_{10},\ x_{20},\ 0,\ n\pi)$ reduces to

$$J = \begin{bmatrix} 0 & 0 & 0 & (-1)^n \beta_0 x_{20} \\ 0 & 0 & 1 & 0 \\ (-1)^n \alpha_0^2 \cos x_{10} & -k^2 & 0 & 0 \\ (-1)^{n+1}\frac{1}{4}\beta_0 x_{20} \sec^2\left(\frac{x_{10}}{2}\right) & (-1)^{n+1}\frac{1}{2}\beta_0 \tan\left(\frac{x_{10}}{2}\right) & 0 & 0 \end{bmatrix}, \tag{16}$$

where n is either zero or an integer. The characteristic equation for the matrix J [Eq. (16)] is

$$\lambda^4 + \left[k^2 + \frac{1}{4}\beta_0^2 x_{20}^2 \sec^2\left(\frac{x_{10}}{2}\right)\right]\lambda^2 + \delta = 0, \tag{17}$$

where

$$\delta = \beta_0 x_{20}\left[\frac{1}{2}(-1)^n \alpha_0^2 \beta_0 \cos(x_{10}) \tan\left(\frac{x_{10}}{2}\right) + \frac{k^2}{4}\beta_0 x_{20} \sec^2\left(\frac{x_{10}}{2}\right)\right]. \tag{18}$$

We numerically examine the roots of Eq. (17) within the domain $(2/\beta)\left(1-\alpha^2 n_0\right) \lesssim k \lesssim (2/\beta)$ for some fixed values of the other parameters, namely, $\beta = 8$, $\alpha = 0.15$, and $n_0 = 10$. Note that the qualitative features will remain the same for some other set of parameter values fulfilling the restrictions for $\alpha,\ n_0$, and k stated before. Since we are interested in the real parts of the eigenvalues corresponding to the equilibrium points $(x_{10},\ x_{20},\ 0,\ n\pi)$, without loss of generality, we assume that $n = 0$. The real parts of the eigenvalues corresponding to P_1 and P_3 are displayed in the subplots (a) and (b) of Fig. 1. Note that the real eigenvalues corresponding to P_2 and P_4 will remain the same as for P_1 and P_3 respectively. Also, of four eigenvalues, only two distinct are shown for P_1 and P_3. It is noted that depending on the ranges of values of k, the eigenvalues can assume zero, negative and positive values, indicating that the system can be stable (when $\Re\lambda$ is zero or negative) or unstable (when $\Re\lambda > 0$) about the equilibrium points. From the subplots (a) and (b), it is also seen that a critical value of k exists near $k = 0.22$, below or above which the system's stability may break down before it again reaches a steady state with a zero or a negative eigenvalue. Since we have seen that the modulational instability of DAWs takes place in $0 < k < k_c$, the domain of k in subplot (b) may provide an initial

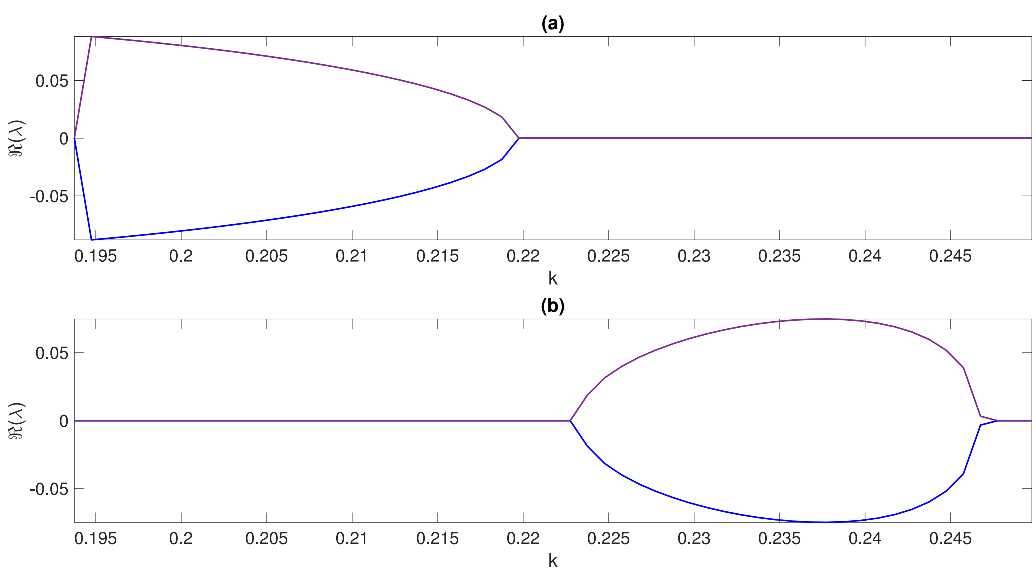

FIG. 1. The real parts of the eigenvalues (λ) corresponding to P_1 [subplot (a)] and P_3 [subplot (b)] are shown against the parameter k. The fixed parameter values are $\beta = 8$, $\alpha = 0.15$, and $n_0 = 10$. The bifurcations indicate that the system [Eq. (14)] can be stable with $\Re\lambda < 0$ or unstable with $\Re\lambda > 0$ around the fixed point P_1 or P_3 in a finite domain of k.

guess for the existence of chaos and periodicity in the wave–wave interactions.

B. Lyapunov exponents, bifurcation diagram, and phase-space portraits

Having predicted the stable and unstable regions of the dynamical system (14) in the domains of the wave number of modulation k as in Sec. III, we proceed to establish the ranges of values of the parameters k, α, and β in which the periodic, quasiperiodic, or chaotic states of plasma waves can exist. To this end, we first calculate the Lyapunov exponents Λ_i, $i = 1, 2, 3, 4$, for the dynamical system [Eq. (14)], to be written in the form $\dot{x}_i = f_i(X)$, with the initial condition: $X(0) = [x_1(0), x_2(0), x_3(0), x_4(0)]$. We are interested in the evolution of attractors and depending on the initial condition, these attractors will be associated with different sets of exponents. The latter, however, describe the behaviors of $X(t)$ in the tangent space of the phase space and are defined by the Jacobian matrix given by

$$J_{ij}(t) = \left.\frac{df_i}{dx_j}\right|_{X(t)}. \tag{19}$$

The evolution of the tangent vectors can then be defined by the matrix A via the following relation:

$$\dot{A} = JA, \tag{20}$$

together with the initial condition $A_{ij}(0) = \delta_{ij}$. Here, δ_{ij} is the Kronecker delta and the matrix A characterizes how a small change of separation distance between two trajectories in phase space develops from the starting point $X(0)$ to the final point $X(t)$. Nonetheless, matrix A is given by

$$A(t) = \begin{bmatrix} 0 & \beta_0 \sin x_4 & 0 & p x_2 \cos x_4 \\ 0 & 0 & 1 & 0 \\ -\alpha_0^2 \cos x_1 \cos x_4 & -k^2 & 0 & -\alpha_0^2 \sin x_1 \sin x_4 \\ -\frac{1}{4}\beta_0 x_2 \sec^2(x_1/2)\cos x_4 & -\frac{1}{2}\beta_0 \tan(x_1/2)\cos x_4 & 0 & -\frac{1}{2}x_2 \tan(x_1/2)\sin x_4 \end{bmatrix}. \tag{21}$$

The Lyapunov exponents Λ_i are, thus, obtained as the eigenvalues of the following matrix:

$$\Lambda = \lim_{t\to\infty} \frac{1}{2t} \log\left[A(t)A^T(t)\right], \tag{22}$$

where A^T denotes the transposed matrix of A. Given an initial condition $X(0)$, the separation distance between two trajectories in phase space or the change of particle's orbit can be obtained by Liouville's formula: $\delta X(t) = \mathrm{tr}\,(J(t))\,|A(t)|$, where $|A(t)| \equiv \det A(t)$ and $\det A(0) = 1 > 0$. Thus, for the dynamical system (14), one obtains $\det A(t) = \exp\left(\int_0^t \mathrm{tr}\,(J(t))\,dt\right) = 1 > 0$. It follows that at least one $\Lambda_i > 0$, implying the existence of a chaotic state in a given time interval $[0, t]$.

Before proceeding further to the analyses of Lyapunov exponent spectra and the bifurcation diagram together with the phase-space portraits, we recapitulate that the MI of Alfvén wave envelopes sets in for $0 < k < k_c$. The growth rate of instability tends to become higher in the interval $0 < k < (3/4)k_c$ and lower in $(3/4)k_c < k < k_c$ with a cut-off at $k = k_c$ at which the pitchfork bifurcation occurs. It follows that the nonlinear dynamics of wave–wave interactions is subsonic in the interval $(3/4)k_c < k < k_c$. However, as k decreases from $(3/4)k_c$, many more unstable wave modes can be excited due to a selection of modes with $M = [k^{-1}]$ and the dynamics may no longer be subsonic. In this situation, a description of nonlinear interactions with three wave modes may be relatively correct. Thus, one may assume that one ($|m| = 1$) Alfvén wave mode is unstable (i.e., the Alfvén waves with $|m| > 1$ are stable) and two driven ion-sound waves of plasma slow response (already excited by the unstable Alfvén mode) remain as they are. This leads to the autonomous system (14). We will investigate how the system behaves as the values of k is successively increased from $(3/4)k_c$ to k_c in the subsonic region and as k reduces from $(3/4)k_c$ to a value so that the three-wave interaction model remains valid (since smaller the values of k, larger is the number of modes M).

In what follows, we calculate the maximum Lyapunov exponent $\lambda_i^{\max}$ for Eq. (14) using the algorithm as stated above in a finite domain of k, i.e., $0 < k < k_c < 1$ and numerically solve Eq. (14) using the fourth order Runge–Kutta scheme with a time step $dt = 10^{-3}$ to obtain the bifurcation diagram of a state variable x_1 and phase-space portraits with the same set of fixed parameter values $\beta = 8$, $\alpha = 0.15$, and $n_0 = 10$ as in Fig. 1. The results are displayed in Figs. 2 and 3. It is noted that, similar to subplot (b) of Fig. 1, two sub-intervals of k exist, namely, $0.2 \lesssim k \lesssim k_1 \approx 0.42$ and $k_1 < k \lesssim 1$. In the former $\lambda_i^{\max} > 0$, while in the latter, it is close to zero, implying that the system may exhibit chaotic states in $0.2 \lesssim k \lesssim k_1$ and quasiperiodic and/or limit cycles in the other sub-interval [see subplot (a) of Fig. 2]. Physically, since lower (higher) values of $k\,(< k_1)$ correspond to a large (small) number of wave modes ($[k^{-1}]$) to participate in the nonlinear wave dynamics, the wave–wave interactions may result into chaos (limit cycles or steady states) by the influence of the nonlinearity associated with the Alfvén wave ponderomotive force (proportional to α) and the nonlinear interactions between the fields (proportional to β). These features can also be verified from the bifurcation diagram of a state variable, e.g., x_1 with respect to the parameter k [see subplot (b) of Fig. 2]. Here, it is seen that as the value of k increases within the domain, a transition from chaotic (dense region) to a periodic or steady (straight line) state can occur. However, the values of k smaller than $k = 0.2$ may not be admissible as those corresponding to a larger number of wave modes and their interactions cannot be described by the low-dimensional system (14) but by the full system of Eqs. (3)

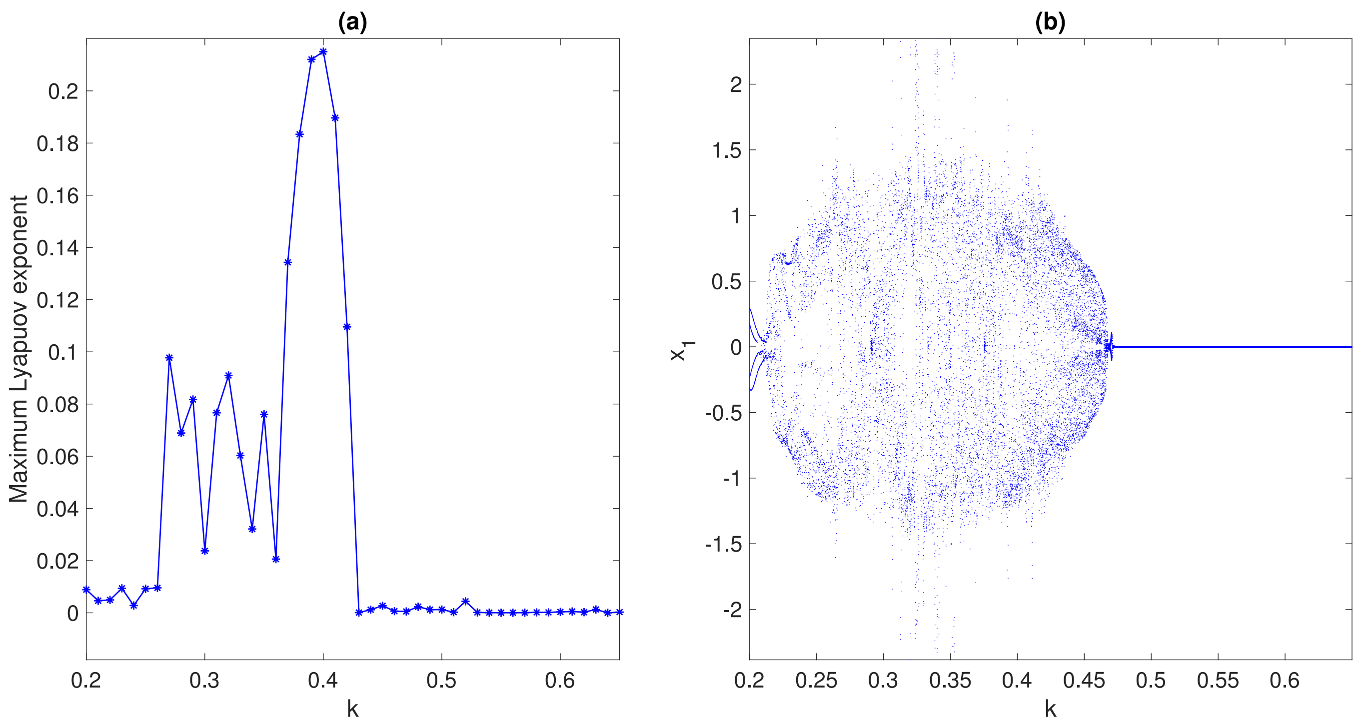

FIG. 2. The maximum Lyapunov exponent [subplot (a)] and the bifurcation diagram [subplot (b)] are shown against the wave number k. The fixed parameter values are the same as in Fig. 1.

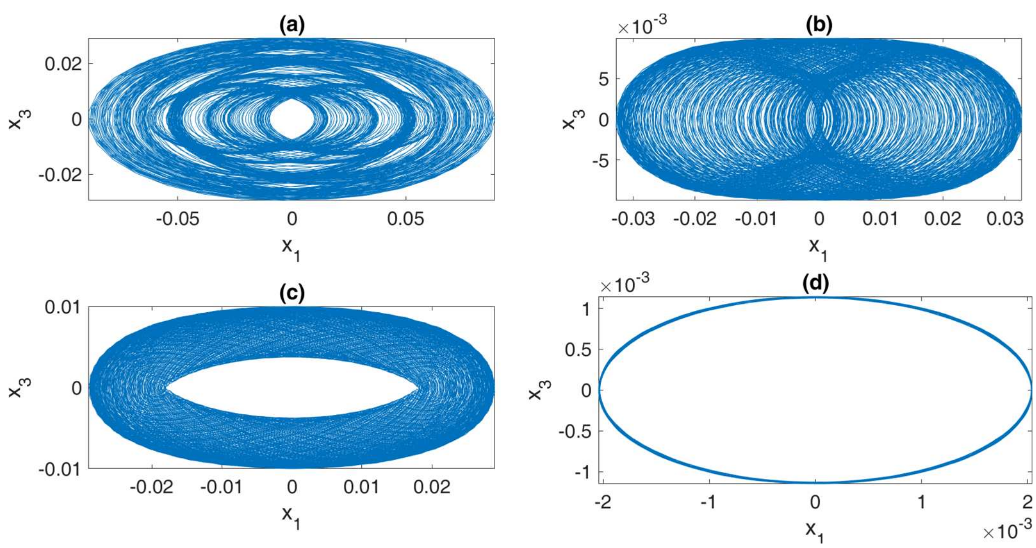

FIG. 3. Different phase-space portraits, formed in the dynamical evolution of Eq. (14), are shown for different values of the wave number of modulation k: (a) $k = 0.21$, (b) $k = 0.42$, (c) $k = 0.45$, and (d) $k = 0.51$. Subplots (a) and (b) show the chaotic states while subplots (c) and (d) exhibit the quasi-periodic and periodic states, respectively. The fixed parameter values are the same as in Fig. 1, i.e., $\beta = 8$, $\alpha = 0.15$, and $n_0 = 10$.

and (4). An investigation of the latter is, however, out of scope of the present work.

In order to further verify the dynamical features so predicted for ranges of values of k and for illustration purpose, different phase-space portraits are also obtained by solving Eq. (14) numerically. From Fig. 3, it is evident that as the values of k increase from smaller [subplots (a) and (b)] to larger ones [subplots (c) and (d)], the chaotic states of AWs transit into quasiperiodic [subplot (c)] and periodic [subplot (d)] states. These features are in agreement with the Lyapunov exponent and the bifurcation diagram shown in Fig. 2.

Thus, it is noted that the nonlinear interaction of a few wave modes of dispersive Alfvén waves and low-frequency plasma density perturbations can exhibit periodic, quasi-periodic, and chaotic states in finite domains of the wave number of modulation due to the finite effects of the nonlinearities associated with the wave electric field driven ponderomotive force and the interactions of the electric field and the plasma density fluctuations. The existence of these states is established by the analyses of Lyapunov exponents, the bifurcation diagram, and the phase-space portraits.

IV. CHARACTERIZATION OF CHAOS: MEASURE OF CHAOS COMPLEXITY

In this section, we study the complexity of the dynamics of wave–wave interactions and thereby measure quantitatively the characteristics of chaotic behaviors of the state variables relating to plasma density or wave electric field perturbations. Although several formulas have been developed in the literature to characterize chaos, we focus mainly on the measures of embedding dimension estimation,[25] correlation dimension,[26] and the approximate entropy.[27–29]

A. Estimation of embedding parameters

Many well known and efficient techniques, e.g., Recurrence quantification technique for the analysis of nonlinear time series require the construction of phase-space profiles of the time series since those techniques are applicable to the phase-space profiles but not to the time series themselves. The method of embedding dimension estimation is one such which also requires the reconstruction of successive phase spaces of chaotic processes with the effects of time delay.

1. Phase-space reconstruction

Reconstruction of phase space has become useful to extract information of a chaotic time series in nonlinear dynamical systems. Let $X = [x(1),\ x(2), \ldots, x(n)]^T$ represent a uniformly sampled univariate time signal, i.e., an observed sequence of the chaotic state variable $x(t)$ (which may be any one of x_1, x_2, x_3, and x_4) with $t = 1, 2, \ldots, n$. Then, to reconstruct a phase space by embedding the dimension m, we construct a time series $Y(t)$ of length m (i.e., m-dimensional points) from the original time series $X(t)$ by considering an appropriate time delay τ as

$$Y(t) = \{X(t),\ X(t+\tau), \ldots, X[t+(m-1)\tau]\}^T, \qquad (23)$$

where $t = 1, 2, \ldots, n-(m-1)\tau$ and τ is a positive integer. Thus, the phase spaces of $M = n-(m-1)\tau$ state variables are reconstructed. Generalizing this result, one can reconstruct the phase spaces for multivariate time signals. So, in order to perform the phase-space reconstruction, one must know the two embedding parameters, namely, the time delay parameter τ, which is the lag at which the time series has to be plotted against itself, and the embedding dimension parameter m, where $m-1$ is the number of times that the time series has to be plotted against itself using the delay τ. Having known these two parameters, one can then reconstruct an approximate phase space of the original one from a given time series. In Secs. IV A 2 and IV A 3, we estimate these two parameters by the methods of computing the two functions, namely, the average mutual information (AMI) and the false nearest neighbors (FNNs)[25] in which the first local minima (or the points of cut-off) of these functions can be estimated as the time delay and the embedding dimension, respectively.

2. Average mutual information (AMI): Estimation of time delay

In AMI, the mutual information is computed between the original time series of a state variable $X(t)$ and a time shifted version of the same time series, i.e., $X(t+\tau)$. This average or auto mutual information can be considered as a nonlinear generalization of the autocorrelation function given by

$$I[X(t), X(t+\tau)] = \sum_{i,j} p_{ij}(\tau)\log\left(\frac{p_{ij}(\tau)}{p_i p_j}\right), \qquad (24)$$

where p_i is the probability that $X(t)$ is in the ith rectangle of the histogram to be constructed from the data points of $X(t)$ and p_{ij} is the probability that $X(t)$ is in the ith rectangle and $X(t+\tau)$ in the jth rectangle.

3. False nearest neighbors (FNNs): Estimation of embedding dimension

Typically, the embedding dimension m for phase-space reconstruction is estimated by inspecting the change in distance between two nearest points in phase space as one gradually embeds the original time series $X(t)$ into higher dimensional ones $Y(t)$. The use of FNN, as prescribed by Kennel *et al.*,[30] is based on the following logic: Initially, we have the one-dimensional time series $X(t)$, $t = 1, 2, \ldots, n$ and the distance between two of its neighboring points are noted. Then, we embed $X(t)$ into two dimensions $Y(t) = \{X(t),\ X(t+\tau)\}$ with some time delay τ and examine whether there is any considerable change in the distance between any two neighboring data points of $Y(t)$. If so, these data points are said to be false neighbors, and the data points need to be embedded further. Otherwise, if the change is not significant, the data points are called true neighbors and the embedding retains the shape of the phase-space attractor, implying that the present embedding dimension is sufficient. This process of successively increasing the embedding dimension m can be continued until the number of FNN reduces to zero, or the subsequent embedding does not alter the number of FNNs, or the number of FNNs starts to increase again. A working algorithm for calculating FNN for our system can be stated below.

1. Identify the nearest point in the Euclidean sense to a given point of the time delay coordinates. That is, for a given time series $Y(t) = \{X(t), X(t+\tau), \ldots, X[t+(m-1)\tau]\}^T$, find a point Y_i in the data set such that the distance $m = \|Y_i - Y_j\|_2$ is minimized, where Y_i and Y_j denote the nearest neighboring data points of $Y(t)$.
2. Determine whether the following expression is true or false:

$$\frac{|X_i - X_j|}{\|Y_i - Y_j\|} \leq \text{Distance threshold (R)}, \tag{25}$$

   where X_i and X_j denote the nearest neighboring data points of $X(t)$. If the condition in Eq. (25) is satisfied, then the neighbors are true nearest neighbors, otherwise they are false nearest neighbors.
3. Perform step 2 for all points i in the data set and calculate the percentage of points in the data set that have false nearest neighbors.
4. Increase embedding the dimension until the percentage of false nearest neighbors drops to zero or an admissible small number.

Following Ref. 25 and using MATLAB, we estimate the embedding parameters, namely, the time lag τ and the embedding dimension m for the four-dimensional time series (x_1, x_2, x_3, x_4) formed by all four variables of system (14). The results are shown in Fig. 4. From subplot (a), we find that all the auto mutual information (AMI) curves, obtained for different time series, cut the threshold line at different values of τ. It is seen that for these curves, the AMI first drops below the threshold value $(1/e)$ after the time lags $\tau = 4.25$, 5.75, 13.1, and 19.66, and we have considered the maximum time delay as $\tau = 25$. Thus, a mean value of τ for each dimension can be obtained as $\tau = (4.25 + 5.75 + 13.1 + 19.66)/4 = 10.7$ for which we obtain an estimate for τ as $\tau = 11$. On the other hand, subplot (b) displays the FNN function against the embedding dimension m. It is clear that the available four-dimensional time series is sufficient and no further time-delayed embedding of dimension is required. Next, having estimated the time delay $\tau = 11$ and the embedding dimension $m > 2$, a reconstruction of phase space is shown in Fig. 5 for all the variables x_1, x_2, x_3, and x_4 of Eq. (14). Here, the time series for the variables are plotted against each other with the time lag $\tau = 11$. It is seen that the resulting phase space with the time lag is approximately the same as the original one.

B. Correlation dimension estimation

One of the most important measures of the complexity of chaotic attractors is the correlation (or fractal) dimension. It has been shown by many researchers that the correlation dimension is more pertinent to experimental data than the capacity dimension as it simply calibrates the geometrical structure of an attractor and is insufficient for higher dimensional systems. Moreover, the correlation dimension is a (or close to the) lower bound on the Hausdorff fractal dimension, which is infinite for noise; positive and finite for a deterministic system; integer for integrable systems; and non-integer for a chaotic deterministic system. The derivation of the correlation dimension also requires the reconstruction of vectors from the time series $X(t)$, i.e.,

$$\begin{aligned} Y_1(t) &= \{X(t), X(t+\tau), \ldots, X[t+(m-1)\tau]\}^T, \\ Y_2(t) &= \left\{X(t+p), \ldots, X[t+p+(m-1)\tau]\right\}^T, \\ &\vdots \\ Y_M(t) &= \left\{X(t+Mp), \ldots, X[t+Mp+(m-1)\tau]\right\}^T, \end{aligned} \tag{26}$$

where p (a positive integer) and τ, respectively, stand for the inter-vector and intra-vector spacing.

After the reconstruction of phase space of a chaotic signal $X(t)$ with M vectors and computing the correlated vector pairs, its proportion in all possible pairs in M^2 is the correlation integral $C(l)$ given by

$$C(l) = \lim_{M\to\infty}\left[\frac{2}{M^2}\sum_{i=1}^{M-k}\sum_{j=i+k}^{M}\Theta\left(l - |X(i) - X(j)|\right)\right], \tag{27}$$

where $X(i)$ and $X(j)$ are the position vectors of pints on an attractor, l is the distance under inspection, k is the summation offset used to prevent proximate vectors being counted, and $\Theta(x)$ is the Heaviside step function defined by

$$\Theta(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases} \tag{28}$$

The correlation dimension d is then calculated from the correlation integral as

$$d = \lim_{l\to 0}\frac{\log C(l)}{\log l}. \tag{29}$$

Next, using Eq. (29), we plot a graph of $\log C(l)$ vs $\log l$ for the time series $X(t)$ of the dynamical system (14) with a fixed embedding dimension $m = 4$ and time delay $\tau = 11$. The system is turned to be higher dimensional by the method described above. As a comparison, we have also obtained graphs of the correlation integrals for the Hénon map with the embedding dimension $m = 2$ and the Lorenz system with $m = 3$. The results are displayed in Fig. 6. The fixed parameter values considered here are the same as in Fig. 1, i.e., $\alpha = 0.15$, $\beta = 8$, and $n_0 = 10$. The slopes of the straight-line portions (obtained using the least-square curve fitting) of the graphs represent the correlation dimensions. For the present system [Eq. (14)], the results as in subplot (a) appear similar to those for the Hénon map [subplot (b)] and the Lorenz system [subplot (c)]. However, the correlation dimension obtained for our system is $d = 1.0776$, while for the Hénon map and the Lorenz system, they are $d = 1.25$ and $d = 2.06$, respectively. It follows that system (14) is chaotic and possesses a strange attractor characterized by $d = 1.0776$.

In Fig. 7, we plot $\log C(l)$ vs $\log l$ for increasing values of the embedding dimension, namely, $m = 8$, 12, 16, and 20. The time series is taken to have consisted of 10 000 points separated by the time lag $\tau = 11$. One can then obtain the correlation dimensions as $d = 1.0777$, 1.0781, 1.0788, respectively. Thus, a series of straight lines indeed exist with slopes $d \approx 1.07 \pm 0.01$ and $\log C(l)/\log l$ is nearly a constant value for large m.

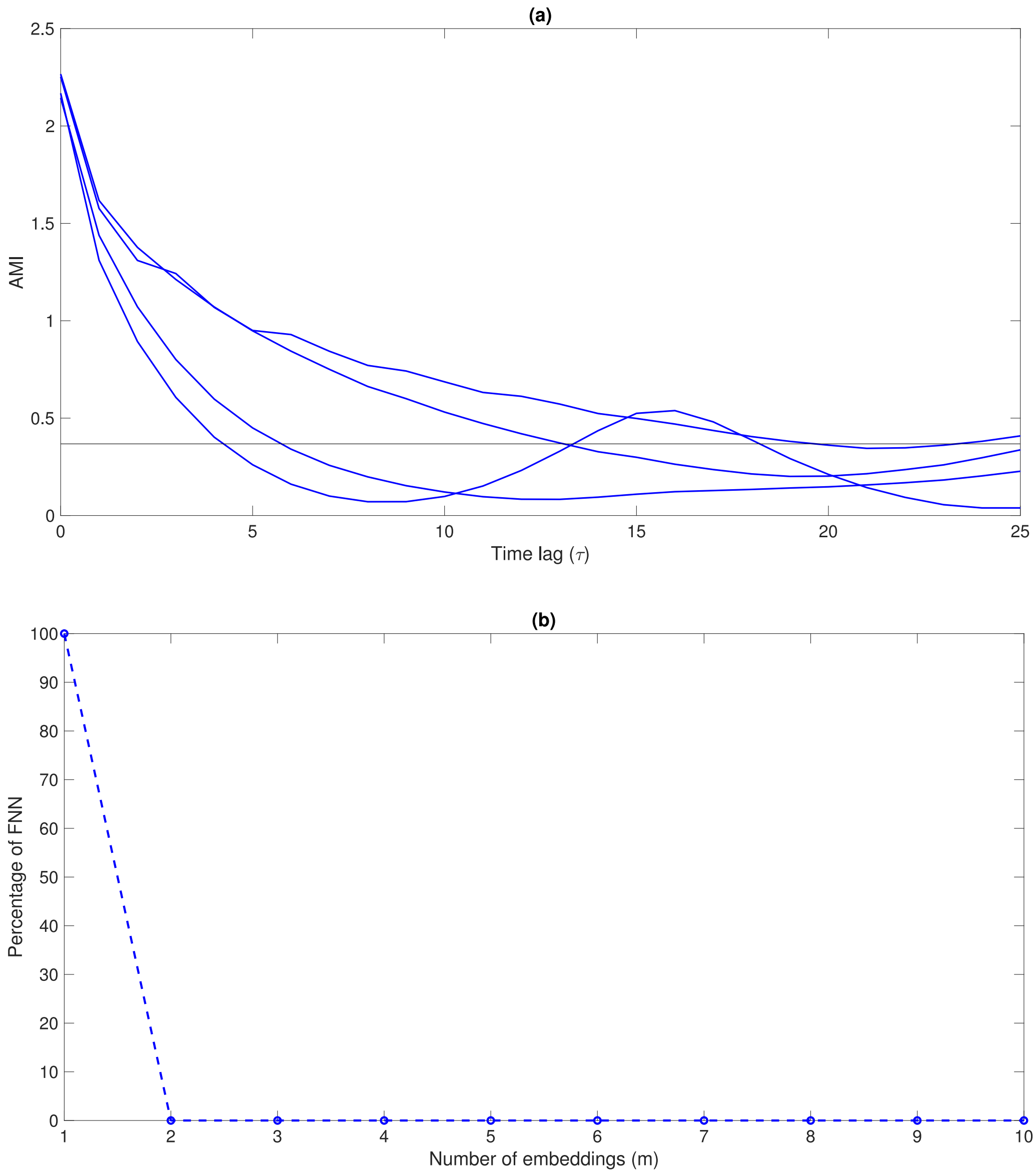

FIG. 4. The graphical output(s) of the average mutual information (AMI) function [subplot (a)] and percentage of false nearest neighborhoods (FNN) function [subplot (b)] are shown against the time lag τ and the embedding dimension m, respectively, for the four-dimensional time series taken from Eq. (14). In the upper panel, the default threshold value $(1/e)$ is shown by the horizontal line. The dashed line in the lower panel shows an immediate drop-off in the percentage of false nearest neighbors to zero, indicating that no additional embedding is necessary for the time series.

In what follows, we calculate the correlation dimension (d) and the Hausdroff fractal dimension (D) (for details, see, e.g., Ref. 31) of a time series of Eq. (14) with different values of the control parameter k and the embedding dimension m. The results are compared with those of the Hénon map and the Lorenz system. A summary of the results is presented in Table I. It is noted that even with an increasing value of the embedding dimension and a change in the value of parameter k, the correlation dimension d converges to a constant value. The bounds for the Hausdroff dimension of the chaotic time series are also calculated. It is seen that the correlation dimension lies within the bounds of the Hausdroff dimension.

C. Approximate entropy (ApEn)

Although a number of techniques are used to measure the complexity of a chaotic system, not all are applicable to limited,

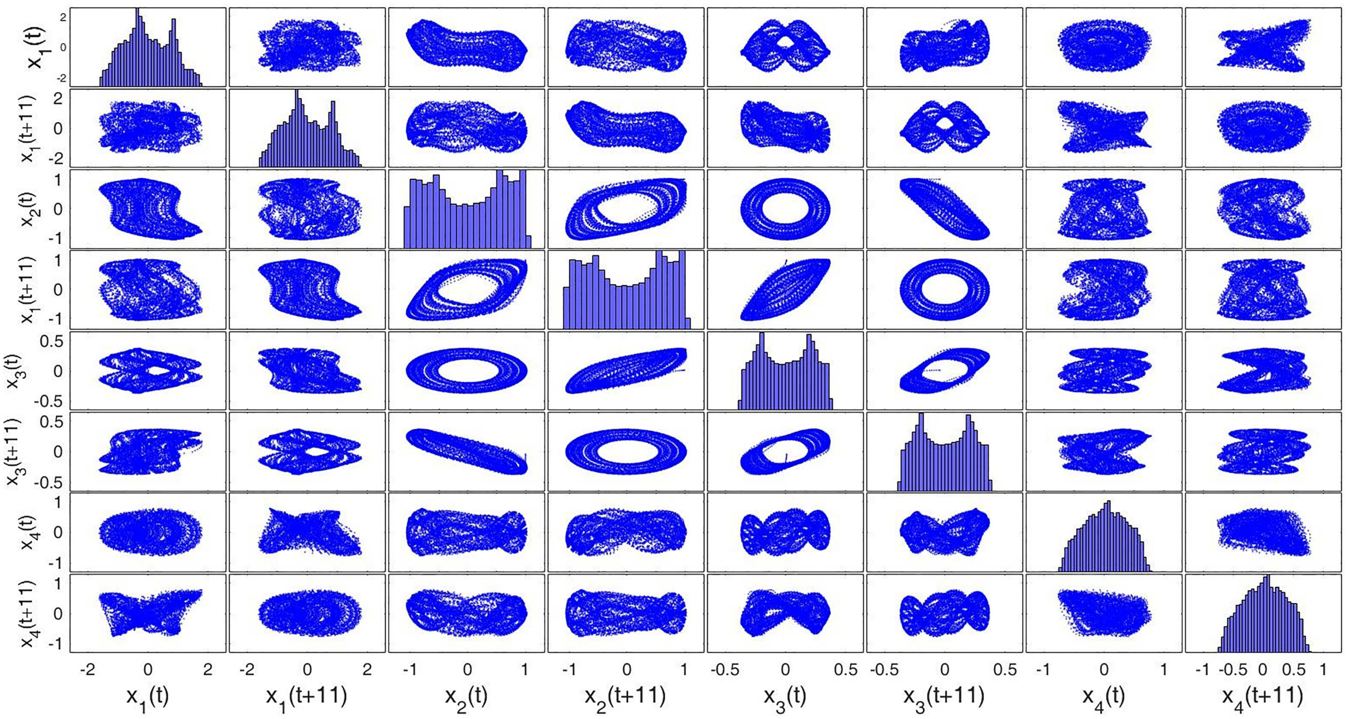

FIG. 5. Reconstruction of phase space with the time lag, $\tau = 11$ and embedding dimension, $m = 4$ from the chaotic time series of Eq. (14).

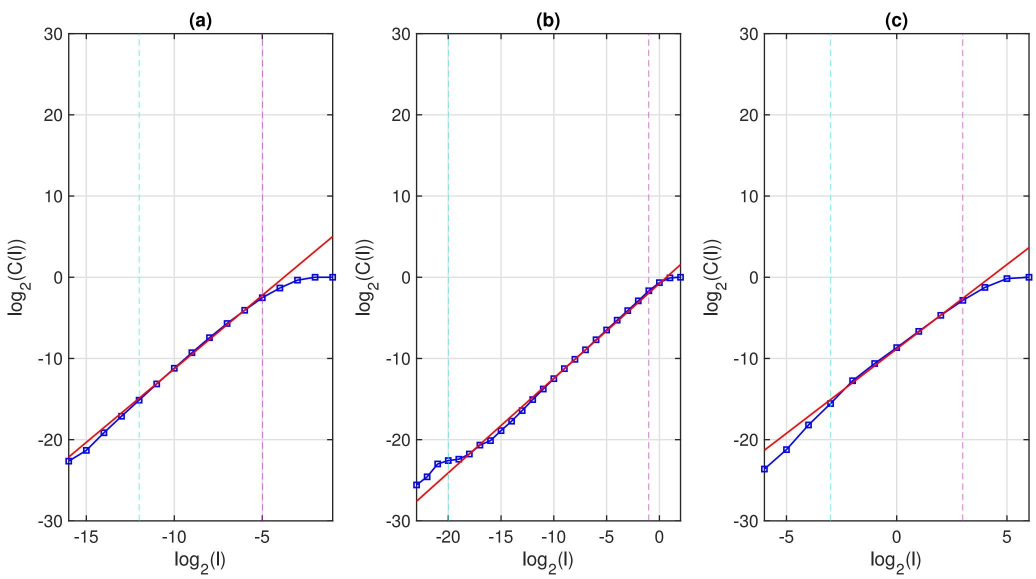

FIG. 6. Plots of the log-correlation integral [log $C(l)$] vs the log-correlation dimension (log l) are shown for (a) the present model [Eq. (14)], (b) for the Hénon map, and (c) for the Lorenz system. The correlation dimensions obtained for the subplots (a)–(c) are $d = 1.0776$, 1.25, and 2.06, respectively.

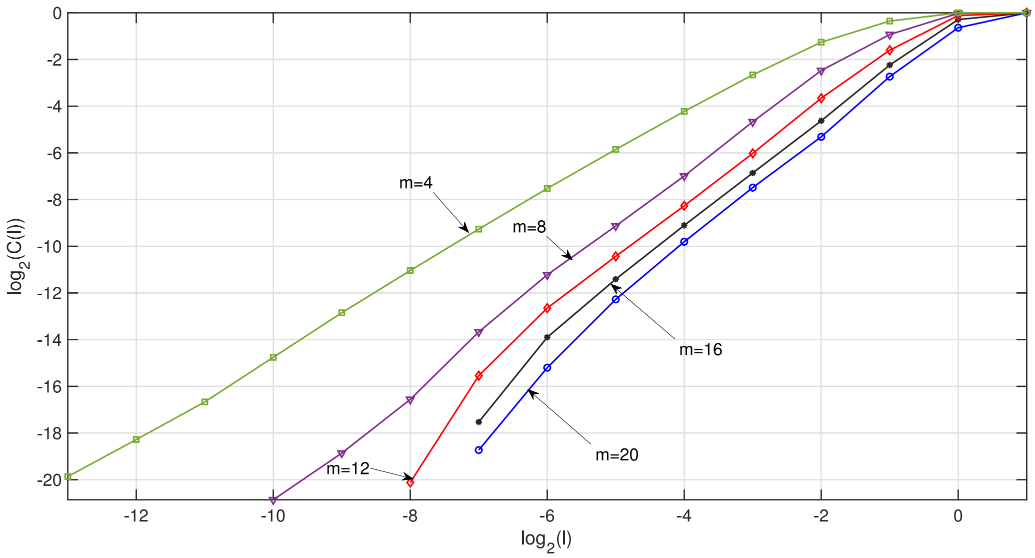

FIG. 7. Plot of the log-correlation integral [log $C(l)$] vs the log-correlation dimension (log l) with different embedding dimensions: $m = 4,\ 8,\ 12,\ 16,\ 20$, and 10 000 observations is shown for Eq. (14). The parameter values are the same as in Fig. 1, i.e., $\beta = 8$, $\alpha = 0.15$, and $n_0 = 10$ together with $k = 0.35$.

noisy, and stochastically derived time series. For example, the Kolmogorov–Sinai (KS) entropy works well for real dynamical systems but not for systems with noise.[29] Also, the finite correlation dimension value discussed before cannot guarantee that the system under consideration is deterministic. Furthermore, the Pincus technique fails for systems dealing with stochastic components. In this situation, the Approximate Entropy (ApEn) is more applicable to measure the system's complexity compared to others in which the

TABLE I. Estimations for the correlation dimension (d) and the Hausdroff fractal dimension (D) are shown for the present model, the Hénon map, and the Lorenz system with different values of m and the control parameter k and with a fixed 10 000 observations.

		Correlation dimension(d) for different values of k and m					
Model	Control parameter	d (m = 4)	d (m = 8)	d (m = 12)	d (m = 16)	d (m = 20)	Hausdorff fractal dimension (D)
Our dynamical model	k = 0.13	1.077 6	1.077 7	1.078 1	1.078 8	1.081 1	1.0646 ± 0.013 997
	k = 0.23	1.045 3	1.046 5	1.047 3	1.048 3	1.053 5	
	k = 0.28	1.071 8	1.072 4	1.073 3	1.074 0	1.074 7	
	k = 0.33	1.072 0	1.070 5	1.069 4	1.068 6	1.076 7	
	k = 0.38	1.072 1	1.070 6	1.069 6	1.068 4	1.067 7	
	k = 0.5	1.000 69	1.000 68	1.000 65	1.000 63	1.000 61	1.000 47 ± 0.000 682 62
Henon map	a = 1.4	1.254 1	1.485 0	1.644 5	2.088 1	3.184 5	1.260 303 ± 0.003
	a = 1.3	1.251 0	1.451 6	1.656 5	2.183 4	2.629 4	
	a = 1.25	1.250 6	1.461 5	1.675 6	2.122 5	2.625 2	
	a = 1.2	1.250 1	1.458 9	1.694 6	2.104 5	2.605 2	
	a = 1.15	1.240 8	1.456 8	1.672 5	2.076 8	2.646 6	
	a = 1.1	1.265 9	1.458 8	1.676 7	2.047 94	2.624 6	
Lorenz system	$\rho = 28$	2.086 6	2.776 3	3.012 1	3.124 6	3.362 3	2.06 ± 0.01
	$\rho = 26$	2.167 0	2.376 7	2.567 9	2.684 4	2.856 3	
	$\rho = 24$	1.828 8	2.022 3	1.951 6	2.123 5	2.487 4	
	$\rho = 22$	2.199 9	2.041 3	1.960 8	2.091 5	2.459 9	
	$\rho = 20$	0.010 47	0.077 9	0.047 8	0.014 8	0.004 8	2.0002 ± 0.000 480 23
	$\rho = 15$	0.007 9	0.007 6	0.007 5	0.007 2	0.007 1	

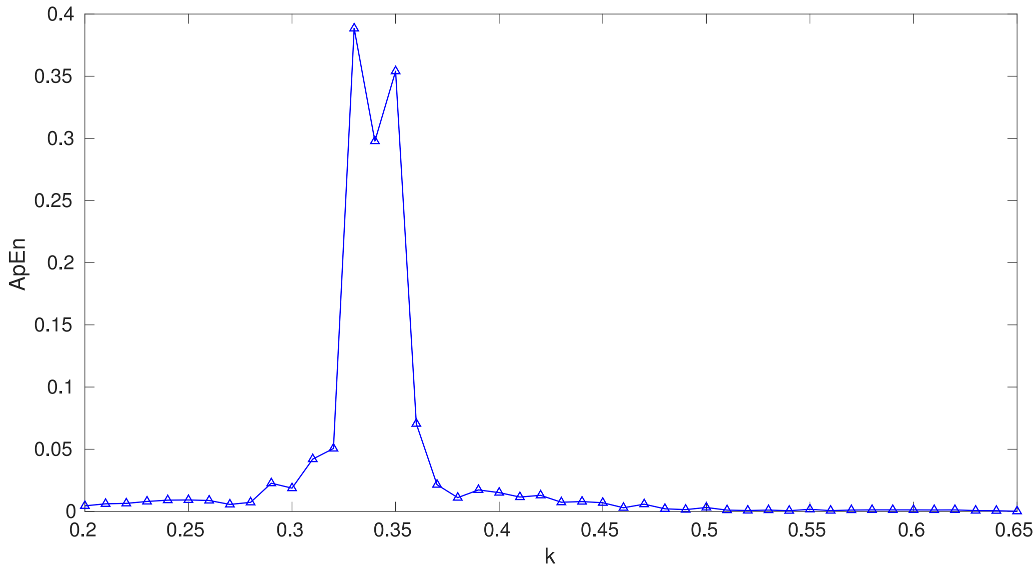

FIG. 8. Approximate entropy (ApEn) is shown against the parameter k for Eq. (14) with $m = 4$, $l = 0.1$, and $N = 5000$. The other parameter values are the same as in Fig. 1, i.e., $\beta = 8$, $\alpha = 0.15$, and $n_0 = 10$.

statistical precision is compromised.[29] The ApEn estimates uniformly sampled time-domain signals through phase-space reconstruction and then measures the amount of regularity and unpredictability of fluctuations in a time series. For an N given data points together with the embedding dimension m and the correlation integral $C(l)$, the ApEn is defined by

$$\mathrm{ApEn}(m, l, N) = \Phi^m(l) - \Phi^{m+1}(l), \tag{30}$$

where

$$\Phi^m(l) = \frac{\sum_{i=1}^{N-m+1} \log C_i^m(l)}{N - m + 1}. \tag{31}$$

We have calculated the ApEn against the controlling parameter k (the other parameter values are the same as Fig. 1, i.e., $\beta = 8$, $\alpha = 0.15$, and $n_0 = 10$) and for a given set of values, namely, $m = 4$ and $l = 0.1$ together with 5000 data points. The results are shown graphically in Fig. 8. It is noted that while the ApEn assumes high values in the subdomain $0.275 \lesssim k \lesssim 0.4$ in which the Lyapunov exponent is found to be positive (*cf.* Figure 2) its values are low in the rest of the domain where the Lyapunov exponents are close to zero. Thus, low values of ApEn predict that the system is steady, tedious, and predictive, while high values imply the independence between the data, a low number of repeated patterns, and randomness.

V. CONCLUSION

We have investigated the dynamical properties of dispersive Alvén waves coupled to plasma slow response of electron and ion density perturbations in a uniform magnetoplasma. By restricting the nonlinear wave–wave interactions to a few numbers of active wave modes, a low-dimensional autonomous system is constructed, which is shown to exhibit periodic, quasiperiodic, and chaotic states by means of the analyses of Lyapunov exponent spectra, bifurcation diagram, and phase-space portraits. The low-dimensional autonomous system can be a good approximation for the nonlinear interaction of Alfvén waves coupled to driven ion-sound waves associated with plasma slow response of density fluctuations in the stable or plane wave region $(3/4)k_c < k < k_c < 1$. In the latter, the modulational instability growth rate of Alvén wave envelopes is low. The model can be relatively accurate in the region $0.2 \lesssim k < (3/4)k_c$ (in which the condition for the subsonic region is relaxed and the instability growth rate is relatively high) where the low-dimensional model exhibits chaos for given values of the pump electric field E_0 as well as the parameters α, β, and n_0, associated with the relative speeds of the Alvén waves compared to the speed of light in vacuum and the ion-sound speed, and the conserved plasmon number, respectively. However, for values of $k < 0.2$, the low-dimensional model will no longer be valid for the description of wave–wave interactions as smaller values of k correspond to the excitation of a large number of unstable modes.

The complexity of chaotic phase-space structures of chaotic time series is also measured quantitatively by means of the correlation dimension and the approximate entropy through the reconstruction of phase spaces and estimation of embedding parameters, namely, the time lag and the embedding dimension. It is found that even with an increasing value of the embedding dimension and with a slightly different set of values of the parameters α, β, n_0, and k, the correlation dimension converges to a constant value. The bounds for the Hausdroff fractal dimension of the chaotic time series are also calculated to show that the correlation dimension lies in between the bounds. Furthermore, the results are shown to be a good qualitative agreement with those for the Hénon map and the Lorenz system.

To conclude, the existence of chaos and its complexity in the low-dimensional interaction model can be a good signature for the emergence of spatiotemporal chaos in the full system of Eqs. (1) and (2) where the participation of many more wave modes (more than three) in the nonlinear interactions can be possible. Such chaotic aspects of Alvén waves can be relevant for the onset of turbulence due to the flow of energy from lower to higher harmonic modes (i.e., with large to small spatial length scales) in the Earth's ionosphere and magnetosphere.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Subhrajit Roy: Formal analysis (equal); Investigation (equal); Writing – original draft (equal). **Animesh Roy:** Formal analysis (equal); Investigation (equal); Methodology (equal); Software (equal); Visualization (equal); Writing – original draft (equal). **Amar P. Misra:** Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Supervision (equal); Validation (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: DERIVATION OF THE DISPERSION RELATION [Eq. (6)]

Here, we give some relevant details for the derivation of the dispersion relation (6) for the modulated DAW envelope. We rewrite Eqs. (3) and (4) as

$$\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial z^2} = -\alpha^2 \frac{\partial^2}{\partial z^2}|E|^2, \tag{A1}$$

$$\frac{\partial E}{\partial t} + \beta\frac{\partial E}{\partial z} - \frac{\beta}{2}\frac{\partial}{\partial z}(nE) + i\gamma\frac{\partial^2 E}{\partial z^2} = 0. \tag{A2}$$

We assume the wave electric field envelope to be of the form $E = \tilde{E}e^{i\theta(z,t)}$ and the density perturbation as $n = \tilde{n}(z,t)$. Then, Eqs. (A1) and (A2) reduce to

$$\frac{\partial^2 \tilde{n}}{\partial t^2} - \frac{\partial^2 \tilde{n}}{\partial z^2} = -\alpha^2 \frac{\partial^2}{\partial z^2}|\tilde{E}|^2, \tag{A3}$$

$$\frac{\partial \tilde{E}}{\partial t} + i\tilde{E}\frac{\partial\theta}{\partial t} + \beta\left(\frac{\partial \tilde{E}}{\partial z} + i\tilde{E}\frac{\partial\theta}{\partial t}\right) - \frac{\beta}{2}\left[\frac{\partial}{\partial z}(\tilde{n}\tilde{E}) + i\tilde{E}\tilde{n}\frac{\partial\theta}{\partial z}\right] + i\gamma\left[\frac{\partial^2\tilde{E}}{\partial z^2} + 2i\frac{\partial\tilde{E}}{\partial z}\frac{\partial\theta}{\partial z} + i\tilde{E}\frac{\partial^2\theta}{\partial z^2} - \tilde{E}\left(\frac{\partial\theta}{\partial z}\right)^2\right] = 0. \tag{A4}$$

Separating the real and imaginary parts of Eq. (A4), we get

$$\frac{\partial \tilde{E}}{\partial t} + \beta\frac{\partial \tilde{E}}{\partial z} - \frac{\beta}{2}\frac{\partial}{\partial z}(\tilde{n}\tilde{E}) - \gamma\left(\tilde{E}\frac{\partial^2\theta}{\partial z^2} + 2\frac{\partial\tilde{E}}{\partial z}\frac{\partial\theta}{\partial z}\right) = 0, \tag{A5}$$

$$\tilde{E}\frac{\partial\theta}{\partial t} + \beta\tilde{E}\frac{\partial\theta}{\partial z} - \frac{\beta}{2}\tilde{E}\tilde{n}\frac{\partial\theta}{\partial z} + \gamma\left[\frac{\partial^2\tilde{E}}{\partial z^2} - \tilde{E}\left(\frac{\partial\theta}{\partial z}\right)^2\right] = 0. \tag{A6}$$

Looking for the modulation of the Alfvén wave envelope, we make the following ansatz:

$$\begin{aligned}\tilde{E}(z,t) &= E_0 + E_1\cos(kz-\omega t) + E_2\sin(kz-\omega t),\\ \tilde{n}(z,t) &= n_1\cos(kz-\omega t) + n_2\sin(kz-\omega t),\\ \theta(z,t) &= \theta_0 + \theta_1\cos(kz-\omega t) + \theta_2\sin(kz-\omega t),\end{aligned} \tag{A7}$$

where $E_0,\ E_1,\ E_2,\ n_1,\ n_2,\ \theta_0,\ \theta_1$, and θ_2 are real constants.

Substituting Eq. (A7) into Eqs. (A3), (A5), and (A6) and linearizing (retaining only the first harmonic terms), we get

$$\left[(\omega^2-k^2)n_1 + 2\alpha^2k^2E_0E_1\right]\cos(kz-\omega t) + \left[(\omega^2-k^2)n_2 + 2\alpha^2k^2E_0E_2\right]\sin(kz-\omega t) = 0, \tag{A8}$$

$$\left[(\omega-\beta k)E_2 + \frac{\beta}{2}E_0kn_2 - \gamma E_0k^2\theta_1\right]\cos(kz-\omega t) - \left[(\omega-\beta k)E_1 + \frac{\beta}{2}E_0kn_1 + \gamma E_0k^2\theta_2\right]\sin(kz-\omega t) = 0, \tag{A9}$$

$$[(\omega-\beta k)E_0\theta_2 + \gamma k^2E_1]\cos(kz-\omega t) - [(\omega-\beta k)E_0\theta_1 - \gamma k^2E_2]\sin(kz-\omega t) = 0. \tag{A10}$$

Equating the coefficients of different harmonics proportional to $\cos(kz-\omega t)$ and $\sin(kz-\omega t)$ to zero, we successively obtain

$$(\omega^2-k^2)n_1 + 2\alpha^2k^2E_0E_1 = 0, \tag{A11}$$

$$(\omega^2-k^2)n_2 + 2\alpha^2k^2E_0E_2 = 0, \tag{A12}$$

$$(\omega-\beta k)E_2 + \frac{\beta}{2}E_0kn_2 - \gamma E_0k^2\theta_1 = 0, \tag{A13}$$

$$(\omega-\beta k)E_1 + \frac{\beta}{2}E_0kn_1 + \gamma E_0k^2\theta_2 = 0, \tag{A14}$$

$$(\omega-\beta k)E_0\theta_2 + \gamma k^2E_1 = 0, \tag{A15}$$

$$(\omega-\beta k)E_0\theta_1 - \gamma k^2E_2 = 0. \tag{A16}$$

Next, eliminating θ_1 and θ_2 from Eqs. (A13)–(A16), we get

$$[(\omega-\beta k)^2 - \gamma^2k^4]E_1 + \frac{\beta}{2}(\omega-\beta k)E_0kn_1 = 0, \tag{A17}$$

$$[(\omega-\beta k)^2-\gamma^2k^4]E_2+\frac{\beta}{2}(\omega-\beta k)E_0kn_2=0. \quad \text{(A18)}$$

Furthermore, eliminating either n_1 from Eqs. (A11) and (A17) or eliminating n_2 from Eqs. (A12) and (A18), and noting that E_1, $E_2 \neq 0$, we obtain

$$(\omega^2-k^2)[(\omega-\beta k)^2-\gamma^2k^4]-\alpha^2\beta k^3|E_0|^2(\omega-\beta k)=0. \quad \text{(A19)}$$

From Eq. (A19), it is noted that while the first term represents a coupling between the Alfvén wave and the ion-acoustic density perturbation, the second term proportional to $|E_0|^2$ appears due to the Alfvén wave driven ponderomotive force. In absence of the latter, we have the usual acoustic mode $\omega = k$ and the following Alfvén wave dispersion equation:

$$\omega-\beta k=-\gamma k^2, \quad \text{(A20)}$$

where the negative sign (on the right-hand side) is considered in order to satisfy Eq. (A2) for the wave eigenmode. So, treating the term proportional to $|E_0|^2$ as the correction term in Eq. (A19) and replacing $(\omega-\beta k)$ by $-\gamma k^2$ therein, we obtain the following dispersion law for the modulated Alfvén wave envelope:

$$(\omega^2-k^2)[(\omega-\beta k)^2-\gamma^2k^4]+\alpha^2\beta\gamma k^5|E_0|^2=0. \quad \text{(A21)}$$

Next, to obtain the growth rate of instability, we assume $\omega \approx \beta k + i\Gamma$ with $\beta k \gg \Gamma$, γk^2, Thus, Eq. (A21) gives

$$[(\beta^2-1)k^2-\Gamma^2+2i\beta k\Gamma](\Gamma^2+\gamma^2k^4)-\alpha^2\beta\gamma k^5|E_0|^2=0. \quad \text{(A22)}$$

Since the term proportional to i in Eq. (A22) does not give any admissible result, we equate the real part to zero. Thus, we obtain

$$[(\beta^2-1)k^2-\Gamma^2](\Gamma^2+\gamma^2k^4)-\alpha^2\beta\gamma k^5|E_0|^2=0. \quad \text{(A23)}$$

Using $\beta k \gg \gamma k^2$ and neglecting the terms containing higher orders (than the second order) of Γ, we obtain from Eq. (A23) the following expression for the growth rate of instability:

$$\Gamma^2=\gamma k^3\left(\frac{\beta\alpha^2|E_0|^2}{\beta^2-1}-\gamma k\right). \quad \text{(A24)}$$

APPENDIX B: DERIVATION OF THE LOW-DIMENSIONAL MODEL [Eqs. (11)–(13)]

We recast Eqs. (3) and (4) as

$$\frac{\partial^2 n}{\partial t^2}-\frac{\partial^2 n}{\partial z^2}=-\alpha^2\frac{\partial^2}{\partial z^2}|E|^2, \quad \text{(B1)}$$

$$\frac{\partial E}{\partial t}+\beta\frac{\partial E}{\partial z}-\frac{\beta}{2}\frac{\partial}{\partial z}(nE)+i\gamma\frac{\partial^2 E}{\partial z^2}=0. \quad \text{(B2)}$$

Next, we consider an one-dimensional spectrum for each of the wave electric field E and the plasma density perturbation n, which describe the general solution of Eqs. (B1) and (B2) as a superposition of a set of normal modes, i.e.,

$$E(z,t)=E_0(t)+E_{-1}(t).e^{-ikz}+E_1(t)\cdot e^{ikz}, \quad \text{(B3)}$$

$$n(z,t)=n_0(t)+n_1(t).e^{ikz}+n_1^*(t)\cdot e^{-ikz}, \quad \text{(B4)}$$

where $E_{-1}(0)=E_1(0)$.

Substituting these expressions for E and n into (B2) and following the same approach as in Refs. 18 and 22, we obtain

$$i\dot{E}_0=0, \quad \text{(B5)}$$

$$\dot{E}_1+ik\beta E_1-ik^2\gamma E_1=i\frac{k\beta}{2}(n_0E_1+n_1E_0), \quad \text{(B6)}$$

$$\dot{E}_{-1}-ik\beta E_{-1}-ik^2\gamma E_{-1}=-i\frac{k\beta}{2}(n_0E_{-1}+n_1^*E_0), \quad \text{(B7)}$$

where the dot denotes the differentiation with respect to t and the asterisk denotes the complex conjugate. Multiplying Eq. (B5) by E_0^*, we obtain

$$i|\dot{E}_0|^2=0. \quad \text{(B8)}$$

Also, multiplying Eqs. (B6) and (B6) successively by E_1^* and E_{-1}^* and subtracting the complex conjugate of the resulting equations from themselves, we get

$$i|\dot{E}_{-1}|^2=\frac{k\beta}{2}(n_1^*E_0E_{-1}^*-n_1E_0E_{-1}), \quad \text{(B9)}$$

$$i|\dot{E}_1|^2=\frac{k\beta}{2}(n_1^*E_0E_1-n_1E_0E_1^*), \quad \text{(B10)}$$

where $|\dot{E}|^2=\frac{d}{dt}|E|^2$. Equations (B8)–(B10) can be added to yield

$$|E_{-1}|^2+|E_1|^2+|E_0|^2=N. \quad \text{(B11)}$$

Next, we assume $n_1=n_1^*$, $n_0=N$, the plasmon number, and introduce the new variables ρ_0, ρ_1, θ_0, and θ_1 according to $E_0=\rho_0e^{i\theta_0}$, $E_{-1}=E_0=\rho_1e^{i\theta_1}$, $\rho_0=\sqrt{n_0}\sin w$, $\rho_1=\sqrt{n_0}\cos w$, $\psi=2w$.

Substituting expressions (B3) and (B4) into Eq. (B1) and using the new variables as defined above, we get

$$\ddot{n}_1-k^2n_1=\alpha^2k^2n_0\sin\psi\cos\phi. \quad \text{(B12)}$$

Also, using the newly defined variables, from Eqs. (B6) and (B7), we obtain

$$\dot{\psi}=\beta kn_1\sin\phi, \quad \text{(B13)}$$

$$\dot{\phi}=k(\beta-\gamma k)-\frac{1}{2}\beta kn_1\tan\frac{\psi}{2}\cos\phi, \quad \text{(B14)}$$

where $\phi=\theta_0-\theta_1$. Equations (B12)–(B14) constitute the required low-dimensional model.

APPENDIX C: EQUILIBRIUM POINTS OF EQ. (14)

To find the equilibrium points of Eq. (14), we equate the right-hand side expression of each of Eq. (14) to zero. Thus, we successively obtain

$$\begin{aligned}\beta_0x_2\sin x_4&=0,\\ x_3&=0,\\ -k^2x_2+\alpha_0^2\sin x_1\cos x_4&=0,\\ \gamma_0-\frac{1}{2}\beta_0x_2\tan\frac{x_1}{2}\cos x_4&=0,\end{aligned} \quad \text{(C1)}$$

where $\beta_0 = \beta k$, $\alpha_0 = \alpha k\sqrt{n_0}$, and $\gamma_0 = k(\beta - \gamma k)$. Since $(0, 0, 0, 0)$ is not an equilibrium point (as explained in Sec. III A), we have $x_2 \neq 0$. So, the first equation of Eq. (C1) gives $\sin x_4 = 0$, i.e., $x_4 = n\pi$. Using this value of x_4 in the third equation of Eq. (C1), one obtains $x_2 = (-1)^n(\alpha_0^2/k^2)\sin x_1$, since $\cos(n\pi) = (-1)^n$, n being zero or an integer. Having obtained the values of x_2 and x_4, and using those in the fourth equation of Eq. (C1), we get $x_1 = 4n\pi \pm 2\sin^{-1}\left(\sqrt{\gamma_0 k^2/\alpha_0^2\beta_0}\right)$. Thus, the equilibrium points of Eq. (14) can be obtained as $(x_{10}, x_{20}, 0, n\pi)$, where $x_{10} = 4n\pi \pm 2\sin^{-1}\left(\sqrt{\gamma_0 k^2/\alpha_0^2\beta_0}\right)$ and $x_{20} = \pm(-1)^n\left(2k/\beta_0\right)\sqrt{\left(\beta - \gamma k\right)\left(\alpha^2 n_0\beta - \beta + \gamma k\right)}$, where n is zero or an integer.

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Dynamical properties of acoustic-gravity waves in the atmosphere

Animesh Roy, Subhrajit Roy, A.P. Misra*

Department of Mathematics, Siksha Bhavana, Visva-Bharati University, Santiniketan, 731 235, India

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ABSTRACT

We study the dynamical behaviors of a system of five coupled nonlinear equations that describes the dynamics of acoustic-gravity waves in the atmosphere. A linear stability analysis together with the analysis of Lyapunov exponents spectra are performed to show that the system can develop from ordered structures to chaotic states. Numerical simulation of the system of equations reveals that an interplay between the order and chaos indeed exists depending on whether the control parameter , associated with the density scale height of acoustic-gravity waves, is below or above its critical value.

1. Introduction

The nonlinear dynamics of low-frequency finite amplitude acoustic-gravity waves has been studied by a number of authors because of their relevance in atmospheric disturbances (Stenflo, 1987, 1991, 1996; Stenflo and Stepanyants, 1995; Jovanovic et al., 2002; Mendonca and Stenflo, 2015; Kaladze et al., 2008). The latter appear due to various meteorological conditions including different pressure and density gradients, as well as the presence of shear flows (Jovanovic et al., 2002). It has been shown that the nonlinear acoustic-gravity waves can appear in the forms of localized solitary vortices (Stenflo, 1987; Jovanovic et al., 2002), ordered structures (Park et al., 2016), as well as chaos (Banerjee et al., 2001) and turbulence (Shaikh et al., 2008).

In a paper (Stenflo, 1996), Stenflo deduced a system of five coupled equations that describes the essential features of low-frequency atmospheric disturbances. His starting point was the most commonly used model equations for two-dimensional acoustic-gravity waves of the form (Stenflo and Stepanyants, 1995)

$$D_t\left(\nabla^2\psi - \frac{1}{4H^2}\psi\right) = -\partial_x\chi, \quad (1)$$

$$D_t\chi = \omega_g^2\partial_x\psi, \quad (2)$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial z^2$, $D_t \equiv \partial_t + \mathbf{v}\cdot\nabla$, H is the density scale height, ω_g is the Brunt-Väisälä frequency, $\psi(x, z)$ is the velocity potential in which z represents the vertical direction, and $\chi(x, z)$ is the normalized density perturbation.

Substitution of the expression for the velocity, i.e., $\mathbf{v} = -\partial_z\psi\hat{x} + \partial_x\psi\hat{z}$ into Eqs. (1) and (2) results in

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{1}{4H^2}\frac{\partial\psi}{\partial t} = -J(\psi, \nabla^2\psi) - \frac{\partial\chi}{\partial x},$$
$$\frac{\partial\chi}{\partial t} = -J(\psi, \chi) + \omega_g^2\frac{\partial\psi}{\partial x}, \quad (3)$$

where $J(f, g) = (\partial f/\partial x)(\partial g/\partial z) - (\partial g/\partial x)(\partial f/\partial z)$ is the Jacobian. For a class of solutions of Eq. (3) of the form

$$\psi = [a(t)\sin(k_0x) + b(t)\cos(k_0x) + \omega_0]z/k_0,$$
$$\chi = [\alpha(t)\sin(k_0x) + \beta(t)\cos(k_0x) + \gamma(t)]z, \quad (4)$$

where k_0 and ω_0 are constants, Stenflo (1996) derived the following set of coupled equations for acoustic-gravity waves, given by,

$$\partial_t a + \widetilde{\omega}_0 b + s_1\beta = -\nu_1 a,$$
$$\partial_t b - \widetilde{\omega}_0 a - s_1\alpha = -\nu_1 b,$$
$$\partial_t\alpha + \omega_0\beta - s_2 b\gamma + \omega_g^2 b = -\nu_2\alpha,$$
$$\partial_t\beta - \omega_0\alpha + s_2 a\gamma - \omega_g^2 a = -\nu_2\beta,$$
$$\partial_t\gamma + a\beta - \alpha b = -\nu_3\gamma. \quad (5)$$

Here, the terms containing ν_1 and ν_2 appear when one considers, in addition with the other effects, the dissipative terms proportional to $\nabla^4\psi$ and $\nabla^2\psi$ respectively in Eqs. (1) and (2), and the term proportional to ν_3 corresponds to the damping term. Also, as in Ref. (Stenflo, 1996), ω_0 and ω_g are two control parameters with $\tilde{\omega}_0 = \omega_0/(1 + 1/4Hk_0^2) = s_1\omega_0$, $s_1 = (1 + 1/4H^2k_0^2)^{-1}$ and $s_2 = 1$.

In this paper, we numerically study the dynamical behaviors of Eq. (5) in absence of the dissipative and damping effects. By means of the linear stability analysis and the Lyapunov exponent spectra, it is seen that the nonlinear interaction of acoustic-gravity waves can result into an ordered structure or chaos depending on whether the parameter s_1, associated with the density scale height H, is below or above its critical

* Corresponding author.
E-mail addresses: aroyiitd@gmail.com (A. Roy), suvo.math88@gmail.com (S. Roy), apmisra@visva-bharati.ac.in, apmisra@gmail.com (A.P. Misra).

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value.

2. Dynamical properties

In this section, we numerically study the dynamical properties of Eq. (5). We focus mainly on the development of chaos as well as the tendency to form ordered structures in absence of the dissipative effects (i.e., terms proportional to ν_1, ν_2 and ν_3). Thus, setting $\nu_1 = \nu_2 = \nu_3 = 0$ and for convenience, redefining the variables, namely, $a = u$, $b = v$, $\alpha = x$, $\beta = y$ and $\gamma = z$, Eq. (5) can be recast as

$$\begin{aligned}
\frac{du}{dt} &= -\tilde{\omega}_0 v - s_1 y,\\
\frac{dv}{dt} &= \tilde{\omega}_0 u + s_1 x,\\
\frac{dx}{dt} &= -\omega_0 y + vz - \omega_g^2 v,\\
\frac{dy}{dt} &= \omega_0 x - uz + \omega_g^2 u,\\
\frac{dz}{dt} &= -uy + xv.
\end{aligned} \tag{6}$$

2.1. Linear stability analysis

In order to perform the stability analysis of system (6), we first find its fixed points $(u_0, v_0, x_0, y_0, z_0)$. These can be obtained by equating the right-hand sides of Eq. (6) to zero and finding solutions for u, v, x, y and z. Thus, the fixed points so obtained are the origin $O = (0,0,0,0,0)$ and $P = (0,0,0,0, \omega_0^2 + \omega_g^2)$. Next, around the fixed points, we apply the perturbations of the forms: $u' = u - u_0$, $v' = v - v_0$, $x' = x - x_0$, $y' = y - y_0$ and $z' = z - z_0$ to obtain a linearized system of perturbation equations: $d\mathbf{X}/dt = J\mathbf{X}$, where $\mathbf{X} = (u', v', x', y', z')$ and J is the Jacobian matrix. For each fixed point, the eigenvalues λ can be obtained from the corresponding eigenvalue problem $J\mathbf{X} = \lambda\mathbf{X}$. The stability of system (6) about the fixed points can then be studied by the nature of these eigenvalues.

The Jacobian matrix corresponding to the fixed point O is given by

$$J_O = \begin{bmatrix} 0 & -\tilde{\omega}_0 & 0 & -s_1 & 0\\ \tilde{\omega}_0 & 0 & s_1 & 0 & 0\\ 0 & -\omega_g^2 & 0 & -\omega_0 & 0\\ \omega_g^2 & 0 & \omega_0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

and the corresponding eigenvalues of the matrix J_O are given by $\lambda = 0$ and

$$\lambda = \pm\frac{1}{\sqrt{2}}\left(-B \pm \sqrt{B^2 - 4C}\right)^{1/2}, \tag{8}$$

where $B = \omega_0^2 + \tilde{\omega}_0^2 + 2s_1\omega_g^2$ and $C = (s_1\omega_g^2 - \omega_0\tilde{\omega}_0)^2$. We note that since $C > 0$ and $B > 0$ for $0 < s_1 < 1$, the values of λ in Eq. (8) are purely imaginary, i.e., $\Re\lambda = 0$, implying that the fixed point O corresponds to a stable center.

Next, for the stability of system (6) around the fixed point P, we apply the similar perturbations as discussed before, i.e., $u' = u$, $v' = v$, $x' = x$, $y' = y$ but with $z' = z - (\omega_0^2 + \omega_g^2)$. The corresponding Jacobian matrix J_P and the corresponding eigenvalues λ are, respectively, given by

$$J_P = \begin{bmatrix} 0 & -\tilde{\omega}_0 & 0 & -s_1 & 0\\ \tilde{\omega}_0 & 0 & s_1 & 0 & 0\\ 0 & \omega_0^2 & 0 & -\omega_0 & 0\\ -\omega_0^2 & 0 & \omega_0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

$\lambda = 0$ and

$$\lambda = \pm\frac{\omega_0}{\sqrt{2}}\left[-(1 - s_1)^2 \pm (1 + s_1)\sqrt{(1 - s_1)^2 - 4s_1}\right]^{1/2}, \tag{10}$$

in which we have used the expression $\tilde{\omega}_0 = s_1\omega_0$. From Eq. (10), we note that the values of λ become purely imaginary for $0 < s_1 \lesssim 0.17$, and in this case, the fixed point P corresponds to a stable center. However, for values of s_1 in $0.17 < s_1 < 1$, λ has complex conjugate values with positive and negative real parts. Thus, it turns out that the system may be unstable (with at least one $\Re\lambda > 0$) around the fixed point P for $0.17 < s_1 < 1$. From the above analysis it follows that the parameter s_1, which typically depends on the density scale height H for acoustic-gravity waves, plays a crucial role for the stability and instability of system (6) about the fixed points O and P. In fact, as the density scale height H increases and so is s_1, the system's stability tends to break down, which can lead to the development of chaos as will be shown later.

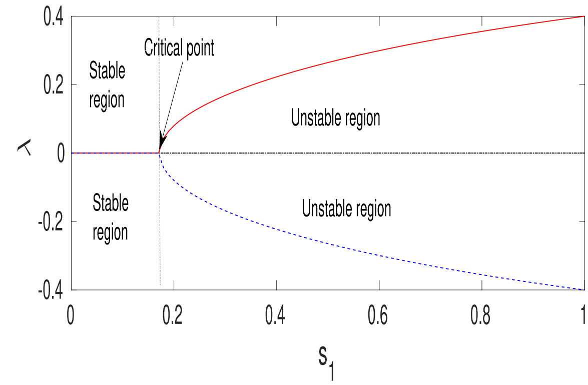

Fig. 1. Pitchfork bifurcation diagram showing the stable and unstable regions of system (6) around the fixed points O and P. While the system is stable in the region $0 < s_1 \lesssim 0.17$ where $\lambda = 0$ or $\Re\lambda = 0$, it exhibits instability in the domain $0.17 \lesssim s_1 < 1$ with $\Re\lambda > 0$. The upper (solid red line) and lower (dashed blue line) branches are corresponding to the $\pm$ sign in the square brackets of the expression for λ [Eq. (10)]. The other parameter values are $\omega_0 = 0.4$, $\omega_g = 1.01$, and $\tilde{\omega}_0 = s_1\omega_0$.

Fig. 1 shows the bifurcation diagram for stable and unstable regions corresponding to the fixed points O and P. We plot λ, given by Eqs. (8) and (10), with respect to the parameter s_1 $(0 < s_1 < 1)$. The dash-dotted line represents $\lambda = 0$ corresponding to the fixed point O. Also, for the fixed point P, $\Re\lambda = 0$ in the interval $0 < s_1 \lesssim 0.17$. So, the system is stable around both the fixed points in the domain $0 < s_1 \lesssim 0.17$. However, beyond this domain, i.e., in $0.17 < s_1 < 1$, the system is shown to be unstable around the fixed point P. In Fig. 1, the upper (solid line) and lower (dashed line) branches are the plots of λ corresponding to the $\pm$ sign in the square brackets in Eq. (10).

In the next subsection 2.2, we will calculate the Lyapunov exponents spectra to verify the existence of chaos with variations of the parameters s_1, ω_0 and ω_g.

2.2. Lyapunov exponents

In order to calculate the Lyapunov exponents, we solve the system of Eq. (6) with the initial condition $X(0) = (u(0), v(0), x(0), y(0), z(0))$. If system (6) is recast as $\dot{X} = (\dot{u}(t), \dot{v}(t), \dot{x}(t), \dot{y}(t), \dot{z}(t))$, its variational form of equation is given by

$$\frac{d}{dt}DX(t) = J_L(t)DX(t), \tag{11}$$

where $D \equiv d/dt$, $DX(0) = I_5$ with I_5 denoting the identity matrix of order 5 and $J_L(t)$ the Jacobian matrix evaluated at the initial value $X(0)$, given by,

$$J_L(t) = \begin{bmatrix} 0 & -\tilde{\omega}_0 & 0 & -s_1 & 0\\ \tilde{\omega}_0 & 0 & s_1 & 0 & 0\\ 0 & z(t) - \omega_g^2 & 0 & -\omega_0 & v(t)\\ \omega_0^2 - z(t) & 0 & \omega_0 & 0 & u(t)\\ -y(t) & x(t) & v(t) & -u(t) & 0 \end{bmatrix}. \tag{12}$$

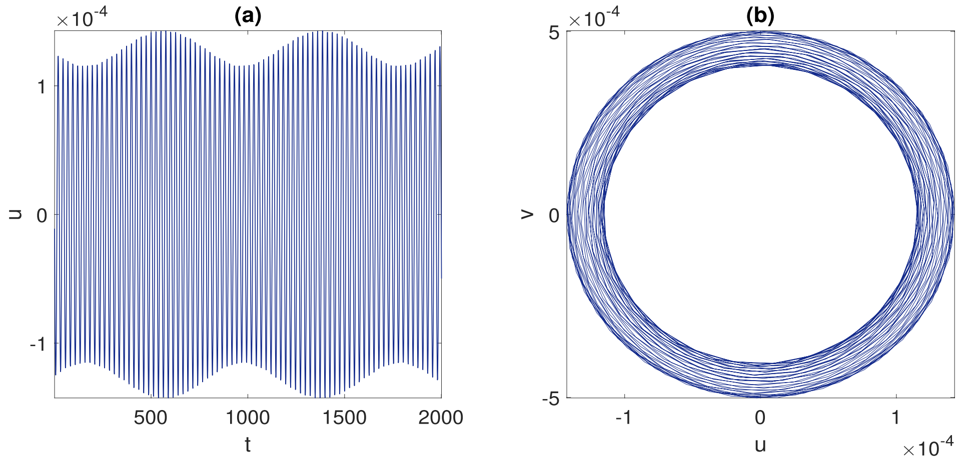

Fig. 2. Stable oscillations: (a) the time series and (b) the phase-space diagram showing that the equilibrium point O corresponds to the stable center. The parameter values are $\omega_0 = 0.01$, $s_1 = 0.61$, $\omega_g = 1.01$ and $\tilde{\omega}_0 = s_1.\ \omega_0 = 0$.

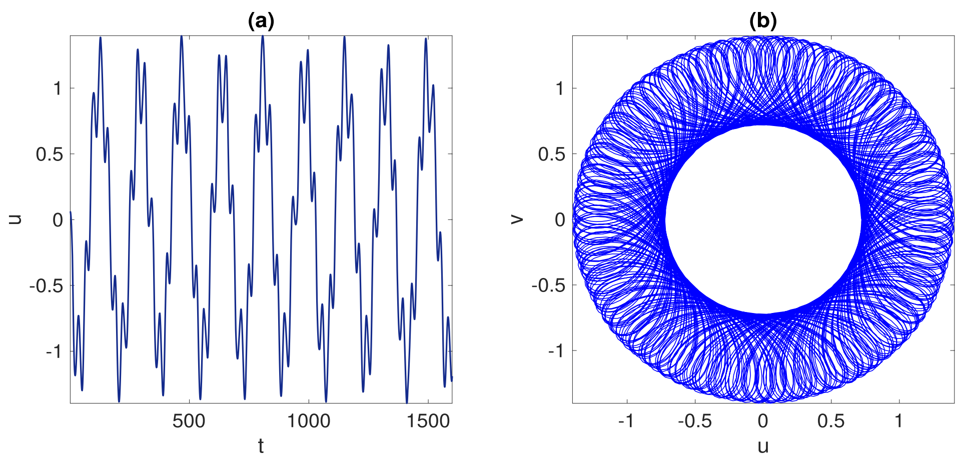

Fig. 3. Subplots (a) and (b) are, respectively, the time series and the phase space (torus) showing the quasi-periodicity of system (6) with parameter values $\omega_0 = 1.5$, $s_1 = 0.31$, $\omega_g = 1.01$ and $\tilde{\omega}_0 = s_1\omega_0 = 0.465$.

Since $DX(0)$ is non-singular and so is $DX(t)$, the solution of Eq. (11) is given by

$$\Lambda = \lim_{t\to\infty} \frac{1}{2t} \ln[(DX)(t)^T DX(t)] \tag{13}$$

from which the Lyapunov exponents are obtained as the eigenvalues λ_i, $i = 1, \dots, 5$ of the matrix Λ. Given a fixed initial condition $X(0)$ of the dynamical system, the change of particle's orbit can be found by the Liouville's formula: $\Delta(t) = \mathrm{tr}(J_L(t))\Delta_t$, where $\Delta_t \equiv \det DX(t)$, $\Delta_0 \equiv \det DX(0) = \det I_5 = 1 > 0$ and 'tr' denotes the trace of the matrix $J_L(t)$. Thus, for system (6) we have $\det DX(t) = \exp(\int_0^t \mathrm{tr}(J_L(t))dt = 1 > 0$, implying that at least one eigenvalue λ_i is positive, and so a chaotic orbit exists for a certain time period $[0\ t]$.

3. Numerical analysis

We study the dynamical behaviors of solutions of system (6). To this end, we numerically integrate Eq. (6) by using the 4-th order Runge-Kutta scheme with a time step $\Delta t = 10^{-3}$. The results are displayed in Figs. 2–4. We note that for certain ranges of values of the parameters ω_0, ω_g and s_1, the system can exhibit stable solutions together with the quasi-periodic and chaotic states. We study these behaviors in three different cases as follows.

Stable Center: We note that for $\omega_0 = 0$, and any values of ω_g and s_1 in $0 < s_1 < 1$, the eigenvalues corresponding to the fixed point O are zero and purely imaginary, while those about the fixed point P are all zero. In this case, the system exhibits stable solutions about the fixed points O and P. The system also possesses a class of stable solutions for $\omega_0 > 0$, $\omega_g > 0$ and $0 < s_1 \lesssim 0.17$ (cf. Sec. 2.1 and the bifurcation diagram in Fig. 1). The corresponding time series (a) and the phase space plots (b) are shown in Fig. 2.

Quasi-periodicity: From the linear stability analysis and the bifurcation diagram (See Fig. 1) it is evident that the system tends to loose its stability for $s_1 > 0.17$ and any positive values of the frequencies ω_0 and ω_g. In fact, there are two subregions of the parameter s_1: $0.17 < s_1 \lesssim s_2$ and $s_2 \lesssim s_1 < 1$. In the former, the system exhibits quasi-periodicity while in the latter it has chaotic behaviors. However, it is very difficult to find a particular region of s_1 in which the quasi-periodicity transits into the chaotic states. Usually, in the quasi-periodic region, we observe a stable torus whereas in the chaotic region, the torus structure breaks down, giving rise to a chaotic structure. For a suitable choice of the initial condition $(-\omega_0 k_1, -\omega_0 k, k, k_1, -\omega_0^2 - \omega_g^2)$, where $k = 0.9$ and $k_1 = 0.8$ together with the parameters $\omega_0 = 1.5$, $\omega_g = 1.01$ and $s_1 = 0.31$ with $\tilde{\omega}_0 = s_1\omega_0 = 0.465$, Fig. 3 shows that the torus structure forms at $s_1 = 0.31$.

Chaotic property: We note that of the two fixed points O and P, the point O always gives a stable center in every possible regions of the parameters and the initial conditions. However, for the other fixed point P, we have a stable center in the region of $0 < s_1 \lesssim 0.17$, while in the other region $0.17 < s_1 < 1$, the system exhibits either quasi-periodicity or chaos. For a suitable choice of the initial condition and the parameters, namely, $(-\omega_0 k_1, -\omega_0 k, k, k_1, -\omega_0^2 - \omega_g^2)$ with $k = 5$,

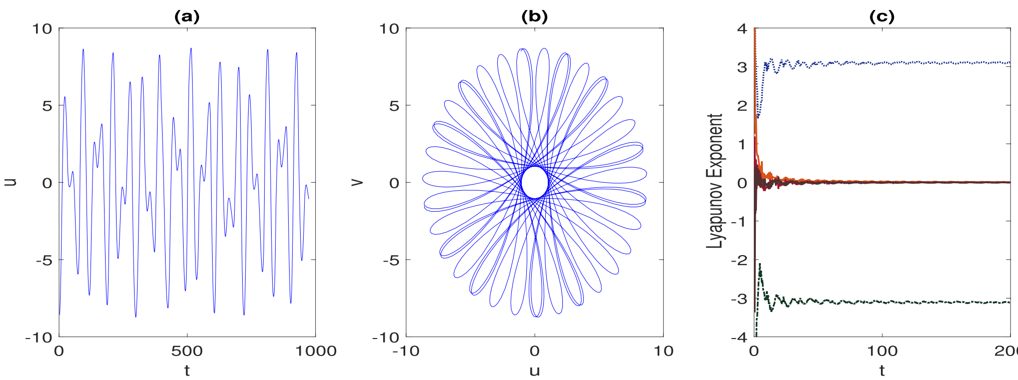

Fig. 4. The chaotic time series (a), the chaotic phase space (b) and the Lyapunov exponents (c) are shown with parameters $\omega_0 = 1.5$, $s_1 = 0.91$, $\omega_g = 1.01$, and $\tilde{\omega}_0 = s_1\omega_0 = 1.365$.

$k_1 = 5.8$, $\omega_0 = 1.5$, $\omega_g = 1.01$, $s_1 = 0.91$, and $\tilde{\omega}_0 = s_1\omega_0 = 1.365$, we show that system (6), indeed, exhibits chaos, i.e., the torus which forms at $s_1 = 0.31$ (see Fig. 3) breaks down at a higher value of $s_1 = 0.91$ (Fig. 4). The corresponding time series (a), the phase space (b) and the Lyapunov exponents (c) are shown in Fig. 4. Here, the appearance of at least one positive Lyapunov exponent ensures the existence of chaos.

4. Conclusion

We have investigated the dynamical properties of five nonlinear coupled Stenflo equations (Stenflo, 1996) that describe the evolution of acoustic-gravity waves in atmospheric disturbances. A linear stability analysis together with the analysis of Lyapunov exponents spectra are carried out for different values of the control parameters. It is found that the parameter s_1, which typically depends on the density scale height of acoustic-gravity waves, plays a vital role for the existence of ordered structures as well as chaos of the Stenflo equations. While the system exhibits stable solutions in the region $0 < s_1 \lesssim 0.17$, it can describe chaotic behaviors in the other region $0.17 < s_1 < 1$. The present results should be useful for understanding the chaotic properties of the atmospheres of the Earth and other planets.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.jastp.2019.02.009.

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Assessing the triage and efficacy of strategies of SAARC to improve regional integrity of South Asia using multicriteria group decision making under *q*-rung orthopair hesitant fuzzy environment

Souvik Gayen [a], Debamitra Banerjee [b], Arun Sarkar [c,*], Animesh Biswas [a]

[a] Department of Mathematics, University of Kalyani, Kalyani, 741235, India
[b] Department of Economics, Sarsuna College, Kolkata, 700061, India
[c] Department of Mathematics, Heramba Chandra College, Kolkata, 700029, India

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A B S T R A C T

The South Asian Association for Regional Cooperation (SAARC) plays a crucial role in fostering regional integration and cooperation among the countries of South Asia. However, the organization faces various challenges in achieving its objectives due to the complex and dynamic nature of the region. This study aims to assess the triage and efficacy of strategies employed by SAARC in pursuit of regional integrity in South Asia, using a Multi-Criteria Group Decision Making (MCGDM) technique. The assessment process involves several uncertainties which are resolved using *q*-rung orthopair hesitant fuzzy (*q*-ROHF) set. A distance measure is developed for *q*-ROHF sets and is applied to find the best strategy among six well known strategies, viz., SAARC Agreement on Trade in Services (SATIS), SAARC Development Fund (SDF), Establishment of South Asian University (SAU), SAARC Arbitration Council (SARCO), South Asian Preferential Trade Agreement (SAPTA) and South Asian Free Trade Area (SAFTA) through SWARA-TOPSIS based MCGDM technique. The MCGDM method is updated by developing several aggregation operators, viz., Archimedean *q*-ROHF weighted average, *q*-ROHF Einstein weighted average, *q*-ROHF Hamacher weighted average, *q*-ROHF Frank weighted average along with their geometric forms to combine decision makers' individual decision. The result shows that the most effective strategy for the economic integration of SAARC is SDF; SATIS comes next as best strategy for economic integration of SAARC. The study reveals that SAU has least impact on the regional economic integration of SAARC. The achieved results reveals the age old proverb that "play on the stomach and sit on the back" – the members of SAARC who are by nature economically poor can afford to take initiative for a successful effective regionalization if it is planned to bring socio economic development of themselves.

1. Introduction

The eight-membered South Asian Association for Regional Cooperation (SAARC) was the brainchild of the late president of Bangladesh, Zia-Ur-Rehman, with India, Pakistan, Bangladesh, Sri Lanka, Maldives, Nepal, Bhutan and Afghanistan as its members, aiming at mutual autonomy and development with a peaceful, friendly and cooperative socio-economic environment around the region. Professor Bela Balassa [1] defined the five stages of economic integration and evinced that economic regionalization has a great impact on trade and development of the region. Some eminent researchers and economists [2–5] also claimed a positive relationship among economic integration, trade and development. According to Balassa [1], the final stage of economic integration for SAARC is SAEU (South Asian Economic Union) which is yet to be achieved after initiating so many strategies for it since the day of its initiation. Researchers from SAARC chambers of commerce and industry also showed that SAARC has achieved regional trade integration of one-third of its potential till date [6]. Lack of political will and poor infrastructure engendered some non-sensible and non-effective bilateral trade agreements in the region making the regionalization an impediment to growth in spite of facilitating it. This study wants to review the strategies taken by SAARC for strengthening regional integration along with the economic development of the individual members as well as of the region as a whole. Instead of applying statistical and

* Corresponding author.
E-mail addresses: souvikgayen99@gmail.com (S. Gayen), banerjee.debamitra731@gmail.com (D. Banerjee), asarkarmth@gmail.com (A. Sarkar), abiswaskln@rediffmail.com (A. Biswas).

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econometric techniques to judge the efficacy of the SAARC strategies, the study uses the Multicriteria Group Decision Making (MCGDM) method in an imprecise environment as the determination of the efficacy and urgency of the strategies seems very difficult due to the existence of interdependent overlapping factors involved in it and contains uncertain data.

It is evident that q-rung orthopair fuzzy set (q-ROFS) [7] possesses the greater capability of capturing uncertainties than other existing variants of fuzzy sets. The benefits of the use of q-ROFS is that it considers the membership values and also non-membership values in a large domain. It must be noted that the sum of q^{th} power of membership and non-membership is less than 1.

Sometimes, DMs feel hesitant to assign suitable decision value by putting a single membership degree corresponding to some alternatives. In order to deal with such situations, in 2010, Torra [8] introduced hesitant fuzzy set (HFS) as a generalization of fuzzy sets [9] which permits the membership function to consider a set of possible values in $[0,1]$. HFS are more effective than other traditional fuzzy set extensions for reflecting human perceptions and hesitancies with pessimistic and optimistic attitudes.

Combining the concepts of HFS and q-ROFS, Wang et al. [10] suggested q-rung orthopair hesitant fuzzy (q-ROHF) set (q-ROHFS). q-ROHFS becomes a powerful tool to deal with imprecision and hesitancy. Based on power average (PA) operator and generalised Heronian mean (GHM) operator, Wang et al. [10] introduced q-ROHF weighted power generalised Heronian mean (q-ROHFWPGHM) and q-ROHF weighted power generalised geometric Heronian mean (q-ROHFWPGGHM) operators. More study on q-ROHF is deeply needed to apply it on various domains. Taking the advantages of q-ROHFSs and aggregation operators (AOs) [11–15] based on various t-norms and t-conorms (t-N&t-CNs) for data fusion process would be some significant works.

In MCGDM contexts, the weight of the DMs and criteria plays a very important role in making reasonable decisions. The practical and theoretical knowledge of the DMs in different fields may not be the same. As a consequence, different DMs would consider different importance levels of criteria in assigning decision values corresponding to some alternatives based on such criteria. In such conditions finding the appropriate weight of criteria by considering the individual importance of criteria of all the DM's as well as weights of the DMs is a complex problem. To overcome this situation Stepwise Weight Assessment Ratio Analysis (SWARA) method [16–18] is used. In this article, an extended technique is introduced for finding weights of the DMs in q-ROHF environment.

Also, Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method [19] is a popular Multicriteria decision making (MCDM) technique used to select the most suitable alternative from a set of options. It considers both positive and negative aspects of alternatives and determines their relative closeness or similarity to an ideal solution. By incorporating multiple criteria, the TOPSIS method allows DMs to make informed decisions that account for various factors.

Utilizing the benefits of the AOs and SWARA-TOPSIS method, few researchers [20–23] used this method to solve decision making problems in various fuzzy domains.

A SWARA-TOPSIS based MCGDM method is developed in this article for selecting the specific priority of options for handling MCGDM problems using the capability and potentiality of the indicated operators. To establish the application potentiality of the developed method a case study relating to assess the triage and efficacy of strategies adopted by SAARC for regional integrity of south Asia has been performed.

1.1. Literature review

Narayan et al. [24] applied MCDM technique to find the performance ranking of the SAARC nations. Balassa [1] defined five stages of economic integration in the year 1963. Sultana and Asrat [25] showed that South Asian Free Trade Area (SAFTA) has great potential to bring economic development in SAARC. Mujaffar et al. [26] showed the importance of integration of SAARC and proved that the arch rivalry between India and Pakistan is the main reason for poor integration of SAARC. A fuzzy logic-based approach of strategic planning for next-generation SAARC was presented by Ahmad et al. [27]. In another research work, Kharel et al. [28] presented challenges of the implementation of e-governance in SAARC using fuzzy logic. In compare to other areas, research work for the development of SAARC by considering possibilistic uncertainties is very limited in the literature.

The key contributions of this study are as follows:

1. From the theoretical perspective, this research makes a substantial contribution by utilizing the parametric and adaptable Einstein, Hamacher, Frank operational framework in a q-ROHF context to process complicated data associated with decision-making.
2. Based on Einstein, Hamacher, Frank t-N& t-CNs, various classes of Archimedean t-N&t-CNs (At-N&t-CNs), certain fundamental operational laws, viz., addition, multiplication, scaler multiplication and exponential are introduced.
3. A series of AOs viz., q-ROHF weighted average (q-ROHFWA), q-ROHF weighted geometric (q-ROHFWG), q-ROHF Einstein weighted average (q-ROHFEWA), q-ROHF Einstein weighted geometric (q-ROHFEWG), q-ROHF Hamacher weighted average (q-ROHFHWA), q-ROHF Hamacher weighted geometric (q-ROHFHWG), q-ROHF Frank weighted average (q-ROHFFWA), q-ROHF Frank weighted geometric (q-ROHFFWG) operator have been developed to aggregate uncertain data.
4. A distance measure is introduced to calculate the proximity or similarity between alternatives or criteria.
5. A SWARA-TOPSIS based MCGDM method is developed for selecting the specific priority of options for handling MCGDM problems using the capability and potential of the indicated operators.
6. From the application viewpoint, a problem related to triage and efficacy of strategies adopted by SAARC for regional integrity of south Asia has been performed using the developed method.
7. The superiority of the proposed method is demonstrated through a comparative analysis with existing methods.

1.2. Organization of the study

This article is organized in such a manner that some basic definitions and properties are discussed in Section 2. In Section 3, Some AOs viz., q-ROHFWA, q-ROHFWG, q-ROHFEWA, q-ROHFEWG, q-ROHFHWA, q-ROHFHWG, q-ROHFFWA and q-ROHFFWG for aggregating q-ROHF information are presented along with their desired properties. Also, here a distance measure for q-ROHFS is proposed. Section 4 elegantly describes an q-ROHF-SWARA-TOPSIS MCGDM method that uses the developed AOs and distance measure. A case study related to assess the triage and efficacy of strategies adopted by SAARC for regional integrity of south Asia is considered to show the validity and superiority of the proposed method in Section 5. Also, the sensitivity analysis is investigated in Section 6. A brief comparative analysis is presented in Section 7. Finally, conclusion and future studies are stated in Section 8.

2. Preliminaries

In this section, fundamental definitions related to q-ROFS [7], HFS [8], q-ROHFS [10] and the operations performed on q-ROHFS are reviewed.

Definition 1. ([7]). Let $\mathscr{U}$ be a universe of discourse, a q-ROFS $\widetilde{A}$ on $\mathscr{U}$ is known by

$$\widetilde{A}=\{(\mathscr{u},\gamma_{\widetilde{A}}(\mathscr{u}),\eta_{\widetilde{A}}(\mathscr{u}))|\mathscr{u}\in\mathscr{U}\},$$

where $\gamma_{\tilde{A}}: \mathscr{U} \to [0,1]$ and $\eta_{\tilde{A}}: \mathscr{U} \to [0,1]$ indicates, respectively, the degree of membership and the degree of non-membership of the element $u \in \mathscr{U}$ to the set $\tilde{A}$, and satisfies the condition that

$0 \leq \gamma_{\tilde{A}}^q(u) + \eta_{\tilde{A}}^q(u) \leq 1, q \geq 1.$

The degree of indeterminacy is given by $\pi_{\tilde{A}}(u) = [1 - \gamma_{\tilde{A}}^q(u) - \eta_{\tilde{A}}^q(u)]^{1/q}$.

For convenience, Yager [7] called $(\gamma_{\tilde{A}}(u), \eta_{\tilde{A}}(u))$ as a q-ROFN and is denoted by $\tilde{\alpha} = (\gamma, \eta)$.

Definition 2. ([8]). Let X be any set. An HFS E on X can be represented as

$$E = \{\langle x, h_E(x)\rangle | x \in X\}.$$

in which $h_E(x)$ represents a collection of possible finite values lying within $[0,1]$, signifying membership degrees for $x \in X$ to the set E.

Definition 3. ([10]). The q-ROHFS $\widetilde{\mathscr{P}}$ on X is expressed by

$$\widetilde{\mathscr{P}} = \{\langle h, \widehat{h}_{\widetilde{\mathscr{P}}}(x)\rangle | x \in X\},$$

in which $\widehat{h}_{\widetilde{\mathscr{P}}}(x)$ represents a collection of a number of possible q-ROFNs (γ, η). For convenience, Wang et al. [10] named $\widehat{h}_{\widetilde{\mathscr{P}}}(x) = \bigcup_{(\gamma,\eta)\in \widehat{h}_{\widetilde{\mathscr{P}}}(x)}\{(\gamma,\eta)\}$ a q-ROHF number (q-ROHFN) and is denoted by $\tilde{\wp} = (\mu,\nu) = \bigcup_{(\gamma,\eta)\in(\mu,\nu)}\{(\gamma,\eta)\}$. The indeterminacy of $\tilde{\wp}$ is defined by

$$\pi_{\tilde{\wp}} = \frac{1}{|\tilde{\wp}|}\sum_{(\gamma,\eta)\in(\mu,\nu)}(1 - (\gamma^q + \eta^q))^{1/q}.$$

The score function, $S(\tilde{\wp})$ and accuracy function, $A(\tilde{\wp})$ of any q-ROHFN $\tilde{\wp} = (\mu,\nu)$ to compare between two q-ROHFNs as follows:

Definition 4. ([10]). The score function, $S(\tilde{\wp})$ and accuracy function, $A(\tilde{\wp})$ of any q-ROHFN $\wp = (\mu,\nu)$ are presented as, respectively

$$S(\tilde{\wp}) = \frac{1}{2}\left(1 + \frac{1}{|\tilde{\wp}|}\sum_{(\gamma,\eta)\in(\mu,\nu)}(\gamma^q - \eta^q)\right), \text{and } A(\wp^{\sim}) = \frac{1}{|\wp^{\sim}|}\sum_{(\gamma,\eta)\in(\mu,\nu)}(\gamma^q + \eta^q), \tag{1}$$

where $|\tilde{\wp}|$ denotes the number of elements in $\tilde{\wp}$.

Definition 5. ([10]). The ordering of two q-ROHFNs $\tilde{\wp}_i = (\mu_i, \nu_i)$ ($i = 1,2$) are given by

(1) If $S(\tilde{\wp}_1) > S(\tilde{\wp}_2)$, then $\tilde{\wp}_1$ is superior to $\tilde{\wp}_2$, denoted by $\tilde{\wp}_1 \succ \tilde{\wp}_2$;
(2) If $S(\tilde{\wp}_1) = S(\tilde{\wp}_2)$, then $\tilde{\wp}_1 \succ \tilde{\wp}_2$ if $A(\tilde{\wp}_1) > A(\tilde{\wp}_2)$ and if $A(\tilde{\wp}_1) = A(\tilde{\wp}_2)$, then $\tilde{\wp}_1 \approx \tilde{\wp}_2$.

Definition 6. ([10]). Let $\tilde{\wp} = (\mu,\nu)$ be any q-ROHFN. The indeterminacy of $\tilde{\wp}$ is defined as follows:

$$\pi_{\tilde{\wp}} = (1/|\tilde{\wp}|)\sum_{(\gamma,\eta)\in(\mu,\nu)}(1 - (\gamma^q + \eta^q))^{1/q}.$$

Definition 7. ([10]). Let $\tilde{\wp} = (\mu,\nu)$, $\tilde{\wp}_1 = (\mu_1,\nu_1)$ and $\tilde{\wp}_2 = (\mu_2,\nu_2)$ represent three q-ROHFNs. Four fundamental operations are presented as follows:

(1) $\tilde{\wp}_1 \oplus \tilde{\wp}_2 = \bigcup_{(\gamma_i,\eta_i)\in(\mu_i,\nu_i), i=1,2}\{((\gamma_1{}^q + \gamma_2{}^q - \gamma_1{}^q\gamma_2{}^q)^{1/q}, \eta_1\eta_2)\}$,

(2) $\tilde{\wp}_1 \otimes \tilde{\wp}_2 = \bigcup_{(\gamma_i,\eta_i)\in(\mu_i,\nu_i), i=1,2}\left\{\left(\gamma_1\gamma_2, (\eta_1{}^q + \eta_2{}^q - \eta_1{}^q\eta_2{}^q)^{\frac{1}{q}}\right)\right\}$,

(3) $\lambda\tilde{\wp} = \bigcup_{(\gamma,\eta)\in(\mu,\nu)}\{((1 - (1 - \gamma^q)^\lambda)^{1/q}, \eta^\lambda)\}$, $(\lambda > 0)$,

(4) $\tilde{\wp}^\lambda = \bigcup_{(\gamma,\eta)\in(\mu,\nu)}\{(\gamma^\lambda, (1 - (1 - \eta^q)^\lambda)^{1/q})\}$, $(\lambda > 0)$.

3. Development of At-N&t-CNs and AOs on q-ROHFNs and a distance measure

In this section, several At-N&t-CNs are defined using increasing and decreasing generators to represent four fundamental operations of q-ROHFNs. Using those operations several AOs are introduced in this section. Also, a distance measure is defined on the q-ROHF environment.

3.1. Different operations corresponding to generating functions

Let $\tilde{\wp} = (\mu, \nu)$, $\tilde{\wp}_1 = (\mu_1,\nu_1)$ and $\tilde{\wp}_2 = (\mu_2,\nu_2)$ represent three q-ROHFNs. Several classes of operations are defined as follows:

- Hamacher classes

Hamacher t-N&t-CNs-based operational laws can be achieved by considering $f(t) = \log\left(\frac{\rho+(1-\rho)t}{t}\right)$, $\rho > 0$, as

(1) $$\tilde{\wp}_1 \oplus_H \tilde{\wp}_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}}\left\{\left(\left(\frac{\gamma_1^q+\gamma_2^q-\gamma_1^q\gamma_2^q-(1-\rho)\gamma_1^q\gamma_2^q}{1-(1-\rho)\gamma_1^q\gamma_2^q}\right)^{1/q}, \left(\frac{\eta_1^q\eta_2^q}{\rho+(1-\rho)(\eta_1^q+\eta_2^q-\eta_1^q\eta_2^q)}\right)^{1/q}\right)\right\};$$

(2) $$\tilde{\wp}_1 \otimes_H \tilde{\wp}_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}}\left\{\left(\left(\frac{\gamma_1^q\gamma_2^q}{\rho+(1-\rho)(\gamma_1^q+\gamma_2^q-\gamma_1^q\gamma_2^q)}\right)^{1/q}, \left(\frac{\eta_1^q+\eta_2^q-\eta_1^q\eta_2^q-(1-\rho)\eta_1^q\eta_2^q}{1-(1-\rho)\eta_1^q\eta_2^q}\right)^{1/q}\right)\right\};$$

(3) $$\lambda\tilde{\wp} = \bigcup_{(\gamma,\eta)\in(\mu,\nu)}\left\{\left(\left(\frac{(1+(\rho-1)\gamma^q)^\lambda-(1-\gamma^q)^\lambda}{(1+(\rho-1)\gamma^q)^\lambda+(\rho-1)(1-\gamma^q)^\lambda}\right)^{1/q}, \left(\frac{\rho\eta^{\lambda q}}{(1+(\rho-1)(1-\eta^q))^\lambda+(\rho-1)\eta^{\lambda q}}\right)^{1/q}\right)\right\};$$

(4) $$\tilde{\wp}^\lambda = \bigcup_{(\gamma,\eta)\in(\mu,\nu)}\left\{\left(\left(\frac{\rho\gamma^{\lambda q}}{(1+(\rho-1)(1-\gamma^q))^\lambda+(\rho-1)\gamma^{\lambda q}}\right)^{1/q}, \left(\frac{(1+(\rho-1)\eta^q)^\lambda-(1-\eta^q)^\lambda}{(1+(\rho-1)\eta^q)^\lambda+(\rho-1)(1-\eta^q)^\lambda}\right)^{1/q}\right)\right\}.$$

Algebraic and Einstein t-N&t-CNs based operational laws of q-ROHFNs are generated from Hamacher operation if $\rho = 1$ and 2 are considered, respectively, on those operations.

- Frank classes

For $f(t) = \log\left(\frac{\zeta-1}{\zeta^t-1}\right)$, the operations are defined as follows ($\zeta > 1$ and $\lambda > 0$):

(1) $$\tilde{\wp}_1 \oplus_F \tilde{\wp}_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}}\left\{\left(\left(1-\log_\zeta\left(1+\frac{(\zeta^{1-\gamma_1^q}-1)(\zeta^{1-\gamma_2^q}-1)}{\zeta-1}\right)\right)^{1/q}, \left(\log_\zeta\left(1+\frac{(\zeta^{\eta_1^q}-1)(\zeta^{\eta_2^q}-1)}{\zeta-1}\right)\right)^{1/q}\right)\right\};$$

(2) $$\tilde{\wp}_1 \otimes_F \tilde{\wp}_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}}\left\{\left(\left(\log_\zeta\left(1+\frac{(\zeta^{\gamma_1^q}-1)(\zeta^{\gamma_2^q}-1)}{\zeta-1}\right)\right)^{1/q}, \left(1-\log_\zeta\left(1+\frac{(\zeta^{1-\eta_1^q}-1)(\zeta^{1-\eta_2^q}-1)}{\zeta-1}\right)\right)^{1/q}\right)\right\};$$

(3) $\lambda\widetilde{\wp} = \bigcup_{(\gamma,\eta)\in(\mu,\nu)}\left\{\left(\left(1-\log_\zeta\left(1+\frac{(\zeta^{1-\gamma^q}-1)^\lambda}{(\zeta-1)^{\lambda-1}}\right)\right)^{1/q}, \left(\log_\zeta\left(1+\frac{(\zeta^{\eta^q}-1)^\lambda}{(\zeta-1)^{\lambda-1}}\right)\right)^{1/q}\right)\right\}$;

(4) $\widetilde{\wp}^\lambda = \bigcup_{(\gamma,\eta)\in(\mu,\nu)}\left\{\left(\left(\log_\zeta\left(1+\frac{(\zeta^{\gamma^q}-1)^\lambda}{(\zeta-1)^{\lambda-1}}\right)\right)^{1/q}, \left(1-\log_\zeta\left(1+\frac{(\zeta^{1-\eta^q}-1)^\lambda}{(\zeta-1)^{\lambda-1}}\right)\right)^{1/q}\right)\right\}$.

3.2. Development of *q*-ROHF archimedean AOs

In this subsection, *q*-ROHF WA and WG AOs are proposed using At-N&*t*-CNs. Following that, different forms of AOs are generated for various types of decreasing generators *f*.

3.2.1. Archimedean operation-based *q*-ROHF WA (A*q*-ROHFWA) operators

Definition 8. Let $\{\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n\}$ represent a set of n *q*-ROHFNs, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ represents the corresponding weight vector such that $\sum_{i=1}^n \omega_i = 1$ with $\omega_i \in [0, 1]$. Then, A*q*-ROHFWA operator is defined as

$$Aq-ROHFWA(\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n) = \oplus_{A\,i=1}^{\;n}(\omega_i \odot_A \widetilde{\wp}_i),$$

Several properties of the above-defined A*q*-ROHFWA operator are described below.

Theorem 1. *Let $\widetilde{\wp}_i = (\mu_i, \nu_i)$ $(i = 1, 2, \ldots, n)$ represent a collection of n q-ROHFNs. Then based on Aq-ROHFWA AO, the fused value also becomes a q-ROHFN such that*

$$Aq-ROHFWA(\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n) = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i),\\ i=1,2,\ldots,n}}\left\{\left(\left(g^{-1}\left(\sum_{i=1}^n \omega_i g(\gamma_i^q)\right)\right)^{1/q}, \left(f^{-1}\left(\sum_{i=1}^n \omega_i f(\eta_i^q)\right)\right)^{1/q}\right)\right\} \quad (2)$$

proof. Please follow Appendix A for proof of this theorem.

3.2.2. Different forms of A*q*-ROHFWA operator

In this subsection, several classes of AOs are discussed.

- Hamacher Operation based

The A*q*-ROHFWA operator transformed to the *q*-ROHF Hamacher WA (*q*-ROHFHWA) operator if $f(t) = \log\left(\frac{\rho+(1-\rho)t}{t}\right)$, $\rho > 0$, defined as:

$$q-ROHFHWA(\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n) = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}}\left\{\left(\left(\frac{\prod_{i=1}^n(1+(\rho-1)\gamma_i^q)^{\omega_i} - \prod_{i=1}^n(1-\gamma_i^q)^{\omega_i}}{\prod_{i=1}^n(1+(\rho-1)\gamma_i^q)^{\omega_i} + (\rho-1)\prod_{i=1}^n(1-\gamma_i^q)^{\omega_i}}\right)^{1/q}, \left(\frac{\rho\prod_{i=1}^n \eta_i^{q\omega_i}}{\prod_{i=1}^n(1+(\rho-1)(1-\eta_i^q))^{\omega_i} + (\rho-1)\prod_{i=1}^n \eta_i^{q\omega_i}}\right)^{1/q}\right)\right\}. \quad (3)$$

It is mentioned that in Eq. (3) for $\rho = 1$ and 2, *q*-ROHFHWA operator is changed into *q*-ROHFWA and *q*-ROHFEWA operator, respectively.

- Frank Operation based

If $f(t) = \log\left(\frac{\zeta-1}{\zeta^t-1}\right)$, $\zeta > 1$, the A*q*-ROHFWA operator changes to *q*-ROHF Frank WA (*q*-ROHFFWA) operator presented as:

$$q-ROHFFWA(\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n) = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}}\left\{\left(\left(1-\frac{\log\left(1+\prod_{i=1}^n\left(\zeta^{1-\gamma_i^q}-1\right)^{\omega_i}\right)}{\log\zeta}\right)^{1/q}, \left(\frac{\log\left(1+\prod_{i=1}^n\left(\zeta^{\eta_i^q}-1\right)^{\omega_i}\right)}{\log\zeta}\right)^{1/q}\right)\right\}. \quad (4)$$

3.2.3. *q*-ROHF Archimedean geometric (A*q*-ROHFWG) AOs

In the following subsection, some *q*-ROHF Archimedean geometric AOs are proposed.

Definition 9. Let $\{\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n\}$ represent a group of *q*-ROHFNs, and the weight vector be $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ where $\sum_{i=1}^n \omega_i = 1$ with $\omega_i \in [0, 1]$. Then, A*q*-ROHFWG operator is a mapping $\widetilde{\wp}^n \to \widetilde{\wp}$, defined as

$$Aq-ROHFWG(\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n) = \otimes_{A\,i=1}^{\;n}(\wp_i^{\,\omega_i}),$$

Theorem 2. *Let $\wp_i = (\mu_i, \nu_i)$ $(i = 1, 2, \ldots, n)$ be a collection of q-ROHFNs. The fused value obtained by Aq-ROHFWG operator also becomes a q-ROHFN such that*

$$Aq-ROHFWG(\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n) = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}}\left\{\left(\left(f^{-1}\left(\sum_{i=1}^n \omega_i f(\gamma_i^q)\right)\right)^{1/q}, \left(g^{-1}\left(\sum_{i=1}^n \omega_i g(\eta_i^q)\right)\right)^{1/q}\right)\right\}. \quad (5)$$

proof. Theorem 2's proof is same as Appendix A.

3.2.4. Different forms of A*q*-ROHFWG operator

If different forms are assigned to the decreasing generator *f*, the following *q*-ROHF WG operators can be deduced:

- Hamacher Operation-based AO

The A*q*-ROHFWG operator becomes *q*-ROHF Hamacher WG (*q*-ROHFHWG) operator when $f(t) = \log\left(\frac{\rho+(1-\rho)t}{t}\right)$, $\rho > 0$,

$$q-ROHFHWG(\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n) = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}}\left\{\left(\left(\frac{\rho\prod_{i=1}^n(\gamma_i)^{q\omega_i}}{\prod_{i=1}^n(1+(\rho-1)(1-\gamma_i^q))^{\omega_i} + (\rho-1)\prod_{i=1}^n(\gamma_i^q)^{\omega_i}}\right)^{1/q}, \left(\frac{\prod_{i=1}^n(1+(\rho-1)\eta_i^q)^{\omega_i} - \prod_{i=1}^n(1-\eta_i^q)^{\omega_i}}{\prod_{i=1}^n(1+(\rho-1)\eta_i^q)^{\omega_i} + (\rho-1)\prod_{i=1}^n(1-\eta_i^q)^{\omega_i}}\right)^{1/q}\right)\right\}. \quad (6)$$

If the above equation $\rho = 1$, 2 is considered *q*-ROHFHWG operator turned, respectively, into *q*-ROHFWG and *q*-ROHFEWG operators.

- *Frank Operation-based AO*

If $f(t) = \log\left(\frac{\zeta-1}{\zeta^t-1}\right)$, $\zeta > 1$, then the A*q*-ROHFWG operator turns into the *q*-ROHF Frank WG (*q*-ROHFFWG) operator defined as:

$$q-ROHFFWG(\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n)$$

$$= \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}} \left\{ \left(\left(\frac{\log\left(1+\prod_{i=1}^{n}\left(\zeta^{\gamma_i^q}-1\right)^{\omega_i}\right)}{\log\zeta} \right)^{1/q}, \left(1-\frac{\log\left(1+\prod_{i=1}^{n}\left(\zeta^{1-\eta_i^q}-1\right)^{\omega_i}\right)}{\log\zeta}\right)^{1/q} \right) \right\}. \tag{7}$$

The above-proposed operators satisfy Boundary, Idempotency, and Additivity properties (See Appendix B-D).

3.3. Distance measures for q-ROHFNs

In this subsection, the normalized distance measure and weighted normalized distance measure between two q-ROHFSs are defined and their necessary properties are stated.

Definition 10. Let $\widetilde{\mathscr{P}} = \{\widetilde{\wp}_1, \widetilde{\wp}_2, \ldots, \widetilde{\wp}_n\}$ and $\widetilde{\mathscr{Q}} = \{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \ldots, \widetilde{\gamma}_n\}$ denote two sets of q-ROHFNs, where $\widetilde{\wp}_i = \bigcup_{(\gamma_{\widetilde{\wp}_i},\eta_{\widetilde{\wp}_i})\in(\mu_{\widetilde{\wp}_i},\nu_{\widetilde{\wp}_i})}\{(\gamma_{\widetilde{\wp}_i},\eta_{\widetilde{\wp}_i})\}$ and $\widetilde{\gamma}_i = \bigcup_{(\gamma_{\widetilde{\gamma}_i},\eta_{\widetilde{\gamma}_i})\in(\mu_{\widetilde{\gamma}_i},\nu_{\widetilde{\gamma}_i})}\{(\gamma_{\widetilde{\gamma}_i},\eta_{\widetilde{\gamma}_i})\}$, and $\{\omega_i\colon 0\le\omega_i\le 1\}$ $(i=1,2,\ldots,n)$ be the respective weight vectors of q-ROHFNs with $\sum_{i=1}^{n}\omega_i = 1$. The normalized and weighted normalized distance between $\widetilde{\mathscr{P}}$ and $\widetilde{\mathscr{Q}}$ is formulated as follows:

- Normalized distanced measure

$$d(\widetilde{\mathscr{P}},\widetilde{\mathscr{Q}}) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2|\widetilde{\wp}_i|}\sum_{s=1}^{|\widetilde{\wp}_i|}\min_l\left(\frac{1}{2}\left(\left|\gamma^q_{\widetilde{\wp}_{is}}-\gamma^q_{\widetilde{\gamma}_{il}}\right|+\left|\eta^q_{\widetilde{\wp}_{is}}-\eta^q_{\widetilde{\gamma}_{il}}\right|+\left|\left(\gamma^q_{\widetilde{\gamma}_{il}}-\gamma^q_{\widetilde{\wp}_{is}}\right)+\left(\eta^q_{\widetilde{\gamma}_{il}}-\eta^q_{\widetilde{\wp}_{is}}\right)\right|\right)\right)+\frac{1}{2|\widetilde{\gamma}_i|}\sum_{l=1}^{|\widetilde{\gamma}_i|}\min_s\left(\frac{1}{2}\left(\left|\gamma^q_{\widetilde{\wp}_{is}}-\gamma^q_{\widetilde{\gamma}_{il}}\right|+\left|\eta^q_{\widetilde{\wp}_{is}}-\eta^q_{\widetilde{\gamma}_{il}}\right|+\left|\left(\gamma^q_{\widetilde{\gamma}_{il}}-\gamma^q_{\widetilde{\wp}_{is}}\right)+\left(\eta^q_{\widetilde{\gamma}_{il}}-\eta^q_{\widetilde{\wp}_{is}}\right)\right|\right)\right)\right) \tag{8}$$

- Weighted normalized distance measure

$$d_w(\widetilde{\mathscr{P}},\widetilde{\mathscr{Q}}) = \sum_{i=1}^{n}\omega_i\left(\frac{1}{2|\widetilde{\wp}_i|}\sum_{s=1}^{|\widetilde{\wp}_i|}\min_l\left(\frac{1}{2}\left(\left|\gamma^q_{\widetilde{\wp}_{is}}-\gamma^q_{\widetilde{\gamma}_{il}}\right|+\left|\eta^q_{\widetilde{\wp}_{is}}-\eta^q_{\widetilde{\gamma}_{il}}\right|+\left|\left(\gamma^q_{\widetilde{\gamma}_{il}}-\gamma^q_{\widetilde{\wp}_{is}}\right)+\left(\eta^q_{\widetilde{\gamma}_{il}}-\eta^q_{\widetilde{\wp}_{is}}\right)\right|\right)\right)+\frac{1}{2|\widetilde{\gamma}_i|}\sum_{l=1}^{|\widetilde{\gamma}_i|}\min_s\left(\frac{1}{2}\left(\left|\gamma^q_{\widetilde{\wp}_{is}}-\gamma^q_{\widetilde{\gamma}_{il}}\right|+\left|\eta^q_{\widetilde{\wp}_{is}}-\eta^q_{\widetilde{\gamma}_{il}}\right|+\left|\left(\gamma^q_{\widetilde{\gamma}_{il}}-\gamma^q_{\widetilde{\wp}_{is}}\right)+\left(\eta^q_{\widetilde{\gamma}_{il}}-\eta^q_{\widetilde{\wp}_{is}}\right)\right|\right)\right)\right) \tag{9}$$

The normalized distance measure satisfies following properties:

i) $0\le d(\widetilde{\mathscr{P}},\widetilde{\mathscr{Q}})\le 1$; ii) $d(\widetilde{\mathscr{P}},\widetilde{\mathscr{Q}})=0$ iff $\widetilde{\mathscr{P}}=\widetilde{\mathscr{Q}}$; iii) $d(\widetilde{\mathscr{P}},\widetilde{\mathscr{Q}}) = d(\widetilde{\mathscr{Q}},\widetilde{\mathscr{P}})$; iv) If $\widetilde{\mathscr{P}}\subseteq\widetilde{\mathscr{Q}}\subseteq\widetilde{\mathscr{R}}$, then $d(\widetilde{\mathscr{P}},\widetilde{\mathscr{Q}})\le d(\widetilde{\mathscr{P}},\widetilde{\mathscr{R}})$ and $d(\widetilde{\mathscr{Q}},\widetilde{\mathscr{R}})\le d(\widetilde{\mathscr{P}},\widetilde{\mathscr{R}})$. The properties for d_w is same as properties of d

4. A SWARA-TOPSIS-based MCGDM technique under q-ROHF environment

In this section, a novel SWARA-TOPSIS-based MCGDM method is introduced for q-ROHF environment. The SWARA method incorporates a step for pairwise comparisons of criteria to determine their relative weights. This helps DMs to quantify the importance of each criterion, providing a structured approach to allocate weights and avoid arbitrary or subjective judgments. In this article, weights of assessment criteria are evaluated by SWARA method in q-ROHF environment. Also, TOPSIS method is used to determine the ranking of the alternatives from best to worst solutions. The brief descriptions of the developed method are given as follows:

Step I: Formulate the MCGDM problem and then create the decision matrix:

suppose $\mathfrak{E} = \{\mathfrak{E}^{(1)}, \mathfrak{E}^{(2)}, \ldots, \mathfrak{E}^{(l)}\}$ is a group of l experts, and a collection of alternatives is represented as $\mathscr{Z} = \{\mathscr{Z}_1, \mathscr{Z}_2, \ldots, \mathscr{Z}_m\}$. Let $\mathscr{C} = \{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n\}$ denotes a collection containing attributes. Assume $\widetilde{\wp}_{ij}^{(k)}$ indicates the evaluation value of the alternative $\mathscr{Z}_i$ $(i=1,2,\ldots,m)$ concerning the criteria $\mathscr{C}_j$ $(j=1,2,\ldots,n)$ given by DM $\mathfrak{E}^{(k)}$. Suppose, $P^{(k)} = [\widetilde{\wp}_{ij}^{(k)}]_{m\times n}$ is the q-ROHF decision matrix (q-ROHFDM) which is evaluated by DM $\mathfrak{E}^{(k)}$.

Step II: Calculate the weights of DMs:

Determining the weights of the DMs during the decision-making process is crucial. Let $\widetilde{\mathscr{A}}^{(k)} = (\mu_k, \nu_k)$ be the importance of the k th DM. Then the weight of the k^{th} expert is obtained by using the expression [29]:

$$\varphi(k) = \frac{\left(1/\left|\widetilde{\mathfrak{d}}^{(k)}\right|\right)\sum_{(\gamma_k,\eta_k)\in(\mu_k,\nu_k)}\left(\gamma_k^q + \left(\pi_k^q\gamma_k^q/\left(\gamma_k^q+\eta_k^q\right)\right)\right)}{\sum_{k=1}^{l}\left(1/\left|\widetilde{\mathfrak{d}}^{(k)}\right|\right)\sum_{(\gamma_k,\eta_k)\in(\mu_k,\nu_k)}\left(\gamma_k^q + \left(\pi_k^q\gamma_k^q/\left(\gamma_k^q+\eta_k^q\right)\right)\right)}. \tag{10}$$

Step III: Aggregate the q-ROHFDM.

To generate the aggregated q-ROHFDM, all individual matrices need to be merged into one group using DMs' opinions. The Aq-ROHFHWA operator is applied, and obtain $P = [\widetilde{\wp}_{ij}]_{m\times n}$, where $\widetilde{\wp}_{ij} = (\mu_{ij},\nu_{ij}) = Aq-ROHFHWA(\widetilde{\wp}_{ij}^{(1)}, \widetilde{\wp}_{ij}^{(2)}, \ldots, \widetilde{\wp}_{ij}^{(l)})$.

Step IV: Calculate the criteria weights.

The SWARA model is used to prioritize the criteria and compares pair-wise from upper to lower-ranking criteria. Resulting, a relative coefficient is assessed, and the weight is obtained and evaluated for solving MCGDM problems. Thus, we have the following steps for the SWARA model:

Step IV-A: The importance of criteria is taken differently depending on the DM's knowledge. Let $\widetilde{\mathscr{I}}_{kj} = (\widetilde{\mu}_{kj}, \widetilde{\nu}_{kj})$ for $k= 1,2,\ldots, l$; $j=1,2,\ldots,n$ be the criteria importance provided by DMs. First, aggregate the $\widetilde{\mathscr{I}}_{kj}$ values to combined importance values of each criterion as follows:

$$\widetilde{\mathscr{I}}_j = Aq-ROHFHWA\left(\widetilde{\mathscr{I}}_{1j}, \widetilde{\mathscr{I}}_{2j}, \ldots, \widetilde{\mathscr{I}}_{lj}\right).$$

Calculate score values $S(\widetilde{\mathscr{I}}_j)$ of q-ROHFNs using Eq. (1) is determined.

Step IV-B: Prioritize the criteria. The criteria are organized based on the DM's preference from the most to the least important criterion.
Step IV-C: Evaluate the relative significance, α_j, of the score value. The relative significance is estimated from the criteria that are chosen in the second rank, and succeeding relative significance is evaluated by differencing criterion j and criterion $j-1$.
Step IV-D: Compute the relative coefficient. The coefficient k_j is given by

$$\kappa_j = \begin{cases} 1, & j = 1 \\ \alpha_j + 1, & j \geq 1, \end{cases} \tag{11}$$

where α_j is the relative significance of score degree.

Step IV-E: Assess the weight. The final weight p_j is given by

$$\varrho_j = \begin{cases} 1, & j = 1 \\ \dfrac{\varrho_{j-1}}{\kappa_j}, & j \geq 1, \end{cases} \tag{12}$$

Step IV-F: Calculate the criteria weight. The weights are estimated by

$$w_j = \frac{\varrho_j}{\sum_{j=1}^{n} \varrho_j}. \tag{13}$$

Step V: Define the best value (BV) and the worst value (WV).

In the proposed method, obtaining the best and worst values for each criterion is very important for DMs. Here, the best and worst values are calculated in the form of the q-ROHF-BV and q-ROHF-WV which are denoted by

$$\tilde{\wp}^{+} = \left(\mu_j^{+}, \nu_j^{+}\right) = \left\{ \left(\mu_{ij}, \nu_{ij}\right) \middle| \max_i S\left(\tilde{\wp}_{ij}\right) \right\}, \tag{14}$$

$$\tilde{\wp}^{-} = \left(\mu_j^{-}, \nu_j^{-}\right) = \left\{ \left(\mu_{ij}, \nu_{ij}\right) \middle| \min_i S\left(\tilde{\wp}_{ij}\right) \right\}. \tag{15}$$

Step VI: Calculation of distance measures from best and worst solution.

By Eq. (9), calculate the weighted distance $d_w(\mathscr{Z}_i, \tilde{\wp}^{+})$ between options $\mathscr{Z}_i$ $(i = 1, 2, \ldots, m)$ and the q-ROHF-BV $(\tilde{\wp}^{+})$ and the weighted distance $d_w(\mathscr{Z}_i, \tilde{\wp}^{-})$ between the options $\mathscr{Z}_i$ and the q-ROHF-WV $(\tilde{\wp}^{-})$.

Step VII: Calculation of relative closeness coefficient ($\mathfrak{RC}$):

Ultimately, the relative closeness coefficient can be calculated by using the below formulation:

$$\mathfrak{RC}(\mathscr{Z}_i) = \frac{d_w(\mathscr{Z}_i, \tilde{\wp}^{-})}{d_w(\mathscr{Z}_i, \tilde{\wp}^{-}) + d_w(\mathscr{Z}_i, \tilde{\wp}^{+})}. \tag{16}$$

Step VIII: Select the highest value among the values $\mathfrak{RC}(\mathscr{Z}_i)$, $i = 1, 2, \ldots, m$. The corresponding $\mathscr{Z}_i$ represents the best option.
Step IX: End.

The flowchart of the proposed method is described in Fig. 1.

5. Formulating the problem

From different theoretical and empirical studies of SAARC countries, it is clear that SAARC has great potentiality to become an economically developed integrated region; but the target is not achieved yet after adopting so many strategies during its 38 years of its journey. Various statistical, mathematical and econometric techniques were applied by different researchers at different times to evaluate the integration process of SAARC by using the data of export, import, migration, poverty, inequality and per capita income, Human Development Index (HDI), flow of capital in the member countries of SAARC. Many have invented various indices to measure the extent of integration of SAARC. All the studies have concluded that SAARC has not been able to achieve its target of reaching South Asian Economic Union (SAEU) till date and the progress is very slow compared to its potentiality. This study wants to take the initiative to analyse the strategies taken by SAARC over the years and make a proper assessment of them according to their capability and efficiency not by using the econometric and statistical techniques which are used already by plenty of studies, but by a new manner which acknowledges the fuzziness or the imprecise nature and outcome of the SAARC strategies in an uncertain environment created by always politically conflicting members of SAARC. This study tries to rank the different strategies based on SWARA-TOPSIS-based MCGDM technique in q-ROHF environment and tries to find the best one which should be emphasized to get a successfully integrated regional block of SAARC.

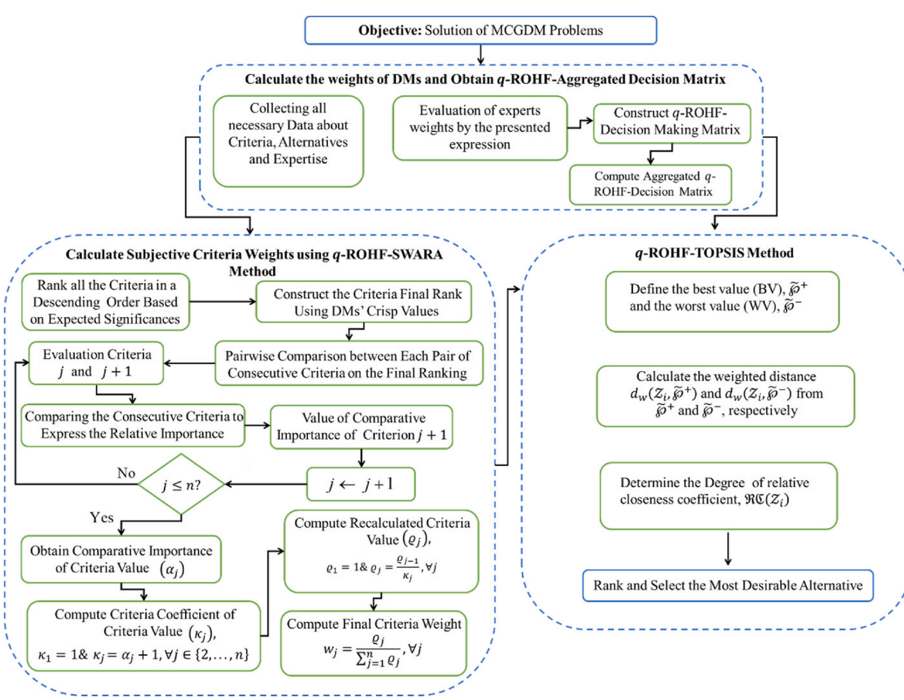

Fig. 1. Flowchart of the proposed methodology.

5.1. Posturing the goal of SAARC as MCGDM

The developed MCGDM method is now applied in this section to assess triage and efficacy of strategies adopted by SAARC for regional integrity of south Asia. The MCGDM problem under this context is developed as follows:

5.1.1. Identify the problem: Achievement of regional economic integration of SAARC

The formation of SAARC and its' strive for achieving an intensely unified regional block in form of South Asian Economic Union (SAEU) might bring a greater scope for intra-regional mobility of labour along with the trade and financial sector mobility in a peaceful, cooperative, collaborative, and coordinated manner among the members who are very much similar in their level of economic development. The end of cross-border violence among members, especially, between India and Pakistan-two big members of SAARC may be both the reason and the consequence of success of SAARC promises.

5.1.2. Identify the alternatives

The different initiatives taken by SAARC secretariat and the governing body for strengthening regional cooperation and economic integration in South Asia are viewed as the different alternatives for MCGDM technique of SAARC secretariat.

SAPTA ($\mathscr{Z}_1$): SAARC Preferential Trade Agreement (SAPTA) was the important initiative for economic integration taken by the SAARC secretariat, first mentioned by Sri Lanka in the sixth summit of SAARC held in December 1991. It was designed to encourage and aggravate mutual trade and economic cooperation within the SAARC through tariff concession. As trade is supposed to be the 'Engine of Growth', emphasis was laid on expansion of intra-regional trade through SAPTA.

SARCO ($\mathscr{Z}_2$): The Agreement for establishment of SAARC Arbitration Council (SARCO) was signed by the SAARC members during thirteenth Summit on 12th November 2005 in Dhaka to pledge an authorized organization within the region for fair and efficient settlement through conciliation and arbitration of commercial, investment and such other disputes as may be referred to the Council by agreement. It was felt that for expansion of trade and having sufficient economic growth in the region, intra-regional capital flow in form of investment is very crucial which can enable to increase in the regional integration of SAARC also. Establishment of SARCO was supposed to make such investment more cost-effective and by resolving the disputes among members it will enhance regional peace and mutual cooperation.

SAFTA ($\mathscr{Z}_3$): The members of SAARC established the South Asian Free Trade Area (SAFTA) to promote and enhance mutual trade and economic cooperation among themselves, through exchanging concessions in accordance with this Agreement. There was a transformation of SAPTA (South Asian Preferential Trade Agreement) to South Asian Free Trade Agreement (SAFTA) in the year 2006 was mostly targeted to elimination of all trade barriers of goods among member countries to accelerate and smoothen the international trade of goods among the members of SAARC and to further enhance the intra-regional economic integration. It tried to bring economic cooperation by reducing the tariff and barriers by and providing special preference to the Least Developed Nations among the SAARC

SATIS ($\mathscr{Z}_4$): SAARC Agreement on Trade in Services (SATIS) was penned by the SAARC members in the year 2010 which came into effect in 2012 was aimed at achieving liberalization in trade in services among members enhancing more dense regional integration by maximizing the recognition of potentiality of the region for trade and development by increasing intra-regional investment and production opportunities.

SAU ($\mathscr{Z}_5$): Establishment of the South Asian University in the year 2010 was a promising step to the way to more deepening regional integration of SAARC. The objective of establishing the South Asian University was to enhance education in the South Asian community encouraging an understanding of one another's view point and strengthen regional consciousness along with providing liberal and humane education to the brightest and the most dedicated students of South Asia so that a new class of quality leadership is nurtured and enhancing the capacity building of the South Asian Nations in science, technology and other areas of higher learning vital for improving their quality of life such as information technology, biotechnology and management sciences, etc.

SDF ($\mathscr{Z}_6$): SAARC Development Fund is another positive step towards the proposal of South Asian Economic Union (SAEU). SDF was established by heads of all eight SAARC member states during 16th SAARC summit at Thimphu, Bhutan in April 2010. Its Governing Council comprises finance ministers of these eight countries. It is the replacement of South Asian Development Fund (SADF), which was launched in 1996 by merging two existing facilities called SAARC Fund for Regional Projects (SFRP) and SAARC Regional Fund. SDF was designed as umbrella financial mechanism for all SAARC developmental projects and programs. Its core objectives are to promote welfare of the people of SAARC region, improve their quality of life and accelerate economic growth, social progress and poverty alleviation in the region. It funds projects in South Asia region via three windows viz. Social Window, Economic Window and Infrastructure Window.

5.1.3. Identify the criteria

There are three main criteria for deciding the best initiative taken by SAARC. First is to see how far these initiatives are capable to bring economic integration ($\mathscr{C}_1$), second is to review how far the initiatives by SAARC can help to bring cultural integration in the region ($\mathscr{C}_2$) and finally to check how they are able to bring societal integration ($\mathscr{C}_3$). All the three decision criteria include several sub-criteria on the basis of which the best alternative selection by SAARC can be made. The sub-criteria can be explained in the following way.

Export Integration ($\mathscr{C}_{11}$) and Import Integration ($\mathscr{C}_{12}$): The primary requirement of a regional integration is the homogenization in terms of its economic variables. Trade of goods and services is the key economic variable proficient in linking the world in a single thread. We can have a clear picture of trade integration of a regional block when viewed in terms of both export (Export Integration – I_{ex}) and imports (Import Integration- I_{im}). Both the export and import integration of the individual members as well as the region as a whole can be viewed together in terms of (a) percentage share of intra-SAARC export (or import) in respective Gross Domestic Product (GDP) (b) percentage share of intra-SAARC export (or import) in total volume of international export (or import) and (c) ratio of connected countries for export (or import) to the total number of members of the region SAARC.

Intra-regional flow of FDI ($\mathscr{C}_{13}$) and intra−regional flow of remittances ($\mathscr{C}_{14}$): Another essential pre-requisite for regional integration is to have integration in financial sector of the member countries and the financial sector integration can be measured by the intra-regional flow of Foreign Direct Investment (I_{FDI}) and intra-regional flow of remittances (I_{rem}) coming from intra-regional migration. FDI is an important source of foreign capital which can stimulate economic development of the economy while remittances from the citizens working abroad is an important source of much-needed fund especially in developing country. FDI and Remittances integration of SAARC are measured in terms of three variables i.e. GDPs, total FDI of the countries and the connecting members of the region for FDI and remittances and the arithmetic mean are calculated to get the value of required integration and higher values of these integration will ensure the homogeneity of the region has reached its advanced level.

Convergence in HDI ($\mathscr{C}_{15}$): The Human Development Index (HDI) is an average measure of health, education, income and over all standard of living of the people of the country and can be conceptualized as the composite statistical index comprising life expectancy, education and per capita income of the economy. The convergence in the level of HDI of the member countries demonstrates the wiping out the disparity in their level of human development and making the whole region more homogenized. This is assumed to be the result of effective integration in trade, finance and labour markets. HDI converge can be measured statistically by both sigma and beta convergence techniques.

Fall in level of poverty and inequality ($\mathscr{C}_{16}$): The reduction in the level of poverty and income inequality measured in terms of GINI coefficient can also be a good indicator of the level of development of the region which is the salient objective of the formation of SAARC.

Intra-regional migration ($\mathscr{C}_{17}$): Intra-Regional migration is another important instrument in the way to have successful regional economic integration and it has also two dimensions-emigration and immigration because a migrant emigrates from the origin country and immigrates to the foreign country. Migration between SAARC countries is very obvious and efficient management of migration can tackle the problem of refugees as well as improve the work force and knowledge transfer mechanism of the region.

Educational Integration ($\mathscr{C}_{12}$): The opportunity to get higher education by all the members of the region is the basic criteria for making the region culturally enriched and integrated.

Knowledge and technology transfer ($\mathscr{C}_{22}$): Knowledge and technology transfer in the field of production can uplift mutual cooperation and bring production efficiency in the region and it is possible when there will be a proper integration.

Abatement of conflict and tussle ($\mathscr{C}_{31}$) and Mutual cooperation and friendship ($\mathscr{C}_{32}$): To have peace and prosperity in the region the political conflict and tussle should be abolished and mutual cooperation and friendship should be established to have an effective integration in the region.

5.1.4. *Assessment or valuation of the alternatives on the basis of fulfilment of the criteria*

To assess the relative importance or effectiveness of the SAARC initiatives taken above, various journals, articles, research papers, interviews and discussions on the performance of SAARC have been reviewed. Based on the experts' comments, researchers' observations and conclusions from SAARC secretariat resolutions, the alternative initiatives of SAARC can be valued in accordance of their effectiveness or importance in fulfilling the mentioned criteria as a linguistic term. Table 1 indicates linguistic variables (LVs), which are subsequently converted into q-ROFNs to illustrate the importance of the DMs and criteria. Table 2 presents the LVs for assessing the performance of each alternative with respect to each criterion.

Three experts are approached with the alternative strategies and the criteria for their linguistic assessment or the preferences of the alternatives and termed them as the DMs for this purpose. Based on the experiences and knowledge, the importance of the DMs is presented in Table 3. Table 4 defines the importance of the criteria in perspective with each DM to evaluate the alternatives concerning each criterion.

5.2. *Application of the developed MCGDM methods*

Now, the developed methods are applied on the formulated problem. The step-by-step execution of that method is presented below.

Step 1. DMs expressed their opinion in q-ROHF environment with the help of LVs as given in Table 2. The q-ROHF decision matrices $P^{(k)} = \left[\tilde{\wp}_{ij}^{(k)}\right]_{6\times 11}$ ($k = 1, 2, 3$) are constructed, and the detailed data of the q-ROHF evaluative rating $\tilde{\wp}_{ij}^{(k)}$ for all $\mathscr{Z}_i$ with respect to $\mathscr{C}_j \in \mathscr{C}$ are presented in Tables 5–7.

Table 1
Linguistic preferences for the importance of DMs and Criteria.

Linguistic terms	Abbreviation	Linguistic values/label
Extremely Unimportant	EU	$(0.2, 0.95)$
Very Unimportant	VU	$(0.35, 0.78)$
Unimportant	U	$(0.42, 0.61)$
Moderate	M	$(0.5, 0.6)$
Important	I	$(0.65, 0.3)$
Very Important	VI	$(0.8, 0.15)$
Extremely Important	EI	$(0.9, 0.1)$

Table 2
Linguistic preferences of alternative strategies.

Linguistic terms	Abbreviation	Linguistic values/label
Absolutely Insignificant	AI	$(0.2, 0.95)$
Insignificant	I	$(0.35, 0.78)$
Slightly Insignificant	SI	$(0.42, 0.61)$
Moderate	M	$(0.5, 0.6)$
Slightly Significant	SS	$(0.65, 0.3)$
Significant	S	$(0.8, 0.15)$
Extremely Significant	ES	$(0.9, 0.1)$

Table 3
Importance level of Experts.

Experts	Linguistic term	Linguistic values/label
First DM ($\mathbf{E}^{(1)}$)	EI	$(0.9, 0.1)$
Second DM ($\mathbf{E}^{(2)}$)	I	$(0.65, 0.3)$
Third DM ($\mathbf{E}^{(3)}$)	VI	$(0.8, 0.15)$

Step II: In accordance with Eq. (10), the DMs' weights are computed as

$$\varphi_1 = 0.3441, \varphi_2 = 0.3137 \text{ and } \varphi_3 = 0.3423.$$

Step III: To generate the aggregated q-ROHFDM, all individual matrices need to be merged into one decision matrix. Considering rung parameter, $q = 3$ and Hamacher parameter, $t = 3$, the q-ROHFHWA operator is applied on Tables 5–7 to aggregate q-ROHFDMs into a single decision matrix $\tilde{\wp}_{ij}$. Aggregate the q-ROHFDM, $\tilde{\wp}_{ij}$ is given as follows and represented in Table 8.

$$\tilde{\wp}_{ij} = (\mu_{ij}, \nu_{ij}) = q - ROHFHWA\left(\tilde{\wp}_{ij}^{(1)}, \tilde{\wp}_{ij}^{(2)}, \ldots, \tilde{\wp}_{ij}^{(l)}\right).$$

Step IV: For finding criteria weights, the SWARA model begins to prioritize the criteria and compares pair-wise from upper to lower-ranking criteria. Resulting, a relative coefficient is assessed, and the weight is obtained and evaluated for solving MCGDM problems. Thus, we have the following steps for the SWARA model:
Step IV-A: For computing each criterion's weight using SWARA, the DMs' role is of high significance in assessing and calculating the weights, which are already depicted in Table 9. Each DM was asked to select each barrier's importance. Aggregated the criteria importance and score values of each aggregated criteria importance are given in Table 9.

With the use of Steps *IV-B* to *IV-F*, the experts ordered all the criteria from the first one to the last one. Based on SWARA technique, all weights of the criteria are presented in Table 10 as w_j column.

Step V: The best value and the worst value for each criterion are calculated and denoted by q-ROHF-BV ($\tilde{\wp}^+$) and q-ROHF-WV ($\tilde{\wp}^-$), given in Table 11.

In the proposed method, obtaining the best and worst values for each criterion is very important for DMs. Here, the best and worst values are calculated by Eqs. 14 and 15 are denoted by q-ROHF-BV ($\tilde{\wp}^+$) and q-ROHF-WV ($\tilde{\wp}^-$).

Table 4
Criteria importance provided by DM's.

	$\mathscr{C}_{11}$	$\mathscr{C}_{12}$	$\mathscr{C}_{13}$	$\mathscr{C}_{14}$	$\mathscr{C}_{15}$	$\mathscr{C}_{16}$	$\mathscr{C}_{17}$	$\mathscr{C}_{21}$	$\mathscr{C}_{22}$	$\mathscr{C}_{31}$	$\mathscr{C}_{32}$
$\mathbf{E}^{(1)}$	EI	EI	EI	EI	VI	M	VI	I	VI	VI	VI
$\mathbf{E}^{(2)}$	EI	EI	EI	EI	I	M	I	U	I	VI	VI
$\mathbf{E}^{(3)}$	EI	EI	EI	EI	M	M	AI	M	I	VI	VI

Table 5
Assessment matrix acquired from $\mathbf{E}^{(1)}$.

	$\mathscr{C}_{11}$	$\mathscr{C}_{12}$	$\mathscr{C}_{13}$	$\mathscr{C}_{14}$	$\mathscr{C}_{15}$	$\mathscr{C}_{16}$	$\mathscr{C}_{17}$	$\mathscr{C}_{21}$	$\mathscr{C}_{22}$	$\mathscr{C}_{31}$	$\mathscr{C}_{32}$
$\mathscr{Z}_1$	$\{ES\}$	$\{ES\}$	$\{SS\}$	$\{S,ES\}$	$\{M\}$	$\{M,S\}$	$\{SS\}$	$\{I\}$	$\{AI,SI\}$	$\{M\}$	$\{M\}$
$\mathscr{Z}_2$	$\{SS,S\}$	$\{M,S\}$	$\{M,S\}$	$\{S\}$	$\{AI,SI\}$	$\{I\}$	$\{SI\}$	$\{SI,M\}$	$\{M\}$	$\{S,ES\}$	$\{SS,ES\}$
$\mathscr{Z}_3$	$\{ES\}$	$\{M,ES\}$	$\{ES\}$	$\{ES\}$	$\{M\}$	$\{SS\}$	$\{ES\}$	$\{SI\}$	$\{SS\}$	$\{I\}$	$\{S\}$
$\mathscr{Z}_4$	$\{S,ES\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{SS,ES\}$	$\{ES\}$	$\{ES\}$	$\{I\}$	$\{I\}$
$\mathscr{Z}_5$	$\{AI\}$	$\{AI\}$	$\{AI\}$	$\{M\}$	$\{S\}$	$\{I\}$	$\{ES\}$	$\{S\}$	$\{ES\}$	$\{SS\}$	$\{SS,S\}$
$\mathscr{Z}_6$	$\{ES\}$	$\{ES\}$	$\{SS,ES\}$	$\{ES\}$	$\{S,ES\}$	$\{ES\}$	$\{ES\}$	$\{S\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$

Step VI: The weighted distance, $d_w(\mathscr{Z}_i, \widetilde{\wp}^+)$ between options $\mathscr{Z}_i$ ($i = 1, 2, \ldots, 6$) and best value, $\widetilde{\wp}^+$ and weighted distance, $d_w(\mathscr{Z}_i, \widetilde{\wp}^-)$ between the options $\mathscr{Z}_i$ ($i = 1, 2, \ldots, 6$) and the worst value $\widetilde{\wp}^-$ are calculated and presented in Table 12.

Step VII: Ultimately, the relative closeness coefficient ($\mathfrak{RC}$) is calculated by Eq. (16) and given as follows:

$$\mathfrak{RC}(\mathscr{Z}_1) = 0.4771, \mathfrak{RC}(\mathscr{Z}_2) = 0.4765, \mathfrak{RC}(\mathscr{Z}_3) = 0.6306, \mathfrak{RC}(\mathscr{Z}_4) = 0.6690, \mathfrak{RC}(\mathscr{Z}_5) = 0.4301, \text{ and } \mathfrak{RC}(\mathscr{Z}_6) = 0.7340.$$

Step VIII: Since $\mathfrak{RC}(\mathscr{Z}_6) > \mathfrak{RC}(\mathscr{Z}_4) > \mathfrak{RC}(\mathscr{Z}_3) > \mathfrak{RC}(\mathscr{Z}_1) > \mathfrak{RC}(\mathscr{Z}_2) > \mathfrak{RC}(\mathscr{Z}_5)$ among the values $\mathfrak{RC}(\mathscr{Z}_i), i = 1, 2, \ldots, 6$, $\mathscr{Z}_i$ is ranked as $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$. As a result, $\mathscr{Z}_6$ i.e., SDF emerges as the best option.

The above results are determined by the proposed methodology with q-ROHFHWA operator. The archived results through proposed methodology with other developed operators are presented in Table 13.

From Table 13, it is very cleared that all geometric operators give the ranking of the alternative as $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$ and all averaging operators give the ranking of the alternatives as $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$. In both cases alternative $\mathscr{Z}_6$ (i.e., SDF) is the best choice over the other alternatives.

The result establishes the fact that SAARC Development Fund is the most effective instrument to integrate the region most successfully along with the improvement in the socio-economic development of the region followed by South Asian Trade in Services and SAFTA. The financial support and protection in three basic grounds like social, infrastructural and economic projects of the members are essentially the "remedy for all diseases" as all the impediments to the economic development of the SAARC members which are all suffering from economic and structural bottlenecks can be taken care of by SDF efficiently and its success will naturally integrate and increase the cooperation of the members. The data from official cite of SDF [30] shows how the fund is contributing to the members of SAARC as shown in pie chart in Fig. 2.

Table 6
Assessment matrix acquired from $\mathbf{E}^{(2)}$.

	$\mathscr{C}_{11}$	$\mathscr{C}_{12}$	$\mathscr{C}_{13}$	$\mathscr{C}_{14}$	$\mathscr{C}_{15}$	$\mathscr{C}_{16}$	$\mathscr{C}_{17}$	$\mathscr{C}_{21}$	$\mathscr{C}_{22}$	$\mathscr{C}_{31}$	$\mathscr{C}_{32}$
$\mathscr{Z}_1$	$\{S\}$	$\{SS,ES\}$	$\{M\}$	$\{I\}$	$\{SI\}$	$\{I\}$	$\{M\}$	$\{AI,SI\}$	$\{AI,I,SI\}$	$\{M\}$	$\{SS\}$
$\mathscr{Z}_2$	$\{SI,SS\}$	$\{SS\}$	$\{SS\}$	$\{M,SS\}$	$\{AI\}$	$\{AI\}$	$\{SI\}$	$\{I\}$	$\{I\}$	$\{ES\}$	$\{SS,ES\}$
$\mathscr{Z}_3$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{SS\}$	$\{S\}$	$\{S\}$	$\{AI\}$	$\{M\}$	$\{M,SS\}$	$\{S\}$
$\mathscr{Z}_4$	$\{ES\}$	$\{ES\}$	$\{S,ES\}$	$\{ES\}$	$\{S\}$	$\{S,ES\}$	$\{S\}$	$\{ES\}$	$\{ES\}$	$\{M\}$	$\{M\}$
$\mathscr{Z}_5$	$\{AI,I\}$	$\{AI\}$	$\{AI\}$	$\{SS\}$	$\{S\}$	$\{I\}$	$\{M,SS\}$	$\{ES\}$	$\{S,ES\}$	$\{S\}$	$\{SS\}$
$\mathscr{Z}_6$	$\{ES\}$	$\{ES\}$	$\{S,ES\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{SS\}$	$\{S\}$	$\{S\}$	$\{S\}$

Table 7
Assessment matrix acquired from $\mathbf{E}^{(3)}$.

	$\mathscr{C}_{11}$	$\mathscr{C}_{12}$	$\mathscr{C}_{13}$	$\mathscr{C}_{14}$	$\mathscr{C}_{15}$	$\mathscr{C}_{16}$	$\mathscr{C}_{17}$	$\mathscr{C}_{21}$	$\mathscr{C}_{22}$	$\mathscr{C}_{31}$	$\mathscr{C}_{32}$
$\mathscr{Z}_1$	$\{S,ES\}$	$\{ES\}$	$\{S,ES\}$	$\{M,S\}$	$\{I\}$	$\{SI\}$	$\{M\}$	$\{AI\}$	$\{M\}$	$\{SI,M,SS\}$	$\{M,S\}$
$\mathscr{Z}_2$	$\{S\}$	$\{M,S\}$	$\{SI,S\}$	$\{M,S\}$	$\{I\}$	$\{I\}$	$\{M\}$	$\{SI\}$	$\{AI\}$	$\{S,ES\}$	$\{ES\}$
$\mathscr{Z}_3$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{SI,SS\}$	$\{S\}$	$\{SS\}$	$\{I\}$	$\{M\}$	$\{SI\}$	$\{S\}$
$\mathscr{Z}_4$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{SS\}$	$\{SS\}$	$\{S\}$	$\{S,ES\}$	$\{ES\}$	$\{AI\}$	$\{M,SS\}$
$\mathscr{Z}_5$	$\{AI\}$	$\{AI\}$	$\{M\}$	$\{SS\}$	$\{SS\}$	$\{I\}$	$\{SS,S\}$	$\{ES\}$	$\{M\}$	$\{M\}$	$\{M\}$
$\mathscr{Z}_6$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{S,ES\}$	$\{ES\}$	$\{ES\}$	$\{ES\}$	$\{S\}$	$\{SS\}$	$\{M\}$	$\{S\}$

Table 8
Aggregated decision matrix $[\widetilde{\wp}_{ij}]_{6\times 11}$.

	$\mathscr{C}_{11}$	$\mathscr{C}_{12}$	$\mathscr{C}_{13}$	$\mathscr{C}_{14}$	$\mathscr{C}_{15}$	$\mathscr{C}_{16}$
$\mathscr{Z}_1$	$\{(0.8416, 0.1305), (0.8751, 0.1136)\}$	$\{(0.8497, 0.1413), (0.9000, 0.1000)\}$	$\{(0.6785, 0.2979), (0.7432, 0.2597)\}$	$\{(0.6164, 0.4212), (0.7179, 0.2609), (0.6944, 0.3684), (0.7751, 0.2272)\}$	$\{(0.4319, 0.6638)\}$	$\{(0.4335, 6590), (0.6001, 7284)\}$
$\mathscr{Z}_2$	$\{(0.6663, 0.2997), (0.7107, 0.2369), (0.7244, 0.2364), (0.7612, 0.1866)\}$	$\{(0.5555, 0.4879), (0.6752, 0.3045), (0.6757, 0.3037), (0.7612, 0.1866)\}$	$\{(0.5354, 0.4909), (0.6752, 0.3045), (0.6622, 0.3057), (0.7612, 0.1866)\}$	$\{(0.6393, 0.3806), (0.7345, 0.2349), (0.6757, 0.3037), (0.7612, 0.1866)\}$	$\{(0.2706, 0.8972), (0.3487, 0.7807)\}$	$\{(0.3169, 0.8379)\}$
$\mathscr{Z}_3$	$\{(0.9000, 0.1000)\}$	$\{(0.8237, 0.1883), (0.9000, 0.1000)\}$	$\{(0.9000, 0.1000)\}$	$\{(0.9000, 0.1000)\}$	$\{(0.5354, 0.4909), (0.6062, 0.3842)\}$	$\{(0.7571, 0.1906)\}$
$\mathscr{Z}_4$	$\{(0.8724, 0.1150), (0.9000, 0.1000)\}$	$\{(0.9000, 0.1000)\}$	$\{(0.8751, 0.1136), (0.9000, 0.1000)\}$	$\{(0.9000, 0.1000)\}$	$\{(0.8071, 0.1656)\}$	$\{(0.8071, 0.1656), (0.8442, 0.1459)\}$
$\mathscr{Z}_5$	$\{(0.2000, 0.9500), (0.2660, 0.9019)\}$	$\{(0.2000, 0.9500)\}$	$\{(0.3607, 0.8409)\}$	$\{(0.6062, 0.3842)\}$	$\{(0.7574, 0.1904)\}$	$\{(0.3500, 0.7800)\}$
$\mathscr{Z}_6$	$\{(0.9000, 0.1000)\}$	$\{(0.9000, 0.1000)\}$	$\{(0.8067, 0.1660), (0.8438, 0.1462), (0.8751, 0.1136), (0.9000, 0.1000)\}$	$\{(0.8726, 0.1149), (0.9000, 0.1000)\}$	$\{(0.8724, 0.1150), (0.9000, 0.1000)\}$	$\{(0.9000, 0.1000)\}$

	$\mathscr{C}_{17}$	$\mathscr{C}_{21}$	$\mathscr{C}_{22}$	$\mathscr{C}_{31}$	$\mathscr{C}_{32}$
$\mathscr{Z}_1$	$\{(0.5604, 0.4779)\}$	$\{(0.2709, 0.8969), (0.3432, 0.7912)\}$	$\{(0.3607, 0.8409), (0.3874, 0.7832), (0.4081, 0.7296), (0.4121, 0.7186), (0.4333, 0.6590), (0.4504, 0.6066),\}$	$\{(0.4754, 0.6034), (0.5000, 0.6000), (0.5601, 0.4785)\}$	$\{(0.5555, 0.4879), (0.6752, 0.3045)\}$
$\mathscr{Z}_2$	$\{(0.4504, 0.6066)\}$	$\{(0.4005, 0.6625), (0.4335, 0.6590)\}$	$\{(0.3878, 0.7825)\}$	$\{(0.8383, 0.1321), (0.8724, 0.1150), (0.8726, 0.1149), (0.9000, 0.1000)\}$	$\{(0.7691, 0.2064), (0.8438, 0.1462), (0.8497, 0.1413), (0.9000, 0.1000)\}$
$\mathscr{Z}_3$	$\{(0.8071, 0.1656)\}$	$\{(0.3487, 0.7807)\}$	$\{(0.5604, 0.4779)\}$	$\{(0.4290, 0.6644), (0.4992, 0.5454)\}$	$\{(0.8000, 0.1500)\}$
$\mathscr{Z}_4$	$\{(0.7571, 0.1906), (0.8416, 0.1305)\}$	$\{(0.8726, 0.1149), (0.9000, 0.1000)\}$	$\{(0.9000, 0.1000)\}$	$\{(0.3823, 0.7880)\}$	$\{(0.4583, 0.6609), (0.5270, 0.5320)\}$
$\mathscr{Z}_5$	$\{(0.7437, 0.2592), (0.7885, 0.2045), (0.7696, 0.2060), (0.8102, 0.1624)\}$	$\{(0.8724, 0.1150)\}$	$\{(0.7830, 0.2130), (0.8243, 0.1877)\}$	$\{(0.6702, 0.3101)\}$	$\{(0.6064, 0.3837), (0.6757, 0.3037)\}$
$\mathscr{Z}_6$	$\{(0.9000, 0.1000)\}$	$\{(0.7612, 0.1866)\}$	$\{(0.8071, 0.1656)\}$	$\{(0.7830, 0.2130)\}$	$\{(0.8416, 0.1305)\}$

Table 9
Aggregated criteria importance and score values.

Criteria	$E^{(1)}$	$E^{(2)}$	$E^{(3)}$	Aggregated q-ROHFNs	Score Values
$\mathscr{C}_{11}$	ES	ES	ES	$\langle\{0.9000\}, \{0.1000\}\rangle$	0.8640
$\mathscr{C}_{12}$	ES	ES	ES	$\langle\{0.9000\}, \{0.1000\}\rangle$	0.8640
$\mathscr{C}_{13}$	ES	ES	ES	$\langle\{0.9000\}, \{0.1000\}\rangle$	0.8640
$\mathscr{C}_{14}$	ES	ES	ES	$\langle\{0.9000\}, \{0.1000\}\rangle$	0.8640
$\mathscr{C}_{15}$	S	SS	M	$\langle\{0.6757\}, \{0.3037\}\rangle$	0.6403
$\mathscr{C}_{16}$	M	M	M	$\langle\{0.5000\}, \{0.6000\}\rangle$	0.4545
$\mathscr{C}_{17}$	S	I	AI	$\langle\{0.5765\}, \{0.5246\}\rangle$	0.5236
$\mathscr{C}_{21}$	I	SI	M	$\langle\{0.4316\}, \{0.6641\}\rangle$	0.3937
$\mathscr{C}_{22}$	S	SS	SS	$\langle\{0.7110\}, \{0.2366\}\rangle$	0.6731
$\mathscr{C}_{31}$	S	S	S	$\langle\{0.8000\}, \{0.1500\}\rangle$	0.7543
$\mathscr{C}_{32}$	S	S	S	$\langle\{0.8000\}, \{0.1500\}\rangle$	0.7543

Table 10
Results of SWARA method for finding weights of the criteria.

Criteria	Score values	Comparative significance of criteria (α_j)	CC (κ_j)	Initial weight (ϱ_j)	Final weight (w_j)
$\mathscr{C}_{11}$	0.8640	–	1.0000	1.0000	0.1055
$\mathscr{C}_{12}$	0.8640	0	1.0000	1.0000	0.1055
$\mathscr{C}_{13}$	0.8640	0	1.0000	1.0000	0.1055
$\mathscr{C}_{14}$	0.8640	0	1.0000	1.0000	0.1055
$\mathscr{C}_{31}$	0.7543	0.1097	1.1097	0.9012	0.0951
$\mathscr{C}_{32}$	0.7543	0	1.0000	0.9012	0.0951
$\mathscr{C}_{22}$	0.6731	0.0812	1.0812	0.8335	0.0897
$\mathscr{C}_{15}$	0.6403	0.0329	1.0329	0.8070	0.0851
$\mathscr{C}_{17}$	0.5236	0.1166	1.1166	0.7227	0.0762
$\mathscr{C}_{16}$	0.4545	0.0691	1.0691	0.6760	0.0713
$\mathscr{C}_{21}$	0.3937	0.0608	1.0608	0.6372	0.0672

6. Sensitivity analysis

The proposed methods depend on two parameters, rung parameter, q and hamacher parameter, τ or frank parameter, ζ which is associated with the proposed operators. If the rung parameter, q is varied from 2 to 10 and fixing hamacher parameter, $\tau = 3$ or frank parameter, $\zeta = 3$, then the ranking through different proposed operators is shown in Figs. 3–6.

- Importance of parameter q.

Firstly, if q-ROHFHWA operator is used in the methodology and considering Hamacher parameter, $\tau = 3$ and varying rung parameter, q from 2 to 10. The obtained results are shown in Fig. 3. From Fig. 3, it is clear that SDF ($\mathscr{Z}_6$) is the best alternative. It's also observed that two types of ranking are possible. If $q \in [2, 3.115]$, the obtained ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$ and for $q \in [3.115, 10]$, ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$.

Similarly, if q-ROHFHWG operator is used, the obtained results are shown in Fig. 4. From Fig. 4, it is clear that SDF ($\mathscr{Z}_6$) is the best alternative. Here it is seen that three types of ranking are possible. If $q \in [2, 2.358]$, the obtained ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$. If $q \in [2.358, 9.795]$, the obtained ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$. And for $q \in [9.795, 10]$ the ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$.

Again, if q-ROHFFWA operator is used in the methodology and

Table 11
Best value and the worst value.

Criteria	q-ROHF-BV ($\tilde{\wp}^{+}$)	q-ROHF-WV ($\tilde{\wp}^{-}$)
$\mathscr{C}_{11}$	$\{(0.9000, 0.1000)\}$	$\{(0.2000, 0.9500), (0.2660, 0.9019)\}$
$\mathscr{C}_{12}$	$\{(0.9000, 0.1000)\}$	$\{(0.2000, 0.9500)\}$
$\mathscr{C}_{13}$	$\{(0.9000, 0.1000)\}$	$\{(0.3607, 0.8409)\}$
$\mathscr{C}_{14}$	$\{(0.9000, 0.1000)\}$	$\{(0.6062, 0.3842)\}$
$\mathscr{C}_{15}$	$\{(0.8724, 0.1150), (0.9000, 0.1000)\}$	$\{(0.2706, 0.8972), (0.3487, 0.7807)\}$
$\mathscr{C}_{16}$	$\{(0.9000, 0.1000)\}$	$\{(0.3169, 0.8379)\}$
$\mathscr{C}_{17}$	$\{(0.9000, 0.1000)\}$	$\{(0.4504, 0.6066)\}$
$\mathscr{C}_{21}$	$\{(0.8726, 0.1149), (0.9000, 0.1000)\}$	$\{(0.2709, 0.8969), (0.3432, 0.7912)\}$
$\mathscr{C}_{22}$	$\{(0.9000, 0.1000)\}$	$\{(0.3878, 0.7825)\}$
$\mathscr{C}_{31}$	$\{(0.8383, 0.1321), (0.8724, 0.1150), (0.8726, 0.1149), (0.9000, 0.1000)\}$	$\{(0.3823, 0.7880)\}$
$\mathscr{C}_{32}$	$\{(0.7691, 0.2064), (0.8438, 0.1462), (0.8497, 0.1413), (0.9000, 0.1000)\}$	$\{(0.4583, 0.6609), (0.5270, 0.5320)\}$

Table 12
Weighted distance from best value and worst value.

	$\mathscr{Z}_1$	$\mathscr{Z}_2$	$\mathscr{Z}_3$	$\mathscr{Z}_4$	$\mathscr{Z}_5$	$\mathscr{Z}_6$
$d_w(\mathscr{Z}_i, \tilde{\wp}^{+})$	0.4510	0.4852	0.3983	0.3784	0.5608	0.3171
$d_w(\mathscr{Z}_i, \tilde{\wp}^{-})$	0.4116	0.4416	0.6799	0.7647	0.4233	0.8751

considering Frank parameter, $\zeta = 3$ and varying rung parameter, q from 2 to 10. The obtained results are shown in Fig. 5. This result is very similar to the results of q-ROHFHWA operator. From Fig. 5, it is also clear that two types of ranking are possible. If $q \in [2, 3.215]$, the obtained ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$ and for $q \in [3.215, 10]$, ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$. So, it is clear that SDF ($\mathscr{Z}_6$) is the best alternative.

Similarly, if q-ROHFHWG operator is used, the obtained results are shown in Fig. 6. This result is quite similar to the results of q-ROHFHWG operator. From Fig. 6, it is clear that SDF ($\mathscr{Z}_6$) is the best alternative. Here it is seen that three types of ranking are possible. If $q \in [2, 2.325]$, the obtained ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$. If $q \in [2.325, 8.65]$, the obtained ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$. And for $q \in [8.65, 10]$ the ranking is $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$.

More over if Hamacher parameter (τ) and Frank parameter (ζ) is varied from 2 to 10 and fixing the rung parameter, q for some values in $[2, 10]$ then the ranking of the alternatives remains same i.e., $\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$. So, for all cases, it is investigated that $\mathscr{Z}_6$, i.e., SDF is the best alternative which needs to be improved firstly.

7. Comparative studies

To determine the validity and efficiency of the proposed method, this case study is compared with several existing methods through different AOs, viz., q-ROHFWPGHM [10], q-ROHFWPGGHM [10], PHFWA [11], PHFWG [11], PHFEWA [11], PHFEWG [11], PHFHWA [11], PHFHWG [11], PHFFWA [11], PHFFWA [11]. Comparisons are carried out using two different approaches, which offer diverse insights and allow for validation and cross-checking of results. By employing distinct methods, researchers can identify limitations, complement findings, enhance robustness, and address biases. Transparency about the chosen methods and their rationale is crucial for scrutiny and validation by the scientific community and interested parties. At first, the comparisons are established based on characteristic of the operators. Following that, the comparisons shift towards evaluating the achieved results of the operators, focusing on their performance in practical applications or experiments.

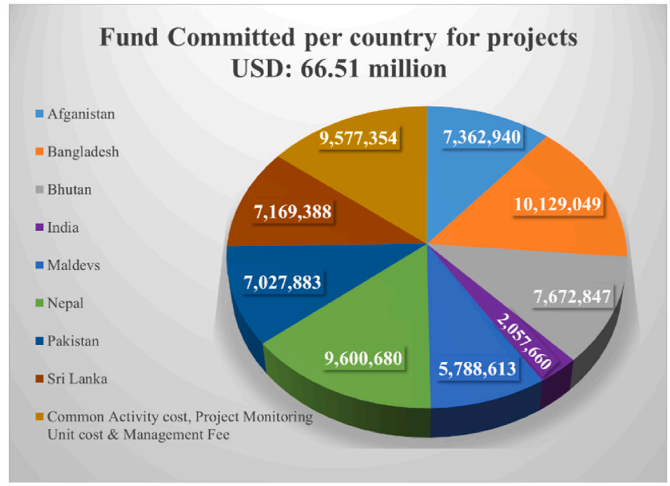

Fig. 2. SDF Contribution as per session report of SDF, April 2023.

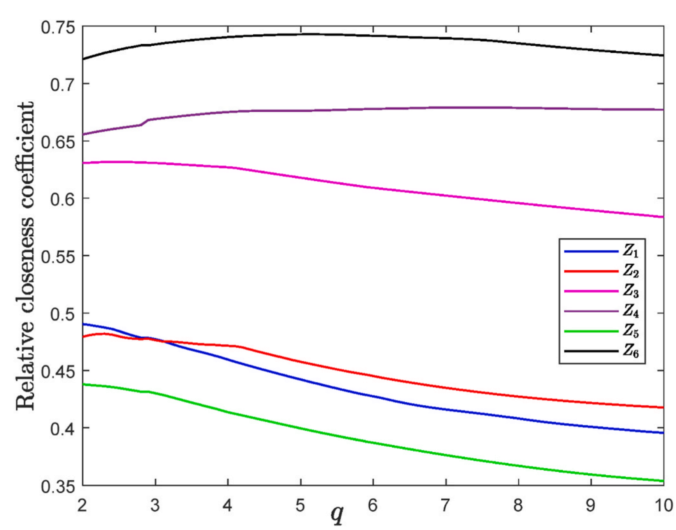

Fig. 3. Relative closeness coefficient of alternatives using q-ROHFHWA operator varying q.

Table 13
Results through different developed operators.

Operators	$\mathfrak{RC}(\mathscr{Z}_1)$	$\mathfrak{RC}(\mathscr{Z}_2)$	$\mathfrak{RC}(\mathscr{Z}_3)$	$\mathfrak{RC}(\mathscr{Z}_4)$	$\mathfrak{RC}(\mathscr{Z}_5)$	$\mathfrak{RC}(\mathscr{Z}_6)$	Rankings
q-ROHFWA	0.4779	0.4761	0.6297	0.6695	0.4359	0.7340	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
q-ROHFWG	0.4541	0.4697	0.6286	0.6595	0.4078	0.7333	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$
q-ROHFEWA	0.4771	0.4759	0.6303	0.6692	0.4322	0.7340	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
q-ROHFEWG	0.4556	0.4694	0.6258	0.6594	0.4101	0.7331	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$
q-ROHFHWG	0.4565	0.4696	0.6253	0.6594	0.4111	0.7330	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$
q-ROHFFWA	0.4780	0.4763	0.6301	0.6692	0.4327	0.7339	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
q-ROHFFWG	0.4556	0.4699	0.6262	0.6593	0.4093	0.7330	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$

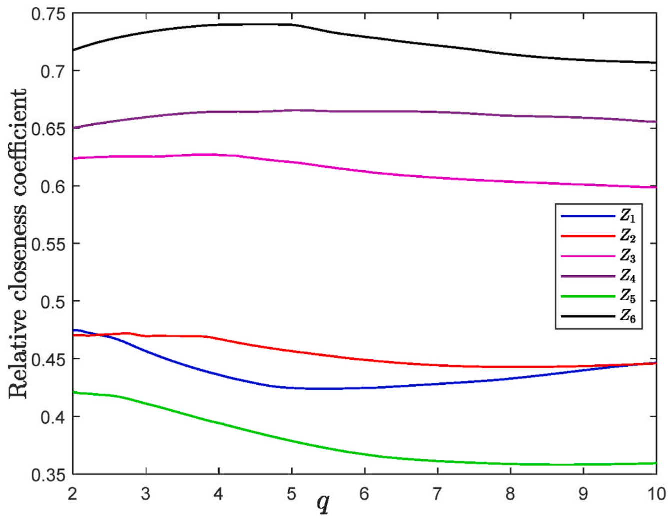

Fig. 4. Relative closeness coefficient of alternatives using q-ROHFHWG operator varying q.

when comparing the approach based on the features of the operators, it is crucial to keep in mind that all of the produced operators and the aforementioned existing operators are capable of capturing hesitant hazy information. The AOs viz., q-ROHFWPGHM, q-ROHFWPGGHM proposed by Wang et al. [10] are based on some algebraic t-N&t-CNs, which are not general and flexible in nature. The proposed aggregation method employs a family of At-N&t-CNs including algebraic, Einstein, Hamacher, Frank, etc., classes. Thus, the developed operators possess the ability to make the aggregation process more robust and smoother by including various types of At-N&t-CNs in the aggregation functions. Further, the proposed AOs, viz., q-ROHFHWA, q-ROHFHWG, q-ROHFFWA, and q-ROHFFWG, include various flexible parameters that can reflect the attitudes of DMs allowing their risk preferences. Moreover, the designed AOs in this paper can generate a list of AOs as their special cases by considering specific decreasing generators. Hence the proposed method is more general than the existing method [10]. Again, the operators proposed by Sarkar and Biswas [11] are in PHFS environment which is a lower domain of the q-ROHFS. So, operator proposed by Sarkar and Biswas [11] can be considered as a special case of proposed operators by considering $q = 2$.

Now the achieved results are compared with the results obtained through wang et al. [10] and Sarkar and Biswas [11] methods given in Table 14. It is worthy to mention here, the methodology proposed by wang et al. [10] and Sarkar and Biswas [11] depends on known weights. So, this case study cannot solve directly as the weights of the criteria and DMs are completely unknown here. So, proposed method can overcome this drawback of the existing operators. More over to solve the case study with those methods, the weights which obtained by proposed methodology are considered. From Table 14, it is found that the ranking of the alternatives is almost same and $\mathscr{Z}_6$ (SDF) is the best alternative which validates the proposed method.

Moreover, the difference of score values between two consecutive alternatives (rank-wise) obtained through method by Wang et al. [10] and proposed method are shown in Fig. 7. From Fig. 7 it is very clear that the score values between two consecutive alternatives in method by Wang et al. [10] is lower than proposed method. Therefore, the proposed method is superior to Wang et al. [10] in terms of choosing the optimal choice.

8. Conclusions and future studies

q-ROHFS is a generalization of the other variants of fuzzy sets viz., Intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (PFSs), q-ROFSs, HFSs. So, q-ROHFS cover more info (both membership and non-membership) than IFSs, PFSs, q-ROFSs, HFSs. The SWARA-TOPSIS methodology offers a robust and comprehensive framework for decision-making in complex and multi-criteria scenarios. By combining the strengths of the SWARA and TOPSIS approaches, this methodology provides a systematic way to evaluate alternatives, assign appropriate weights to criteria, and select the most suitable option. In view of this type of flexibility, a SWARA-TOPSIS-based MCGDM method is developed on q-ROHF environment. To aggregate q-ROHF data, several At-CN&t-Ns-based AOs, viz., q-ROHFWA, q-ROHFWG, q-ROHFEWA, q-ROHFEWG, q-ROHFHWA, q-ROHFHWG, q-ROHFFWA, and q-ROHFFWG are introduced. Also, to perform TOPSIS method, a novel distance measure is proposed. Additionally, by changing the Hamacher and Frank parameters, all generalised cases are taken into account. Since these operators are far more trustworthy than other existing aggregation operators on such sets, they may be used to solve many decision-making issues more successfully. The recommended operators have the ability to detect moments of human uncertainty and comprehend the interconnections between combined points of view. Furthermore, the suggested approaches might autonomously adapt the parameter's value based on the risk tolerance thresholds of the decision-maker.

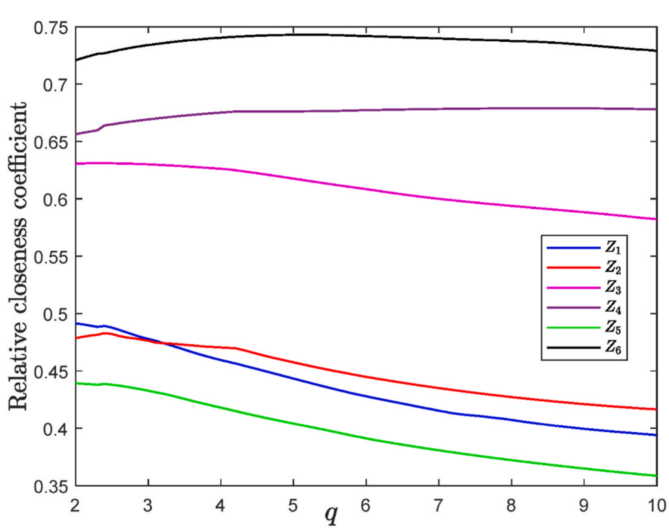

Fig. 5. Relative closeness coefficient of alternatives using q-ROHFFWA operator varying q.

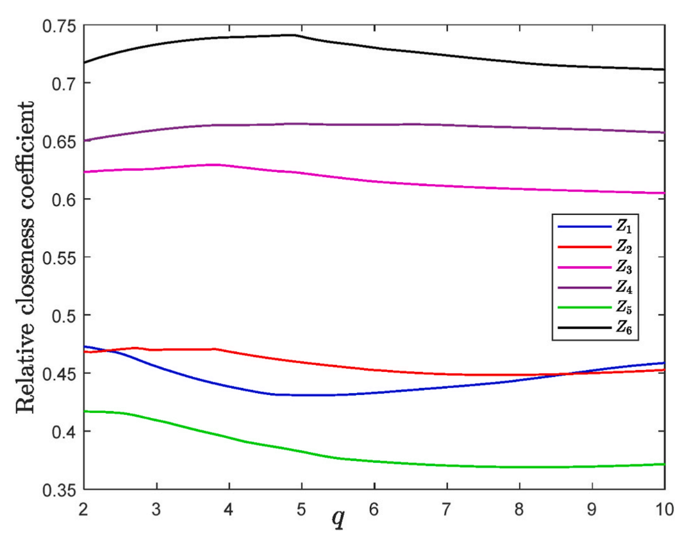

Fig. 6. Relative closeness coefficient of alternatives using q-ROHFFWG operator varying q.

Table 14
Comparison results with existing methods.

Operators	$S(\mathscr{Z}_1)$	$S(\mathscr{Z}_2)$	$S(\mathscr{Z}_3)$	$S(\mathscr{Z}_4)$	$S(\mathscr{Z}_5)$	$S(\mathscr{Z}_6)$	Rankings
q-ROHF WPGHM [10]	0.0425	0.0450	0.0670	0.0771	0.0402	0.0856	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$
q-ROHF WPGGHM [10]	0.7970	0.7910	0.8358	0.8566	0.7775	0.8764	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
PHFWA [11]	0.6691	0.6677	0.7974	0.8392	0.6261	0.8692	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
PHFWG [11]	0.4722	0.4509	0.6424	0.6827	0.3310	0.8378	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
PHFEWA [11]	0.6498	0.6491	0.7868	0.8318	0.5985	0.8670	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
PHFEWG [11]	0.4973	0.4817	0.6691	0.7164	0.3660	0.8437	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
PHFHWA [11]	0.6397	0.6390	0.7810	0.8276	0.5826	0.8660	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
PHFHWG [11]	0.5113	0.4985	0.6837	0.7338	0.3854	0.8470	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
PHFFWA [11]	0.6595	0.6588	0.7921	0.8355	0.6123	0.8680	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
PHFFWG [11]	0.4843	0.4661	0.6549	0.6987	0.3479	0.8403	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
q-ROHFWA	0.4779	0.4761	0.6297	0.6695	0.4359	0.7340	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
q-ROHFWG	0.4541	0.4697	0.6286	0.6595	0.4078	0.7333	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$
q-ROHFEWA	0.4771	0.4759	0.6303	0.6692	0.4322	0.7340	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
q-ROHFEWG	0.4556	0.4694	0.6258	0.6594	0.4101	0.7331	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$
q-ROHFHWA	0.4771	0.4765	0.6306	0.6690	0.4301	0.7340	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
q-ROHFHWG	0.4565	0.4696	0.6253	0.6594	0.4111	0.7330	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$
q-ROHFFWA	0.4780	0.4763	0.6301	0.6692	0.4327	0.7339	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_1 \succ \mathscr{Z}_2 \succ \mathscr{Z}_5$
q-ROHFFWG	0.4556	0.4699	0.6262	0.6593	0.4093	0.7330	$\mathscr{Z}_6 \succ \mathscr{Z}_4 \succ \mathscr{Z}_3 \succ \mathscr{Z}_2 \succ \mathscr{Z}_1 \succ \mathscr{Z}_5$

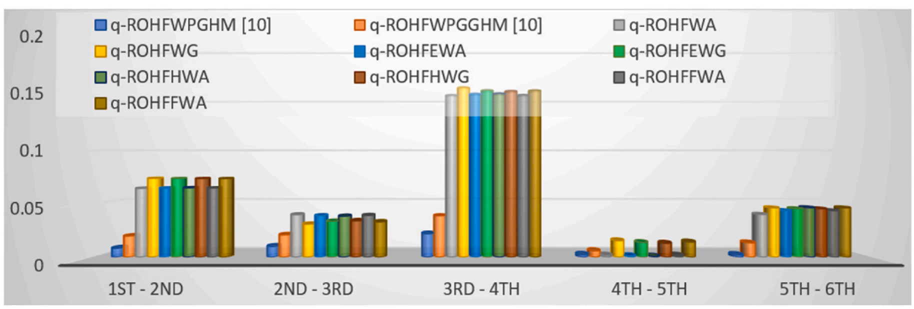

Fig. 7. Difference of Score values between two consecutive alternatives (rank-wise).

To establish application potentiality of the developed method a case study relating to assess the triage and efficacy of strategies adopted by SAARC for regional integrity of south Asia has been performed. Using *q*-ROHFS to find the efficacy of some of the strategies taken by SAARC for the integration and development of the region is an independent and disparate inspection of the results derived from the usual econometric and statistical method of analysis. The proposed method shows that SAARC Development Fund has the capability to integrate and develop the region most among other strategies. Practically, the final and complete stage of economic integration (SAEU) can only be achieved if there arises transparency in all the sectors in the region with financial cooperation among the members. The next best weapon in the hand of SAARC is SATIS which tries to integrate trade in service sector. The third best strategy is SAFTA as derived by this study. Establishment of South Asian University can be a better approach to have higher education in the region but without fulfilling the other conditions of peace, integration, and cooperation it may not work effectively and that is obvious from the result of the study.

The limitation of this paper is that this method cannot capture dual hesitant *q*-ROF, Type 2 fuzzy, *T*-Spherical fuzzy information also proposed method cannot capture the intra relationship between criteria, in those cases, results may be differ. To overcome this, several developments may be performed in future: Choquet Integral-based AOs. The proposed operators can be developed to address scenarios encompassing complete or vague probabilistic linguistic preference relationships within the context of group decision-making. multi-criteria group decision making with large-scale data in dual hesitant *q*-ROF, Type 2 fuzzy, *T*-Spherical fuzzy environment.

Policy implications

- SDF can be viewed as best instrument in solving poverty, inequality and unemployment of all the SAARC countries which are the root causes of underdevelopment of them and therefore the main obstacles in their way of regionalization, mutual cooperation. So, the member countries of SAARC should make a proper planning to make a strong developmental fund for themselves and work on it authentically.
- SAARC should also give importance to service sector trade because service sector integrity will make a strong base for financial integration of all the similar countries of South Asia.
- MCDGM can be viewed as a good technique in taking the decision of socio-economic planning not only for SAARC countries but also for other developing countries where there exists uncertainties and unavailability of proper data along with the presence of interdependent factors.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

CRediT authorship contribution statement

Souvik Gayen: Methodology, Validation, Visualization, Software, Writing - Original Draft, Review & Editing.

Debamitra Banerjee: Economic issues selection, Validation, Investigation, Writing - Original Draft, Review & Editing.

Arun Sarkar: Conceptualization, Software, Visualization, Investigation, Writing - Original Draft, Review & Editing.

Animesh Biswas: Investigation, organization, Writing - Original Draft, Review & Editing.

Declaration of competing interest

All the authors declare that they have no conflict of interest.

Data availability

Data is available in the article

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.seps.2023.101766.

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Mr. Souvik Gayen is acting as a Research scholar in the Department of Mathematics, University of Kalyani, India. He received M.Sc. degree in Mathematics from IIT Kanpur, India in 2017. He has published some research articles in leading international journals and attended several international and national conferences. The area of his research includes Multicriteria decision making, Data clustering, Pattern recognition, etc, in various fuzzy environment. Mr. Gayen is also a member of the Operational Research Society of India.

Dr. Debamitra Banerjee is presently serving as lecturer in Economics in Sarsuna College, Behala, Kolkata, since 2007. She has completed her Master's degree in 1996 with International Trade as special paper from University of Calcutta and got her PhD degree in the year 2022 from West Bengal State University, Barasat. Her research interest is based on international trade and international labour migration in SAARC countries. Some of her research work have been published in 'Vidyasagar University Journal of Economics', 'Arthaniti Katha'-Journal of Calcutta University.

Dr. Arun Sarkar is now working as an Assistant Professor in the Department of Mathematics at Heramba Chandra College, Kolkata 700,029, India, since 2017. He received M. Sc. degree in Applied Mathematics from University of Kalyani, India, in 2009. He was awarded the Ph.D. degree in Mathematics by University of Kalyani, India. His research interests include the development of aggregation operators in different variants of fuzzy sets, fuzzy decision making and their applications, etc. He has produced 25 research articles in leading International Journals and Conferences. He is a member of Operational Research Society of India and Indian Statistical Institute, Kolkata, India.

Dr. Animesh Biswas is rendering his service as a Professor in the Department of Mathematics, University of Kalyani, India. He is an active researcher in the fields of Mathematical Modelling, Optimization Techniques, Decision Making under imprecise environments, Fuzzy Logic, Soft Computing, etc. Eight research students received Ph.D. degree under his supervision and five are pursuing research towards Ph.D. degree. Dr. Biswas produced about 100 research articles in different reputed international journals. He attended various international conferences within India and abroad, delivered invited talks, presented papers. Dr. Biswas is a member of Indian Statistical Institute, Kolkata, India, Operational Research Society of India, Soft Computing Research Society, India, Tripura Mathematical Society, India, Computational Intelligence Society, IEEE, World Academy of Science, Engineering and Technology (WASET), International Association of Engineers (IAENG), International Economics Development Research Center (IEDRC), Multicriteria Decision Making Society, Germany, etc.

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Weighted dual hesitant *q*-rung orthopair fuzzy sets and their application in multicriteria group decision making based on Hamacher operations

Arun Sarkar[1] · **Nayana Deb**[2] · **Animesh Biswas**[2]

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Abstract

Dual hesitant q-rung orthopair fuzzy set has already been appeared as a useful tool to express fuzzy and ambiguous information more precisely than other variants of fuzzy sets. Usually, equal weights of the possible membership as well as non-membership values in a dual hesitant q-rung orthopair fuzzy set, are considered in modelling decision making problems, which is quite unreasonable. Because, in ascertaining possible membership or non-membership values for an alternative under some criteria, the frequency level of appearing those values frequently differs. Thus, employing same weights/ degrees of importance to each of the assigned membership and non-membership values would affect overall process of decision making. To overcome such situation, this paper introduces the notion of weighted dual hesitant q-rung orthopair fuzzy set which allows decision makers to assign different weights of possible arguments in details. Taking advantage of Hamacher t-norms and t-conorms as a generalization of algebraic and Einstein operations, some operational laws for weighted dual hesitant q-rung orthopair fuzzy sets are investigated in this paper. Further, based on those defined operational rules, a series of weighted aggregation operators are proposed to aggregate the weighted dual hesitant q-rung orthopair fuzzy information effectively. Next, applying the proposed operators, a methodology for solving real-life group decision making problems under weighted dual hesitant q-rung orthopair fuzzy context is developed. Lastly, the aptness of the introduced method is illustrated by solving few numerical examples.

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✉ Arun Sarkar
asarkarmth@gmail.com

Nayana Deb
nayanadeb1@gmail.com

Animesh Biswas
abiswaskln@rediffmail.com

1 Department of Mathematics, Heramba Chandra College, Kolkata 700029, India

2 Department of Mathematics, University of Kalyani, Kalyani 741235, India

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1 Introduction

To deal with uncertainties and imprecision in different complicated environments, Zadeh (1965) introduced the concept of fuzzy sets. Later on, many researchers extended the idea of fuzzy sets to cope with vagueness more rigorously. Atanassov (1986) developed intuitionistic fuzzy sets (IFS), which have membership along with non-membership degrees having their sum less than or equal to 1. Later on, Yager (2013, 2014) established the concept of Pythagorean fuzzy (PF) sets (PFSs), which is a generalization of IFS. PFS consists of membership and non-membership degrees satisfying the condition that their square sum is less than or equal to 1. After the inception of PFS, it is successfully implemented on various types of real-life based problems (Sarkar and Biswas 2019, 2020; Biswas and Deb 2020; Gayen and Biswas 2021; Deb and Biswas 2021; Sarkar et al. 2021). But, PFS contains some drawbacks. It fails to compute the decision arguments like (0.6, 0.9) where the square sum of membership and non-membership parts becomes greater than 1. To address such difficulties, Yager (2017) further extended PFSs to a more generalized set, called q-rung orthopair fuzzy (q-ROF) set (q-ROFS). The distinguishing characteristic of q-ROFS is that the sum of the qth power of membership and non-membership degrees is not exceeding 1. For $q = 1$ and 2, q-ROFS becomes IFS and PFS, respectively. So q-ROFS is more extensively applied to handle decision arguments than IFS and PFS. Also for a clearer understanding of the satisfactory region for IFS, PFS, and q-ROFS, Fig. 1 is provided. Recently Senapati and Yager (2020) introduced Fermatean fuzzy (FF) sets, which is also a special case of q-ROFS, with $q = 3$. Due to this broader flexibility of q-ROFS, numerous researchers investigated and developed plenty of theories on it. Liu and Wang (2017) introduced some fundamental theories of q-ROFS and proposed weighted averaging (WA) and weighted geometric (WG) aggregation operators: q-ROF WA (q-ROFWA) and q-ROF WG (q-ROFWG) aggregation operators. Wei et al. (2018a, b) introduced generalized geometric Heronian mean in q-ROFSs and developed several aggregation operators to solve multicriteria decision making (MCDM) problems. Peng et al. (2018) proposed a new score function for q-ROF number (q-ROFN) and derived q-ROF weighted exponential aggregation operator by introducing exponential operational laws on q-ROFS. Wang et al. (2019b) presented a family of q-ROF Muirhead mean (MM) operators, viz., q-ROF MM, q-ROF weighted MM, q-ROF dual MM and q-ROF weighted dual MM operators. Combining Bonferroni mean (BM) with q-ROFNs, Liu and Wang (2019) defined some aggregation operators based on Archimedean t-norms and t-conorms. Further, Yang and Pang (2019) introduced partitioned BM in q-ROF environment, and they also developed two q-ROF aggregation operators, viz., q-ROF partitioned BM and geometric BM operators. Wang et al. (2019a) proposed several aggregation operators based on Hamy mean, dual Hamy mean and their weighted variants. Jana et al. (2019) introduced Dombi operations to construct q-ROF Dombi aggregation operators, such as q-ROF-Dombi WA, Dombi ordered WA, Dombi hybrid WA, and their geometric counterparts. To aggregate q-ROFNs, Xing et al. (2019) developed a class of q-ROF point WA and WG aggregation

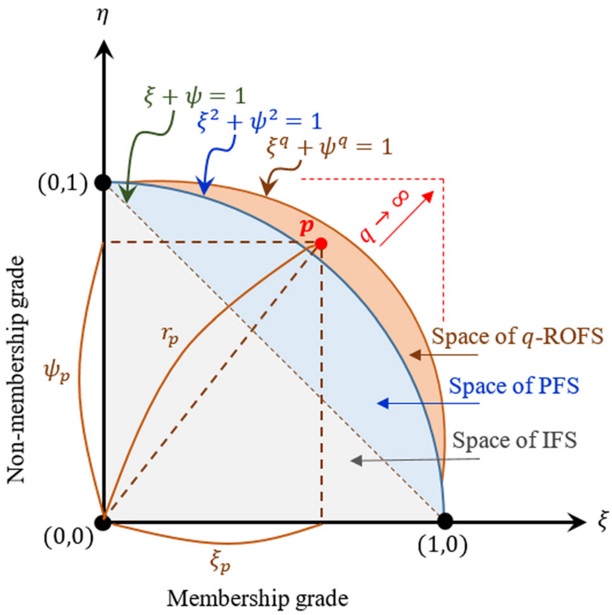

Fig. 1 Satisfying areas of IFS, PFS, and q-ROFS

operators by introducing point operators, viz., q-ROF point WA, q-ROF point WG, generalized q-ROF point WA and generalized q-ROF point WG operators. Considering fair or neutral decisions, Garg and Chen (2020) presented q-ROF weighted neutral averaging, ordered neutral averaging and hybrid neutral averaging aggregation operators. Darko and Liang (2020a) developed some new Hamacher operations for q-ROFS, and applied those operations to develop q-ROF Hamacher average, q-ROF Hamacher Maclaurin symmetric mean and corresponding weighted aggregation operators.

In daily life, decision makers (DMs) often become irresolute and hesitant among several possible judgement values while making decisions. To capture these instances, Torra (2010) proposed a new variant of fuzzy set called hesitant fuzzy set (HFS), which ease peoples' difficulty in expressing their ambiguous decision values. It allows the DMs to provide their evaluation values corresponding to each element using a set of possible assessing values instead of a single membership value in [0, 1]. Further, the scarcity of non-membership values in HFSs was identified by Zhu et al. (2012). They accomplished this need and developed the concept of dual hesitant fuzzy (DHF) sets (DHFSs). DHFS can capture both the membership and non-membership grades with different possible decision values. Subsequently, the theory and its applications are investigated by numerous scholars (Wang et al. 2016; Biswas and Sarkar 2019; Darko and Liang 2020b). Fusing HFS (Torra 2010) with q-ROFS (Yager 2017), Xu et al. (2018) introduced the concept of dual hesitant q-ROF (DHq-ROF) set (DHq-ROFS). In DHq-ROFS, the membership and non-membership values of objects are represented by two sets of several possible numbers. The significance of DHq-ROFSs in dealing with decision making problems is that they can portray inherent hesitancy and complexity in a more extensive way by changing the q parameter according to their needs. Wang et al. (2019c) proposed some MM and dual MM (DMM) operators under DHq-ROF environment, viz., DHq-ROF MM, DHq-ROF DMM and their weighted operators. Based on

Hamacher operations, Wang et al. (2019d) developed several WA, ordered WA and hybrid average operators under DHq-ROF environment along with geometric operators. For solving multi-attribute decision making (MADM) problems, Akram et al. (2021a) introduced a new approach by extending the concept of q-ROF graphs to DHq-ROF context, and based on Hamcher operation, they proposed DHq-ROF Hamacher graphs. Recently, Sarkar and Biswas (2021) developed BM operators for aggregating DHq-ROF information based on Dombi t-conorms and t-norms to solve multicriteria group decision making (MCGDM) problems.

However, in MCDM, the level of importance of various possible evaluation values for a specific object may vary depending on various situations. Thus, it is necessary to assign degrees of individual importance to the evaluation values associated with the process of decision making to prevent loss of essential information.

It is crucial to observe that, in the above-mentioned hesitant fuzzy contexts, the possible membership degrees or non-membership degrees maintain an equal level of importance, which may result an unreasonable consequence. To overcome such drawbacks in HFS, Zhang and Wu (2014) developed a remarkable perception, called weighted HFS (WHFS), by incorporating individual possible importance degrees to membership values. Also, they introduced some operations on WHFS using Archimedean t-conorms and t-norms and delivered some Archimedean operation-based WHF WA and WG aggregation operators. Further, Zeng et al. (2019a) designed different weights to the possible interval-valued (IV) membership and non-membership terms of IV HFS (IVHFS) and introduced the concept of weighted IVHFS (WIVHFS). An MCGDM model in accordance with the weights of DMs (whether known or unknown) is also presented by them. Again, Zeng et al. (2019b) developed weighted hesitant fuzzy linguistic (WHFL) term sets considering different confidences of DMs. They also provided a list of aggregation operators, viz., WHFL WA, WHFL WG, generalized WHFL WA and the generalized WHFL WG operators for fusing WHFL data. The concept of weighted dual hesitant (WDH) fuzzy (WDHF) set (WDHFS) was generated by Zeng et al. (2020) by taking into account the important degrees of several possible membership and non-membership grades of each element. They investigated the operations and properties of WDHFSs, and presented several WDHF aggregation operators, viz., WDHF WA and WDHF WG operators, to apply them in group decision making contexts. Recently, Ali et al. (2021) introduced weighted IVDHF sets (WIVDHFSs) and developed WIVDHF aggregation operators based on Archimedean operations.

As per the authors' knowledge, importance degrees related to hesitant membership and non-membership degrees of DHq-ROFS are not found yet in the existing literature. It is the fact that ignoring individual weightage or degree of importance corresponding to hesitant membership and non-membership values for assessing a variable may cause a distortion of information. To validate that point, the following example is provided for a better understanding of the necessity of the incorporation of importance degrees.

Suppose an educational institution wants to recruit an efficient teacher. This task would be performed by evaluating three aspirants T_i $(i = 1, 2, 3)$ on the basis of some relevant criteria. For this purpose, the institute hired ten DMs to anonymously provide their opinions about the candidates in view of the criteria. The experts provided their assessment values using q-ROFNs. For candidate T_1, suppose five experts provided assessment value as $(0.5, 0.2)$, two experts provided $(0.8, 0.1)$, and three experts $(0.7, 0.2)$. Similarly, for T_2, five experts provided $(0.8, 0.1)$, two experts provided $(0.5, 0.2)$, and three experts provided $(0.7, 0.2)$ as assessment values. For T_3, five experts provided $(0.7, 0.2)$, two experts give $(0.8, 0.1)$ and three experts give $(0.5, 0.2)$ as assessment values. Now, if the decision values provided by the experts are represented by a DHq-ROFN, the number would be

$\widetilde{d} = (\{0.5, 0.7, 0.8\}, \{0.1, 0.2\})$ for every aspirant T_i $(i = 1, 2, 3)$. So, it is now difficult to select the best one. If the weight/importance degrees of membership and non-membership values are assigned by counting the number of experts who provided the same assessment values, the problem can be resolved. Thus, assigning individual weights to hesitant membership and non-membership degrees of DHq-ROFS would produce a more prominent and rational result in practical decision cases.

Being motivated by the above interpretation, in this article, weighted DHq-ROFS (WDHq-ROFS) is developed as a significant tool for relevant research. The proposed WDHq-ROFS consists of membership part together with the non-membership part characterized by their independent weights/importance degrees. It is noteworthy to mention here that several existing fuzzy variants viz., DHq-ROFS, dual hesitant Pythagorean fuzzy set, DHFS, HFS etc., can be accomplished from the proposed concept. In this manner, the proposed WDHq-ROFS is recognized as a powerful means for assessing uncertainties connected with realistic instances.

For aggregating WDHq-ROF information, Hamacher t-norms and t-conorms (Ht-N&t-CNs) are considered in this paper. In the past few years, Hamacher operation-based aggregation operators became a leading research topic (Darko and Liang 2020a; Wei et al. 2018a, b; Wang et al. 2021; Akram et al. 2021b; Hadi et al. 2021; Shahzadi et al. 2021; Gayen et al. 2022). The basic algebraic and Einstein t-norms and t-conorms can be easily obtained from Ht-N&t-CNs, and also a comprehensive class of t-norms together with t-conorms can be generated by changing the value of Hamacher parameter. Thus, employing Ht-N&t-CNs for the yielding of WDHq-ROF information would be an impactful development that would enrich the information aggregation premise.

So the motivation and objective of this paper are:

1. to introduce the notion of WDHq-ROFSs and WDHq-ROFNs by incorporating different weightage values/importance degrees among possible membership and non-membership terms.
2. to develop some Hamacher operational rule-based aggregation operators, viz., WDHq-ROF Hamacher WA (WDHq-ROFHWA), and WDHq-ROF Hamacher WG (WDHq-ROFHWG) operators for the purpose of solving MCGDM.
3. to propose a systematic approach to enrich the MCGDM method in WDHq-ROF environment.
4. to apply the suggested approach in solving real-world MCGDM problems to demonstrate its feasibility and to provide a comparative analysis to highlight its benefits.

The main difference between the proposed method and the existing methods (Darko and Liang 2020a; Garg 2018; Wang et al. 2014; Wei and Lu 2017) is that the proposed method can reflect the weightage of importance of each decision information, whereas the existing methods use equal weightage values/importance degrees among possible membership and non-membership terms. But, in real-life decision-making contexts, preference degrees of different possible membership and non-membership values may deviate from each other. Therefore, the proposed method is more appropriate than existing method for dealing with MCGDM problems under uncertain contexts.

The remainder of this paper is structured as follows. Section 2 briefly recalls basic concepts related to WDHFS, q-ROFS and DHq-ROFS. In Sect. 3, the concept of WDHq-ROFS is introduced. Also, the score and accuracy functions are defined for ordering WDHq-ROFNs. Further, some algebraic operational rules of WDHq-ROFNs are explored based on Ht-N&t-CNs. Hamacher operations-based aggregation operators: WDHq-ROFHWA and WDHq-ROFHWG operators are developed in Sect. 4. In Sect. 5, a novel method for solving

MCGDM problems under WDHq-ROF context is presented utilizing the proposed WDHq-ROF aggregation operators. Section 6 comprises some numerical examples to explore the rationality and applicability of the proposed approach. The next section compares the developed model with several well-known existing approaches to show the validity and effectiveness of the proposed model. Finally, Sect. 8 summarizes the paper. Finally, the conclusion is made and the scope for future works is provided in Sect. 8.

2 Preliminaries: concepts and definitions

This section is comprised of some fundamental definitions and concepts related to WDHFS, q-ROFS, DHq-ROFS and Ht-N&t-CNs.

2.1 WDHFS

Definition 2.1.1 (Zeng et al. 2020) Let X be a fixed set, then a WDHFS, $\mathbb{D}^{\mathfrak{w}}$ on X is defined as:

$$\mathbb{D}^{\mathfrak{w}} = \left\{\left\langle x, \mathcal{Y}^{\mathfrak{w}}(x), \mathcal{G}^{\mathfrak{w}}(x)\right\rangle | x \in X\right\}, \tag{1}$$

in which $\mathcal{Y}^{\mathfrak{w}}(x) = \bigcup_{\langle\gamma, \mathfrak{w}_\gamma\rangle \in \mathcal{Y}^{\mathfrak{w}}(x)} \left\{\left\langle\gamma, \mathfrak{w}_\gamma\right\rangle\right\}$, and $\mathcal{G}^{\mathfrak{w}}(x) = \bigcup_{\langle\eta, \mathfrak{w}_\eta\rangle \in \mathcal{G}^{\mathfrak{w}}(x)} \left\{\left\langle\eta, \mathfrak{w}_\eta\right\rangle\right\}$ represent the collection of possible membership and non-membership degrees of an element $x \in X$ to the set $\mathbb{D}^{\mathfrak{w}}$, respectively. Where $\gamma, \eta \in [0, 1]$ with $0 \le \gamma + \eta \le 1$, and $\sum \mathfrak{w}_\gamma = 1$, $\sum \mathfrak{w}_\eta = 1$, $\mathfrak{w}_\gamma, \mathfrak{w}_\eta \in [0, 1]$. In WDHFS $\mathbb{D}^{\mathfrak{w}}$, $\mathfrak{w}_\gamma$ and $\mathfrak{w}_\eta$ represent the importance degrees corresponding to the possible membership and non-membership values γ and η, respectively. For convenience, Zeng et al. (2020) called $d^{\mathfrak{w}} = \left(\mathcal{Y}^{\mathfrak{w}}(x), \mathcal{G}^{\mathfrak{w}}(x)\right)$ as a WDHF element (WDHFE), and simply denoted as $d^{\mathfrak{w}} = \left(\mathcal{Y}^{\mathfrak{w}}, \mathcal{G}^{\mathfrak{w}}\right)$. It is noted that, $d = (\mathcal{Y}, \mathcal{G})$ represents the original DHF element of $d^{\mathfrak{w}}$.

2.2 q-ROFS

Definition 2.2.1 (*Yager* 2017) Let X be a fixed set. A q-ROFS, $\widetilde{\wp}$ on X is represented by.

$$\widetilde{\wp} = \left\{\left\langle x, \xi_{\widetilde{\wp}}(x), \psi_{\widetilde{\wp}}(x)\right\rangle | x \in X\right\}, \tag{2}$$

where $\xi_{\widetilde{\wp}} : X \to [0, 1]$ and $\psi_{\widetilde{\wp}} : X \to [0, 1]$ denote the membership and non-membership grades of the element $x \in X$ to the set $\widetilde{\wp}$ satisfying the condition that.

$$0 \le \left(\xi_{\widetilde{\wp}}(x)\right)^q + \left(\psi_{\widetilde{\wp}}(x)\right)^q \le 1, q \ge 1. \tag{3}$$

The degree of indeterminacy $\pi_{\widetilde{\wp}}(x)$ of x is given as.

$$\pi_{\widetilde{\wp}}(x) = \left[\left(\xi_{\widetilde{\wp}}(x)\right)^q + \left(\psi_{\widetilde{\wp}}(x)\right)^q - \left(\xi_{\widetilde{\wp}}(x)\right)^q \left(\psi_{\widetilde{\wp}}(x)\right)^q\right]^{\frac{1}{q}}$$

For simplicity, $\left(\xi_{\widetilde{\wp}}(x), \psi_{\widetilde{\wp}}(x)\right)$ is called a q-ROFN and is denoted by $\widetilde{\wp} = (\xi, \psi)$.

2.3 DHq-ROFS

Combining q-ROFSs (Yager 2017) with DHFS (Zhu et al. 2012), Xu et al. (2018) proposed the notion of DHq-ROFSs, which permits the DMs in providing their evaluation values with two sets of possible membership and non-membership values.

Definition 2.3.1 (*Xu et al.* 2018) Let X be a fixed set. A DHq-ROFS $\mathcal{R}$ on X is described as:

$$\mathcal{R} = (\langle x, \mathfrak{h}_{\mathcal{R}}(x), \mathfrak{g}_{\mathcal{R}}(x)\rangle | x \in X), \tag{4}$$

where $\mathfrak{h}_{\mathcal{R}}(x) = \bigcup_{\zeta \in \mathfrak{h}_{\mathcal{R}}(x)}\{\zeta : 0 \leq \zeta \leq 1\}$ and $\mathfrak{g}_{\mathcal{R}}(x) = \bigcup_{\vartheta \in \mathfrak{g}_{\mathcal{R}}(x)}\{\vartheta : 0 \leq \vartheta \leq 1\}$ represent the set of possible membership and non-membership values, respectively, of the element $x \in X$ to the set $\mathcal{R}$ satisfying the conditions:

$$0 \leq \left(\max_{\zeta \in h_R(x)}\{\zeta\}\right)^q + \left(\max_{\vartheta \in g_R(x)}\{\vartheta\}\right)^q \leq 1,\ q \geq 1.$$

For convenience, Xu et al. (2018) called the pair $\mathcal{R}(x) = (\mathfrak{h}_{\mathcal{R}}(x), \mathfrak{g}_{\mathcal{R}}(x))$ as a DHq-ROFN, and denoted it by $\widetilde{\mathfrak{r}} = \langle \mathfrak{h}, \mathfrak{g}\rangle$.

Definition 2.3.2 (*Xu et al.* 2018) Suppose $\widetilde{\mathfrak{r}} = \langle \mathfrak{h}, \mathfrak{g}\rangle$ be a DHq-RFN. Then the score function $S(\widetilde{\mathfrak{r}})$ of $\widetilde{\mathfrak{r}}$ is expressed as:

$$S(\widetilde{\mathfrak{r}}) = \frac{1}{n_{\mathfrak{h}}}\sum\nolimits_{\zeta \in \mathfrak{h}} \zeta^q - \frac{1}{n_{\mathfrak{g}}}\sum\nolimits_{\vartheta \in \mathfrak{g}} \vartheta^q,$$

And, the accuracy function of $\widetilde{\mathfrak{r}}$, denoted by $A(\widetilde{\mathfrak{r}})$, is defined by.

$$A(\widetilde{\mathfrak{r}}) = \frac{1}{n_{\mathfrak{h}}}\sum\nolimits_{\zeta \in \mathfrak{h}} \zeta^q + \frac{1}{n_{\mathfrak{g}}}\sum\nolimits_{\vartheta \in \mathfrak{g}} \vartheta^q,$$

Where $n_{\mathfrak{h}}$ and $n_{\mathfrak{g}}$ are the number of elements in $\mathfrak{h}$ and $\mathfrak{g}$, respectively.

The ordering of DHq-RFNs can be processed by the following rule:

Let $\widetilde{\mathfrak{r}}_i = (\mathfrak{h}_i, \mathfrak{g}_i)$ $(i = 1, 2)$ be any two DHq-RFNs.

- If $S(\widetilde{\mathfrak{r}}_1) > S(\widetilde{\mathfrak{r}}_2)$, then $\widetilde{\mathfrak{r}}_1$ is superior to $\widetilde{\mathfrak{r}}_2$, denoted by $\widetilde{\mathfrak{r}}_1 \succ \widetilde{\mathfrak{r}}_2$;
- If $S(\widetilde{\mathfrak{r}}_1) = S(\widetilde{\mathfrak{r}}_2)$, then
- for $A(\widetilde{\mathfrak{r}}_1) > A(\widetilde{\mathfrak{r}}_2)$, $\widetilde{\mathfrak{r}}_1 \succ \widetilde{\mathfrak{r}}_2$;
- for $A(\widetilde{\mathfrak{r}}_1) = A(\widetilde{\mathfrak{r}}_2)$, $\widetilde{\mathfrak{r}}_1$ is equivalent to $\widetilde{\mathfrak{r}}_2$, denoted by $\widetilde{\mathfrak{r}}_1 \approx \widetilde{\mathfrak{r}}_2$.

2.4 Ht-N&t-CNs

In fuzzy aggregation theory, t-norms and t-conorms perform a significant role, which is capable of formulating generalized intersection and union of fuzzy sets and its extensions. In 1978, Hamacher (1978) introduced one of generalized t-norms and t-conorms, which is known as Ht-N&t-CNs, and expressed as ($\varrho > 0$):

Hamacher t-norms: $T_{\varrho}^{H}(x, y) = \frac{xy}{\varrho + (1-\varrho)(x+y-xy)}$,

Hamacher t-conorms: $S_{\varrho}^{H}(x, y) = \frac{x+y-xy-(1-\varrho)xy}{1-(1-\varrho)xy}$.

3 Development of WDHq-ROFS

The preference degrees of different possible membership and non-membership values may deviate from each other in real-life decision-making contexts. It is worthy to mention here that DHq-ROFS overlooks the variation of the importance of degrees of possible evaluation values. To handle this issue and acquire more relevant and rational results, WDHq-ROFS is introduced in this section by incorporating respective weights of possible membership and non-membership degrees.

Definition 3.1 A WDHq-ROFS, $\mathcal{K}^{\omega}$ on a fixed set, X, is defined as.

$$\mathcal{K}^{\omega} = \left\{\left\langle x, \mathfrak{h}^{\omega}(x), \mathfrak{g}^{\omega}(x)\right\rangle | x \in X\right\} \tag{5}$$

where $\mathfrak{h}^{\omega}(x) = \bigcup_{\langle\zeta|\omega_{\zeta}\rangle\in\mathfrak{h}^{\omega}(x)}\left\{\left\langle\zeta|\omega_{\zeta}\right\rangle\right\}$, $\mathfrak{g}^{\omega}(x) = \bigcup_{\langle\vartheta|\omega_{\vartheta}\rangle\in\mathfrak{g}^{\omega}(x)}\{\langle\vartheta|\omega_{\vartheta}\rangle\}$, represent the collection of possible membership and non-membership degrees of the element $x \in X$ to the set $\mathcal{K}^{\omega}$, respectively, and $\omega_{\zeta}, \omega_{\vartheta} \in [0, 1]$ represent the weights/degrees of importance corresponding to ζ and ϑ, respectively. The elements of the sets $\mathfrak{h}^{\omega}(x)$ and $\mathfrak{g}^{\omega}(x)$ satisfy the condition that,

$$0 \le \left(\max_{\langle\zeta|\omega_{\zeta}\rangle\in h^{\omega}(x)}\{\zeta\}\right)^{q} + \left(\max_{\langle\vartheta|\omega_{\vartheta}\rangle\in g^{\omega}(x)}\{\vartheta\}\right)^{q} \le 1,$$

where $q \ge 1$ and $\zeta, \vartheta \in [0, 1]$ along with $\sum_{\langle\zeta|\omega_{\zeta}\rangle\in\mathfrak{h}^{\omega}(x)}^{\ell(\mathfrak{h}^{\omega})}\omega_{\zeta} = 1$, and $\sum_{\langle\vartheta|\omega_{\vartheta}\rangle\in\mathfrak{g}^{\omega}(x)}^{\ell(\mathfrak{g}^{\omega})}\omega_{\vartheta} = 1$.

Here $\ell(\mathfrak{h}^{\omega})$ and $\ell(\mathfrak{g}^{\omega})$ denotes the cardinality of $\mathfrak{h}^{\omega}(x)$ and $\mathfrak{g}^{\omega}(x)$, respectively.

For convenience, $\hbar^{\omega} = (\mathfrak{h}^{\omega}(x), \mathfrak{g}^{\omega}(x))$ is called a WDHq-ROF number (WDHq-ROFN), and denoted by $\hbar^{\omega} = (\mathfrak{h}^{\omega}, \mathfrak{g}^{\omega})$. Here, $\hbar = (\mathfrak{h}, \mathfrak{g}) = \langle\bigcup_{\zeta\in\mathfrak{h}(x)}\{\zeta\}, \bigcup_{\vartheta\in\mathfrak{g}(x)}\{\vartheta\}\rangle$ represents the corresponding DHq-ROFN of WDHq-ROFN $\hbar^{\omega}$.

Remark For instance, WDHq-ROFS is converted into DHq-ROFS if all the degrees of importance ω_{ζ} and ω_{ϑ} attached with possible membership and non-membership values turn into an equal value. Thus, DH q-ROFS is a particular case of WDHq-ROFS.

Definition 3.2 Let $\hbar^{\omega} = \langle\mathfrak{h}^{\omega}, \mathfrak{g}^{\omega}\rangle$ be any WDHq-ROFN, then its score function $S\left(\hbar^{\omega}\right)$ and accuracy function $A\left(\hbar^{\omega}\right)$ are defined as follows:

$$S\left(\hbar^{\omega}\right) = \sum_{\langle\zeta|\omega_{\zeta}\rangle\in\mathfrak{h}^{\omega}}\zeta^{q}\omega_{\zeta} - \sum_{\langle\vartheta|\omega_{\vartheta}\rangle\in\mathfrak{g}^{\omega}}\vartheta^{q}\omega_{\vartheta},$$
$$A\left(\hbar^{\omega}\right) = \sum_{\langle\zeta|\omega_{\zeta}\rangle\in\mathfrak{h}^{\omega}}\zeta^{q}\omega_{\zeta} + \sum_{\langle\vartheta|\omega_{\vartheta}\rangle\in\mathfrak{g}^{\omega}}\vartheta^{q}\omega_{\vartheta}.$$

Based on the defined score and accuracy functions of WDHq-ROFN, the following rule is delivered to compare any two WDHq-ROFNs.

Definition 3.3 The ordering of WDHq-ROFNs can be performed as follows:

Let $\hbar_i^{\omega} = \left(\mathfrak{h}_i^{\omega}, \mathfrak{g}_i^{\omega}\right)$ $(i = 1, 2)$ be any two WDHq-ROFNs,

If $S\left(\hbar_1^{\omega}\right) > S\left(\hbar_2^{\omega}\right)$, then $\hbar_1^{\omega}$ is superior to $\hbar_2^{\omega}$, denoted by $\hbar_1^{\omega} \succ \hbar_2^{\omega}$;
If $S\left(\hbar_1^{\omega}\right) = S\left(\hbar_2^{\omega}\right)$, then

(i) if $A\left(\hbar_1^{\omega}\right) > A\left(\hbar_2^{\omega}\right)$, then $\hbar_1^{\omega} \succ \hbar_2^{\omega}$;
(ii) if $A\left(\hbar_1^{\omega}\right) = A\left(\hbar_2^{\omega}\right)$, then $\hbar_1^{\omega}$ is analogous to $\hbar_2^{\omega}$, denoted by $\hbar_1^{\omega} \approx \hbar_2^{\omega}$.

3.1 Hamacher operations on WDHq-ROFNs

Utilizing the Ht-N&t-CNs, some basic operational rules for WDHq-ROFNs are defined as follows:

Definition 3.1.1 Let $\hbar_1^\omega = \left(\mathfrak{h}_1^\omega, \mathfrak{g}_1^\omega\right)$, $\hbar_2^\omega = \left(\mathfrak{h}_2^\omega, \mathfrak{g}_2^\omega\right)$ and $\hbar^\omega = (\mathfrak{h}^\omega, \mathfrak{g}^\omega)$ be any three given WDHq-ROFNs, and consider $\lambda > 0$. Then in the following, some Hamacher operational rules are defined as.

1. $$\hbar_1^\omega \oplus_H \hbar_2^\omega = \left(\bigcup_{\langle \zeta_i | \omega_{\zeta_i}\rangle \in \mathfrak{h}_i^\omega, i=1,2}\left\{\left\langle\left(\frac{{\zeta_1}^q + {\zeta_2}^q - {\zeta_1}^q{\zeta_2}^q - (1-\varrho){\zeta_1}^q{\zeta_2}^q}{1-(1-\varrho){\zeta_1}^q{\zeta_2}^q}\right)^{1/q}, \omega_{\zeta_1}\omega_{\zeta_2}\right\rangle\right\},\right.$$
$$\left.\bigcup_{\langle \vartheta_i | \omega_{\vartheta_i}\rangle \in \mathfrak{g}_i^\omega, i=1,2}\left\{\left\langle\left(\frac{{\vartheta_1}^q{\vartheta_2}^q}{\varrho + (1-\varrho)({\vartheta_1}^q + {\vartheta_2}^q - {\vartheta_1}^q{\vartheta_2}^q)}\right)^{1/q}, \omega_{\vartheta_1}\omega_{\vartheta_2}\right\rangle\right\}\right);$$
2. $$\hbar_1^\omega \otimes_H \hbar_2^\omega = \left(\bigcup_{\langle \zeta_i | \omega_{\zeta_i}\rangle \in \mathfrak{h}_i^\omega, i=1,2}\left\{\left\langle\left(\frac{{\zeta_1}^q{\zeta_2}^q}{\varrho + (1-\varrho)({\zeta_1}^q + {\zeta_2}^q - {\zeta_1}^q{\zeta_2}^q)}\right)^{1/q}, \omega_{\zeta_1}\omega_{\zeta_2}\right\rangle\right\},\right.$$
$$\left.\bigcup_{\langle \vartheta_i | \omega_{\vartheta_i}\rangle \in \mathfrak{g}_i^\omega, i=1,2}\left\{\left\langle\left(\frac{{\vartheta_1}^q + {\vartheta_2}^q - {\vartheta_1}^q{\vartheta_2}^q - (1-\varrho){\vartheta_1}^q{\vartheta_2}^q}{1-(1-\varrho){\vartheta_1}^q{\vartheta_2}^q}\right)^{1/q}, \omega_{\vartheta_1}\omega_{\vartheta_2}\right\rangle\right\}\right);$$
3. $$\lambda \odot_H \hbar^\omega = \left(\bigcup_{\langle \zeta | \omega_{\zeta}\rangle \in \mathfrak{h}^\omega}\left\{\left\langle\left(\frac{(1+(\varrho-1)\zeta^q)^\lambda - (1-\zeta^q)^\lambda}{(1+(\varrho-1)\zeta^q)^\lambda + (\varrho-1)(1-\zeta^q)^\lambda}\right)^{1/q}, \omega_{\zeta}\right\rangle\right\},\right.$$
$$\left.\bigcup_{\langle \vartheta | \omega_{\vartheta}\rangle \in \mathfrak{g}^\omega}\left\{\left\langle\left(\frac{\varrho\vartheta^{q\lambda}}{(1+(\varrho-1)(1-\vartheta^q))^\lambda + (\varrho-1)\vartheta^{q\lambda}}\right)^{1/q}, \omega_{\vartheta}\right\rangle\right\}\right);$$
4. $$\left(\hbar^\omega\right)^\lambda = \left(\bigcup_{\langle \zeta | \omega_{\zeta}\rangle \in \mathfrak{h}^\omega}\left\{\left\langle\left(\frac{\varrho\zeta^{q\lambda}}{(1+(\varrho-1)(1-\zeta^q))^\lambda + (\varrho-1)\zeta^{q\lambda}}\right)^{1/q}, \omega_{\zeta}\right\rangle\right\},\right.$$
$$\left.\bigcup_{\langle \vartheta | \omega_{\vartheta}\rangle \in \mathfrak{g}^\omega}\left\{\left\langle\left(\frac{(1+(\varrho-1)\vartheta^q)^\lambda - (1-\vartheta^q)^\lambda}{(1+(\varrho-1)\vartheta^q)^\lambda + (\varrho-1)(1-\vartheta^q)^\lambda}\right)^{1/q}, \omega_{\vartheta}\right\rangle\right\}\right).$$

In particular, when $\varrho = 1$, the above-mentioned Hamacher operational rules reduce to the algebraic operational rules on WDHq-ROFNs presented as follows:

- **Algebraic operational rules on WDHq-ROFNs:**

(A1) $\hbar_1^\omega \oplus \hbar_2^\omega = \left(\bigcup_{\langle \zeta_i | \omega_{\zeta_i}\rangle \in \mathfrak{h}_i^\omega, i=1,2}\left\{\left\langle({\zeta_1}^q + {\zeta_2}^q - {\zeta_1}^q{\zeta_2}^q)^{\frac{1}{q}}, \omega_{\zeta_1}\omega_{\zeta_2}\right\rangle\right\},\right.$ $\left.\bigcup_{\langle \vartheta_i | \omega_{\vartheta_i}\rangle \in \mathfrak{g}_i^\omega,} i=1,2\left\{\left\langle\vartheta_1\vartheta_2, \omega_{\vartheta_1}\omega_{\vartheta_2}\right\rangle\right\}\right);$

(A2) $\hbar_1^\omega \otimes \hbar_2^\omega = \left(\bigcup_{\langle \zeta_i | \omega_{\zeta_i}\rangle \in \mathfrak{h}_i^\omega, i=1,2}\left\{\left\langle\zeta_1\zeta_2, \omega_{\zeta_1}\omega_{\zeta_2}\right\rangle\right\},\right.$ $\left.\bigcup_{\langle \vartheta_i | \omega_{\vartheta_i}\rangle \in \mathfrak{g}_i^\omega, i=1,2}\left\{\left\langle({\vartheta_1}^q + {\vartheta_2}^q - {\vartheta_1}^q{\vartheta_2}^q)^{\frac{1}{q}}, \omega_{\vartheta_1}\omega_{\vartheta_2}\right\rangle\right\}\right);$

(A3) $\lambda\hbar^\omega = \left(\bigcup_{\langle \zeta | \omega_{\zeta}\rangle \in \mathfrak{h}^\omega}\left\{\left\langle\left(1-(1-\zeta^q)^\lambda\right)^{1/q}, \omega_\zeta\right\rangle\right\}, \bigcup_{\langle \vartheta | \omega_{\vartheta}\rangle \in \mathfrak{g}^\omega}\left\{\left\langle\vartheta^\lambda, \omega_\vartheta\right\rangle\right\}\right);$

(A4) $\left(\hbar^\omega\right)^\lambda = \left(\bigcup_{\langle \zeta | \omega_{\zeta}\rangle \in \mathfrak{h}^\omega}\left\{\left\langle\zeta^\lambda, \omega_\zeta\right\rangle\right\}, \bigcup_{\langle \vartheta | \omega_{\vartheta}\rangle \in \mathfrak{g}^\omega}\left\{\left\langle\left(1-(1-\vartheta^q)^\lambda\right)^{1/q}, \omega_\vartheta\right\rangle\right\}\right).$

Moreover, the Hamacher operational rules reduce to the Einstein operational rules on WDHq-ROFNs for $\varrho = 2$ presented as follows:

- **Einstein operational rules on WDHq-ROFNs:**

(E1) $\hbar_1^\omega \oplus_E \hbar_2^\omega = \left(\bigcup_{\langle \zeta_i | \omega_{\zeta_i}\rangle \in \mathfrak{h}_i^\omega, i=1,2} \left\{\left\langle \left(\frac{\zeta_1{}^q + \zeta_2{}^q}{1+\zeta_1{}^q \zeta_2{}^q}\right)^{1/q}, \omega_{\zeta_1}\omega_{\zeta_2}\right\rangle\right\},\right.$

$$\left.\bigcup_{\langle \vartheta_i | \omega_{\vartheta_i}\rangle \in \mathfrak{g}_i^\omega, i=1,2} \left\{\left\langle \left(\frac{\vartheta_1{}^q \vartheta_2{}^q}{1+(1-\vartheta_1{}^q)(1-\vartheta_2{}^q)}\right)^{1/q}, \omega_{\vartheta_1}\omega_{\vartheta_2}\right\rangle\right\}\right);$$

(E2) $\hbar_1^\omega \otimes_E \hbar_2^\omega = \left(\bigcup_{\langle \zeta_i | \omega_{\zeta_i}\rangle \in \mathfrak{h}_i^\omega, i=1,2} \left\{\left\langle \left(\frac{\zeta_1{}^q \zeta_2{}^q}{1+(1-\zeta_1{}^q)(1-\zeta_2{}^q)}\right)^{1/q}, \omega_{\zeta_1}\omega_{\zeta_2}\right\rangle\right\},\right.$

$$\left.\bigcup_{\langle \vartheta_i | \omega_{\vartheta_i}\rangle \in \mathfrak{g}_i^\omega, i=1,2} \left\{\left\langle \left(\frac{\vartheta_1{}^q + \vartheta_2{}^q}{1+\vartheta_1{}^q \vartheta_2{}^q}\right)^{1/q}, \omega_{\vartheta_1}\omega_{\vartheta_2}\right\rangle\right\}\right);$$

(E3) $\lambda\hbar^\omega = \left(\bigcup_{\langle \zeta | \omega_{\zeta}\rangle \in \mathfrak{h}^\omega} \left\{\left\langle \left(\frac{(1+\zeta^q)^\lambda - (1-\zeta^q)^\lambda}{(1+\zeta^q)^\lambda + (1-\zeta^q)^\lambda}\right)^{1/q}, \omega_{\zeta}\right\rangle\right\},\right.$

$$\left.\bigcup_{\langle \vartheta | \omega_{\vartheta}\rangle \in \mathfrak{g}^\omega} \left\{\left\langle \left(\frac{2\vartheta^{q\lambda}}{(1+(1-\vartheta^q))^\lambda + \vartheta^{q\lambda}}\right)^{1/q}, \omega_{\vartheta}\right\rangle\right\}\right);$$

(E4) $\left(\hbar^\omega\right)^\lambda = \left(\bigcup_{\langle \zeta | \omega_{\zeta}\rangle \in \mathfrak{h}^\omega} \left\{\left\langle \left(\frac{2\zeta^{q\lambda}}{(1+(1-\zeta^q))^\lambda + \zeta^{q\lambda}}\right)^{1/q}, \omega_{\zeta}\right\rangle\right\},\right.$

$$\left.\bigcup_{\langle \vartheta | \omega_{\vartheta}\rangle \in \mathfrak{g}^\omega} \left\{\left\langle \left(\frac{(1+\vartheta^q)^\lambda - (1-\vartheta^q)^\lambda}{(1+\vartheta^q)^\lambda + (1-\vartheta^q)^\lambda}\right)^{1/q}, \omega_{\vartheta}\right\rangle\right\}\right).$$

With the aid of the above Hamacher-based operations, in the next section, some new WDHq-ROF Hamacher aggregation operators, viz., WDHq-ROFHWA, WDHq-ROFHWG operators and their exceptional cases are derived. Further, several desirable characteristics are also discussed in detail.

4 Development of aggregation operators under WDHq-ROF environment

In the MCGDM context, the advantages of Ht-N&t-CNs have already been discussed in the introduction section. Utilizing the benefits of Hamacher operations, several aggregation operators under WDHq-ROF environment are developed as follows:

4.1 WDHq-ROFHWA operator

Definition 4.1.1 Let $\left\{\hbar_1^\omega, \hbar_2^\omega, \ldots, \hbar_n^\omega\right\}$ be a set of WDHq-ROFNs, and $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_n)^T$ be the weight vector with $\sum_{i=1}^n \Omega_i = 1$ and $\Omega_i \in [0, 1]$. Then WDHq-ROFHWA operator is a function: $(\mathcal{K}^\omega)^n \to \mathcal{K}^\omega$, given by.

$$\mathrm{WDHq-ROFHWA}\left(\hbar_1^\omega, \hbar_2^\omega, \ldots, \hbar_n^\omega\right) = \oplus_{H}{}_{i=1}^{n}\left(\Omega_i \hbar_i^\omega\right) \tag{6}$$

Theorem 4.1.1 *Let* $\hbar_i^\omega = \left(\mathfrak{h}_i^\omega, \mathfrak{g}_i^\omega\right)$ $(i = 1, 2, \ldots, n)$ *be a set of WDHq-ROFNs, then the aggregated outcome employing WDHq-ROFHWA operator is also a WDHq-ROFN, and.*

$$\text{WDHq-ROFHWA}(\hbar_1^{\omega}, \hbar_2^{\omega}, \ldots, \hbar_n^{\omega})$$

$$= \left(\bigcup_{\substack{\zeta_i|\omega_{\zeta_i} \in \mathfrak{h}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{ \left(\frac{\prod_{i=1}^{n}\left(1+(\varrho-1)\zeta_i^q\right)^{\Omega_i} - \prod_{i=1}^{n}\left(1-\zeta_i^q\right)^{\Omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\zeta_i^q\right)^{\Omega_i} + (\varrho-1)\prod_{i=1}^{n}\left(1-\zeta_i^q\right)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\zeta_i} \right\}, \right. \tag{7}$$
$$\left. \bigcup_{\substack{\vartheta_i|\omega_{\vartheta_i} \in \mathfrak{g}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{ \left(\frac{\varrho\prod_{i=1}^{n}\vartheta_i^{q\Omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\left(1-\vartheta_i^q\right)\right)^{\Omega_i} + (\varrho-1)\prod_{i=1}^{n}\left(\vartheta_i^q\right)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\vartheta_i} \right\} \right)$$

Proof For $n = 2$,

$$\Omega_1\hbar_1^{\omega} = \left(\bigcup_{\langle\zeta_1|\omega_{\zeta_1}\rangle\in\mathfrak{h}_1^{\omega}} \left\{ \left\langle \left(\frac{(1+(\varrho-1)\zeta_1{}^q)^{\Omega_1} - (1-\zeta_1{}^q)^{\Omega_1}}{(1+(\varrho-1)\zeta_1{}^q)^{\Omega_1} + (\varrho-1)(1-\zeta_1{}^q)^{\Omega_1}} \right)^{1/q}, \omega_{\zeta_1} \right\rangle \right\}, \right.$$
$$\left. \bigcup_{\langle\vartheta_1|\omega_{\vartheta_1}\rangle\in\mathfrak{g}_1{}^{\omega}} \left\{ \left\langle \left(\frac{\varrho\vartheta_1{}^{q\Omega_1}}{(1+(\varrho-1)(1-\vartheta_1{}^q))^{\Omega_1} + (\varrho-1)\vartheta_1{}^{q\Omega_1}} \right)^{1/q}, \omega_{\vartheta_1} \right\rangle \right\} \right).$$

and $\Omega_2\hbar_2^{\omega} = \left(\bigcup_{\langle\zeta_2|\omega_{\zeta_2}\rangle\in\mathfrak{h}_2^{\omega}} \left\{ \left\langle \left(\frac{(1+(\varrho-1)\zeta_2{}^q)^{\Omega_2} - (1-\zeta_2{}^q)^{\Omega_2}}{(1+(\varrho-1)\zeta_2{}^q)^{\Omega_2} + (\varrho-1)(1-\zeta_2{}^q)^{\Omega_2}} \right)^{1/q}, \omega_{\zeta_2} \right\rangle \right\}, \right.$

$$\left. \bigcup_{\langle\vartheta_2|\omega_{\vartheta_2}\rangle\in\mathfrak{g}_2{}^{\omega}} \left\{ \left\langle \left(\frac{\varrho\vartheta_2{}^{q\Omega_2}}{(1+(\varrho-1)(1-\vartheta_2{}^q))^{\Omega_2} + (\varrho-1)\vartheta_2{}^{q\Omega_2}} \right)^{1/q}, \omega_{\vartheta_2} \right\rangle \right\} \right).$$

Now, $\Omega_1\hbar_1^{\omega} \oplus_H \Omega_2\hbar_2^{\omega} =$

$$= \left(\bigcup_{\substack{\langle\zeta_i|\omega_{\zeta_i}\rangle \in \mathfrak{h}_i^{\omega} \\ i=1,2}} \left\{ \left(\left(\frac{\prod_{i=1}^{2}(1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} - \prod_{i=1}^{2}(1-\zeta_i{}^q)^{\Omega_i}}{\prod_{i=1}^{2}(1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} + (\varrho-1)\prod_{i=1}^{2}(1-\zeta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{2}\omega_{\zeta_i} \right) \right\}, \right.$$
$$\left. \bigcup_{\substack{\langle\vartheta_i|\omega_{\vartheta_i}\rangle \in \mathfrak{g}_i^{\omega} \\ i=1,2}} \left\{ \left(\left(\frac{\varrho\prod_{i=1}^{2}\vartheta_i{}^{q\Omega_i}}{\prod_{i=1}^{2}(1+(\varrho-1)(1-\vartheta_i{}^q))^{\Omega_i} + (\varrho-1)\prod_{i=1}^{2}(\vartheta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{2}\omega_{\vartheta_i} \right) \right\} \right)$$

i.e., the theorem holds for $n = 2$. Assume now that the theorem holds for $n = t$. Hence, $\text{WDHq-ROFHWA}\left(\hbar_1^{\omega}, \hbar_2^{\omega}, \ldots, \hbar_t^{\omega}\right)$

$$= \left(\bigcup_{\substack{\langle\zeta_i|\omega_{\zeta_i}\rangle \in \mathfrak{h}_i^{\omega} \\ i=1,2,\ldots,t}} \left\{ \left(\left(\frac{\prod_{i=1}^{t}(1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} - \prod_{i=1}^{t}(1-\zeta_i{}^q)^{\Omega_i}}{\prod_{i=1}^{t}(1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} + (\varrho-1)\prod_{i=1}^{t}(1-\zeta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{t}\omega_{\zeta_i} \right) \right\}, \right.$$
$$\left. \bigcup_{\substack{\langle\vartheta_i|\omega_{\vartheta_i}\rangle \in \mathfrak{g}_i^{\omega} \\ i=1,2,\ldots,t}} \left\{ \left(\left(\frac{\varrho\prod_{i=1}^{t}\vartheta_i{}^{q\Omega_i}}{\prod_{i=1}^{t}(1+(\varrho-1)(1-\vartheta_i{}^q))^{\Omega_i} + (\varrho-1)\prod_{i=1}^{t}(\vartheta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{t}\omega_{\vartheta_i} \right) \right\} \right);$$

Then for $n = t + 1$,

$$\text{WDHq-ROFHWA}\left(\hbar_1^{\omega}, \hbar_2^{\omega}, \ldots, \hbar_t^{\omega}, \hbar_{t+1}^{\omega}\right)$$

$$= \text{WDHq-ROFHWA}\left(\hbar_1^\omega, \hbar_2^\omega, \ldots, \hbar_t^\omega\right) \oplus_H \Omega_{t+1} \hbar_{t+1}^\omega$$

$$= \left(\bigcup_{\substack{\langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^\omega \\ i=1,2,\ldots,t}} \left\{ \left(\left(\frac{\prod_{i=1}^t (1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} - \prod_{i=1}^t (1-\zeta_i{}^q)^{\Omega_i}}{\prod_{i=1}^t (1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} + (\varrho-1)\prod_{i=1}^t (1-\zeta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^t \omega_{\zeta_i} \right) \right\}, \right.$$

$$\left. \bigcup_{\substack{\langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^\omega \\ i=1,2,\ldots,t}} \left\{ \left(\left(\frac{\varrho \prod_{i=1}^t \vartheta_i{}^{q\Omega_i}}{\prod_{i=1}^t (1+(\varrho-1)(1-\vartheta_i{}^q))^{\Omega_i} + (\varrho-1)\prod_{i=1}^t (\vartheta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^t \omega_{\vartheta_i} \right) \right\} \right)$$

$$\oplus_H \left(\bigcup_{\langle \zeta_{t+1} | \omega_{\zeta_{t+1}} \rangle \in \mathfrak{h}_{t+1}^\omega} \left\{ \left\langle \left(\frac{(1+(\varrho-1)\zeta_{t+1}{}^q)^{\Omega_{t+1}} - (1-\zeta_{t+1}{}^q)^{\Omega_{t+1}}}{(1+(\varrho-1)\zeta_{t+1}{}^q)^{\Omega_{t+1}} + (\varrho-1)(1-\zeta_{t+1}{}^q)^{\Omega_{t+1}}} \right)^{1/q}, \omega_{\zeta_{t+1}} \right\rangle \right\}, \right.$$

$$\left. \bigcup_{\langle \vartheta_{t+1} | \omega_{\vartheta_{t+1}} \rangle \in \mathfrak{g}_{t+1}{}^\omega} \left\{ \left\langle \left(\frac{\varrho \vartheta_{t+1}{}^{q\Omega_{t+1}}}{(1+(\varrho-1)(1-\vartheta_{t+1}{}^q))^{\Omega_{t+1}} + (\varrho-1)\vartheta_{t+1}{}^{q\Omega_{t+1}}} \right)^{1/q}, \omega_{\vartheta_{t+1}} \right\rangle \right\} \right)$$

$$= \left(\bigcup_{\substack{\langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^\omega \\ i=1,2,\ldots,t}} \left\{ \left(\left(\frac{(1+(\varrho-1)\zeta_{t+1}{}^q)^{\Omega_{t+1}} \prod_{i=1}^t (1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} - (1-\zeta_{t+1}{}^q)^{\Omega_{t+1}} \prod_{i=1}^t (1-\zeta_i{}^q)^{\Omega_i}}{(1+(\varrho-1)\zeta_{t+1}{}^q)^{\Omega_{t+1}} \prod_{i=1}^t (1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} + (\varrho-1)(1-\zeta_{t+1}{}^q)^{\Omega_{t+1}} \prod_{i=1}^t (1-\zeta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \right. \right.$$

$$\left. \left. \omega_{\zeta_{t+1}} \prod_{i=1}^t \omega_{\zeta_i} \right) \right\},$$

$$\bigcup_{\substack{\langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^\omega \\ i=1,2,\ldots,t}} \left\{ \left(\left(\frac{\varrho \vartheta_{t+1}{}^{q\Omega_{t+1}} \prod_{i=1}^t \vartheta_i{}^{q\Omega_i}}{(1+(\varrho-1)(1-\vartheta_{t+1}{}^q))^{\Omega_{t+1}} \prod_{i=1}^t (1+(\varrho-1)(1-\vartheta_i{}^q))^{\Omega_i} + (\varrho-1)(\vartheta_{t+1}{}^q)^{\Omega_{t+1}} \prod_{i=1}^t (\vartheta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \right. \right.$$

$$\left. \left. \left. \omega_{\vartheta_{t+1}} \prod_{i=1}^t \omega_{\vartheta_i} \right) \right\} \right)$$

$$= \left(\bigcup_{\substack{\langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^\omega \\ i=1,2,\ldots,t,t+1}} \left\{ \left(\left(\frac{\prod_{i=1}^{t+1} (1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} - \prod_{i=1}^{t+1} (1-\zeta_i{}^q)^{\Omega_i}}{\prod_{i=1}^{t+1} (1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} + (\varrho-1)\prod_{i=1}^{t+1} (1-\zeta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{t+1} \omega_{\zeta_i} \right) \right\}, \right.$$

$$\left. \bigcup_{\substack{\langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^\omega \\ i=1,2,\ldots,t,t+1}} \left\{ \left(\left(\frac{\varrho \prod_{i=1}^{t+1} \vartheta_i{}^{q\Omega_i}}{\prod_{i=1}^{t+1} (1+(\varrho-1)(1-\vartheta_i{}^q))^{\Omega_i} + (\varrho-1)\prod_{i=1}^{t+1} (\vartheta_i{}^q)^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{t+1} \omega_{\vartheta_i} \right) \right\} \right).$$

Therefore, the theorem is true for $n = t + 1$ also; and hence is true for all n.
This completes the proof.

- Subject to particular values of Hamacher parameter ϱ, some particular variants of the developed WDHq-ROFHWA operator can be established.

If $\varrho = 1$, then WDHq-ROFHWA operator is converted to WDHq-ROFWA operator as follows:

$$WDHq - ROFWA\left(\hbar_1^\omega, \hbar_2^\omega, \ldots, \hbar_n^\omega\right) = \oplus_{i=1}^n \left(\Omega_i \hbar_i^\omega\right)$$

$$= \left(\bigcup_{\substack{\langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^\omega \\ i=1,2,\ldots,n}} \left\{ \left(\left(1 - \prod_{i=1}^n \left(1-\zeta_i{}^q\right)^{\Omega_i} \right)^{\frac{1}{q}}, \prod_{i=1}^n \omega_{\zeta_i} \right) \right\}, \right.$$

$$\left. \bigcup_{\substack{\langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^\omega \\ i=1,2,\ldots,n}} \left\{ \left(\prod_{i=1}^n \vartheta_i{}^{q\Omega_i}, \prod_{i=1}^n \omega_{\vartheta_i} \right) \right\} \right).$$

If $\varrho = 2$, the WDHq-ROFHWA operator is converted to WDHq-ROFEWA operator as follows:

$$WDHq-ROFEWA\left(\mathscr{k}_1^{\omega}, \mathscr{k}_2^{\omega}, \ldots, \mathscr{k}_n^{\omega}\right) = \oplus_E{}_{i=1}^{n}\left(\Omega_i \mathscr{k}_i^{\omega}\right)$$

$$= \left(\bigcup_{\substack{\langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{\left(\left(\frac{\prod_{i=1}^{n}(1+\zeta_i{}^q)^{\Omega_i} - \prod_{i=1}^{n}(1-\zeta_i{}^q)^{\Omega_i}}{\prod_{i=1}^{n}(1+\zeta_i{}^q)^{\Omega_i} + \prod_{i=1}^{n}(1-\zeta_i{}^q)^{\Omega_i}}\right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\zeta_i}\right)\right\},\right.$$

$$\left.\bigcup_{\substack{\langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{\left(\left(\frac{2\prod_{i=1}^{n}\vartheta_i{}^{q\Omega_i}}{\prod_{i=1}^{n}(1+(1-\vartheta_i{}^q))^{\Omega_i} + \prod_{i=1}^{n}(\vartheta_i{}^q)^{\Omega_i}}\right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\vartheta_i}\right)\right\}\right),$$

Now, some fundamental properties of the proposed WDHq-ROFHWA operator are stated in follow-up.

4.2 The properties of the WDHq-ROFHWA operator

The proposed aggregation operators meet some stimulating properties, including idempotency, monotonicity, and boundedness, which are presented below:

Theorem 4.2.1 (Idempotency) *Suppose $\left\{\mathscr{k}_1^{\omega}, \mathscr{k}_2^{\omega}, \ldots, \mathscr{k}_n^{\omega}\right\}$ is a collection of WDHq-ROFNs, if all $\mathscr{k}_i^{\omega} = \langle \mathfrak{h}_i^{\omega}, \mathfrak{g}_i^{\omega} \rangle$ are equal, i.e., $\mathscr{k}_i^{\omega} = \mathscr{k}^{\omega} = \langle \mathfrak{h}^{\omega}, \mathfrak{g}^{\omega} \rangle$ for all i, then.*

$$\text{WDHq-ROFHWA}\left(\mathscr{k}_1^{\omega}, \mathscr{k}_2^{\omega}, \ldots, \mathscr{k}_n^{\omega}\right) = \mathscr{k}^{\omega}. \tag{8}$$

Proof

$$\text{WDHq-ROFHWA}\left(\mathscr{k}_1^{\omega}, \mathscr{k}_2^{\omega}, \ldots, \mathscr{k}_n^{\omega}\right)$$

$$= \left(\bigcup_{\substack{\langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{\left(\left(\frac{\prod_{i=1}^{n}(1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} - \prod_{i=1}^{n}(1-\zeta_i{}^q)^{\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\zeta_i{}^q)^{\Omega_i} + (\varrho-1)\prod_{i=1}^{n}(1-\zeta_i{}^q)^{\Omega_i}}\right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\zeta_i}\right)\right\},\right.$$

$$\left.\bigcup_{\substack{\langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{\left(\left(\frac{\varrho\prod_{i=1}^{n}\vartheta_i{}^{q\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)(1-\vartheta_i{}^q))^{\Omega_i} + (\varrho-1)\prod_{i=1}^{n}(\vartheta_i{}^q)^{\Omega_i}}\right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\vartheta_i}\right)\right\}\right). \tag{9}$$

Now, since $\mathscr{k}_i^{\omega} = \mathscr{k}^{\omega} = \langle \mathfrak{h}^{\omega}, \mathfrak{g}^{\omega} \rangle$ for $i = 1, 2, \ldots, n$, then.
$\zeta_i = \zeta$ and $\vartheta_i = \vartheta$ for $i = 1, 2, \ldots, n$.
Therefore, WDHq-ROFHWA$\left(\mathscr{k}^{\omega}, \mathscr{k}^{\omega}, \ldots, \mathscr{k}^{\omega}\right)$

$$= \left(\bigcup_{\langle \zeta | \omega_{\zeta} \rangle \in \mathfrak{h}^{\omega}} \left\{\left(\left(\frac{\prod_{i=1}^{n}(1+(\varrho-1)\zeta^q)^{\Omega_i} - \prod_{i=1}^{n}(1-\zeta^q)^{\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\zeta^q)^{\Omega_i} + (\varrho-1)\prod_{i=1}^{n}(1-\zeta^q)^{\Omega_i}}\right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\zeta}\right)\right\},\right.$$

$$\left.\bigcup_{\langle \vartheta | \omega_{\vartheta} \rangle \in \mathfrak{g}^{\omega}} \left\{\left(\left(\frac{\varrho\prod_{i=1}^{n}\vartheta^{q\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)(1-\vartheta^q))^{\Omega_i} + (\varrho-1)\prod_{i=1}^{n}(\vartheta^q)^{\Omega_i}}\right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\vartheta}\right)\right\}\right)$$

$$= \left(\bigcup_{\langle \zeta | \omega_{\zeta} \rangle \in \mathfrak{h}^{\omega}} \left\{\left\langle\left(\frac{\varrho\zeta^q}{\varrho}\right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\zeta}\right\rangle\right\}, \bigcup_{\langle \vartheta | \omega_{\vartheta} \rangle \in \mathfrak{g}^{\omega}} \left\{\left\langle\left(\frac{\varrho\vartheta^q}{\varrho}\right)^{\frac{1}{q}}, \prod_{i=1}^{n}\omega_{\vartheta}\right\rangle\right\}\right)$$

$$= \left(\bigcup_{\langle \zeta | \omega_\zeta \rangle \in \mathfrak{h}^\omega} \{ \langle \zeta, \omega_\zeta{}^* \rangle \}, \bigcup_{\langle \vartheta | \omega_\vartheta \rangle \in \mathfrak{g}^\omega} \{ \langle \vartheta, \omega_\vartheta{}^* \rangle \} \right)$$

$= \langle \mathfrak{h}^\omega, \mathfrak{g}^\omega \rangle = \mathscr{k}^\omega$.

Here, $\omega_\zeta{}^* = \prod_{i=1}^n \omega_\zeta$, $\omega_\vartheta{}^* = \prod_{i=1}^n \omega_\vartheta$.

Hence the theorem.

Theorem 4.2.2 (Monotonicity) *Let* $\{\mathscr{k}_1^\omega, \mathscr{k}_2^\omega, \ldots, \mathscr{k}_n^\omega\}$ *and* $\left\{\mathscr{k}_1^{\omega'}, \mathscr{k}_2^{\omega'}, \ldots, \mathscr{k}_n^{\omega'}\right\}$ *be two collections of WDHq-ROFNs,*

$$\mathscr{k}_i^\omega = \langle \mathfrak{h}_i^\omega, \mathfrak{g}_i^\omega \rangle = \left(\cup_{\substack{\langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^\omega \\ i=1,2,\ldots,n}} \{ \langle \zeta_i, \omega_{\zeta_i} \rangle \}, \cup_{\substack{\langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^\omega \\ i=1,2,\ldots,n}} \{ \langle \vartheta_i, \omega_{\vartheta_i} \rangle \} \right) \textit{and.}$$

$$\mathscr{k}_i^{\omega'} = \langle \mathfrak{h}_i^{\omega'}, \mathfrak{g}_i^{\omega'} \rangle = \left(\cup_{\substack{\langle \zeta_i' | \omega_{\zeta_i'} \rangle \in \mathfrak{h}_i^\omega \\ i=1,2,\ldots,n}} \left\{ \left\langle \zeta_i', \omega_{\zeta_i'} \right\rangle \right\}, \cup_{\substack{\langle \vartheta_i' | \omega_{\vartheta_i'} \rangle \in \mathfrak{g}_i^{\omega'} \\ i=1,2,\ldots,n}} \left\{ \left\langle \vartheta_i', \omega_{\vartheta_i'} \right\rangle \right\} \right)$$

If $\zeta_i \le \zeta_i^{'}$, $\vartheta_i \ge \vartheta_i'$, $\omega_{\zeta_i} \le \omega_{\zeta_i'}$ and $\omega_{\vartheta_i} \ge \omega_{\vartheta_i'}$ $\forall i = 1, 2, \ldots, n$, then.

$$\text{WDHq-ROFHWA}\left(\mathscr{k}_1^\omega, \mathscr{k}_2^\omega, \ldots, \mathscr{k}_n^\omega\right) \le \text{WDHq-ROFHWA}\left(\mathscr{k}_1^{\omega'}, \mathscr{k}_2^{\omega'}, \ldots, \mathscr{k}_n^{\omega'}\right)$$

where $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_n)^T$ representing the weight vector and $\sum_{i=1}^n \Omega_i = 1$, $\Omega_i \in [0, 1]$.

Proof Let $f(x) = \frac{1+(\varrho-1)x}{1-x}$, $x \in [0, 1)$, then $f'(x) = \frac{\varrho}{(1-x)^2} > 0$, thus f is an increasing function. Since for every $\mathscr{k}_i^\omega$ and $\mathscr{k}_i^{\omega'}$, $\zeta_i \le \zeta_i'$,$(i = 1, 2, \ldots, n)$

$$\frac{(1 + (\varrho - 1)\zeta_i{}^q)}{(1 - \zeta_i{}^q)} \le \frac{\left(1 + (\varrho - 1)\zeta_i'^q\right)}{\left(1 - \zeta_i'^q\right)}.$$

So, $\left(\frac{(1+(\varrho-1)\zeta_i{}^q)}{(1-\zeta_i{}^q)}\right)^{\Omega_i} \le \left(\frac{\left(1+(\varrho-1)\zeta_i'^q\right)}{\left(1-\zeta_i'^q\right)}\right)^{\Omega_i}$

$$\iff \prod_{i=1}^n \left(\frac{(1 + (\varrho - 1)\zeta_i{}^q)}{(1 - \zeta_i{}^q)}\right)^{\Omega_i} \le \prod_{i=1}^n \left(\frac{\left(1 + (\varrho - 1)\zeta_i'^q\right)}{\left(1 - \zeta_i'^q\right)}\right)^{\Omega_i}$$

$$\iff \prod_{i=1}^n \left(\frac{(1 + (\varrho - 1)\zeta_i{}^q)}{(1 - \zeta_i{}^q)}\right)^{\Omega_i} + (\varrho - 1) \le \prod_{i=1}^n \left(\frac{\left(1 + (\varrho - 1)\zeta_i'^q\right)}{\left(1 - \zeta_i'^q\right)}\right)^{\Omega_i} + (\varrho - 1)$$

$$\iff \frac{1}{\prod_{i=1}^n \left(\frac{(1+(\varrho-1)\zeta_i{}^q)}{(1-\zeta_i{}^q)}\right)^{\Omega_i} + (\varrho - 1)} \ge \frac{1}{\prod_{i=1}^n \left(\frac{\left(1+(\varrho-1)\zeta_i'^q\right)}{\left(1-\zeta_i'^q\right)}\right)^{\Omega_i} + (\varrho - 1)}$$

$$\iff \frac{\varrho \prod_{i=1}^n (1 - \zeta_i{}^q)^{\Omega_i}}{\prod_{i=1}^n (1 + (\varrho - 1)\zeta_i{}^q)^{\Omega_i} + (\varrho - 1)\prod_{i=1}^n (1 - \zeta_i{}^q)^{\Omega_i}}$$

$$
\begin{aligned}
&\geq \frac{\varrho\prod_{i=1}^{n}\left(1-\zeta_i'^{q}\right)^{\Omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\zeta_i'^{q}\right)^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\zeta_i'^{q}\right)^{\Omega_i}}\\
&\iff 1-\frac{\varrho\prod_{i=1}^{n}(1-\zeta_i{}^{q})^{\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\zeta_i{}^{q})^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}(1-\zeta_i{}^{q})^{\Omega_i}}\\
&\leq 1-\frac{\varrho\prod_{i=1}^{n}\left(1-\zeta_i'^{q}\right)^{\Omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\zeta_i'^{q}\right)^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\zeta_i'^{q}\right)^{\Omega_i}}\\
&\iff \frac{\prod_{i=1}^{n}(1+(\varrho-1)\zeta_i{}^{q})^{\Omega_i}-\prod_{i=1}^{n}(1-\zeta_i{}^{q})^{\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\zeta_i{}^{q})^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}(1-\zeta_i{}^{q})^{\Omega_i}}\\
&\leq \frac{\prod_{i=1}^{n}\left(1+(\varrho-1)\zeta_i'^{q}\right)^{\Omega_i}-\prod_{i=1}^{n}\left(1-\zeta_i'^{q}\right)^{\Omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\zeta_i'^{q}\right)^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\zeta_i'^{q}\right)^{\Omega_i}}
\end{aligned}
\tag{10}
$$

Again let $\mathrm{g}(y)=\frac{(1+(\varrho-1)(1-y))}{y}$, $y\in(0,1]$, $\varrho>0$, then $\mathrm{g}'(y)=-\frac{\varrho}{y^2}<0$, thus $\mathrm{g}(y)$ is a decreasing function.

Since for all i, $\vartheta_i{}^{q}\geq\vartheta_i'^{q}$, then.

$\frac{1+(\varrho-1)(1-\vartheta_i{}^{q})}{\vartheta_i{}^{q}}\leq\frac{1+(\varrho-1)\left(1-\vartheta_i'^{q}\right)}{\vartheta_i'^{q}}$,

Thus, $\left(\frac{1+(\varrho-1)(1-\vartheta_i{}^{q})}{\vartheta_i{}^{q}}\right)^{\Omega_i}\leq\left(\frac{1+(\varrho-1)\left(1-\vartheta_i'^{q}\right)}{\vartheta_i'^{q}}\right)^{\Omega_i}$

$$
\iff \prod_{i=1}^{n}\left(\frac{1+(\varrho-1)(1-\vartheta_i{}^{q})}{\vartheta_i{}^{q}}\right)^{\Omega_i}\leq\prod_{i=1}^{n}\left(\frac{1+(\varrho-1)\left(1-\vartheta_i'^{q}\right)}{\vartheta_i'^{q}}\right)^{\Omega_i}
$$

$$
\begin{aligned}
&\iff \prod_{i=1}^{n}\left(\frac{1+(\varrho-1)(1-\vartheta_i{}^{q})}{\vartheta_i{}^{q}}\right)^{\Omega_i}+(\varrho-1)\\
&\leq \prod_{i=1}^{n}\left(\frac{1+(\varrho-1)\left(1-\vartheta_i'^{q}\right)}{\vartheta_i'^{q}}\right)^{\Omega_i}+(\varrho-1)
\end{aligned}
$$

$$
\iff \frac{1}{\prod_{i=1}^{n}\left(\frac{1+(\varrho-1)(1-\vartheta_i{}^{q})}{\vartheta_i{}^{q}}\right)^{\Omega_i}+(\varrho-1)}\geq\frac{1}{\prod_{i=1}^{n}\left(\frac{1+(\varrho-1)\left(1-\vartheta_i'^{q}\right)}{\vartheta_i'^{q}}\right)^{\Omega_i}+(\varrho-1)}
$$

$$
\begin{aligned}
&\iff \frac{\varrho\prod_{i=1}^{n}\vartheta_i{}^{q\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)(1-\vartheta_i{}^{q}))^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}\vartheta_i{}^{q\Omega_i}}\\
&\geq \frac{\varrho\prod_{i=1}^{n}\vartheta_i'^{q\Omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\left(1-\vartheta_i'^{q}\right)\right)^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}\vartheta_i'^{q\Omega_i}}.
\end{aligned}
\tag{11}
$$

From (10) and (11) and using the relations $\omega_{\zeta_i}\leq\omega_{\zeta_i'}$ and $\omega_{\vartheta_i}\geq\omega_{\vartheta_i'}$, it is clear that.

$S\left(\text{WDHq-ROFHWA}\left(\hbar_1^{\omega},\hbar_2^{\omega},\ldots,\hbar_n^{\omega}\right)\right)\leq S\left(\text{WDHq-ROFHWA}\left(\hbar_1^{\omega'},\hbar_2^{\omega'},\ldots,\hbar_n^{\omega'}\right)\right)$

Therefore, $\text{WDHq-ROFHWA}\left(\hbar_1^{\omega},\hbar_2^{\omega},\ldots,\hbar_n^{\omega}\right)\leq\text{WDHq-ROFHWA}\left(\hbar_1^{\omega'},\hbar_2^{\omega'},\ldots,\hbar_n^{\omega'}\right)$.

Theorem 4.2.3 (boundedness) *Let* $\hbar_i^{\omega} = \left(\mathfrak{h}_i^{\omega}, \mathfrak{g}_i^{\omega}\right)$ $(i = 1, 2, \ldots, n)$ *be a collection of WDHq-ROFNs, and let.*

$$\zeta^- = min\left\{\zeta_i | \langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^{\omega}\right\}, \zeta^+ = max\left\{\zeta_i | \langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^{\omega}\right\},$$

$$\vartheta^- = min\left\{\vartheta_i | \langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^{\omega}\right\}, \vartheta^+ = max\left\{\vartheta_i | \langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^{\omega}\right\},$$

$$\omega_{\zeta^-} = min\left\{\omega_{\zeta_i} | \langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^{\omega}\right\}, \omega_{\zeta^+} = max\left\{\omega_{\zeta_i} | \langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^{\omega}\right\},$$

$$\omega_{\vartheta^-} = min\left\{\omega_{\vartheta_i} | \langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^{\omega}\right\}, \omega_{\vartheta^+} = max\left\{\omega_{\vartheta_i} | \langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^{\omega}\right\}.$$

If $\hbar_-^{\omega} = \left(\langle \zeta^- | \omega_{\zeta^-} \rangle, \langle \vartheta^+ | \omega_{\vartheta^+} \rangle\right)$ *and* $\hbar_+^{\omega} = \left(\langle \zeta^+ | \omega_{\zeta^+} \rangle, \langle \vartheta^- | \omega_{\vartheta^-} \rangle\right)$,
then $\hbar_-^{\omega} \le \text{WDHq-ROFHWA}\left(\hbar_1^{\omega}, \hbar_2^{\omega}, \ldots, \hbar_n^{\omega}\right) \le \hbar_+^{\omega}$.

Proof Let $f(x) = \frac{1+(\varrho-1)x}{1-x}$, $x \in [0, 1)$, then $f'(x) = \frac{\varrho}{(1-x)^2} > 0$, thus f is an increasing function.

Since, $\zeta^- \le \zeta_i \le \zeta^+$, $(i = 1, 2, \ldots, n)$
i.e., $\left(\zeta^-\right)^q \le \zeta_i{}^q \le \left(\zeta^+\right)^q$ $(i = 1, 2, \ldots, n)$ then $f\left(\left(\zeta^-\right)^q\right) \le f(\zeta_i{}^q) \le f\left(\left(\zeta^+\right)^q\right)$
i.e., $\frac{1+(\varrho-1)\zeta^-}{1-\zeta^-} \le \frac{1+(\varrho-1)\zeta_i{}^q}{1-\zeta_i{}^q} \le \frac{1+(\varrho-1)\left(\zeta^+\right)^q}{1-(\zeta^+)^q}$.
Let $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_n)^{\mathrm{T}}$ is the weight vector satisfying $\sum_{i=1}^{n} \Omega_i = 1$ and $\Omega_i \in [0, 1]$.
Then for all i, $\left(\frac{1+(\varrho-1)\left(\zeta^-\right)^q}{1-\left(\zeta^-\right)^q}\right)^{\Omega_i} \le \left(\frac{1+(\varrho-1)\zeta_i{}^q}{1-\zeta_i{}^q}\right)^{\Omega_i} \le \left(\frac{1+(\varrho-1)\left(\zeta^+\right)^q}{1-(\zeta^+)^q}\right)^{\Omega_i}$

$$\Longleftrightarrow \prod_{i=1}^{n}\left(1 + \frac{\varrho\left(\zeta^-\right)^q}{1 - \left(\zeta^-\right)^q}\right)^{\Omega_i} \le \prod_{i=1}^{n}\left(\frac{1 + (\varrho - 1)\zeta_i{}^q}{1 - \zeta_i{}^q}\right)^{\Omega_i}$$

$$\le \prod_{i=1}^{n}\left(1 + \frac{\varrho\left(\zeta^+\right)^q}{1 - (\zeta^+)^q}\right)^{\Omega_i}$$

$$\Longleftrightarrow 1 + \frac{\varrho\left(\zeta^-\right)^q}{1 - \left(\zeta^-\right)^q} \le \prod_{i=1}^{n}\left(\frac{1 + (\varrho - 1)\zeta_i{}^q}{1 - \zeta_i{}^q}\right)^{\Omega_i} \le 1 + \frac{\varrho\left(\zeta^+\right)^q}{1 - (\zeta^+)^q}$$

$$\Longleftrightarrow \frac{1}{\varrho + \frac{\varrho(\zeta^+)^q}{1-(\zeta^+)^q}} \le \frac{1}{\prod_{i=1}^{n}\left(\frac{1+(\varrho-1)\zeta_i{}^q}{1-\zeta_i{}^q}\right)^{\Omega_i} + (\varrho - 1)} \le \frac{1}{\varrho + \frac{\varrho\left(\zeta^-\right)^q}{1-\left(\zeta^-\right)^q}}$$

$$\Longleftrightarrow \frac{1 - \left(\zeta^+\right)^q}{\varrho} \le \frac{\prod_{i=1}^{n}(1 - \zeta_i{}^q)^{\Omega_i}}{\prod_{i=1}^{n}(1 + (\varrho - 1)\zeta_i{}^q)^{\Omega_i} + (\varrho - 1)\prod_{i=1}^{n}(1 - \zeta_i{}^q)^{\Omega_i}} \le \frac{1 - \left(\zeta^-\right)^q}{\varrho}$$

$$\Longleftrightarrow \zeta^- \le \left(\frac{\prod_{i=1}^{n}(1 + (\varrho - 1)\zeta_i{}^q)^{\Omega_i} - \prod_{i=1}^{n}(1 - \zeta_i{}^q)^{\Omega_i}}{\prod_{i=1}^{n}(1 + (\varrho - 1)\zeta_i{}^q)^{\Omega_i} + (\varrho - 1)\prod_{i=1}^{n}(1 - \zeta_i{}^q)^{\Omega_i}}\right)^{\frac{1}{q}} \le \zeta^+ \tag{12}$$

Again let $\mathrm{g}(y) = \frac{(1+(\varrho-1)(1-y))}{y}$, $y \in (0, 1]$, $\varrho > 0$, then $\mathrm{g}'(y) = -\frac{\varrho}{y^2} < 0$, thus $\mathrm{g}(y)$ is a decreasing function.
Since for all i, $\left(\vartheta^-\right)^q \le \vartheta_i{}^q \le \left(\vartheta^+\right)^q$, then $\mathrm{g}\left(\left(\vartheta^-\right)^q\right) \ge \mathrm{g}(\vartheta_i{}^q) \ge \mathrm{g}\left(\left(\vartheta^+\right)^q\right)$, i.e.,

$$\frac{\left(1 + (\varrho - 1)\left(1 - \left(\vartheta^+\right)^q\right)\right)}{(\vartheta^+)^q} \le \frac{(1 + (\varrho - 1)(1 - \vartheta_i{}^q))}{\vartheta_i{}^q} \le \frac{\left(1 + (\varrho - 1)\left(1 - \left(\vartheta^-\right)^q\right)\right)}{\left(\vartheta^-\right)^q}$$

$$\Longleftrightarrow \prod_{i=1}^{n}\left(\frac{\left(1 + (\varrho - 1)\left(1 - \left(\vartheta^+\right)^q\right)\right)}{(\vartheta^+)^q}\right)^{\Omega_i} \le \prod_{i=1}^{n}\left(\frac{(1 + (\varrho - 1)(1 - \vartheta_i{}^q))}{\vartheta_i{}^q}\right)^{\Omega_i}$$

$$\leq \prod_{i=1}^{n}\left(\frac{\left(1+(\varrho-1)\left(1-\left(\vartheta^{-}\right)^{q}\right)\right)}{\left(\vartheta^{-}\right)^{q}}\right)^{\Omega_i}$$

$$\Longleftrightarrow \frac{\varrho}{(\vartheta^{+})^{q}}-(\varrho-1)\leq \prod_{i=1}^{n}\left(\frac{(1+(\varrho-1)(1-{\vartheta_i}^{q}))}{{\vartheta_i}^{q}}\right)^{\Omega_i}\leq \frac{\varrho}{\left(\vartheta^{-}\right)^{q}}-(\varrho-1)$$

$$\Longleftrightarrow \frac{1}{\frac{\varrho}{(\vartheta^{-})^{q}}}\leq \frac{\prod_{i=1}^{n}{\vartheta_i}^{q\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)(1-{\vartheta_i}^{q}))^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}({\vartheta_i}^{q})^{\Omega_i}}\leq \frac{1}{\frac{\varrho}{(\vartheta^{+})^{q}}}$$

$$\Longleftrightarrow \vartheta^{-}\leq \left(\frac{\varrho\prod_{i=1}^{n}{\vartheta_i}^{q\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)(1-{\vartheta_i}^{q}))^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}{\vartheta_i}^{q\Omega_i}}\right)^{\frac{1}{q}}\leq \vartheta^{+}. \tag{13}$$

Using (12) and (13) and based on given ω_{ζ^-}, ω_{ζ^+}, ω_{ϑ^-} and ω_{ϑ^+} it is found that.

$$\hbar_{-}^{\omega}\leq \text{WDHq-ROFHWA}\left(\hbar_1^{\omega},\hbar_2^{\omega},\ldots,\hbar_n^{\omega}\right)\leq \hbar_{+}^{\omega}$$

Hence the theorem.

4.3 WDHq-ROFHWG operator

In this subsection, WDHq-ROFWG operator is developed based on Hamacher operational rules.

Definition 4.3.1 Let $\left\{\hbar_1^{\omega},\hbar_2^{\omega},\ldots,\hbar_n^{\omega}\right\}$ be a set of WDHq-ROFNs, Then WDHq-ROFWG operator is a function: $(K^{w})^{n}\rightarrow K^{w}$, given by.

$$WDHq-ROFHWG\left(\hbar_1^{\omega},\hbar_2^{\omega},\ldots,\hbar_n^{\omega}\right)=\otimes_{H}{}_{i=1}^{n}\left(\left(\hbar_i^{\omega}\right)^{\Omega_i}\right), \tag{14}$$

where $\Omega=(\Omega_1,\Omega_2,\ldots,\Omega_n)^{T}$ representing the weight vector with $\sum_{i=1}^{n}\Omega_i=1$ and $\Omega_i\in[0,1]$.

Theorem 4.3.1 *Let $\hbar_i^{\omega}=\left(\mathfrak{h}_i^{\omega},\mathfrak{g}_i^{\omega}\right)$ $(i=1,2,\ldots,n)$ be a set of WDH q-ROFNs, then the aggregating value using WDHq-ROFHWG operator is also an WDHq-ROFN and.*

$$WDHq-ROFHWG\left(\hbar_1^{\omega},\hbar_2^{\omega},\ldots,\hbar_n^{\omega}\right)$$

$$=\langle\bigcup_{\substack{\langle\zeta_i|\omega_{\zeta_i}\rangle\in\mathfrak{h}_i^{\omega}\\ i=1,2,\ldots,n}}\left\{\left(\left(\frac{\varrho\prod_{i=1}^{n}{\zeta_i}^{q\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)(1-{\zeta_i}^{q}))^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}({\zeta_i}^{q})^{\Omega_i}}\right)^{\frac{1}{q}},\prod_{i=1}^{n}\omega_{\zeta_i}\right)\right\},$$
$$\bigcup_{\substack{\langle\vartheta_i|\omega_{\vartheta_i}\rangle\in\mathfrak{g}_i^{\omega}\\ i=1,2,\ldots,n}}\left\{\left(\left(\frac{\prod_{i=1}^{n}(1+(\varrho-1){\vartheta_i}^{q})^{\Omega_i}-\prod_{i=1}^{n}(1-{\vartheta_i}^{q})^{\Omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1){\vartheta_i}^{q})^{\Omega_i}+(\varrho-1)\prod_{i=1}^{n}(1-{\vartheta_i}^{q})^{\Omega_i}}\right)^{\frac{1}{q}},\prod_{i=1}^{n}\omega_{\vartheta_i}\right)\right\}\rangle. \tag{15}$$

Proof This proof is analogous to the proof of Theorem 4.1.1.

- Subject to particular values of Hamacher parameter ϱ, some special variants of the developed WDHq-ROFHWG operator can be established.

If $\varrho=1$, then WDHq-ROFHWG operator is converted to WDHq-ROFWG operator as follows:

$$\text{WDHq-ROFWG}\left(\hbar_1^{\omega},\hbar_2^{\omega},\ldots,\hbar_n^{\omega}\right)=\oplus_{i=1}^{n}\left(\Omega_i\hbar_i^{\omega}\right)$$

$$= \left(\bigcup_{\substack{\langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{ \left(\prod_{i=1}^{n} {\zeta_i}^{q\Omega_i}, \prod_{i=1}^{n} \omega_{\zeta_i} \right) \right\}, \right.$$

$$\left. \bigcup_{\substack{\langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{ \left(\left(1 - \prod_{i=1}^{n} (1 - {\vartheta_i}^{q})^{\Omega_i} \right)^{\frac{1}{q}}, \prod_{i=1}^{n} \omega_{\vartheta_i} \right) \right\} \right)$$

If $\varrho = 2$, the WDHq-ROFHWG operator is converted to WDH q-ROFEWG operator as follows:

$$\text{WDHq-ROFEWG} \left(\hbar_1^{\omega}, \hbar_2^{\omega}, \ldots, \hbar_n^{\omega} \right) = \oplus_E{}_{i=1}^{n} \left(\Omega_i \hbar_i^{\omega} \right)$$

$$= \left(\bigcup_{\substack{\langle \zeta_i | \omega_{\zeta_i} \rangle \in \mathfrak{h}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{ \left(\left(\frac{2\prod_{i=1}^{n} {\zeta_i}^{q\Omega_i}}{\prod_{i=1}^{n} (1 + (1 - {\zeta_i}^{q}))^{\Omega_i} + \prod_{i=1}^{n} ({\zeta_i}^{q})^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{n} \omega_{\zeta_i} \right) \right\}, \right.$$

$$\left. \bigcup_{\substack{\langle \vartheta_i | \omega_{\vartheta_i} \rangle \in \mathfrak{g}_i^{\omega} \\ i=1,2,\ldots,n}} \left\{ \left(\left(\frac{\prod_{i=1}^{n} (1 + {\vartheta_i}^{q})^{\Omega_i} - \prod_{i=1}^{n} (1 - {\vartheta_i}^{q})^{\Omega_i}}{\prod_{i=1}^{n} (1 + {\vartheta_i}^{q})^{\Omega_i} + \prod_{i=1}^{n} (1 - {\vartheta_i}^{q})^{\Omega_i}} \right)^{\frac{1}{q}}, \prod_{i=1}^{n} \omega_{\vartheta_i} \right) \right\} \right)$$

It is clear that the WDHq-ROFHWG operator also satisfies the properties idempotency, monotonicity, and boundedness.

Moreover, a list of some valuable aggregation operators is presented in Fig. 2, which can be easily constructed from the developed operators by setting particular values of Hamachar and rung parameters.

5 An approach to multicriteria group decision making with WDHq-ROF information

In the present section, the proposed aggregation operators are employed to develop an MCGDM approach under WDHq-ROF environment.

In an MCGDM problem, suppose $Z = \{Z_1, Z_2, \ldots, Z_m\}$ denotes a set of distinct alternatives and $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\}$ represents the set of n criteria along with their weight vector $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_n)^T$, satisfying $\Omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \Omega_i = 1$. There are p number of experts, $E^{(1)}, E^{(2)}, \ldots, E^{(p)}$, who provide their judgement values by evaluating the alternatives on the basis of the criteria as mentioned above. Let a q-ROFN, ${\wp_{ij}}^{(k)} = \langle {\zeta_{ij}}^{(k)}, {\vartheta_{ij}}^{(k)} \rangle$ represents the assessment value given by an expert, $E^{(k)}$ under the criteria, $\mathcal{C}_j$ for the alternative, Z_i, and the individual decision matrices $D_{m\times n}^{(k)} = \left[{\wp_{ij}}^{(k)} \right]_{m\times n}$ $(k = 1, 2, \ldots, p)$ are constructed. The goal of the DMs is to find the ranking of alternatives to identify the best choice.

Then, the developed WDHq-ROF aggregation operators are applied to the MCGDM method which involves the following steps:

Step 1: Formulate a collective decision matrix, $\mathfrak{R}^{\omega}{}_{m\times n} = \left[r_{ij}^{\omega} \right]_{m\times n}$, based on the individual decision matrices $D_{m\times n}^{(k)} = \left[{\wp_{ij}}^{(k)} \right]_{m\times n}$, provided by the DMs, where the elements

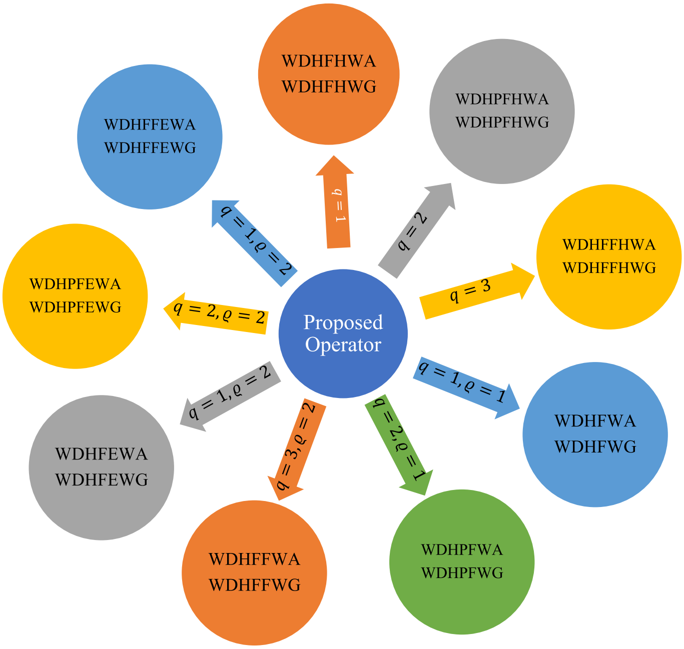

Fig. 2 List of several reduced aggregation operators. WDH PF (WDHPF); WDH FF (WDHFF); Hamacher WA (HWA); Hamacher WG (HWG); Einstein WA (EWA); Einstein WG (EWG)

are in the form of WDHq-ROFNs. In this context, two different cases may occur, respectively, in computing the collective decision matrix $\mathfrak{R}^{\omega}{}_{m\times n} = \left[\varkappa_{ij}^{\omega}\right]_{m\times n}$ by means of different WDHq-ROFNs, $\varkappa_{ij}^{\omega}$:

Case 1: If the experts' weights are unknown,

$$\varkappa_{ij}^{\omega} = \left\{\left(\left\{\left\langle \zeta_{ij}|\omega_{\zeta_{ij}}\right\rangle\right\}, \left\{\left\langle \vartheta_{ij}|\omega_{\vartheta_{ij}}\right\rangle\right\}\right) : \zeta_{ij} \in \bigcup_k \left\{\zeta_{ij}{}^{(k)}\right\}, \vartheta_{ij} \in \bigcup_k \left\{\vartheta_{ij}{}^{(k)}\right\}, \omega_{\zeta_{ij}} = \frac{l\left(\zeta_{ij}\right)}{p}, \omega_{\vartheta_{ij}} = \frac{l\left(\vartheta_{ij}\right)}{p}\right\}, \tag{16}$$

where $l\left(\zeta_{ij}\right)$ represents the number of experts providing satisfaction degree ζ_{ij}, and $l\left(\vartheta_{ij}\right)$ is the number of experts providing dissatisfaction degree ϑ_{ij}.

Case 2. If the weight vector of the DMs is given as $w = \left(w_1, w_2, \ldots, w_p\right)^T$, $w_i \geq 0$ with $\sum_{i=1}^{p} w_i = 1$. Then,

$$\varkappa_{ij}^{\omega} = \left\{\left(\left\{\left\langle \zeta_{ij}|\omega_{\zeta_{ij}}\right\rangle\right\}, \left\{\left\langle \vartheta_{ij}|\omega_{\vartheta_{ij}}\right\rangle\right\}\right) : \zeta_{ij} \in \bigcup_k \left\{\zeta_{ij}{}^{(k)}\right\}, \vartheta_{ij} \in \bigcup_k \left\{\vartheta_{ij}{}^{(k)}\right\}, \omega_{\zeta_{ij}} = \sum_{E^{(k)} \in N\left(\zeta_{ij}\right)} w_k, \omega_{\vartheta_{ij}} = \sum_{E^{(k)} \in N\left(\vartheta_{ij}\right)} w_k\right\}, \tag{17}$$

where $N\left(\zeta_{ij}\right)$ and $N\left(\vartheta_{ij}\right)$ denote the collection of the experts who provided the satisfaction and dissatisfaction degrees, ζ_{ij} and ϑ_{ij}, respectively.

Step 2: Compute the overall assessments of the alternatives. Aggregate WDH q-ROFNs ν_{ij}^{ω} for each alternative Z_i using WDH q-ROFHWA (or WDH q-ROFHWG) operator to obtain ν_i^{ω} as follows:

$$\nu_i^{\omega} = \left(\cup_{\substack{\langle \zeta_{ij} | \omega_{\zeta_{ij}} \rangle \in \mathfrak{h}_{ij}^{\omega} \\ j=1,2,\ldots,n}} \left\{ \left\langle \left(\frac{\prod_{j=1}^{n}\left(1+(\varrho-1)\zeta_{ij}^{q}\right)^{\Omega_{ij}} - \prod_{j=1}^{n}\left(1-\zeta_{ij}^{q}\right)^{\Omega_{ij}}}{\prod_{j=1}^{n}\left(1+(\varrho-1)\zeta_{ij}^{q}\right)^{\Omega_{ij}} + (\varrho-1)\prod_{j=1}^{n}\left(1-\zeta_{ij}^{q}\right)^{\Omega_{ij}}} \right)^{\frac{1}{q}}, \prod_{j=1}^{n}\omega_{\zeta_{ij}} \right\rangle \right\}, \right.$$
$$\left. \cup_{\substack{\langle \vartheta_{ij} | \omega_{\vartheta_{ij}} \rangle \in \mathfrak{g}_{ij}^{\omega} \\ j=1,2,\ldots,n}} \left\{ \left\langle \left(\frac{\varrho \prod_{j=1}^{n} \vartheta_{ij}^{q\Omega_{ij}}}{\prod_{j=1}^{n}\left(1+(\varrho-1)\left(1-\vartheta_{ij}^{q}\right)\right)^{\Omega_{ij}} + (\varrho-1)\prod_{j=1}^{n}\left(\vartheta_{ij}^{q}\right)^{\Omega_{ij}}} \right)^{\frac{1}{q}}, \prod_{j=1}^{n}\omega_{\vartheta_{ij}} \right\rangle \right\} \right), \tag{18}$$

or,

$$\nu_i^{\omega} = \left(\cup_{\substack{\langle \zeta_{ij} | \omega_{\zeta_{ij}} \rangle \in \mathfrak{h}_{ij}^{\omega} \\ j=1,2,\ldots,n}} \left\{ \left\langle \left(\frac{\varrho \prod_{j=1}^{n} \zeta_{ij}^{q\Omega_{ij}}}{\prod_{j=1}^{n}\left(1+(\varrho-1)\left(1-\zeta_{ij}^{q}\right)\right)^{\Omega_{ij}} + (\varrho-1)\prod_{j=1}^{n}\left(\zeta_{ij}^{q}\right)^{\Omega_{ij}}} \right)^{\frac{1}{q}}, \prod_{j=1}^{n}\omega_{\zeta_{ij}} \right\rangle \right\}, \right.$$
$$\left. \cup_{\substack{\langle \vartheta_{ij} | \omega_{\vartheta_{ij}} \rangle \in \mathfrak{g}_{ij}^{\omega} \\ j=1,2,\ldots,n}} \left\{ \left\langle \left(\frac{\prod_{j=1}^{n}\left(1+(\varrho-1)\vartheta_{ij}^{q}\right)^{\Omega_{ij}} - \prod_{j=1}^{n}\left(1-\vartheta_{ij}^{q}\right)^{\Omega_{ij}}}{\prod_{j=1}^{n}\left(1+(\varrho-1)\vartheta_{ij}^{q}\right)^{\Omega_{ij}} + (\varrho-1)\prod_{j=1}^{n}\left(1-\vartheta_{ij}^{q}\right)^{\Omega_{ij}}} \right)^{\frac{1}{q}}, \prod_{j=1}^{n}\omega_{\vartheta_{ij}} \right\rangle \right\} \right), \tag{19}$$

for $i = 1, 2, \ldots, m \& j = 1, 2, \ldots, n$.

Step 3: Calculate the the scores values $S\left(\nu_i^{\omega}\right)$ and accuracy values $A\left(\nu_i^{\omega}\right)$ of the aggregated WDHq-ROFNs ν_i^{ω} for each alternative Z_i $(i = 1, \ldots, m)$.

Step 4: Rank the alternatives Z_i, $(i = 1, 2, \ldots, m)$ in descending order using the comparison rule of WDHq-ROFNs, and finally, select the most desirable alternative.

6 Illustrative examples

In this section, WDHq-ROFWA and WDHq-ROFWG operators are implemented practically to demonstrate the effectiveness of the proposed approach. Two practical examples with WDHq-ROF data are considered to describe the implementation process. In the first example, a manufacturing company wants to select an assembling parts supplier and second one, a supplier chain management desires to select a suitable supplier from a group of prospect suppliers.

Example 1. Supplier selection is the procedure, a company uses to find, assess, and work with suppliers. The process of choosing a supplier effectively uses a significant part of a company's financial resources and is essential to the success of any corporation. Suppose a manufacturing company wants to select the most suitable supplier to supply critical parts in their assembling process (adapted from Zeng et al. (2020)). There are five available suppliers Z_i $(i = 1, 2, \ldots, 5)$ from which the most preferred supplier is to be identified based on five criteria, viz., $\mathcal{C}_1$: overall cost of the product; $\mathcal{C}_2$: quality of the product; $\mathcal{C}_3$: service performance of supplier; $\mathcal{C}_4$: supplier's profile and $\mathcal{C}_5$: risk factor. The weight vector corresponding to the criteria $\mathcal{C}$ is $\Omega = (\Omega_1, \ldots, \Omega_5)^T = (0.2, 0.15, 0.2, 0.3, 0.15)^T$. There are four experts $E^{(k)}$ $(k = 1, 2, 3, 4)$ in the decision making committee. The experts $E^{(k)}$ $(k = 1, 2, 3, 4)$ evaluated the alternatives Z_i $(i = 1, 2, \ldots, 5)$ with respect to the attributes $\mathcal{C}_j$ $(j = 1, 2, \ldots, 5)$ using q-ROFNs. Based on the experts' assessments, four q-ROF decision matrices (q-ROFDMs), $D^{(k)}_{m\times n} = \left[\wp_{ij}{}^{(k)}\right]_{m\times n}$ $(k = 1, 2, 3, 4)$ (see Tables 1, 2, 3, 4) are constructed.

The weights of the experts are given as $w = (0.3, 0.2, 0.3, 0.2)^T$. Using the proposed method, the task is to determine the best supplier which is presented below.

Step 1: Since the weight of the experts is known, the collective WDH q-ROF decision matrix $\mathfrak{R}^{\omega} = \left[r^{\omega}_{ij}\right]_{5\times 5}$ is constructed using Eq. (17) as shown in Table 5.

In the matrix, $\mathfrak{R}^{\omega}$, the $(1, 1)$th entry, $r^{\omega}_{11} = \begin{pmatrix} \{\langle 0.3|0.2\rangle, \langle 0.4|0.3\rangle, \langle 0.5|0.2\rangle, \langle 0.6|0.3\rangle\}, \\ \{\langle 0.3|0.5\rangle, \langle 0.4|0.2\rangle, \langle 0.8|0.3\rangle\} \end{pmatrix}$ corresponding to $(Z_1, \mathcal{C}_1)$ is represented on the basis of the membership degrees $(0.4, 0.5, 0.6, 0.3)$ and non-membership degrees $(0.8, 0.3, 0.3, 0.4)$ provided by the respective DMs having weights $(0.3, 0.2, 0.3, 0.2)^{\mathrm{T}}$. Thus the membership values

Table 1 q-ROFDM $D^{(1)}_{m\times n}$ provided by $E^{(1)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
Z_1	$\langle 0.4, 0.8\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.6, 0.8\rangle$	$\langle 0.7, 0.5\rangle$
Z_2	$\langle 0.6, 0.2\rangle$	$\langle 0.9, 0.2\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.8, 0.2\rangle$
Z_3	$\langle 0.7, 0.4\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.6, 0.3\rangle$
Z_4	$\langle 0.3, 0.4\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.9, 0.1\rangle$
Z_5	$\langle 0.8, 0.1\rangle$	$\langle 0.3, 0.4\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.9, 0.2\rangle$	$\langle 0.5, 0.2\rangle$

Table 2 q-ROFDM $D^{(2)}_{m\times n}$ provided by $E^{(2)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
Z_1	$\langle 0.5, 0.3\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.7, 0.5\rangle$	$\langle 0.5, 0.6\rangle$	$\langle 0.8, 0.2\rangle$
Z_2	$\langle 0.7, 0.2\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.4, 0.4\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.7, 0.3\rangle$
Z_3	$\langle 0.5, 0.6\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.4, 0.2\rangle$	$\langle 0.6, 0.1\rangle$
Z_4	$\langle 0.5, 0.4\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.4, 0.2\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.9, 0.3\rangle$
Z_5	$\langle 0.9, 0.3\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.1\rangle$

Table 3 q-ROFDM $D^{(3)}_{m\times n}$ provided by $E^{(3)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
Z_1	$\langle 0.6, 0.3\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.6, 0.8\rangle$	$\langle 0.8, 0.5\rangle$	$\langle 0.7, 0.4\rangle$
Z_2	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.6, 0.3\rangle$
Z_3	$\langle 0.6, 0.1\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.8, 0.1\rangle$
Z_4	$\langle 0.7, 0.3\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.8, 0.3\rangle$	$\langle 0.5, 0.2\rangle$
Z_5	$\langle 0.9, 0.1\rangle$	$\langle 0.6, 0.2\rangle$	$\langle 0.9, 0.3\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.7, 0.1\rangle$

Table 4 q-ROFDM $D^{(4)}_{m\times n}$ provided by $E^{(4)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
Z_1	$\langle 0.3, 0.4\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.8, 0.4\rangle$	$\langle 0.3, 0.7\rangle$	$\langle 0.4, 0.6\rangle$
Z_2	$\langle 0.7, 0.1\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.4, 0.2\rangle$	$\langle 0.9, 0.2\rangle$	$\langle 0.3, 0.1\rangle$
Z_3	$\langle 0.4, 0.1\rangle$	$\langle 0.5, 0.2\rangle$	$\langle 0.8, 0.1\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.6, 0.3\rangle$
Z_4	$\langle 0.8, 0.2\rangle$	$\langle 0.5, 0.1\rangle$	$\langle 0.9, 0.4\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.8, 0.2\rangle$
Z_5	$\langle 0.6, 0.1\rangle$	$\langle 0.9, 0.2\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.9, 0.3\rangle$	$\langle 0.8, 0.1\rangle$

$\{0.3, 0.4, 0.5, 0.6\}$ possesses the weights $\{0.3, 0.4, 0.5, 0.6\}$, respectively and the non-membership values $\{0.3, 0.4, 0.8\}$, bears the weights $\{0.5, 0.2, 0.3\}$, respectively.

Step 2: Utilise WDH q-ROFHWA operator (Eq. (18)) (by considering $q = 3, \varrho = 3$) to aggregate all the preference values r^{ω}_{ij} $(j = 1, 2, 3, 4, 5)$ for the ith alternative, Z_i, and derive the overall performance value r^{ω}_i $(i = 1, 2, \ldots, 5)$ corresponding to that alternative as

$$r^{\omega}_1 = (\{\langle 0.4373|0.0029\rangle, \langle 0.4993|0.0086\rangle, \langle 0.5324|0.0029\rangle, \langle 0.4841|0.0029\rangle,$$

$$\langle 0.5371|0.0086\rangle, \langle 0.5663|0.0029\rangle, \langle 0.5197|0.0043\rangle, \langle 0.5670|0.0130\rangle, \langle 0.5936|0.0043\rangle,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\langle 0.6808|0.0022\rangle, \langle 0.6995|0.0007\rangle, \langle 0.7144|0.0007\rangle, \langle 0.7403|0.0022\rangle, \langle 0.7556|0.0007\rangle\},$$

$$\{\langle 0.2590|0.0036\rangle, \langle 0.2876|0.0054\rangle, \langle 0.2978|0.0054\rangle, \langle 0.3070|0.0036\rangle, \langle 0.2753|0.0024\rangle,$$

$$\langle 0.3055|0.0036\rangle, \langle 0.3164|0.0036\rangle, \langle 0.3260|0.0024\rangle, \langle 0.2911|0.0024\rangle, \langle 0.3229|0.0033\rangle,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\langle 0.6178|0.0022\rangle, \langle 0.5586|0.0032\rangle, \langle 0.6123|0.0049\rangle, \langle 0.6309|0.0049\rangle, \langle 0.6471|0.0032\rangle\})$$

In a similar manner r^{ω}_2, r^{ω}_3, r^{ω}_4 and r^{ω}_5 can be calculated.

Step 3: Based on Definition 3.2, the score values $S(r^{\omega}_i)$ $(i = 1, 2, 3, 4, 5)$ are evaluated as.

$S(r^{\omega}_1) = 0.1504$, $S(r^{\omega}_2) = 0.2881$, $S(r^{\omega}_3) = 0.2065$, $S(r^{\omega}_4) = 0.3278$, $S(r^{\omega}_5) = 0.4057$.

Step 4: Since $S(r^{\omega}_5) > S(r^{\omega}_4) > S(r^{\omega}_2) > S(r^{\omega}_3) > S(r^{\omega}_1)$, the ordering of the alternatives is obtained as $Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$. So, the best alternative is found as Z_5.

In a similar manner, if the given MCGDM is solved using WDHq-ROFHWG operator, the score values are found as.

$S(r^{\omega}_1) = 0.0129$, $S(r^{\omega}_2) = 0.2202$, $S(r^{\omega}_3) = 0.1191$, $S(r^{\omega}_4) = 0.2356$, $S(r^{\omega}_5) = 0.3051$

Following the principle of ordering as described in Definition 3.3, the ranking of alternatives becomes $Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$. So, the best alternative is identified as Z_5.

Table 5 The collective WDHq-ROFDM $\mathfrak{R}^{\omega}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
Z_1	$(\{\langle 0.3\mid 0.2\rangle, \langle 0.4\mid 0.3\rangle, \langle 0.5\mid 0.2\rangle, \langle 0.6\mid 0.3\rangle\}, \{\langle 0.3\mid 0.5\rangle, \langle 0.4\mid 0.2\rangle, \langle 0.8\mid 0.3\rangle\})$	$(\{\langle 0.5\mid 0.6\rangle, \langle 0.6\mid 0.2\rangle, \langle 0.7\mid 0.2\rangle\}, \{\langle 0.1\mid 0.4\rangle, \langle 0.2\mid 0.6\rangle\})$	$(\{\langle 0.6\mid 0.6\rangle, \langle 0.7\mid 0.2\rangle, \langle 0.8\mid 0.2\rangle\}, \{\langle 0.2\mid 0.3\rangle, \langle 0.4\mid 0.2\rangle, \langle 0.5\mid 0.2\rangle, \langle 0.8\mid 0.3\rangle\})$
Z_2	$(\{\langle 0.6\mid 0.6\rangle, \langle 0.7\mid 0.4\rangle\}, \{\langle 0.1\mid 0.2\rangle, \langle 0.2\mid 0.8\rangle\})$	$(\{\langle 0.5\mid 0.3\rangle, \langle 0.6\mid 0.2\rangle, \langle 0.7\mid 0.2\rangle, \langle 0.9\mid 0.3\rangle\}, \{\langle 0.2\mid 0.5\rangle, \langle 0.3\mid 0.5\rangle\})$	$(\{\langle 0.3\mid 0.3\rangle, \langle 0.4\mid 0.4\rangle, \langle 0.6\mid 0.3\rangle\}, \{\langle 0.2\mid 0.2\rangle, \langle 0.4\mid 0.8\rangle\})$
Z_3	$(\{\langle 0.4\mid 0.2\rangle, \langle 0.5\mid 0.2\rangle, \langle 0.6\mid 0.3\rangle, \langle 0.7\mid 0.3\rangle\}, \{\langle 0.1\mid 0.5\rangle, \langle 0.4\mid 0.3\rangle, \langle 0.6\mid 0.2\rangle\})$	$(\{\langle 0.5\mid 0.2\rangle, \langle 0.7\mid 0.2\rangle, \langle 0.8\mid 0.6\rangle\}, \{\langle 0.1\mid 0.5\rangle, \langle 0.2\mid 0.5\rangle\})$	$(\{\langle 0.5\mid 0.3\rangle, \langle 0.6\mid 0.2\rangle, \langle 0.7\mid 0.3\rangle, \langle 0.8\mid 0.2\rangle\}, \{\langle 0.1\mid 0.2\rangle, \langle 0.3\mid 0.5\rangle, \langle 0.5\mid 0.3\rangle\})$
Z_4	$(\{\langle 0.3\mid 0.3\rangle, \langle 0.5\mid 0.2\rangle, \langle 0.7\mid 0.3\rangle, \langle 0.8\mid 0.2\rangle\}, \{\langle 0.2\mid 0.2\rangle, \langle 0.3\mid 0.3\rangle, \langle 0.4\mid 0.5\rangle\})$	$(\{\langle 0.5\mid 0.2\rangle, \langle 0.6\mid 0.3\rangle, \langle 0.7\mid 0.3\rangle, \langle 0.8\mid 0.2\rangle\}, \{\langle 0.1\mid 1.0\rangle\})$	$(\{\langle 0.4\mid 0.2\rangle, \langle 0.5\mid 0.3\rangle, \langle 0.8\mid 0.3\rangle, \langle 0.9\mid 0.2\rangle\}, \{\langle 0.1\mid 0.3\rangle, \langle 0.2\mid 0.5\rangle, \langle 0.4\mid 0.2\rangle\})$
Z_5	$(\{\langle 0.6\mid 0.2\rangle, \langle 0.8\mid 0.3\rangle, \langle 0.9\mid 0.5\rangle\}, \{\langle 0.1\mid 0.8\rangle, \langle 0.3\mid 0.2\rangle\})$	$(\{\langle 0.3\mid 0.3\rangle, \langle 0.5\mid 0.2\rangle, \langle 0.6\mid 0.3\rangle, \langle 0.9\mid 0.2\rangle\}, \{\langle 0.2\mid 0.5\rangle, \langle 0.4\mid 0.5\rangle\})$	$(\{\langle 0.4\mid 0.3\rangle, \langle 0.6\mid 0.2\rangle, \langle 0.7\mid 0.2\rangle, \langle 0.9\mid 0.3\rangle\}, \{\langle 0.2\mid 0.2\rangle, \langle 0.3\mid 0.5\rangle, \langle 0.5\mid 0.3\rangle\})$

	$\mathcal{C}_4$	$\mathcal{C}_5$
Z_1	$(\{\langle 0.3\mid 0.2\rangle, \langle 0.5\mid 0.2\rangle, \langle 0.6\mid 0.3\rangle, \langle 0.8\mid 0.3\rangle\}, \{\langle 0.5\mid 0.3\rangle, \langle 0.6\mid 0.2\rangle, \langle 0.7\mid 0.2\rangle, \langle 0.8\mid 0.3\rangle\})$	$(\{\langle 0.4\mid 0.2\rangle, \langle 0.7\mid 0.6\rangle, \langle 0.8\mid 0.2\rangle\}, \{\langle 0.2\mid 0.2\rangle, \langle 0.4\mid 0.3\rangle, \langle 0.5\mid 0.3\rangle, \langle 0.6\mid 0.2\rangle\})$
Z_2	$(\{\langle 0.6\mid 0.2\rangle, \langle 0.7\mid 0.6\rangle, \langle 0.9\mid 0.2\rangle\}, \{\langle 0.1\mid 0.3\rangle, \langle 0.2\mid 0.7\rangle\})$	$(\{\langle 0.3\mid 0.2\rangle, \langle 0.6\mid 0.3\rangle, \langle 0.7\mid 0.2\rangle, \langle 0.8\mid 0.3\rangle\}, \{\langle 0.1\mid 0.2\rangle, \langle 0.2\mid 0.3\rangle, \langle 0.3\mid 0.5\rangle\})$
Z_3	$(\{\langle 0.3\mid 0.6\rangle, \langle 0.4\mid 0.2\rangle, \langle 0.6\mid 0.2\rangle\}, \{\langle 0.2\mid 0.2\rangle, \langle 0.4\mid 0.3\rangle, \langle 0.5\mid 0.5\rangle\})$	$(\{\langle 0.6\mid 0.7\rangle, \langle 0.8\mid 0.3\rangle\}, \{\langle 0.1\mid 0.5\rangle, \langle 0.3\mid 0.5\rangle\})$
Z_4	$(\{\langle 0.4\mid 0.3\rangle, \langle 0.7\mid 0.4\rangle, \langle 0.8\mid 0.3\rangle\}, \{\langle 0.2\mid 0.4\rangle, \langle 0.3\mid 0.3\rangle, \langle 0.6\mid 0.3\rangle\})$	$(\{\langle 0.5\mid 0.3\rangle, \langle 0.8\mid 0.2\rangle, \langle 0.9\mid 0.5\rangle\}, \{\langle 0.1\mid 0.3\rangle, \langle 0.2\mid 0.5\rangle, \langle 0.3\mid 0.2\rangle\})$
Z_5	$(\{\langle 0.5\mid 0.3\rangle, \langle 0.6\mid 0.2\rangle, \langle 0.9\mid 0.5\rangle\}, \{\langle 0.2\mid 0.8\rangle, \langle 0.3\mid 0.2\rangle\})$	$(\{\langle 0.5\mid 0.5\rangle, \langle 0.7\mid 0.3\rangle, \langle 0.8\mid 0.2\rangle\}, \{\langle 0.1\mid 0.7\rangle, \langle 0.2\mid 0.3\rangle\})$

6.1 Impact of parameters on the decision result

To demonstrate the impact of the Hamacher parameter ϱ in the above example, steps 2 and 3 are repeatedly executed with different values of ϱ. For convenience, the rung parameter is fixed at $q = 3$ in this case. The aggregated score values and corresponding ranking results corresponding to WDH q-ROFHWA and WDHq-ROFHWG operators are presented in Tables 6 and 7, respectively. From Tables 6 and 7, it is ascertained that although different score values are obtained for different values of Hamacher parameter ϱ, but the ranking results remain the same as $Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$ for each case. It is significantly noticeable that the score values of the alternatives become smaller with the increasing values of the Hamacher parameter based on WDHq-ROFHWA operator. In Fig. 3, the change in score values of different alternatives is visualized for $q = 3$ as a fixed value and varying ϱ in [1, 10]. A decreasing trend in score values is observed there. Thus it can be ascertained that a DM can take pessimistic or optimistic views based on this conviction. So, DMs having a pessimistic attitude towards an alternative based on some criteria must prefer a higher value of Hamacher parameter ϱ.

On the other hand, using WDHq-ROFHWG operator, an increasing trend of the score values of alternatives is observed with the increase of Hamacher parameter ϱ. This fact has been presented in Fig. 4, in which ϱ varied in [1, 10] and the value of the rung parameter q is fixed at $q = 3$. In this case, also DMs can utilize apply their optimistic or pessimistic standpoint for the evaluation process.

Table 6 Ranking results varying Hamacher parameter ϱ in WDH q-ROFHWA operator

Parameter	$S(Z_1)$	$S(Z_2)$	$S(Z_3)$	$S(Z_4)$	$S(A_5)$	Rankings
$\varrho = 1$	0.171	0.3087	0.2244	0.3542	0.4392	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 2$	0.1579	0.2959	0.2136	0.3377	0.418	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 3$	0.1504	0.2881	0.2065	0.3278	0.4057	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 4$	0.1454	0.2827	0.2013	0.3209	0.3974	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 6$	0.1388	0.2754	0.1939	0.3118	0.3866	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 8$	0.1345	0.2705	0.1888	0.3059	0.3798	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 10$	0.1314	0.267	0.1851	0.3016	0.375	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$

Table 7 Ranking results varying Hamacher parameter ϱ in WDHq-ROFHWG operator

Parameter	$S(Z_1)$	$S(Z_2)$	$S(Z_3)$	$S(Z_4)$	$S(Z_5)$	Rankings
$\varrho = 1$	−0.0161	0.2069	0.1085	0.2172	0.2780	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 2$	0.0019	0.216	0.1156	0.2294	0.2961	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 3$	0.0129	0.2202	0.1191	0.2356	0.3051	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 4$	0.0208	0.2228	0.1214	0.2394	0.3107	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\varrho = 6$	0.0317	0.2258	0.1245	0.2443	0.3173	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\boldsymbol{\varrho} = 8$	0.0393	0.2276	0.1266	0.2473	0.3212	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\boldsymbol{\varrho} = 10$	0.0449	0.2289	0.1283	0.2495	0.3238	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$

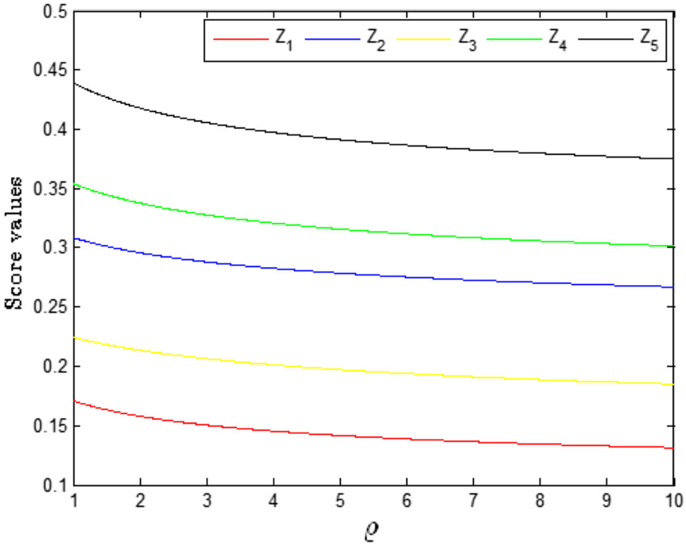

Fig. 3 Effect of Hamacher parameter on score values based on WDHq-ROFHWA operator

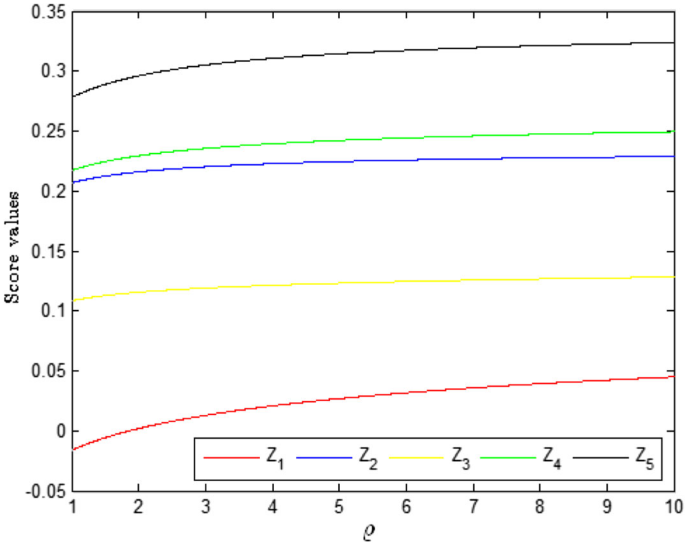

Fig. 4 Effect of Hamacher parameter on score values based on WDHq-ROFHWG operator

If the above problem is solved based on algebraic and Einstein operations, i.e. for $\varrho = 1$ and 2, the achieved scores and rankings of the alternatives are listed in Table 8.

On the contrary, keeping the Hamacher parameter fixed at $\varrho = 3$**,** adopting different values of rung parameter q the consequences in score values and orderings of the preferences are manifested in Tables 9 and 10, respectively, with the aid of WDHq-ROFHWA and WDHq-ROFHWG operators.

Table 8 Ranking results using algebraic and Einstein operations-based WDHq-ROF AO

Operators($q = 3$)	Scores	Rankings
WDHq-ROFWA ($\varrho = 1$)	$S(Z_1) = 0.1710$, $S(Z_2) = 0.3087$, $S(Z_3) = 0.2244$, $S(Z_4) = 0.3542$, $S(Z_5) = 0.4392$	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
WDHq-ROFEWA ($\varrho = 2$)	$S(Z_1) = 0.1579$, $S(Z_2) = 0.2959$, $S(Z_3) = 0.2136$, $S(Z_4) = 0.3377$, $S(Z_5) = 0.4180$	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
WDHq-ROFWG ($\varrho = 1$)	$S(Z_1) = -0.0161$, $S(Z_2) = 0.2069$, $S(Z_3) = 0.1085$, $S(Z_4) = 0.2172$, $S(Z_5) = 0.2780$	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
WDHq-ROFEWG ($\varrho = 2$)	$S(Z_1) = 0.0019$, $S(Z_2) = 0.2160$, $S(Z_3) = 0.1156$, $S(Z_4) = 0.2294$, $S(Z_5) = 0.2961$	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$

Table 9 Ranking results in varying rung parameter q in WDH q-ROFHWA operator

	$S(Z_1)$	$S(Z_2)$	$S(Z_3)$	$S(Z_4)$	$S(Z_5)$	Rankings
$q = 1$	0.1570	0.4311	0.3188	0.4446	0.5175	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$q = 2$	0.1736	0.3844	0.2821	0.4161	0.4956	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$q = 3$	0.1504	0.2881	0.2065	0.3278	0.4057	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$q = 4$	0.1192	0.2117	0.1466	0.2532	0.3288	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$q = 6$	0.0688	0.1205	0.076	0.1574	0.2278	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$q = 8$	0.0392	0.0752	0.0419	0.104	0.1681	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$q = 10$	0.0228	0.0508	0.0242	0.0718	0.1289	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$

Table 10 Ranking results in varying rung parameter q in WDH q-ROFHWG operator

	$S(Z_1)$	$S(Z_2)$	$S(Z_3)$	$S(Z_4)$	$S(Z_5)$	Rankings
$q = 1$	0.0743	0.3866	0.2516	0.3795	0.4514	$Z_5 \succ Z_2 \succ Z_4 \succ Z_3 \succ Z_1$
$q = 2$	0.0493	0.323	0.1923	0.3289	0.4063	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$q = 3$	0.0129	0.2202	0.1191	0.2356	0.3051	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$q = 4$	−0.0109	0.1424	0.0702	0.1617	0.2199	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\boldsymbol{q} = 6$	−0.0270	0.05751	0.0238	0.0754	0.1123	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\boldsymbol{q} = 8$	−0.0248	0.0234	0.0083	0.0363	0.0587	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$
$\boldsymbol{q} = 10$	−0.0186	0.0097	0.0034	0.0182	0.0319	$Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$

It is indicated from Table 9 that even if different score values are acquired by the preferences but the leading option is always Z_5 and also a steady ranking $Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$ is found under WDHq-ROFHWA operator. However, from Table 10, it is ascertained that different score values according to different rung parameters q are acquired by the alternatives through WDHq-ROFHWG operator. It is significant to note that ranking results quite differ at $q = 1$. After that, the ranking of alternatives becomes stable. Nevertheless, the leading option remains the same in any case.

- Based on the traditional MCGDM method using $\boldsymbol{q}$-ROFHWA and $\boldsymbol{q}$-ROFHWG operators
  If the above Example 1 is solved using the traditional MCGDM method (Darko and Liang 2020a), then the consequences are checked. Utilizing the q-ROFHWA operator (considering $q = 3$, $\varrho = 3$) under the traditional MCGDM method, the results are calculated as $S(Z_1) = 0.1765$, $S(Z_2) = 0.2897$, $S(Z_3) = 0.2097$, $S(Z_4) = 0.3313$, $S(Z_5) = 0.4115$. The ranking of the alternatives is found as $Z_5 \succ Z_4 \succ Z_2 \succ Z_3 \succ Z_1$. Besides using the q-ROFHWG operator (taking $q = 3$,$\varrho = 3$) under traditional group decision making framework, the score values are evaluated as $S(Z_1) = 0.0141$, $S(Z_2) = 0.2127$, $S(Z_3) = 0.1122$, $S(Z_4) = 0.2193$, $S(Z_5) = 0.2891$. Hence the ranking of the alternatives remains the same as previous.

7 Comparative study

TO elucidate the superiority and efficacy of the suggested WDH q-ROF aggregation operators in realistic problems, a comparative study is performed in this section.

7.1 Comparison with existing methods

The identical ranking result is found using the traditional MCGDM method (Darko and Liang 2020a) (under q-ROF environment) and the proposed method, reflecting the proposed method's validity. It can also be stated that the proposed method can be considered as an alternative way to deal with MCGDM problems. Furthermore, Zeng et al. (2020) also found the same ranking result. Over this also the sustainability of the proposed method is confirmed.

Further, it is to be noted here that Zeng et al. (2020) solved the above example under the WDHF environment, which fails to incorporate decision values with membership and nonmembership grades having sum greater than 1. That was clearly a drawback of Zeng et al.'s (2020) method. While, the proposed method is investigated under WDHq-ROF context, and consequently, it widens the scope of application in the information aggregation process with the aid of a parameter q. It is important to ascertain that Zeng et al.'s method becomes a particular case of the proposed method if Hamacher parameter $\varrho = 1$ and rung parameter $q = 1$, are considered in the process of aggregation. Thus it can be concluded that the proposed WDHq-ROFHWA and WDHq-ROFHWG operators are more flexible and general than existing operators (Zeng et al. 2020) in the information aggregation processes. Thus, the developed method in this paper is more proficient under a realistic decision-making context.

Nevertheless, the same ranking results are not enough to affirm the advantage of the proposed method. So, another example is considered and solved in the next section for the establishment of the benefits of the proposed approach.

Table 11 DHq-ROFDM provided by DM

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
Z_1	$(\{0.3, 0.4\}, \{0.6\})$	$(\{0.4, 0.5\}, \{0.3, 0.4\})$	$(\{0.2, 0.3\}, \{0.7\})$	$(\{0.4, 0.5\}, \{0.5\})$
Z_2	$(\{0.6\}, \{0.4\})$	$(\{0.2, 0.4, 0.5\}, \{0.4\})$	$(\{0.2\}, \{0.6, 0.7, 0.8\})$	$(\{0.5\}, \{0.4, 0.5\})$
Z_3	$(\{0.5, 0.7\}, \{0.2\})$	$(\{0.2\}, \{0.7, 0.8\})$	$(\{0.2, 0.3, 0.4\}, \{0.6\})$	$(\{0.5, 0.6, 0.7\}, \{0.3\})$
Z_4	$(\{0.7\}, \{0.3\})$	$(\{0.6, 0.7, 0.8\}, \{0.2\})$	$(\{0.1, 0.2\}, \{0.3\})$	$(\{0.1\}, \{0.6, 0.7, 0.8\})$
Z_5	$(\{0.6, 0.7\}, \{0.2\})$	$(\{0.2, 0.3, 0.4\}, \{0.5\})$	$(\{0.4, 0.5\}, \{0.2\})$	$(\{0.2, 0.3, 0.4\}, \{0.5\})$

7.2 Further comparative analysis

The proposed method is verified in the above sub-section by comparing it with existing methods (Darko and Liang 2020a; Zeng et al. 2020). Also, some sort of superiority of the developed approach is outlined in the view points of rung parameter q and the Hamacher parameter ϱ. However, to illuminate the advantages of the suggested approach compared to some existing methods (Garg 2018; Wang et al. 2014; Wei and Lu 2017), the following numerical example is considered.

Example 2 In the context of supply chain management Wei and Lu (2017) studied a problem to select the most suitable supplier. There are five prospective suppliers Z_i $(i = 1, 2, 3, 4, 5)$ need to be evaluated on four criteria $\mathcal{C}_j$ $(i = 1, 2, 3, 4)$. The four criteria include product quality $\mathcal{C}_1$, service $\mathcal{C}_2$, delivery $\mathcal{C}_3$ and price $\mathcal{C}_4$, respectively.

Wei and Lu (2017) used DHPF data as presented in Table 11 to indicate the DMs' preference values by considering the same importance degree of each possible membership and non-membership values. However, DMs may have different confidences in ascertaining each possible membership and non-membership values during assessing alternatives under certain criteria. Thus ignoring the different importance degrees of each possible membership and non-membership value might cause a significant loss of information. Next, to show the effectiveness of the suggested approach, the existing problem is revised by considering the different importance degrees of possible membership and non-membership values and solved using the proposed method under WDHq-ROF environment.

Table 11 is revised by incorporating different importance degrees and a WDHq-ROFDM, $D_{m\times n} = \left[\hbar_{ij}^{\omega}\right]_{m\times n}$ is constructed, which is presented in Table 12.

The proposed operators are used to aggregate the WDHq-ROFNs as described in Table 12, and ranking results of the alternatives are obtained. Besides, different existing aggregation methods (Garg 2018; Wang et al. 2014; Wei and Lu 2017) are applied in Table 11 to find the ranking of the alternatives. Then, the achieved outcomes utilizing the existing methods and the proposed method are displayed in Table 13 for representing a summary of comparative analysis. From Table 13, it is understood that a fluctuation in the ranking results is found by applying the proposed method. This happens because the proposed method can consider different importance degrees for each possible membership and non-membership values of alternatives with respect to given criteria. On the other side, existing methods neglect the associating weightage of importance for possible membership and non-membership grades of each alternative in the process of decision-making. This negligence of the importance values causes a severe loss of information, leading to erroneous decision results.

Table 12 Constructed WDHq-ROFDM

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
Z_1	$\left(\left\{\frac{0.3}{0.6}, \frac{0.4}{0.4}\right\}, \left\{\frac{0.6}{1}\right\}\right)$	$\left(\left\{\frac{0.4}{0.7}, \frac{0.5}{0.3}\right\}, \left\{\frac{0.3}{0.4}, \frac{0.4}{0.6}\right\}\right)$	$\left(\left\{\frac{0.2}{0.5}, \frac{0.3}{0.5}\right\}, \left\{\frac{0.7}{1}\right\}\right)$	$\left(\left\{\frac{0.4}{0.4}, \frac{0.5}{0.6}\right\}, \left\{\frac{0.5}{1}\right\}\right)$
Z_2	$\left(\left\{\frac{0.6}{1}\right\}, \left\{\frac{0.4}{1}\right\}\right)$	$\left(\left\{\frac{0.2}{0.3}, \frac{0.4}{0.4}, \frac{0.5}{0.3}\right\}, \left\{\frac{0.4}{1}\right\}\right)$	$\left(\left\{\frac{0.2}{1}\right\}, \left\{\frac{0.6}{0.3}, \frac{0.7}{0.3}, \frac{0.8}{0.4}\right\}\right)$	$\left(\left\{\frac{0.5}{1}\right\}, \left\{\frac{0.4}{0.7}, \frac{0.5}{0.3}\right\}\right)$
Z_3	$\left(\left\{\frac{0.5}{0.9}, \frac{0.7}{0.1}\right\}, \left\{\frac{0.2}{1}\right\}\right)$	$\left(\left\{\frac{0.2}{1}\right\}, \left\{\frac{0.7}{0.9}, \frac{0.8}{0.1}\right\}\right)$	$\left(\left\{\frac{0.2}{0.8}, \frac{0.3}{0.1}, \frac{0.4}{0.1}\right\}, \left\{\frac{0.6}{1}\right\}\right)$	$\left(\left\{\frac{0.5}{0.8}, \frac{0.6}{0.1}, \frac{0.7}{0.1}\right\}, \left\{\frac{0.3}{1}\right\}\right)$
Z_4	$\left(\left\{\frac{0.7}{1}\right\}, \left\{\frac{0.3}{1}\right\}\right)$	$\left(\left\{\frac{0.6}{0.1}, \frac{0.7}{0.1}, \frac{0.8}{0.8}\right\}, \left\{\frac{0.2}{1}\right\}\right)$	$\left(\left\{\frac{0.1}{0.1}, \frac{0.2}{0.9}\right\}, \left\{\frac{0.3}{1}\right\}\right)$	$\left(\left\{\frac{0.1}{1}\right\}, \left\{\frac{0.6}{0.8}, \frac{0.7}{1}, \frac{0.8}{1}\right\}\right)$
Z_5	$\left(\left\{\frac{0.6}{0.8}, \frac{0.7}{0.2}\right\}, \left\{\frac{0.2}{1}\right\}\right)$	$\left(\left\{\frac{0.2}{0.7}, \frac{0.3}{0.2}, \frac{0.4}{0.1}\right\}, \left\{\frac{0.5}{1}\right\}\right)$	$\left(\left\{\frac{0.4}{0.9}, \frac{0.5}{0.1}\right\}, \left\{\frac{0.2}{1}\right\}\right)$	$\left(\left\{\frac{0.2}{0.8}, \frac{0.3}{0.1}, \frac{0.4}{0.1}\right\}, \left\{\frac{0.5}{1}\right\}\right)$

Table 13 Comparison with different existing methods

Method	Score values	Ranking
HPFWA (Garg 2018) ($q = 2$ and $\varrho = 1$)	$S(Z_1) = -0.1622$, $S(Z_2) = -0.0600$, $S(Z_3) = 0.0784$, $S(Z_4) = 0.0913$,$S(Z_5) = 0.1210$	$Z_5 \succ Z_4 \succ Z_3 \succ Z_2 \succ Z_1$
HPFWG (Garg 2018)	$S(Z_1) = -0.2302$, $S(Z_2) = -0.1866$, $S(Z_3) = -0.1162$, $S(Z_4) = -0.1852$,$S(Z_5) = 0.0188$	$Z_5 \succ Z_3 \succ Z_4 \succ Z_2 \succ Z_1$
DHFWA (Wang et al. 2014) ($q = 1$ and $\varrho = 1$)	$S(Z_1) = -0.1846$, $S(Z_2) = -0.0852$, $S(Z_3) = 0.0617$, $S(Z_4) = 0.0392$,$S(Z_5) = 0.1406$	$Z_5 \succ Z_3 \succ Z_4 \succ Z_2 \succ Z_1$
DHFWG (Wang et al. 2014)	$S(Z_1) = -0.2388$, $S(Z_2) = -0.1897$, $S(Z_3) = -0.1008$, $S(Z_4) = -0.2261$,$S(Z_5) = 0.0450$	$Z_5 \succ Z_3 \succ Z_2 \succ Z_4 \succ Z_1$
DHPFHWA (Wei and Lu 2017) ($q = 2$ and $\varrho = 3$)	$S(Z_1) = -0.1743$, $S(Z_2) = -0.0810$, $S(Z_3) = 0.0489$, $S(Z_4) = 0.0426$,$S(Z_5) = 0.1057$	$Z_5 \succ Z_3 \succ Z_4 \succ Z_2 \succ Z_1$
DHPFHWG (Wei and Lu 2017)	$S(Z_1) = -0.2179$, $S(Z_2) = -0.1642$, $S(Z_3) = -0.0822$, $S(Z_4) = -0.1506$,$S(Z_5) = 0.0324$	$Z_5 \succ Z_3 \succ Z_4 \succ Z_2 \succ Z_1$
WDHq-ROFWA ($q = 3, \varrho = 1$)	$S(Z_1) = -0.1130$, $S(Z_2) = -0.0279$, $S(Z_3) = 0.0220$, $S(Z_4) = 0.1231$,$S(Z_5) = 0.0571$	$Z_4 \succ Z_5 \succ Z_3 \succ Z_2 \succ Z_1$
WDHq-ROFWG ($q = 3, \varrho = 1$)	$S(Z_1) = -0.1746$, $S(Z_2) = -0.1489$, $S(Z_3) = -0.1104$, $S(Z_4) = -0.0865$,$S(Z_5) = -0.0217$	$Z_5 \succ Z_4 \succ Z_3 \succ Z_2 \succ Z_1$
WDHq-ROFEWA ($q = 3, \varrho = 2$)	$S(Z_1) = -0.1172$, $S(Z_2) = -0.0349$, $S(Z_3) = 0.0164$, $S(Z_4) = 0.1046$,$S(Z_5) = 0.0530$	$Z_4 \succ Z_5 \succ Z_3 \succ Z_2 \succ Z_1$
WDHq-ROFEWG ($q = 3, \varrho = 2$)	$S(Z_1) = -0.1692$, $S(Z_2) = -0.1373$, $S(Z_3) = -0.1018$, $S(Z_4) = -0.0790$,$S(Z_5) = -0.0193$	$Z_5 \succ Z_4 \succ Z_3 \succ Z_2 \succ Z_1$
WDHq-ROFHWA ($q = 3, t = 3$)	$S(Z_1) = -0.1193$, $S(Z_2) = -0.0391$, $S(Z_3) = 0.0129$, $S(Z_4) = 0.0919$,$S(Z_5) = 0.0501$	$Z_4 \succ Z_5 \succ Z_3 \succ Z_2 \succ Z_1$
WDHq-ROFHWG ($q = 3, t = 3$)	$S(Z_1) = -0.1654$, $S(Z_2) = -0.1299$, $S(Z_3) = -0.0958$, $S(Z_4) = -0.0739$,$S(Z_5) = -0.0175$	$Z_5 \succ Z_4 \succ Z_3 \succ Z_2 \succ Z_1$

It is to be mentioned here that Wei and Lu (2017) solved the problem under DHPF environment and found the ranking of the alternatives as $Z_5 \succ Z_3 \succ Z_4 \succ Z_2 \succ Z_1$ using both DHPFHWA and DHPFHWG operators. While solving this problem using HPFWA and HPFWG operators (Garg 2018), the ordering of the alternatives is found as $Z_5 \succ Z_4 \succ Z_3 \succ Z_2 \succ Z_1$ and $Z_5 \succ Z_3 \succ Z_4 \succ Z_2 \succ Z_1$, respectively. Again, based on DHFWA and DHFWG operators (Wang et al. 2014), the ranking of the alternatives is obtained as $Z_5 \succ Z_3 \succ Z_4 \succ Z_2 \succ Z_1$ and $Z_5 \succ Z_3 \succ Z_2 \succ Z_4 \succ Z_1$, respectively.

It is worthy to mention here that all the above ranking results can be achieved by the proposed method if the different importance degrees are ignored. In this respect, it is to be pointed out here that for $q = 2$ and $\varrho = 1$, the proposed operator will generate same ranking results as achieved by Garg (2018). Further, for $q = 1$ and $\varrho = 1$, in the proposed operator would result same ranking as obtained by Wang et al. (2014), Again, the results obtained by Wei and Lu (2017) can be acquired by the proposed operators if $q = 2$ and $\varrho = 3$ are taken into account. It is to be noted here that all the above rankings are obtained through the proposed operators by ignoring different weights corresponding to possible membership and non-membership grades. Thus it can be concluded that the existing operators under consideration become some specific cases of developed operators. In each of the above cases, the best alternative is Z_5. Whereas, using the proposed method, the best alternative may vary between Z_4 and Z_5. Two different rankings $Z_4 \succ Z_5 \succ Z_3 \succ Z_2 \succ Z_1$ and $Z_5 \succ Z_4 \succ Z_3 \succ Z_2 \succ Z_1$ are achieved by applying the proposed WDHq-ROFHWA and WDHq-ROFHWG operators, respectively. The proposed method can prevent the loss of information by considering different weights associated with each possible membership and non-membership degrees of DH q-ROFNs. Thus, it can be stated that, compared to the existing methods, the proposed weight-based method can produce more rational rankings and generate relevant best choices according to the situation.

8 Conclusions

In this article, the concepts of WDHq-ROFS and WDHq-ROFN are introduced by incorporating different weightage values/importance degrees among possible membership and non-membership terms to control the degree of certainty of the elements. Also, the score function and accuracy function for WDHq-ROFNs are provided to define comparison of WDHq-ROFNs. Some fundamental operational laws of WDHq-ROFNs are defined based on Ht-N&t-CNs. Meanwhile, to aggregate the attribute performances which are in the form of WDHq-ROFNs, WDHq-ROFHWA and WDHq-ROFHWG operators, are proposed to model the complex decision making situations. The proposed operators include a class of weighted averaging and weighted geometric operators for various fuzzy environments which are discussed explicitly in this paper. A new MCGDM model under WDHq-ROFSs environment is developed. Finally, some numerical instances are given, along with some comparative analyses to demonstrate the validity and effectiveness of the proposed approach. Through the achieved outcomes and comparative studies, it is evidenced that the proposed method reflects its robustness in dealing MCGDM problems. In the future, the developed approach may be applied to the fields of medical diagnosis, knowledge representation, cluster analysis and so on. Furthermore, the proposed weighted concept can be incorporated with interval-valued DHq-ROF, neutrosophic fuzzy (Jana et al. 2020), q-rung orthopair hesitant fuzzy uncertain linguistic, linguistic q-rung orthopair fuzzy (Deb et al. 2022), complex fermatean fuzzy (Akram et al. 2022), and complex hesitant q-rung orhopair sets comparatively.

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Declarations

Conflict of interest All the authors declare that they have no conflict of interest.

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Sugeno–Weber triangular norm-based aggregation operators under T-spherical fuzzy hypersoft context

Arun Sarkar [a], Tapan Senapati [b,*], LeSheng Jin [c], Radko Mesiar [d,e], Animesh Biswas [f], Ronald R. Yager [g]

[a] *Department of Mathematics, Heramba Chandra College, Kolkata 700029, India*
[b] *School of Mathematics and Statistics, Southwest University, Beibei 400715, Chongqing, China*
[c] *Business School, Nanjing Normal University, Nanjing, China*
[d] *Faculty of Civil Engg., Slovak University of Technology, Bratislava, Slovakia*
[e] *Dept. Algebra and Geometry, Faculty of Science, Palacký University, Olomouc, 17. listopadu 12, 77 146 Olomouc, Czech Republic*
[f] *Department of Mathematics, University of Kalyani, Kalyani 741235, India*
[g] *Machine Intelligence Institute, Iona College, New Rochelle, NY 10801, USA*

A R T I C L E I N F O

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A B S T R A C T

This research work contributes significantly to the current information field by offering an innovative model named the *T*-spherical fuzzy hypersoft (*T*-SFHS) set (*T*-SFHSS). This framework addresses both aspects of the three-dimensional knowledge implicated in the satisfaction, abstinence, and dissatisfaction inherent in human decision-making. It is an innovative approach to the problem of introducing computer cognition and decision-making in uncertain settings into the real world. The *T*-SFHSS is superior at determining what to do with unclear or imprecise data. The *T*-SFHSS enhances fuzzy sets such as the "intuitionistic fuzzy hypersoft set" and the "Pythagorean fuzzy hypersoft set". It aims to increase the precision of fuzzy set calculations. To aggregate the decision data most effectively, we propose some novel Sugeno-Weber t-norm and t-conorm-based operational rules for *T*-SFHS numbers (*T*-SFHSNs). We then propose some *T*-SFHS aggregation operators with desirable properties in light of these operational laws. We conduct an illustrative study on natural agribusiness to demonstrate the viability and utility of the present methodology. The correctness of the obtained results can be verified by contrasting the proposed SW aggregation operators (AOs) on *T*-SFSS with the approaches already in use. The findings show that the proposed methodology is more consistent and successful than the current procedures.

Abbreviations: DM, Decision maker; FS, Fuzzy set; FSS, Fuzzy soft set; HySS, Hypersoft set; HyS, Hypersoft; IFHySS, Intuitionistic fuzzy hypersoft set; IFS, Intuitionistic fuzzy set; IFSS, Intuitionistic fuzzy soft set; MCDM, Multicriteria decision making; MCGDM, Multicriteria group decision-making; MG, Membership grade; NHSS, Neutrosophic hypersoft Set; NMG, Non-membership grade; PFHySS, Pythagorean fuzzy hypersoft set; PFS, Pythagorean fuzzy set; PFSS, Picture fuzzy soft set; *q*-ROF, *q*-rung orthopair fuzzy; *q*-ROFSS, *q*-rung orthopair fuzzy soft set; SFS, Spherical fuzzy soft; SFSS, Spherical fuzzy soft set; SFHyS, Spherical fuzzy hypersoft set; SS, Soft set; SW, Sugeno-Weber; SW *t*-N&*t*-CN, Sugeno-Weber t-norm and t-conorm; TOPSIS, Technique for order preference by similarity to an ideal solution; *T*-SFHySDM, *T*-SFHyS decision matrix; *T*-SFHySN, *T*-Spherical fuzzy hypersoft number; *T*-SFHySS, *T*-Spherical fuzzy hypersoft set; *T*-SFHyS, *T*-Spherical fuzzy hypersoft; *T*-SFN, *T*-Spherical fuzzy number; *T*-SFSS, *T*-Spherical fuzzy soft set; *T*-SFHySSWWA, *T*-SFHySS Sugeno-Weber weighted averaging; *T*-SFHySSWWG, *T*-SFHySS Sugeno-Weber weighted geometric.

* Corresponding author at: School of Mathematics and Statistics, Southwest University, Beibei 400715, Chongqing, China.
E-mail addresses: asarkarmth@gmail.com (A. Sarkar), math.tapan@gmail.com (T. Senapati), jls1980@163.com (L. Jin), radko.mesiar@stuba.sk (R. Mesiar), abiswaskln@rediffmail.com (A. Biswas), yager@panix.com (R.R. Yager).

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1. Introduction

In our daily lives, conflict resolution is built on intelligence gathering, machine learning, knowledge compilation, and so on. The unavailability of concrete evidence is a major problem in strategic planning. This lack of information can be filled in by a mathematical theory, which makes it possible to use the right decision-making (DM) strategies. Companies might benefit from DM ideology, which can sometimes compile and rate a variety of opinions from good to one of the worst possible choices. Therefore, it enables us to select, categorize, construct our possibilities, and conduct an in-depth analysis. "Multicriteria decision-making (MCDM)" is the best way to find a much better answer based on all possible factors or criteria that could be involved. Throughout history, it has generally been accepted that all evidence about the option concerning attributes and associated weights was expected to be transmitted in crisp numbers. There are a great number of considerations and indications packed into the multiple evaluation and decision-making problems. The primary objective of resolving the assessment and decision-making problem is categorizing and compiling the information for evaluation indicators. However, due to the complexities present in real-life systems, people deal with many MCDM problems where the evaluation information is uncertain. To deal with it, Zadeh [47] introduced fuzzy sets (FS) as a common method for dealing with DM juxtapositions and inconsistencies. A membership value that ranges from 0 to 1 is assigned to each element within this list. Professionals mostly look at "membership and non-membership values" during the DM process, which FS can't do. Atanasov [2] proposed the "intuitionistic fuzzy set (IFS)" as a solution to the problems mentioned above.

The complexity of FS and IFS binds them together to appropriately address concerns such as uncertainty. The main source of these difficulties could be a flaw in the parameter estimation tools. It looks for a theoretical way to deal with situations that don't have any problems. In 1999, the Russian researcher Molodtsov addressed this flaw. Prof. Molodtsov introduced the soft set (SS) theory [27], which is yet another predefined category of groups in the discourse universe. SS theory is useful in many areas, like game theory, measurement theory, making decisions, Riemann integration, smoothness of functions, etc. SS studies have been quite vigorous, and many discoveries have been obtained in fundamental and methodological areas. In Maji et al. [25], the concept of "fuzzy soft sets" was proposed as a fuzzy expansion of "classical soft sets", along with some basic characteristics that were carefully examined. Since then, several investigations have been made on this topic. Jiang et al. [15] introduced entropy on intuitionistic fuzzy soft sets (IFSS) and interval-valued fuzzy soft sets.

Even though it is thought to be possible to imagine the linear discrepancy seen between "membership grade (MG)", μ and "non-membership grade (NMG)", ν, the FSS and IFSS models are unable to accommodate incorrect and ambiguous facts. For instance, the IFSS, as explained previously, cannot handle it if decision-makers choose MG and NMG values of 0.9 and 0.2, respectively. It is because of $0.9 + 0.2 > 1$. Peng et al. [29] integrated SS with a "Pythagorean fuzzy set (PFS)" [43] to directly describe it, and they introduced the concept of "Pythagorean fuzzy SS (PFSS)" by changing the fundamental constraint from $\mu + \nu \leq 1$ to $\mu^2 + \nu^2 \leq 1$. It is clear from their structure that FSS and IFSS are special cases of PFSS.

The PFSS theory is important for solving many important MCDM problems [20] but is imperfect. When the square sum of MG and NMG exceeds 1, the PFSS cannot handle the problem. To address these issues, Hussain et al. [12] proposed a generalization of IFSS and PFSS, known as "q-rung orthopair fuzzy (q-ROF) SS (q-ROFSS)". q-ROFSS gives systemic designers additional leeway in expressing their opinions about MG and NMG by meeting the constraint that the sum of q-th powers of MG and NMG is less than or equal to 1. For example, $\langle 0.7, 0.8 \rangle$ is appropriate for using the q-ROFSS to tackle cases where $q = 3$. Hamid et al. [10] created the q-ROFS TOPSIS and q-ROFS VIKOR methods to solve MCGDM difficulties. Chinram et al. [4] proposed the q-ROFS weighted geometric operator for q-ROFSS with its advantageous characteristics. To resolve MCDM problems, they also employed their newly built operator.

In real life, q-ROFSS can handle representative samples that need to be completed or clarified, but it needs help with data that contradicts itself. For illustration, in Son's work [37], voting results for the election of village director may be broken down into three categories: "vote for", "neutral voting", and "vote against". "Neutral voting" means "no vote." On the white ballot, you can't agree or disagree with the candidate, but you can still vote. This case occurred in reality, but q-ROFSS could not deal with it. In this regard, Yang et al. [44] introduced picture FSS, which is a direct extension of FSS, IFSS, PFSS, and q-ROFSS by incorporating the concepts of positive, μ, negative, ν, and neutral MG, δ of an element. Khan et al. [18] introduced generalized picture FSS and their basic properties. Jan et al. [14] suggested some AOs for the proposed multi-valued picture FSS. These are called "multi-picture FSS-weighted averaging, ordered weighted averaging, and hybrid weighted averaging operators".

Whereas PFSS and q-ROFSS structures have been substantially enhanced by picture FSS structures, picture FSS still has a few restrictions in specific situations; for example, picture FSSs cannot be allocated (μ, δ, ν) when $\mu + \delta + \nu > 1$. To overcome this issue, a new model of spherical fuzzy SS (SFSS) was developed by Perveen et al. [30], and some of its properties have been introduced. SFSS is constructed by $\mu^2 + \delta^2 + \nu^2 \leq 1$. Ahmmad et al. [1] laid out the basic operational laws for SFSS. Based on these operational laws, they then laid out several AOs, such as, "SFS- weighted average, ordered weighted average, and hybrid average AOs", were introduced. Riaz et al. (2022) defined the concepts of SFSS topology and SFSS separation axioms. The investigation of various SFSS-topology properties yields pertinent conclusions. Some fundamental terms, such as "spherical fuzzy soft basis, spherical fuzzy soft subspace, spherical fuzzy soft interior, spherical fuzzy soft closure, and spherical fuzzy soft boundary", were defined by Garg et al. [6]. However, even squaring is not enough as the squared sum of μ, δ, and ν exceeds the unit interval, i.e., $\mu^2 + \delta^2 + \nu^2 > 1$. To deal with this kind of situation, Guleria and Bajaj [7] presented a modification of SFSS which was known as T-spherical fuzzy SS (T-SFSS) based on "T-spherical fuzzy set" [24]. T-SFSS has the condition that $\mu^{\ell} + \delta^{\ell} + \nu^{\ell} \leq 1$ where $\ell \geq 1$. It is mentioned here that, just as the q-ROF is a generalized version of IF and PF context, the T-SF environment is a more generalized version of the picture fuzzy and spherical fuzzy environment. Guleria and Bajaj [7] introduced some averaging and geometric AOs (weighted, ordered, and hybrid) for the "T-spherical fuzzy soft numbers". T-SFSS are a more generic version of IFSSs, PFSSs, and SFSSs.

1.1. Review on hypersoft set theory

In reality, distinct attributes are further partitioned into disjoint attribute–value sets. The existing soft set theory does not apply to such sets. As a result, Florentin Smarandache, an American scientist, first proposed a new structure, hypersoft set theory, in 2018 to address such circumstances.

We briefly look at some research on real-world decision-making with different kinds of extinct contexts, such as fuzzy information, intuitionistic fuzzy information, neutrosophic sets, Pythagorean fuzzy information, hesitant fuzzy information, and linguistic information, all of which use hypersoft set theory.

After the invention of hypersoft sets, several extended versions have been developed as a combination of a set with several sets, such as a fuzzy set, a rough set, an expert set, a cubic set, etc. Different works on these sets have been conducted within a very short span. Saqlain et al. [35] proposed the TOPSIS approach for the neutrosophic HySS. Further, they defined some basic operational laws for NHySS: intersection, union, complement, etc. And the suggested method is applied to the MCDM issue. Saeed et al. [34] introduced fuzzy HySS (FHySS), resulting from the hypersoft set concept. The fact that each FHySS may be viewed as a (fuzzy) information system and can be expressed in a datasheet with values between [0,1] is particularly useful. Yolcu and Ozturk [46] introduced some notions of the fundamental operation of FHySSs. Martin and Smarandache [26] made the plithogenic HySS include the degree to which the elements of the attribute system belong to each other. Rahman et al. [32] defined complex FHySS by hybridizing HySS and complex fuzzy sets. By integrating HySSs with bipolarity, Musa and Asaad [28] produced a new mathematical framework named the bipolar HySS. Saeed et al. [33] presented the novelty of complex multi-fuzzy HySS. Musa and Asaad, in the year 2022, added bipolar hypersoft topological spaces to the group of bipolar HySSs. Yolcu et al. [45] developed a new environment, intuitionistic fuzzy HySS (IFHySS). They also introduce some fundamental operational laws of IFHySSs, viz., union, intersection, complement, etc. Zulqarnain et al. [50] proposed the concept of Pythagorean FHySS. Saqlain et al. [36] first consider distances for NHySS and then propose similarity measures for NHSS. They also consider aggregated operations when aggregating the NHySS decision matrix. Khan et al. [19] introduced a new concept called q-rung orthopair fuzzy HySS (q-ROF Hypersoft sets). We introduce a theme of fundamental operations such as q-ROF Hypersoft-subset, q-ROF Hypersoft null-set, q-ROF Hypersoft absolute-set, union, intersection, and complement. We also present "AND" and "OR" operators.

1.2. Motivations of the paper

Compared to individual decision-making, group decision-making has two benefits: synergy and shared knowledge. The concept of synergy holds that the sum of its parts is greater than the whole. When choosing, a group's collective judgment may be superior to any individual member's. Group members can develop more comprehensive and reliable ideas and recommendations through discussion, inquiry, and collaboration. In decision-making scenarios, the attribution function f, which is unique to the cartesian product with the n attribute, cannot consider many sub-attributes in the extant research. The existing theories, viz., IFSS, PFSS, q-ROFSS, etc., fall short of resolving these kinds of issues when any attribute from a set of parameters includes additional sub-attributes. Several compelling hybridized models with different hypersoft sets have been proposed to tackle such situations, namely, bipolar hypersoft sets, IFHSS, Pythagorean fuzzy HSS, q-ROF hypersoft sets. The authors have argued for their applicability in group decision-making theory. One of the major insufficiencies of these models is that they only study the assignment of MGs and NMGs to the parameterized characterization of the elements by different experts, taking no account of the abstain degrees. From this viewpoint, this paper aims to develop a new hybrid method combining hypersoft theory to represent some problems that are difficult to explain in other extensions of fuzzy set theory, such as human opinions involving four types of answers: yes, abstain, no, and refusal. For this purpose, this paper combines the T-spherical fuzzy set and hypersoft set to obtain a new hypersoft set model named the T-spherical fuzzy hypersoft set. Sometimes, this new model makes descriptions of the real world more realistic and useful. The new notion includes a family of existing and novel fuzzy extensions, viz., IFHSS, Pythagorean fuzzy HSS, q-ROF HSS, picture fuzzy HSS, spherical fuzzy HSS, etc.

Sugeno has introduced a family of nilpotent t-conorms (with asymptotic members drastic sum and probabilistic sum) in his PhD thesis [38]. On the other hand, a family of nilpotent t-norms were introduced by Weber [40] (with asymptotic members product and drastic product). These t-norm and t-conorm (t-N & t–CN) are dual in the sense of families, namely Sugeno t–CN with parameter $\lambda \in]-1,\infty[$ is an operation dual to Weber t-N with parameter $\psi = -\frac{\lambda}{1+\lambda} \in]-1,\infty[$. Due to this duality, both families are given tribute to Sugeno and Weber, thus calling them Sugeno-Weber t-norms and t-conorms (SW t-N & t–CN). For more details, we recommend subsection 4.7 of Klement et al. [21]. Observe also that using a probabilistic approach, considering the product for independent random events A and B, $P(A \cap B) = T_P(P(A),P(B))$ (the product t-norm is given by $T_P(x,y) = x.y$, as well as the valuation-based boundary $P(A \cap B) \geq P(A) + P(B) - 1$, and thus, due to the non-negativity of probability measures, $P(A \cap B) \geq T_L(P(A),P(B))$ (here T_L is the Lukasiewicz t-norm given by $T_L(x,y) = \max(x + y\text{-}1,0)$), one can look on Sugeno-Weber t-norms as a truncated by 0 linear combination with parameters $\frac{\lambda}{1+\lambda}$ and $\frac{1}{1+\lambda}$, $\lambda \in]-1,\infty[$, of the product and binary operation given by $x + y\text{-}1$ generating T_L. SW t-N & t–CN have the properties of general t-norm and t-conorm (t-N & t-CNs). The SW t-N&t–CN incorporates a variable parameter ψ, that provides decision-makers (DM) greater flexibility by allowing them to set the parameter's value appropriately. As a result,

the SW t-N&t-CN appear appropriate for defining T-SFHySSs operations and reducing inaccuracies and data redundancy.

An AO is a methodical mathematical representation in data analysis that combines all the evidence gathered as an argument into a single data form useful for making many important decisions. Sugeno-Weber (SW)-based AOs are well known for being endearing classical AOs. In some cases, the current AOs don't seem eager to use the DM method to mark the exact decision. Several AOs must be modified to address these specific challenges. Consequently, we shall present SW AOs depending on preliminary data to determine whether they will inspire the ongoing study and constraints of T-SFHySSs indicated earlier. The accompanying research's main goals are classified in this manner:

1.3. Contribution of the paper

1. We introduce the T-SFHySS concept.
2. We exhibit comprehensive operating laws for T-spherical fuzzy hypersoft numbers (T-SFHySNs) based on SW t-N&t-CN.
3. The T-SFHSS successfully handles complex challenges while considering DM process factors. We created the SW AOs for T-SFHySS with this benefit in mind.
4. We create the T-SFHyS Sugeno-Weber operators by combining the above-mentioned SW t-N&t-CN operating rules with some essential characteristics.

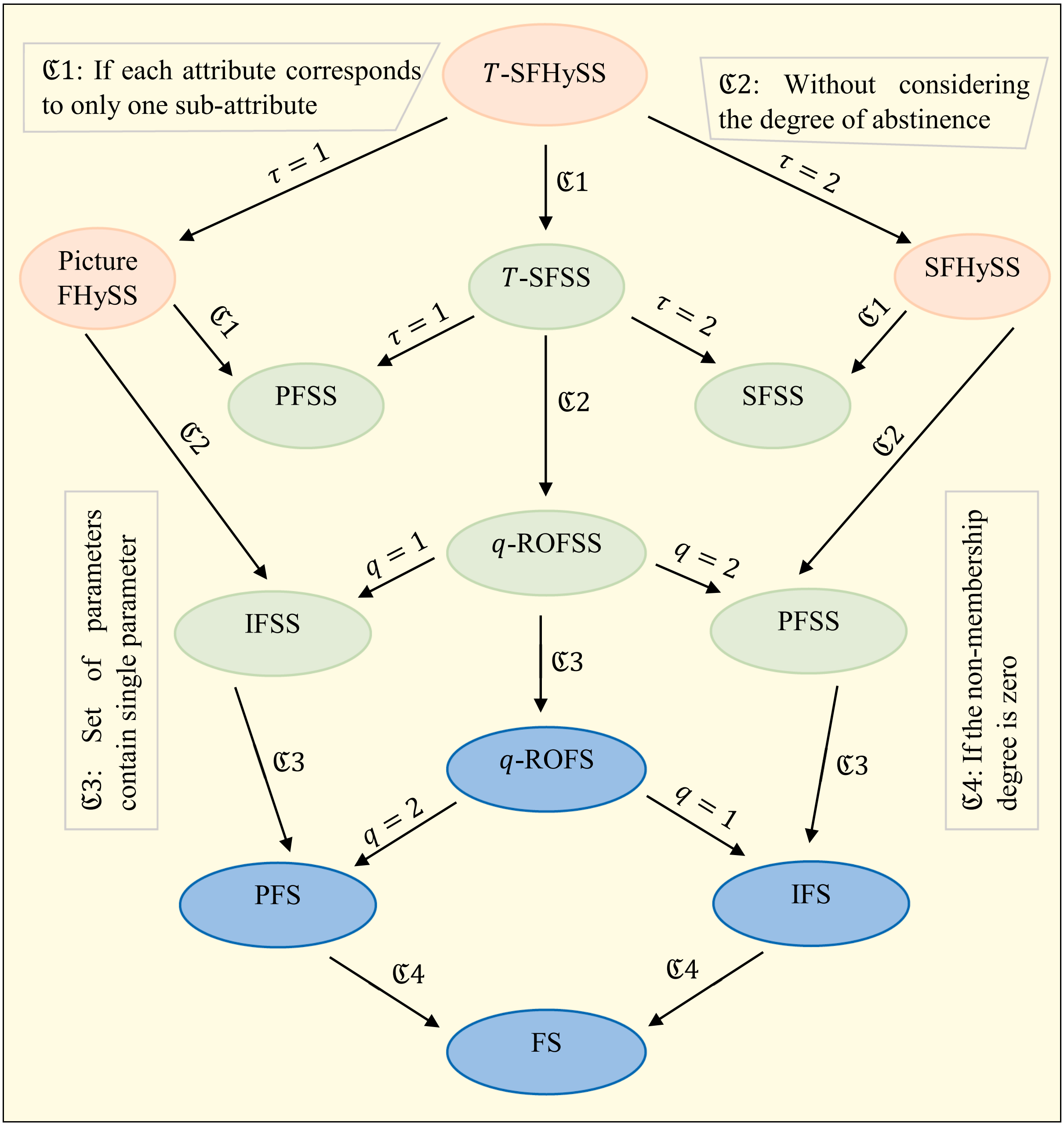

Fig. 1. Classification of T-SFHySS.

5. Following the suggested AOs, we develop an MCDM method to address DM difficulties in the T-SFHyS context and offer a mathematical illustration of natural agriculture.
6. We provide a comparative analysis to evaluate the viability and superiority of the new MCDM methodology.

1.4. Main novelties of the paper

It should be noted that the T-SFHySS were made by generalizing the FSS, IFSS, PSS, q-ROFSS, PFSS, SFSS, and T-SFSS sets that were already known. Severe variants of fuzzy HySS are also a particular case in our developed set. T-SFHySS is the best way to deal with many sub-attributes of assumed parameters connected to SS or other popular frameworks. Our developed set is more powerful at conveying the uncertainty and ambiguity of the DMs than existing sets. A chart is presented in Fig. 1 for a clear understanding. We used Sugeno–Weber triangular norm-based aggregation operators for the first time in a T-spherical fuzzy hypersoft setting to solve MCDM problems.

1.5. Structure of the paper

The following framework has been maintained throughout this investigation: Section 2 discusses the fundamental understanding of various key concepts, including T-SFS, SS, HySS, SW t-N&t-CNs, and PFHySS. Section 3 establishes the concept of T-SFHySS model. Based on SWt-N&t-CNs, Section 4 delineates some fundamental operational laws for T-SFHySS. The next two sections define T-SFHySS Sugeno-Weber weighted averaging (T-SFHySSWWA) and T-SFHySS Sugeno-Weber weighted geometric (T-SFHySSWWG) operators. In these two sections, the authors discuss several adaptable characteristics of the intended operators. In Section 7, the authors construct an MCDM method using the T-SFHySSWWA and T-SFHySSWWG operators. Section 8 provides a case study to elucidate the suggested operators and methodology. Section 9 examines the impact of parameters on decision-making outcomes. Section 10 discusses the advantages and efficacy of the proposed methodology, along with a comparison to several current methods. Section 11 concludes the article by expanding its conclusions and posing queries for future research.

2. Preliminaries

This section reviews and provides some fundamental ideas related to the suggested methodology, which is widely established in the field.

2.1. T-Spherical fuzzy sets

Sometimes decision-makers express their opinion on an alternative, which may include more than two components: favour, abstinence, dislike, and refusal. To overcome this difficulty, Mahmood et al. [24] developed the concept of T-SFS, taking into consideration approval, objection, and waiver degrees corresponding to an object. The definition of T-SFS is presented below.

Definition 1. [24] *For any universal set X, a T-SFS takes the form as*

$$\widetilde{\mathscr{A}} = \left\{ \left\langle x, \left(\mu_{\widetilde{\mathscr{A}}}(x), \delta_{\widetilde{\mathscr{A}}}(x), \nu_{\widetilde{\mathscr{A}}}(x)\right) \right\rangle \middle| x \in X \right\}, \tag{1}$$

where $\mu_{\widetilde{\mathscr{A}}}(x), \delta_{\widetilde{\mathscr{A}}}(x), \nu_{\widetilde{\mathscr{A}}}(x) : X \to [0,1]$ indicates the MG, abstinence, and NMG, respectively, fulfilling the condition that $0 \le \mu_{\widetilde{\mathscr{A}}}{}^{\prime}(x) + \delta_{\widetilde{\mathscr{A}}}{}^{\prime}(x) + \nu_{\widetilde{\mathscr{A}}}{}^{\prime}(x) \le 1$ where $\prime \ge 1$. The "degree of refusal" of x in $\widetilde{\mathscr{A}}$ is represented as $\varsigma_{\widetilde{\mathscr{A}}}(x) = \left(1 - \left(\mu_{\widetilde{\mathscr{A}}}{}^{\prime}(x) + \delta_{\widetilde{\mathscr{A}}}{}^{\prime}(x) + \nu_{\widetilde{\mathscr{A}}}{}^{\prime}(x)\right)\right)^{\frac{1}{\prime}}$. The triplet (μ, δ, ν) is referred to as the "T-SF number (T-SFN)" and is represented by $\widetilde{\alpha} = (\mu, \delta, \nu)$.

For comparison among T-SFNs, Mahmood et al. [24] defined the score and accuracy functions of T-SFNs, and utilizing this tool, they defined a ranking method for T-SFNs as follows:

Definition 2. [24] *Let $\widetilde{\alpha}_1 = (\mu_1, \delta_1, \nu_1)$ and $\widetilde{\alpha}_2 = (\mu_2, \delta_2, \nu_2)$ be any two T-SFNs and $S\left(\widetilde{\alpha}_1\right) = \left(\mu_1^{\prime} - \delta_1^{\prime} - \nu_1^{\prime}\right)$, $S(\widetilde{\alpha}_2) = \left(\mu_2^{\prime} - \delta_2^{\prime} - \nu_2^{\prime}\right)$ representing the score functions of $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ and $A(\widetilde{\alpha}_1) = \left(\mu_1^{\prime} + \delta_1^{\prime} + \nu_1^{\prime}\right)$, $A\left(\widetilde{\alpha}_2\right) = \left(\mu_2^{\prime} + \delta_2^{\prime} + \nu_2^{\prime}\right)$ representing the accuracy functions of $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$, respectively. Then the ordering between $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ is maintained by the following rules:*

(i) If $S(\widetilde{\alpha}_1) > S\left(\widetilde{\alpha}_2\right)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$;

(ii) If $S\left(\widetilde{\alpha}_1\right) = S\left(\widetilde{\alpha}_2\right)$, then

- If $A(\widetilde{\alpha}_1) > A\left(\widetilde{\alpha}_2\right)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$;
- If $A\left(\widetilde{\alpha}_1\right) = A\left(\widetilde{\alpha}_2\right)$, then $\widetilde{\alpha}_1 \approx \widetilde{\alpha}_2$.

2.2. Soft sets

Several useful approaches, such as the theory of probability [22], the fuzzy set theory [47], the theory of interval mathematics [5], the rough set theory [48], etc., have been established for interacting with ambiguous and imprecise situations. However, the parameterization tool associated with any of these theories needs to be improved. In 1999, Molodtsov presented the "soft set (SS)" theory to tackle these issues. SS can be used as a general-purpose mathematical tool for managing uncertainty.

Definition 3. [27] *Consider* $\Im$ *a universal set,* E *be a set of parameters and* $\Sigma \subseteq \mathrm{E}$*. The power set of* $\Im$ *is denoted by* $\mathscr{P}(\Im)$*. A pair* (Λ, Σ) *is referred to as an SS over* $\Im$*, where* Λ *is a mapping given by*

$$\Lambda : \Sigma \rightarrow \mathscr{P}(\Im)$$

Also, it can be represented as follows:

$$(\Lambda, \Sigma) = \{\Lambda(e) \in \mathscr{P}(\Im) | e \in \mathrm{E}\}$$

2.3. Hypersoft sets

In numerous real-world scenarios, distinct attributes are subdivided into separate value sets. While making a decision, decision-makers may have a propensity or a talent for neglecting this categorization of traits. The explanatory study of soft sets does not apply to such sets. To address such circumstances, Smarandache (2018) developed a hypersoft set.

Definition 4. ((*Smarandache, 2018*)) *"Let* $\mathscr{U}$ *be a universe of discourse;* $\mathscr{P}(\mathscr{U})$ *is the power set of* $\mathscr{U}$*." Let* $a_1, a_2, \cdots, a_n$*, for* $n \geq 1$*, be a set of n well-defined, distinct attributes, whose associated sub-attributes are, respectively, the sets* $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ *with* $\mathfrak{A}_i \cap \mathfrak{A}_j = \varnothing$ *for* $i \neq j$*, and* $i, j = 1, 2, \cdots, n$*. Suppose* $\mathfrak{A}_1 \times \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_1 = \dddot{\mathscr{A}} = \{\mathscr{L}_{1h_1} \times \mathscr{L}_{2h_2} \times \ldots \times \mathscr{L}_{nh_n}\}$ *be a collection of multi-attributes, where* $1 \leq h_1 \leq |\mathfrak{A}_1|$*,* $1 \leq h_2 \leq |\mathfrak{A}_2|$ *and so on. Then the pair* $\left(F, \mathfrak{A}_1 \times \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_1 = \dddot{\mathscr{A}}\right)$ *is said to be HySS over* $\mathscr{U}$*, and its mapping is defined as.*

$$F : \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n = \dddot{\mathscr{A}} \rightarrow \mathscr{P}(\mathscr{U})$$

It is also defined as

$$\left(F, \dddot{\mathscr{A}}\right) = \left\{\left(\widehat{\mathscr{L}}, F_{\dddot{\mathscr{A}}}(\widehat{\mathscr{L}})\right) : \widehat{\mathscr{L}} \in \dddot{\mathscr{A}}, F_{\dddot{\mathscr{A}}}(\widehat{\mathscr{L}}) \in \mathscr{P}(\mathscr{U})\right\}. \tag{2}$$

2.4. Pythagorean fuzzy hypersoft sets

Utilizing the ideas of PFS and HySS, Zulqarnain et al. [50] extended the concept of the intuitionistic fuzzy hypersoft set to PFHySS. Compared to the intuitionistic fuzzy hypersoft set, the PFHySS can handle more uncertainty, which is the most important thing to do when making a decision based on fuzzy information.

Definition 5. [50] *Let* $\Im$ *be a universal set. Let* $a_1, a_2, \cdots, a_n$*, for* $n \geq 1$*, be a set of n well-defined, distinct attributes, whose corresponding sub-attributes are, respectively, the sets* $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ *with* $\mathfrak{A}_i \cap \mathfrak{A}_j = \varnothing$ *for* $i \neq j$*, and* $i, j = 1, 2, \cdots, n$*. Assume* $\mathfrak{A}_1 \times \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_1 = \dddot{\mathscr{A}} = \{\mathscr{L}_{1h_1} \times \mathscr{L}_{2h_2} \times \ldots \times \mathscr{L}_{nh_n}\}$ *is a set of multi-attributes, where* $1 \leq h_1 \leq |\mathfrak{A}_1|$*,* $1 \leq h_2 \leq |\mathfrak{A}_2|$ *and so on. Consider* $PFS^{\Im}$ *to be a set of all "Pythagorean fuzzy subsets" over* $\Im$*. The pair* $\left(F, \mathfrak{A}_1 \times \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_1 = \dddot{\mathscr{A}}\right)$ *is then said to be PFHySS over* $\mathscr{U}$*, and its mapping is defined as*

$$F : \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n = \dddot{\mathscr{A}} \rightarrow PyFS^{\Im}.$$

It is also defined as

$$\left(F, \dddot{\mathscr{A}}\right) = \left\{\left(\widehat{a}, F_{\dddot{\mathscr{A}}}(\widehat{a})\right) : \widehat{a} \in \dddot{\mathscr{A}}, F_{\dddot{\mathscr{A}}}(\widehat{a}) \in PyFS^{\Im}\right\}. \tag{3}$$

2.5. Sugeno–Weber t-norms and t-conorm

Siegfried Weber established the SW *t*-Ns family in the early 1980s, and Michio Sugeno established the dual *t*-CNs in the early 1970s.

Definition 6. [17] *The category* $(T_{SW}^{\psi})_{\psi \in [0,\infty)}$ *of SW t-Ns is stated by*

$$T_{SW}^{\psi}(x,y)=\begin{cases}T_D(x,y) & if\psi=-1\\ \max\left(0,\dfrac{x+y-1+\psi xy}{1+\psi}\right) & if-1<\psi<+\infty\\ T_P(x,y) & if\psi=+\infty\end{cases} \quad (4)$$

where $T_D(x,y)$ and $T_P(x,y)$ represent the drastic t-N and product t-N (or, algebraic product), respectively.

The category $\left(S_{SW}^{\psi}\right)_{\psi\in[0,\infty)}$ of SW t-CNs is stated by

$$S_{SW}^{\psi}(x,y)=\begin{cases}S_D(x,y) & if\psi=-1\\ \min\left(1,x+y-\dfrac{\psi}{1+\psi}xy\right) & if-1<\psi<+\infty\\ S_P(x,y) & if\psi=+\infty\end{cases} \quad (5)$$

where $S_D(x,y)$ and $S_P(x,y)$ represent the drastic t-CNs and probabilistic sum (or, algebraic sum), respectively.

3. Development of *T*-Spherical fuzzy hypersoft sets

Definition 7. *Let $\Im$ be a universal set and $\mathscr{P}(\Im)$ be the power set of $\Im$. Assume $a_1,a_2,\cdots,a_n$, for $n\geq 1$, is a set of n well-defined distinct attributes, whose associated sub-attributes are, respectively, the sets $\mathfrak{A}_1$, $\mathfrak{A}_2$, …, $\mathfrak{A}_n$ with $\mathfrak{A}_i\cap\mathfrak{A}_j=\varnothing$ for $i\neq j$, and $i,j=1,2,\cdots,n$. Assume $\mathfrak{A}_1\times\mathfrak{A}_1\times\ldots\times\mathfrak{A}_1=\dddot{\mathscr{A}}=\left\{\widehat{\alpha}_{1h_1}\times\widehat{\alpha}_{2h_2}\times\ldots\times\widehat{\alpha}_{nh_n}\right\}$ be a collection of multi-attributes, where $1\leq h_1\leq|\mathfrak{A}_1|$, $1\leq h_2\leq|\mathfrak{A}_2|$ and so on. Suppose $T-SFS^{\Im}$ is a collection of all T-SF subsets over $\Im$. The pair $\left(F,\mathfrak{A}_1\times\mathfrak{A}_1\times\ldots\times\mathfrak{A}_1=\dddot{\mathscr{A}}\right)$ is then described as T-SFHySS over $\Im$, and its mapping is indicated as*

$$F:\mathfrak{A}_1\times\mathfrak{A}_2\times\ldots\times\mathfrak{A}_n=\dddot{\mathscr{A}}\rightarrow T-SFS^{\Im} \quad (6)$$

It can also be interpreted as follows:

$$\left(F,\dddot{\mathscr{A}}\right)=\left\{\left(\widehat{\alpha},F_{\dddot{\mathscr{A}}}(\widehat{\mathscr{L}})\right):\widehat{\mathscr{L}}\in\dddot{\mathscr{A}},F_{\dddot{\mathscr{A}}}(\widehat{\mathscr{L}})\in T-SFS^{\Im}\right\} \quad (7)$$

where $F_{\dddot{\mathscr{A}}}(\widehat{\mathscr{L}})=\left\{x,\mu_{F(\hat{\mathscr{L}})}(x),\delta_{F(\hat{\mathscr{L}})}(x),\nu_{F(\hat{\mathscr{L}})}(x):x\in\Im\right\}$, in which $\mu_{F(\hat{\mathscr{L}})}(x),\delta_{F(\hat{\mathscr{L}})}(x),\nu_{F(\hat{\mathscr{L}})}(x):\Im\rightarrow[0,1]$ stands for the MG, degree of abstinence, and NMG, respectively, of the sub-attributes of the considered parameters, satisfying the condition that $0\leq\mu_{F(\hat{\mathscr{L}})}{}^{\tau}(x)+\delta_{F(\hat{\mathscr{L}})}{}^{\tau}(x)+\nu_{F(\hat{\mathscr{L}})}{}^{\tau}(x)\leq 1$ where $\tau\geq 1$. The degree of hesitancy of x in $\widetilde{\mathscr{A}}$ is represented as

$$\mathscr{H}_{\hat{\mathscr{L}}_{ij}}=\left(1-\left(\mu_{F(\hat{\mathscr{L}})}{}^{\tau}(x)+\delta_{F(\hat{\mathscr{L}})}{}^{\tau}(x)+\nu_{F(\hat{\mathscr{L}})}{}^{\tau}(x)\right)\right)^{\frac{1}{\tau}} \quad (8)$$

Simply, $\mathfrak{A}_{\hat{\mathscr{L}}_j}(x_i)=\left\{\left(x_i,\mu_j(x_i),\delta_j(x_i),\nu_j(x_i)\right):x_i\in\Im\right\}$ can be written $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}=\langle\mu_{F(\hat{\mathscr{L}}_{ij})},\delta_{F(\hat{\mathscr{L}}_{ij})},\nu_{F(\hat{\mathscr{L}}_{ij})}\rangle$, which is called the T-SF hypersoft number (T-SFHySN).

Example 1. *Consider the universe of discourse $\Im=\{x_1,x_2\}$ and $a=\{a_1$: Teaching methodology; a_2 : Subjects; a_3 : Classes$\}$ be a collection of attributes with the following corresponding attribute values*:

Teaching methodology $\mathfrak{A}_1=\{a_{11}$: project base; a_{11} : class discussion$\}$,
Subjects $\mathfrak{A}_2=\{a_{21}$: mathematics; a_{22} : computer science; a_{23} : statistics$\}$,
Classes $\mathfrak{A}_3=\{a_{31}$: masters; a_{32} : doctoral$\}$.

Let $\dddot{\mathscr{A}}=(\mathfrak{A}_1\times\mathfrak{A}_2\times\mathfrak{A}_3)=\{\widehat{\mathscr{L}}_{11},\widehat{\mathscr{L}}_{12}\}\times\{\widehat{\mathscr{L}}_{21},\widehat{\mathscr{L}}_{22},\widehat{\mathscr{L}}_{23}\}\times\{\widehat{\mathscr{L}}_{31},\widehat{\mathscr{L}}_{32}\}=$

$$\{(a_{11},a_{21},a_{31}),(a_{11},a_{21},a_{32}),(a_{11},a_{22},a_{31}),(a_{11},a_{22},a_{32}),(a_{11},a_{23},a_{31}),(a_{11},a_{23},a_{32}),$$

$$(a_{12},a_{21},a_{31}),(a_{12},a_{21},a_{32}),(a_{12},a_{22},a_{31}),(a_{12},a_{22},a_{32}),(a_{12},a_{23},a_{31}),(a_{12},a_{23},a_{32})\}$$

$$\dddot{\mathscr{A}}=\{\widehat{\mathscr{L}}_1,\widehat{\mathscr{L}}_2,\widehat{\mathscr{L}}_3,\widehat{\mathscr{L}}_4,\widehat{\mathscr{L}}_5,\widehat{\mathscr{L}}_6,\widehat{\mathscr{L}}_7,\widehat{\mathscr{L}}_8,\widehat{\mathscr{L}}_9,\widehat{\mathscr{L}}_{10},\widehat{\mathscr{L}}_{11},\widehat{\mathscr{L}}_{12}\}.$$

Then the T-SFHySS over $\Im$ is given as follows:

$$(\mathscr{F},A)=\left\{\begin{array}{l}(\widehat{\mathscr{L}}_1,(x_1,\langle 0.6,0.25,0.3\rangle),(x_2,\langle 0.5,0.3,0.7\rangle)),(\widehat{\mathscr{L}}_2,(x_1,\langle 0.6,0.15,0.7\rangle),(x_2,\langle 0.7,0.3,0.5\rangle)),\\(\widehat{\mathscr{L}}_3,(x_1,\langle 0.4,0.35,0.8\rangle),(x_2,\langle 0.3,0.2,0.7\rangle)),(\widehat{\mathscr{L}}_4,(x_1,\langle 0.6,0.1,0.5\rangle),(x_2,\langle 0.5,0.2,0.6\rangle)),\\(\widehat{\mathscr{L}}_5,(x_1,\langle 0.7,0.4,0.3\rangle),(x_2,\langle 0.4,0.2,0.8\rangle)),(\widehat{\mathscr{L}}_6,(x_1,\langle 0.5,0.2,0.4\rangle),(x_2,\langle 0.6,0.1,0.5\rangle)),\\(\widehat{\mathscr{L}}_7,(x_1,\langle 0.5,0.25,0.6\rangle),(x_2,\langle 0.4,0.3,0.5\rangle)),(\widehat{\mathscr{L}}_8,(x_1,\langle 0.2,0.3,0.5\rangle),(x_2,\langle 0.3,0.3,0.9\rangle)),\\(\widehat{\mathscr{L}}_9,(x_1,\langle 0.4,0.2,0.6\rangle),(x_2,\langle 0.8,0.2,0.5\rangle)),(\widehat{\mathscr{L}}_{10},(x_1,\langle 0.7,0.4,0.4\rangle),(x_2,\langle 0.7,0.15,0.2\rangle)),\\(\widehat{\mathscr{L}}_{11},(x_1,\langle 0.4,0.35,0.5\rangle),(x_2,\langle 0.5,0.1,0.3\rangle)),(\widehat{\mathscr{L}}_{12},(x_1,\langle 0.5,0.2,0.7\rangle),(x_2,\langle 0.4,0.2,0.7\rangle))\end{array}\right\}.$$

Definition 8. *Let $\widetilde{\wp}_{\hat{\alpha}_{ij}}=\langle\mu_{ij},\delta_{ij},\nu_{ij}\rangle$ be a T-SFHySN, then score function can be defined in the following way:*

$$S\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right)=\mu_{ij}{}^{\tau}-\delta_{ij}{}^{\tau}-\nu_{ij}{}^{\tau}+\left(\frac{e^{\mu_{ij}{}^{\tau}-\delta_{ij}{}^{\tau}-\nu_{ij}{}^{\tau}}}{e^{\mu_{ij}{}^{\tau}-\delta_{ij}{}^{\tau}-\nu_{ij}{}^{\tau}}+1}-\frac{1}{2}\right)\mathscr{H}_{\hat{\mathscr{L}}_{ij}}{}^{\tau} \tag{9}$$

for $q\geq 1$ and $S\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right)\in[-1,1]$.

Let $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}=\langle\mu_{11},\delta_{11},\nu_{11}\rangle$ and $\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}=\langle\mu_{12},\delta_{12},\nu_{12}\rangle$ be two T-SFHySNs. Then

(i) If $S\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\right)\rangle S\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}\right)$, then $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\succ\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}$

(ii) If $S\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\right)\langle S\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}\right)$, then $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\prec\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}$

If $S\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\right)=S\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}\right)$, then

- If $\mathscr{H}_{\hat{\mathscr{L}}_{11}}>\mathscr{H}_{\hat{\mathscr{L}}_{12}}$, then $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\prec\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}$
- If $\mathscr{H}_{\hat{\mathscr{L}}_{11}}=\mathscr{H}_{\hat{\mathscr{L}}_{12}}$, then $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\approx\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}$.

As it is, T-SFHySS can encompass many fuzzy sets. Therefore, if different parameters in T-SFHySS assumed different values, it could largely encompass current fuzzy sets. Concerning this idea, Fig. 1 shows several fuzzy sets (Algebraic, Einstein, Hamacher, Frank, etc.) and environments (like T-SFSS, SFSS, PFSS, q-ROFSS, PyFSS, etc.).

4. Sugeno-Weber operations of *T*-SFHySNs

This section discusses the Sugeno-Weber (SW) operation and its notions in some fundamental operations. Suppose that the t-Ns, T^{ψ}_{SW} and the t-CNs, S^{ψ}_{SW} represent the SW sum and SW product, respectively, and the generalization of intersection and union of T-SFHySS turns into the SW sum $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\oplus_{SW}\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}$ and the SW product $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\otimes_{SW}\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}$ from the two T-SFHySNs, respectively which are presented as

(i) $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\oplus_{SW}\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}=\langle S^{\psi}_{SW}(\mu_{11},\mu_{12}),T^{\psi}_{SW}(\delta_{11},\delta_{12})T^{\psi}_{SW}(\nu_{11},\nu_{12})\rangle$,

(ii) $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\otimes_{SW}\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}=\langle T^{\psi}_{SW}(\mu_{11},\mu_{12}),S^{\psi}_{SW}(\delta_{11},\delta_{12}),S^{\psi}_{SW}(\nu_{11},\nu_{12})\rangle$.

Definition 9. *Let $\widetilde{\wp}_{\hat{\mathscr{L}}}=\langle\mu,\delta,\nu\rangle$, $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}=\langle\mu_{11},\delta_{11},\nu_{11}\rangle$ and $\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}=\langle\mu_{12},\delta_{12},\nu_{12}\rangle$ be any three T-SFHySNs, and λ be a positive real number. Then, some basic operations of T-SFHySNs based on SW t-N&t-CNs are given as*

(i) $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\oplus_{SW}\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}=\langle\sqrt[\tau]{\mu_{11}^{\tau}+\mu_{12}^{\tau}-\frac{\psi}{1+\psi}\mu_{11}^{\tau}\mu_{12}^{\tau}},\sqrt[\tau]{\frac{\delta_{11}^{\tau}+\delta_{12}^{\tau}-1+\psi\delta_{11}^{\tau}\delta_{12}^{\tau}}{1+\psi}},\sqrt[\tau]{\frac{\nu_{11}^{\tau}+\nu_{12}^{\tau}-1+\psi\nu_{11}^{\tau}\nu_{12}^{\tau}}{1+\psi}}\rangle$;

(ii) $\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}\otimes_{SW}\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}=\langle\sqrt[\tau]{\frac{\mu_{11}^{\tau}+\mu_{12}^{\tau}-1+\psi\mu_{11}^{\tau}\mu_{12}^{\tau}}{1+\psi}},\sqrt[\tau]{\delta_{11}^{\tau}+\delta_{12}^{\tau}-\frac{\psi}{1+\psi}\delta_{11}^{\tau}\delta_{12}^{\tau}},\sqrt[\tau]{\nu_{11}^{\tau}+\nu_{12}^{\tau}-\frac{\psi}{1+\psi}\nu_{11}^{\tau}\nu_{12}^{\tau}}\rangle$;

(iii) $\lambda\odot_{SW}\widetilde{\wp}_{\hat{\mathscr{L}}}=\langle\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\left(1-\mu^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\lambda}\right)},\sqrt[\tau]{\left((1+\psi)\left(\frac{\psi\delta^{\tau}+1}{1+\psi}\right)^{\lambda}-1\right)\frac{1}{\psi}},\sqrt[\tau]{\left((1+\psi)\left(\frac{\psi\nu^{\tau}+1}{1+\psi}\right)^{\lambda}-1\right)\frac{1}{\psi}}\rangle$;

(iv) $\widetilde{\wp}_{\hat{\mathscr{L}}}{}^{\lambda}=\langle\left(\frac{1}{\psi}\left((1+\psi)\left(\frac{\psi\mu^{\tau}+1}{1+\psi}\right)^{\lambda}-1\right)\right)^{\frac{1}{\tau}},\left(\frac{1+\psi}{\psi}\left(1-\left(1-\delta^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\lambda}\right)\right)^{\frac{1}{\tau}},\left(\frac{1+\psi}{\psi}\left(1-\left(1-\nu^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\lambda}\right)\right)^{\frac{1}{\tau}}\rangle$.

5. *T*-SFHyS Sugeno-Weber weighted averaging aggregation operators

In a T-SFHyS environment, we now propose the T-SFHySSWWA operator.

Definition 10. *Let* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}} = \langle \mu_{ij}, \delta_{ij}, \nu_{ij} \rangle$ *be an accumulation of T-SFHySNs, where* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$. *If* $T-\mathrm{SFHyS}SWWA : \Delta^n \rightarrow \Delta$, *then T-SFHySSWWA is defined as*

$$T-\mathrm{SFHyS}SWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) = \oplus_{SW}{}_{j=1}^{n}\omega_j\left(\oplus_{SW}{}_{i=1}^{m}\Omega_i \widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right) \tag{10}$$

where $\Omega_i > 0$, $\sum_{i=1}^{m}\Omega_i = 1$ and $\omega_j > 0$, $\sum_{j=1}^{n}\omega_j = 1$ represent the weights of experts and attributes, respectively.

Theorem 1. *Let* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}} = \langle \mu_{ij}, \delta_{ij}, \nu_{ij} \rangle$ *be the collection of T-SFHySNs, where* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$. *Then, the obtained aggregated values is also a T-SFHySN and*

$$T-\mathrm{SFHyS}SWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) = \oplus_{SW}{}_{j=1}^{m}\omega_j\left(\oplus_{SW}{}_{i=1}^{n}\Omega_i \widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right)$$

$$= \left\langle \sqrt{\tau \frac{1+\psi}{\psi}\left(1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \frac{\psi}{1+\psi}\mu_{ij}{}^{\tau}\right)^{\Omega_i}\right)^{\omega_j}\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\frac{\psi\delta_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j} - 1\right)},\right.$$

$$\left.\sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j} - 1\right)}\right\rangle \tag{11}$$

where Ω_i and ω_j represent the weight vectors of the experts and attributes, respectively, such as $\Omega_i > 0$, $\sum_{i=1}^{m}\Omega_i = 1$ and $\omega_j > 0$, $\sum_{j=1}^{n}\omega_j = 1$.

Proof. The proof of the *T*-SFHySSWWA operator can be given by using mathematical induction and some basic operations, such as the following:

We will prove for $m = 2$ and $n = 2$. Then we have

$$T-\mathrm{SFHyS}SWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) = \oplus_{SW}{}_{j=1}^{2}\omega_j\left(\oplus_{SW}{}_{i=1}^{2}\Omega_j \widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right)$$

$$= \oplus_{SW}{}_{j=1}^{2}\omega_j\left(\Omega_1\widetilde{\wp}_{\hat{\mathscr{L}}_{1j}} \oplus_{SW} \Omega_2\widetilde{\wp}_{\hat{\mathscr{L}}_{2j}}\right)$$

$$= \omega_1\left(\Omega_1\widetilde{\wp}_{\hat{\mathscr{L}}_{11}} \oplus_{SW} \Omega_2\widetilde{\wp}_{\hat{\mathscr{L}}_{21}}\right) + \omega_2\left(\Omega_1\widetilde{\wp}_{\hat{\mathscr{L}}_{12}} \oplus_{SW} \Omega_2\widetilde{\wp}_{\hat{\mathscr{L}}_{22}}\right)$$

$$= \omega_1\left(\begin{array}{c}\left\langle \sqrt{\tau\frac{1+\psi}{\psi}\left(1 - \left(1 - \mu_{11}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_1}\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\frac{\psi\delta_{11}{}^{\tau}+1}{1+\psi}\right)^{\Omega_1} - 1\right)},\right. \\ \left.\sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\frac{\psi\nu_{11}{}^{\tau}+1}{1+\psi}\right)^{\Omega_1} - 1\right)}\right\rangle \oplus_{SW} \left\langle \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1 - \left(1 - \mu_{21}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_2}\right)},\right. \\ \left.\sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\frac{\psi\delta_{21}{}^{\tau}+1}{1+\psi}\right)^{\Omega_2} - 1\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\frac{\psi\nu_{21}{}^{\tau}+1}{1+\psi}\right)^{\Omega_2} - 1\right)}\right\rangle\end{array}\right) \oplus_{SW}$$

$$\omega_2\left(\begin{array}{c}\left\langle \sqrt{\tau\frac{1+\psi}{\psi}\left(1 - \left(1 - \mu_{12}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_1}\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\frac{\psi\delta_{12}{}^{\tau}+1}{1+\psi}\right)^{\Omega_1} - 1\right)},\right. \\ \left.\sqrt[\tau]{\left((1+\psi)\left(\frac{\psi\nu_{12}{}^{\tau}+1}{1+\psi}\right)^{\Omega_1} - 1\right)\frac{1}{\psi}}\right\rangle \oplus_{SW} \left\langle \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1 - \left(1 - \mu_{22}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_2}\right)},\right. \\ \left.\sqrt[\tau]{\left((1+\psi)\left(\frac{\psi\delta_{22}{}^{\tau}+1}{1+\psi}\right)^{\Omega_2} - 1\right)\frac{1}{\psi}}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\frac{\psi\nu_{22}{}^{\tau}+1}{1+\psi}\right)^{\Omega_2} - 1\right)}\right\rangle\end{array}\right)$$

$$= \omega_1\left\langle \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1 - \prod_{i=1}^{2}\left(1 - \mu_{i1}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)}, \sqrt[\tau]{\frac{1}{\psi}\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\delta_{i1}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i} - 1\right)},\right.$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\nu_{i1}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}-1\right)}\rangle\oplus_{SW}\omega_2\langle\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{i=1}^{2}\left(1-\mu_{i2}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)}$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\delta_{i2}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}-1\right)},\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\nu_{i2}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}-1\right)}\rangle$$

$$=\left\langle\sqrt{\tau\frac{1+\psi}{\psi}\left(1-\left(\prod_{i=1}^{2}\left(1-\mu_{i1}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_1}\right)},\sqrt[\tau]{\frac{1}{\psi}\left(\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\delta_{i1}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_1}-1\right)}\right.$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\nu_{i1}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_1}-1\right)}\rangle\oplus_{SW}\langle\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\left(\prod_{i=1}^{2}\left(1-\mu_{i2}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_2}\right)}$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\delta_{i2}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_2}-1\right)},\sqrt[\tau]{\frac{1}{\psi}\left(\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\nu_{i2}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_2}-1\right)}\rangle$$

$$=\left\langle\sqrt{\tau\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_j}\right)},\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{2}\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\delta_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)},\right.$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{2}\left(\prod_{i=1}^{2}(1+\psi)\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)}\rangle$$

This demonstrates that the acquired result is correct for $m = 2$ and $n = 2$.
Assume that equation (1) is true for $m = \beta_1$ and $n = \beta_2$. Then

$$T-\text{SFHyS}SWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right)=\oplus_{SW_{j=1}}^{\beta_2}\omega_j\left(\oplus_{SW_{i=1}}^{\beta_1}\Omega_i\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right)$$

$$=\left\langle\sqrt{\tau\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{\beta_2}\left(\prod_{i=1}^{\beta_1}\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_j}\right)},\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2}\left(\left(\prod_{i=1}^{\beta_1}(1+\psi)\left(\frac{\psi\delta_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_{ij}}\right)\right)^{\omega_j}-1\right)},\right.$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2}\left(\left(\prod_{i=1}^{\beta_1}(1+\psi)\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_{ij}}\right)\right)^{\omega_j}-1\right)}\rangle$$

Now, for $m = \beta_1 + 1$ and $n = \beta_2 + 1$, we have

$$T-\text{SFHyS}SWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right)=\oplus_{SW_{j=1}}^{\beta_2+1}\gamma_j\left(\oplus_{SW_{i=1}}^{\beta_1+1}\Omega_i\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right)$$

$$=\oplus_{SW_{j=1}}^{\beta_2+1}\omega_j\left(\oplus_{SW_{i=1}}^{\beta_1}\Omega_i\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\oplus_{SW}\Omega_{\beta_1+1}\widetilde{\wp}_{\hat{\mathscr{L}}_{(\beta_1+1)}}\right)$$

$$=\oplus_{SW_{j=1}}^{\beta_2+1}\omega_j\left(\oplus_{SW_{i=1}}^{\beta_1}\Omega_i\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right)\oplus_{SW_{j=1}}^{\beta_2+1}\gamma_j\Omega_{\beta_1+1}\widetilde{\wp}_{\hat{\mathscr{L}}_{(\beta_1+1)j}}$$

$$=\left\langle\sqrt{\tau\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{\beta_2+1}\left(\prod_{i=1}^{\beta_1}\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_j}\right)},\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2+1}\left(\left(\prod_{i=1}^{\beta_1}(1+\psi)\left(\frac{\psi\delta_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)\right)^{\omega_j}-1\right)}\right.$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2+1}\left(\left(\prod_{i=1}^{\beta_1}(1+\psi)\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_{ij}}\right)\right)^{\omega_j}-1\right)}\rangle\oplus_{SW}\langle\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{\beta_2+1}\left(\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_{\beta_1+1}}\right)^{\omega_j}\right)},$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2+1}\left((1+\psi)\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_{\beta_1+1}}\right)^{\omega_j}-1\right)},\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2+1}\left((1+\psi)\left(\frac{\psi\nu_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_{\beta_1+1}}\right)^{\omega_j}-1\right)}\rangle$$

$$=\left\langle\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{\beta_2+1}\left(\prod_{i=1}^{\beta_1}\left(1-\mu_{ij}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_{\beta_1+1}}\right)^{\omega_j}\right)},\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{\beta_2+1}\left(\left(1-\delta_{ij}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_{\beta_1+1}}\right)^{\omega_j}\right)},\right.$$

$$\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{\beta_2+1}\left(\left(1-\nu_{ij}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_{\beta_1+1}}\right)^{\omega_j}\right)}\rangle\oplus_{SW}\langle\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2+1}\left(\prod_{i=1}^{\beta_1}(1+\psi)\left(\frac{\psi\mu_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)},$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2+1}\left((1+\psi)\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_{\beta_1+1}}\right)^{\omega_j}-1\right)},\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2+1}\left((1+\psi)\left(\frac{\psi\nu_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_{\beta_1+1}}\right)^{\omega_j}-1\right)}\rangle$$

$$=\left\langle\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{\beta_2+1}\left(\prod_{i=1}^{\beta_1+1}\left(1-\mu_{ij}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_j}\right)},\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2+1}\left(\left(\prod_{i=1}^{\beta_1+1}(1+\psi)\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)\right)^{\omega_j}-1\right)},\right.$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{\beta_2+1}\left(\left(\prod_{i=1}^{\beta_1+1}(1+\psi)\left(\frac{\psi\nu_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)\right)^{\omega_j}-1\right)}\rangle$$

This demonstrates that equation (1) holds for every $n \geq 1$ and $m \geq 1$.

The most intriguing aspect is that the aggregated values produced using the T-SFHySSWWA operator are also T-SFHySN. To demonstrate this, assume $\widetilde{\wp}_{\mathcal{L}_{ij}} = \langle\mu_{ij}, \delta_{ij}, \nu_{ij}\rangle$, where $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, $0 \leq \mu_{ij}, \delta_{ij}, \nu_{ij} \leq 1$ and satisfies $0 \leq \mu_{ij}^{\tau} + \delta_{ij}^{\tau} + \nu_{ij}^{\tau} \leq 1$ where Ω_i and ω_j are the weight vectors for experts and attributes, respectively, and having conditions that $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i = 1$ and $\omega_j > 0$, $\sum_{j=1}^{m}\omega_j = 1$.

As we are all aware that

$$0 \leq \mu_{ij} \leq 1 \Rightarrow 0 \leq \mu_{ij}^{\tau} \leq 1 \Rightarrow 0 \leq \left(1-\frac{\psi}{1+\psi}\mu_{ij}^{\tau}\right)^{\Omega_i} \leq 1$$

$$\Rightarrow 0 \leq \prod_{i=1}^{m}\left(1-\frac{\psi}{1+\psi}\mu_{ij}^{\tau}\right)^{\Omega_i} \leq 1$$

$$\Rightarrow 0 \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\frac{\psi}{1+\psi}\mu_{ij}^{\tau}\right)^{\Omega_i}\right)^{\omega_j} \leq 1$$

$$\Rightarrow 0 \leq \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\frac{\psi}{1+\psi}\mu_{ij}^{\tau}\right)^{\Omega_i}\right)^{\omega_j}\right)} \leq 1.$$

Similarly,

$$0 \leq \delta_{ij} \leq 1 \Rightarrow 0 \leq (1+\psi)\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i} \leq 1$$

i.e., $0 \leq \prod_{i=1}^{m}(1+\psi)\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i} \leq 1$

$$0 \leq \prod_{j=1}^{n}\left(\left(\prod_{i=1}^{m}(1+\psi)\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)\right)^{\omega_j} \leq 1$$

$$0 \leq \sqrt[\tau]{\left(\prod_{j=1}^{n}\left(\left(\prod_{i=1}^{m}(1+\psi)\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)\right)^{\omega_j}-1\right)\frac{1}{\psi}} \leq 1.$$

Similarly,

$$0 \leq \nu_{ij} \leq 1 \Rightarrow 0 \leq \nu_{ij}^{\tau} \leq 1 \Rightarrow 0 \leq (1+\psi)\left(\frac{\psi\nu_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i} \leq 1$$

i.e., $0 \le \prod_{i=1}^{m}(1+\psi)\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i} \le 1$

$$0 \le \prod_{j=1}^{n}\left(\left(\prod_{i=1}^{m}(1+\psi)\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)\right)^{\omega_j} \le 1$$

$$0 \le \sqrt[\tau]{\left(\prod_{j=1}^{n}\left(\left(\prod_{i=1}^{m}(1+\psi)\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)\right)^{\omega_j}-1\right)\frac{1}{\psi}} \le 1.$$

Therefore,

$$0 \le \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\frac{\psi\mu_{ij}{}^{\tau}}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}\right)}+\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{m}\left(\left(\prod_{i=1}^{n}(1+\psi)\left(\frac{\psi\delta_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)\right)^{\omega_j}-1\right)}+$$

$$\sqrt[\tau]{\frac{1}{\psi}\left(\prod_{j=1}^{n}\left(\left(\prod_{i=1}^{m}(1+\psi)\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)\right)^{\omega_j}-1\right)} \le 1$$

So, it can be shown that the result of applying the *T*-SFHySSWWA operator is also a *T*-SFHySN.

Example 2. *Let $\mathscr{U} = \{u_1, u_2, u_3, u_4\}$ be a set of experts with weights $\Omega_i = (0.218, 0.304, 0.162, 0.316)^T$, who are going to describe the attractiveness of a house under the defined set of attributes $a = \{a_1 : lawn, a_2 : secutitysystem\}$ with their corresponding sub-attributes*

(i) *$Lawn = a_1 = \{a_{11} = withgrass, a_{12} = withoutgrass\}$;*
(ii) *$secutitysystem = a_2 = \{a_{21} = guards, a_{22} = careras\}$.*

Let $\dddot{\mathscr{A}} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}\}$

$= \{(a_{11}, a_{21}), (a_{11}, a_{22}), (a_{12}, a_{21}), (a_{12}, a_{22})\}$

Let $\dddot{\mathscr{A}} = \{\widehat{\mathscr{L}}_1, \widehat{\mathscr{L}}_2, \widehat{\mathscr{L}}_3, \widehat{\mathscr{L}}_4\}$ be a set of multi-sub-attributes with weights $\omega = (0.2,\ 0.25,\ 0.2, 0.35)^T$. The rating values of the experts for alternatives in the form of *T*-SFHySN $\left(\mathscr{F}, \dddot{\mathscr{A}}\right) = \mu^{(l)}, \delta^{(l)}, \nu^{(l)}_{4\times 4}$ is given as:

$$(\mathscr{F}, A) = \begin{bmatrix} \langle 0.2, 0.2, 0.6\rangle & \langle 0.5, 0.3, 0.2\rangle & \langle 0.5, 0.2, 0.3\rangle & \langle 0.4, 0.3, 0.2\rangle \\ \langle 0.3, 0.2, 0.6\rangle & \langle 0.3, 0.2, 0.5\rangle & \langle 0.4, 0.3, 0.3\rangle & \langle 0.3, 0.3, 0.5\rangle \\ \langle 0.4, 0.5, 0.2\rangle & \langle 0.7, 0.2, 0.2\rangle & \langle 0.3, 0.2, 0.8\rangle & \langle 0.4, 0.2, 0.4\rangle \\ \langle 0.3, 0.2, 0.7\rangle & \langle 0.5, 0.2, 0.3\rangle & \langle 0.8, 0.2, 0.1\rangle & \langle 0.6, 0.3, 0.5\rangle \end{bmatrix}$$

The aggregated value, $T-SFHySSWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{34}}\right)$ (assigning $\tau = 3$, $\psi = 2$) is obtained by using Eq. (5).

$$T-SFHySSWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{34}}\right) = \oplus_{SW_{j=1}^{4}}\omega_j\left(\oplus_{SW_{i=1}^{4}}\Omega_i\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right) =$$

$$\left\langle \sqrt{3\frac{1+2}{2}\left(1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\frac{2\mu_{ij}{}^{3}}{1+2}\right)^{\Omega_i}\right)^{\omega_j}\right)}, \sqrt[3]{\frac{1}{2}\left((1+2)\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\frac{2\delta_{ij}{}^{3}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)},\right.$$

$$\left.\sqrt[3]{\frac{1}{2}\left((1+2)\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\frac{2\nu_{ij}{}^{3}+1}{1+2}\right)^{\Omega_i}\right)^{\omega_j}-1\right)}\right\rangle$$

Aggregating MG

$$\sqrt[3]{\frac{1+2}{2}\left(1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\frac{2\mu_{ij}{}^{3}}{1+2}\right)^{\Omega_i}\right)^{\omega_j}\right)} =$$

$$\sqrt[3]{\frac{3}{2}\left(1-\prod_{j=1}^{4}\left(\left(1-\frac{2\mu_{1j}{}^{3}}{3}\right)^{\Omega_1}\left(1-\frac{2\mu_{2j}{}^{3}}{3}\right)^{\Omega_2}\left(1-\frac{2\mu_{3j}{}^{3}}{3}\right)^{\Omega_3}\left(1-\frac{2\mu_{4j}{}^{3}}{3}\right)^{\Omega_4}\right)^{\omega_j}\right)} = \sqrt[4]{\frac{3}{2}(1-\mathscr{M})}$$

where $\mathscr{M} = \prod_{j=1}^{4}\left(\left(1-\frac{2}{3}\mu_{1j}^{3}\right)^{0.218}\left(1-\frac{2}{3}\mu_{2j}^{3}\right)^{0.304}\left(1-\frac{2}{3}\mu_{3j}^{3}\right)^{0.162}\left(1-\frac{2}{3}\mu_{4j}^{3}\right)^{0.316}\right)^{\omega_j}$.

Now, $A = \left((1-\frac{2}{3}\mu_{11}^{3})^{0.218}(1-\frac{2}{3}\mu_{21}^{3})^{0.304}(1-\frac{2}{3}\mu_{31}^{3})^{0.162}(1-\frac{2}{3}\mu_{41}^{3})^{0.316}\right)^{0.2}\left((1-\frac{2}{3}\mu_{12}^{3})^{0.218}(1-\frac{2}{3}\mu_{22}^{3})^{0.304}(1-\frac{2}{3}\mu_{32}^{3})^{0.162}(1-\frac{2}{3}\mu_{42}^{3})^{0.316}\right)^{0.25}\left((1-\frac{2}{3}\mu_{13}^{3})^{0.218}(1-\frac{2}{3}\mu_{23}^{3})^{0.304}(1-\frac{2}{3}\mu_{33}^{3})^{0.162}(1-\frac{2}{3}\mu_{43}^{3})^{0.316}\right)^{0.2}\left((1-\frac{2}{3}\mu_{14}^{3})^{0.218}(1-\frac{2}{3}\mu_{24}^{3})^{0.304}(1-\frac{2}{3}\mu_{34}^{3})^{0.162}(1-\frac{2}{3}\mu_{44}^{3})^{0.316}\right)^{0.35}$.

$$= \left(\left(1-\frac{2\times 0.2^3}{1+2}\right)^{0.218}\left(1-\frac{2\times 0.3^3}{1+2}\right)^{0.304}\left(1-\frac{2\times 0.4^3}{1+2}\right)^{0.162}\left(1-\frac{2\times 0.3^3}{1+2}\right)^{0.316}\right)^{0.2}\left(\left(1-\frac{2\times 0.5^3}{1+2}\right)^{0.218}\left(1-\frac{2\times 0.3^3}{1+2}\right)^{0.304}\left(1-\frac{2\times 0.7^3}{1+2}\right)^{0.162}\left(1-\frac{2\times 0.5^3}{1+2}\right)^{0.316}\right)^{0.25}\left(\left(1-\frac{2\times 0.5^3}{1+2}\right)^{0.218}\left(1-\frac{2\times 0.4^3}{1+2}\right)^{0.304}\left(1-\frac{2\times 0.3^3}{1+2}\right)^{0.162}\left(1-\frac{2\times 0.8^3}{1+2}\right)^{0.316}\right)^{0.2}\left(\left(1-\frac{2\times 0.4^3}{1+2}\right)^{0.218}\left(1-\frac{2\times 0.3^3}{1+2}\right)^{0.304}\left(1-\frac{2\times 0.4^3}{1+2}\right)^{0.162}\left(1-\frac{2\times 0.6^3}{1+2}\right)^{0.316}\right)^{0.35}$$

$$=(0.6805\times 0.7051\times 0.4001\times 0.7190)^{0.2}(0.4182\times 0.7051\times 0.2126\times 0.5440)^{0.25}(0.4182\times 0.6167\times 0.4784\times 0.2880)^{0.2}(0.4972\times 0.7051\times 0.4001\times 0.4579)^{0.35} = 0.6730\times 0.4297\times 0.5130\times 0.3826 = 0.0568$$

As a result, the aggregating MG $= 0.4997$.
In a similarly way, integrated abstinence and NMG can be obtained as

$$= \langle 0.4997, 0.2768, 0.4749\rangle$$

Using the prior work of the *T*-SFHySSWWA operator, the following basic features for the accumulation of *T*-SFHySNs are provided:

Theorem 2. *(Idempotency) If,* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}} = \widetilde{\wp}_{\hat{\mathscr{L}}} = \langle \mu_{ij}, \delta_{ij}, \nu_{ij}\rangle$ *where* $i = 1,2,\cdots,m$ *and* $j = 1,2,\cdots,n$ *be T-SFHySNs. Then,*

$$T-SFHySSWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) = \widetilde{\wp}_{\hat{\mathscr{L}}}$$

Proof. Let $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}} = \widetilde{\wp}_{\hat{\alpha}} = \langle \mu_{ij}, \delta_{ij}, \nu_{ij}\rangle$ (where $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n$) be set of *T*-SFHSNs. Then,

$$T-SFHySSWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) =$$

$$\left\langle \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\frac{\psi\mu_{ij}^{\tau}}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}\right)}, \sqrt[\tau]{\left((1+\psi)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)\frac{1}{\psi}}, \sqrt[\tau]{\left((1+\psi)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{\psi\nu_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)\frac{1}{\psi}}\right\rangle$$

$$= \left\langle \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\left(\left(1-\frac{\psi\mu_{ij}^{\tau}}{1+\psi}\right)^{\sum_{i=1}^{m}\Omega_i}\right)^{\sum_{j=1}^{n}\omega_j}\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\sum_{i=1}^{m}\Omega_i}\right)^{\sum_{j=1}^{n}\omega_j}-1\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\left(\frac{\psi\nu_{ij}^{\tau}+1}{1+\psi}\right)^{\sum_{i=1}^{m}\Omega_i}\right)^{\sum_{j=1}^{n}\omega_j}-1\right)}\right\rangle$$

$$= \left\langle \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-1+\frac{\psi\mu_{ij}^{\tau}}{1+\psi}\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\frac{(\psi\delta_{ij}^{\tau}+1)}{(1+\psi)}-1\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\frac{(\psi\nu_{ij}^{\tau}+1)}{(1+\psi)}-1\right)}\right\rangle$$

$$= \langle \mu_{ij}, \delta_{ij}, \nu_{ij}\rangle = \widetilde{\wp}_{\hat{\mathscr{L}}}$$

Theorem 3. *(Boundedness) Let* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}} = \langle \mu_{ij}, \delta_{ij}, \nu_{ij}\rangle$ *be the collection of T-SFHySNs, where* $i = 1,2,\cdots,m$ *and* $j = 1,2,\cdots,n$ *and* $\widetilde{\wp}^{-}_{\hat{\alpha}_{ij}} = \langle \min_{j}\min_{i}\{\mu_{ij}\}, \max_{j}\max_{i}\{\delta_{ij}\}, \max_{j}\max_{i}\{\nu_{ij}\}\rangle$ *and* $\widetilde{\wp}^{+}_{\hat{\alpha}_{ij}} = \langle \max_{j}\max_{i}\{\mu_{ij}\}, \min_{j}\min_{i}\{\delta_{ij}\}, \min_{j}\min_{i}\{\nu_{ij}\}\rangle$. *Then,*

$$\widetilde{\wp}^{-}_{\hat{\mathscr{L}}_{ij}} \leq T-SFHySSWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) \leq \widetilde{\wp}^{+}_{\hat{\mathscr{L}}_{ij}}$$

Proof. For each $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n$, we have

$$\min_j\min_i \mu_{ij}{}^{\tau} \leq \mu_{ij}{}^{\tau} \leq \max_j\max_i\{\mu_{ij}{}^{\tau}\}$$

$$\Rightarrow 1-\max_j\max_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\} \leq 1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right) \leq 1-\min_j\min_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\}$$

$$\Leftrightarrow \left(1-\max_j\max_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\}\right)^{\Omega_i} \leq \left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i} \leq \left(1-\min_j\min_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\}\right)^{\Omega_i}$$

i.e., $\left(1-\max_j\max_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\}\right)^{\sum_{i=1}^{n}\Omega_i} \leq \prod_{i=1}^{m}\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i} \leq \left(1-\min_j\min_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\}\right)^{\sum_{i=1}^{n}\Omega_i}$

i.e., $\left(1-\max_j\max_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\}\right)^{\sum_{j=1}^{n}\omega_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{n}\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_j} \leq \left(1-\min_j\min_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\}\right)^{\sum_{j=1}^{n}\omega_j}$

i.e., $1-\max_j\max_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_j} \leq 1-\min_j\min_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\}$

$$\Leftrightarrow \min_j\min_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\} \leq 1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_j} \leq \max_j\max_i\left\{\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right\}$$

$$\Leftrightarrow \min_j\min_i\{\mu_{ij}{}^{\tau}\} \leq \frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_j}\right) \leq \max_j\max_i\{\mu_{ij}{}^{\tau}\}$$

$$\Leftrightarrow \min_j\min_i\{\mu_{ij}{}^{\tau}\} \leq \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\mu_{ij}{}^{\tau}\left(\frac{\psi}{1+\psi}\right)\right)^{\Omega_i}\right)^{\omega_j}\right)} \leq \max_j\max_i\{\mu_{ij}{}^{\tau}\}.$$

Similarly

$$\Leftrightarrow \min_j\min_i\{\delta_{ij}{}^{\tau}\} \leq \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\frac{\psi\delta_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)} \leq \max_j\max_i\{\delta_{ij}{}^{\tau}\}$$

$$\Leftrightarrow \min_j\min_i\{\nu_{ij}{}^{\tau}\} \leq \sqrt[\tau]{\left((1+\psi)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)\frac{1}{\psi}} \leq \max_j\max_i\{\nu_{ij}{}^{\tau}\}$$

Then by comparing two *T*-SFHSNs, we obtain

$$\widetilde{\wp}^{-}_{\hat{\mathscr{L}}_{ij}} \leq T-SFHySSWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) \leq \widetilde{\wp}^{+}_{\hat{\mathscr{L}}_{ij}}$$

Theorem 4. *(Homogeneity) If λ be any positive real number, then*

$$T\text{-}SFHySSWWA\left(\lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) = \lambda T\text{-}SFHySSWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right).$$

Proof. Let $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}$ be a *T*-SFHySNs and $\lambda > 0$. Then, we get

$$\lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}} = \left\langle \sqrt{\tau\frac{1+\psi}{\psi}\left(1-\left(1-\frac{\psi\mu_{ij}{}^{\tau}}{1+\psi}\right)^{\lambda}\right)}, \sqrt[\tau]{\frac{1}{\psi}\left(\left(\frac{\psi\delta_{ij}{}^{\tau}+1}{1+\psi}\right)^{\lambda}-1\right)}, \sqrt[\tau]{\frac{1}{\psi}\left(\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\lambda}-1\right)}\right\rangle$$

So, $T-SFHySSWWA\left(\lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) = \oplus_{SW}{}_{j=1}^{n}\omega_j\left(\oplus_{SW}{}_{i=1}^{m}\lambda\Omega_i\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right)$

$$= \left\langle \sqrt{\tau\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\frac{\psi\mu_{ij}{}^{\tau}}{1+\psi}\right)^{\lambda\Omega_i}\right)^{\omega_j}\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\frac{\psi\delta_{ij}{}^{\tau}+1}{1+\psi}\right)^{\lambda\Omega_i}\right)^{\omega_j}-1\right)}\right\rangle$$

$$\sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\frac{\psi\nu_{ij}^{\tau}+1}{1+\psi}\right)^{\lambda\Omega_i}\right)^{\omega_j}-1\right)}\rangle$$

$$=\left\langle\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\frac{\psi\mu_{ij}^{\tau}}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}\right)^{\lambda}\right)},\sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\frac{\psi\delta_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}\right)^{\lambda}-1\right)},\right.$$

$$\left.\sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\left(\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\frac{\psi\nu_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}\right)^{\lambda}-1\right)}\right\rangle$$

$$=\lambda T-SFHySSWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right)$$

6. *T*-SFHyS Sugeno-Weber weighted geometric aggregation operators

Definition 11. *Let* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}=\langle\mu_{ij},\delta_{ij},\nu_{ij}\rangle$ *be an accumulation of q-ROFHySNs, where* $i=1,2,\cdots,n$ *and* $j=1,2,\cdots,m$. *If* $T-SFHySSWWG:\Delta^{n}\rightarrow\Delta$, *then T-SFHySSWWG is defined as* $T-SFHySSWWG\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right)=\otimes_{SW}{}_{j=1}^{m}\left(\otimes_{SW}{}_{i=1}^{n}\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}{}^{\Omega_i}\right)^{\omega_j}$, *where* $\Omega_i>0$, $\sum_{i=1}^{n}\Omega_i=1$ *and* $\omega_j>0$, $\sum_{j=1}^{m}\omega_j=1$ *represents the weights of experts and attributes.*

Theorem 5. *Let* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}=\langle\mu_{ij},\delta_{ij},\nu_{ij}\rangle$ *be the collection of T-SFHySNs, where* $i=1,2,\cdots,m$ *and* $j=1,2,\cdots,n$. *Then, the acquired aggregated values are also a T-SFHySN and*

$$T-SFHySSWWG\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right)=\otimes_{SW}{}_{j=1}^{m}\left(\otimes_{SW}{}_{i=1}^{n}\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}{}^{\Omega_i}\right)^{\omega_j}$$

$$=\left\langle\sqrt[\tau]{\left((1+\psi)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\frac{\psi\mu_{ij}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)\frac{1}{\psi}},\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\frac{\psi\delta_{ij}^{\tau}}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}\right)}\right\rangle$$

$$\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\frac{\psi\nu_{ij}^{\tau}}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}\right)}\rangle$$

where Ω_i and ω_j represent the weights of the experts and attributes respectively, such as $\Omega_i>0$, $\sum_{i=1}^{m}\Omega_i=1$ and $\omega_j>0$, $\sum_{j=1}^{m}\omega_j=1$.

Proof. Theorem 5 is obtained in the same manner as Theorem 1.

Remark. 1. *If* $\mu_{ij}^{\tau}+\delta_{ij}^{\tau}+\nu_{ij}^{\tau}\leq 1$ *and* $\tau=2$, *then T-SFHySS was reduced to the spherical fuzzy HySS (yet needs to be introduced).*

2. If $\mu_{ij}^{\tau}+\delta_{ij}^{\tau}+\nu_{ij}^{\tau}\leq 1$ and $\tau=1$, then *T*-SFHySS was reduced to the picture fuzzy HySS.
3. If each attribute contains only one sub-attribute and the set contains only one attribute, then *T*-SFHySS is reduced to *T*-SFSS.
4. If each attribute contains only one sub-attribute, the set contains only one attribute and $\tau=2$, then *T*-SFHySS is reduced to SFSS.

Example 3. *The same example is considered here.*

$$T-SFHySSWWG\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}},\widetilde{\wp}_{\hat{\mathscr{L}}_{12}},\cdots,\widetilde{\wp}_{\hat{\mathscr{L}}_{44}}\right)=\otimes_{SW}{}_{j=1}^{4}\gamma_j\left(\otimes_{i=1}^{4}\Omega_i\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}\right)$$

$$=\left\langle\sqrt[3]{\frac{1}{\psi}\left((1+\psi)\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\frac{\psi\mu_{ij}^{3}+1}{1+\psi}\right)^{\Omega_i}\right)^{\gamma_j}-1\right)},\sqrt[3]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\frac{\psi}{1+\psi}\delta_{ij}^{3}\right)^{\Omega_i}\right)^{\gamma_j}\right)},\right.$$

$$\left.\sqrt[3]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\frac{\psi}{1+\psi}\nu_{ij}^{3}\right)^{\Omega_i}\right)^{\gamma_j}\right)}\right\rangle$$

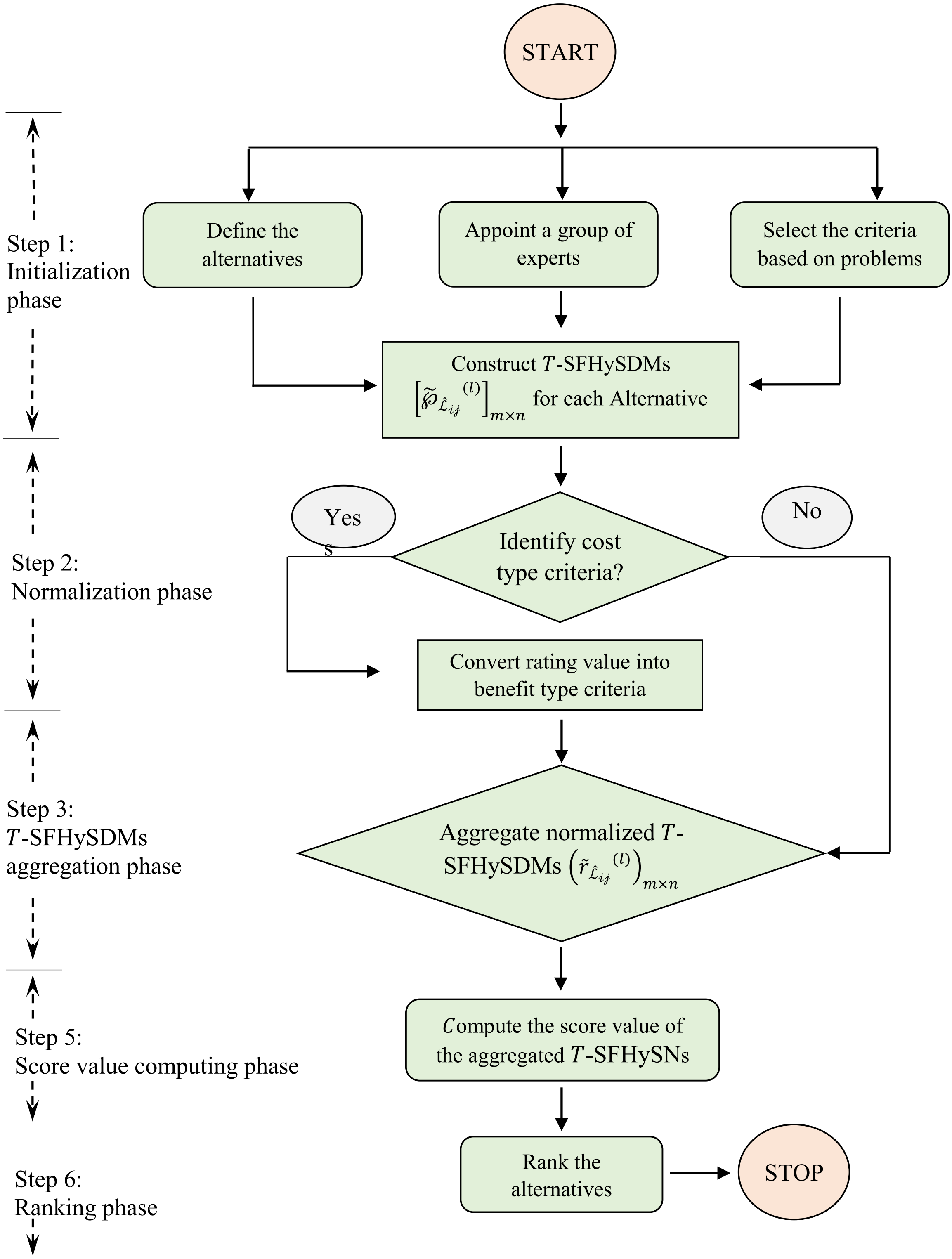

Fig. 2. Frame diagram of the proposed study.

$= \langle 0.4730, 0.2807, 0.5017\rangle$

Using work done by the *T*-SFHySSWWG operator in the past, the following important aspects of the collection of *T*-SFHySNs are taken into account:

Theorem 6. *(Idempotency) If* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}} = \langle \mu_{ij}, \delta_{ij}, \nu_{ij}\rangle$ *is the same as* $\widetilde{\wp}_{\hat{\mathscr{L}}} = \langle \mu, \delta, \nu\rangle$ *for all* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$, *then*

$$T - SFHySSWWG\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) = \widetilde{\wp}_{\hat{\mathscr{L}}}$$

Proof. Theorem 6 is obtained in the same manner as Theorem 2.

Theorem 7. *(Boundedness) Let* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}} = \langle \mu_{ij}, \delta_{ij}, \nu_{ij}\rangle$ *be the collection of T-SFHySNs, where* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$ *and* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ -} = \langle \underset{j}{\min}\underset{i}{\min}\left\{\mu_{ij}\right\}, \underset{j}{\max}\underset{i}{\max}\{\delta_{ij}\}, \underset{j}{\max}\underset{i}{\max}\{\nu_{ij}\}\rangle$ *and* $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ +} = \langle \underset{j}{\max}\underset{i}{\max}\left\{\mu_{ij}\right\}, \underset{j}{\min}\underset{i}{\min}\{\delta_{ij}\}, \underset{j}{\min}\underset{i}{\min}\{\nu_{ij}\}\rangle$, *then*

$$\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ -} \le T - SFHySSWWG\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) \le \widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ +}$$

Proof. Theorem 7 is obtained in the same manner as Theorem 3.

Theorem 8. *(Homogeinity) If* λ *is any positive real number, then*

$$T - SFHySSWWG\left(\lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \lambda\widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right) = \lambda T - SFHySSWWA\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{11}}, \widetilde{\wp}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{\wp}_{\hat{\mathscr{L}}_{mn}}\right).$$

Proof. Theorem 8 is acquired in the same way that Theorem 4 was.

7. Decision-making approach based on Sugeno-Weber aggregation operators in *T*-SFHyS *setting*

The MCDM approach will be built into the section after using our established *T*-SFHyS AOs. In addition, a mathematical formula has also been given to show that the intended operators are correct.

Consider $A = \{A_1, A_2, \cdots, A_p\}$ to be a finite set of alternatives and $\mathscr{D} = \{\mathscr{D}_1, \mathscr{D}_2, \cdots, \mathscr{D}_m\}$ to be m number of experts. The weights of experts are given by $\Omega = (\Omega_1, \Omega_1, \cdots, \Omega_m)^T$ such that $\sum_{i=1}^{m}\Omega_i = 1$ where $\Omega_i \in [0, 1]$. Assume $\mathscr{C} = \{\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n\}$ is a set of criteria and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weights of the criteria such that $\sum_{j=1}^{n}\omega_j = 1$ where $\omega_j \in [0, 1]$. Suppose experts provide the decision matrix, *T*-SFHyS decision matrix (*T*-SFHySDM), $\mathscr{K}^{(l)} = \left[\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ (l)}\right]_{m\times n} = \langle \mu_{ij}^{(l)}, \delta_{ij}^{(l)}, \nu_{ij}^{(l)}\rangle_{m\times n}$ for each alternative in terms of *T*-SFHySNs. Where $\mu_{ij}^{(l)}$, $\delta_{ij}^{(l)}$ and $\nu_{ij}^{(l)}$ indicate, respectively, the MG, abstinence degree and NMG of i^{th} alternative for j^{th} criterion by the l^{th} DM.

Use the *T*-SFHySSWWA and *T*-SFHySSWWG operators to build the aggregated *T*-SFHySNs, $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}$, based on the expert's preference values for each option. Lastly, use Equation (1) to use the order of the options based on the score function. The steps listed below are a summary of the method as mentioned above:

Step 1. Acquire a decision matrix $\left[\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ (l)}\right]_{m\times n} = \langle \mu_{ij}^{(l)}, \delta_{ij}^{(l)}, \nu_{ij}^{(l)}\rangle_{m\times n}$ in the form of *T*-SFHySNs for alternatives in accordance with experts.

Step 2. Use the normalisation formula to turn the rating values of the cost-type parameters into benefit-type parameters. This is the second step of the collective information decision matrix.

$$R^{(l)} = \widetilde{r}_{\hat{\mathscr{L}}_{ij}}^{\ (l)} = \begin{cases} \widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ (l)} \text{for benefit type } \mathscr{C}_j, \\ \left(\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ (l)}\right)^C \quad \text{for cost types } \mathscr{C}_j \end{cases} \tag{12}$$

$\left(\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ (l)}\right)^C = \langle \nu_{ij}^{(l)}, \mu_{ij}^{(l)}, \delta_{ij}^{(l)}\rangle_{m\times n}$ is the complement of $\widetilde{\wp}_{\hat{\mathscr{L}}_{ij}}^{\ (l)} = \langle \mu_{ij}^{(l)}, \delta_{ij}^{(l)}, \nu_{ij}^{(l)}\rangle_{m\times n}$.

Step 3. Aggregate the normalized *T*-SFHySNs $\widetilde{r}_{\hat{\mathscr{L}}_{ij}}^{\ (l)}$ for each alternative $A = \{A_1, A_2, \cdots, A_s\}$ into a collective *T*-SFHySN using the developed *T*-SFHySSWWA (or *T*-SFHySSWWG) operators presented in Definition 10 (or Definition 11).

Step 4. Employing Equation (1), calculate the score values of $\widetilde{r}_{\hat{\mathscr{L}}_{ij}}^{\ (l)}$ for each alternative.

Step 5. Select the alternative with the highest possible score.

Step 6. Examine the rankings.

A frame diagram is shown in Fig. 2 to show how the proposed study will be done.

8. Illustrative example

Agribusiness provides a way to overcome challenges and enhance the diversification of agricultural output. Agribusiness is the science and application of previous and existing activities related to production, processing, marketing, and trade, the distribution of raw and processed foods, feed, and fibre, and the provision of inputs and services for such activities. Green agribusiness emphasizes perceptions of agricultural science that are progressing sustainably, such as increasing food and fibre production, while considering societal and economic constraints to ensure the manufacturing industry's long-term viability. Agribusiness is a vast sector with a wide range of businesses and operations. Agribusinesses include everything from small family farms and food producers to large multinational companies making food for the country.

A revised version of a real-world case study made from an article described by Zulqarnain et al. [49] is known to set up the implementation potential for the simulation model. Five core alternates are interconnected in natural agribusinesses, such as Good crop production (A_1; Environmental protection (A_2); Natural resource availability (A_3); Food security and productivity (A_4); Availability of machines (A_5). In addition, the five alternatives mentioned above are evaluated using four parameters. The attributes of robotic agriculture are as follows: $\varsigma = \{a_1 =$ Quality production, a_2=Completion of a time $-$ consuming project, a_3=Consistency in completing a project, a_4=Limiting the need for manual labour}.

The conforming sub-attributes of the deliberated parameters

Quality production=a_1={$\widehat{\mathscr{L}}_{11}$ = High quality production, $\widehat{\mathscr{L}}_{12}$= low quality production},

Completion of a time-consuming project=a_2={$\widehat{\mathscr{L}}_{21}$ = Short-term, $\widehat{\mathscr{L}}_{22}$=Long-term}

Consistent role in project completion=a_3={$\widehat{\mathscr{L}}_{31}$ = Project budgeting and forecasting}

Reduce the need for manual labor=a_4={$\widehat{\mathscr{L}}_{41}$ = Limiting the need for manual labor}.

Let $\dddot{\mathscr{A}} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_4$ be a set of sub-attributes

$$\dddot{\mathscr{A}} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_4 = \{a_{11}, a_{12}\} \times \{a_{21}, a_{22}\} \times \{a_{31}\} \times \{a_{41}\}$$

$= \{(a_{11}, a_{21}, a_{31}, a_{41}), (a_{11}, a_{22}, a_{31}, a_{41}), (a_{12}, a_{21}, a_{31}, a_{41}), (a_{12}, a_{22}, a_{31}, a_{41})\}$, $\dddot{\mathscr{A}} = \{\widehat{\mathscr{L}}_1, \widehat{\mathscr{L}}_2, \widehat{\mathscr{L}}_3, \widehat{\mathscr{L}}_4\}$ be a set of all sub-attributes with weights $\omega = (0.2, 0.2, 0.2, 0.4)^T$. Let A_1, A_2, A_3 and A_4 be a set of four experts with weights $\Omega = (0.1, 0.3, 0.3, 0.3)^T$.

Due to the fuzzy nature of the available data and the need for more precision in evaluating attributes, evaluating and choosing between different options has become more difficult over the past few years. Hence, decision-making systems must be improved to handle these circumstances. This *T*-SFHyS framework can consider many sub-attributes and both sides of the three-dimensional knowledge involved in satisfaction, abstinence, and dissatisfaction, which are all part of how people make decisions. Considering all these things, experts give their preferences in the form of *T*-SFHySNs to help decide which option is best.

Here is a step-by-step explanation of how to solve the example using the proposed method to find the best alternative:

8.1. By using *T*-SFHySSWWA operators

Step 1: According to the expert, *T*-SFHyS decision matrices for all alternatives are given in Tables 1–5.

Step 2: Because $\widehat{\alpha}_1$ and $\widehat{\alpha}_3$ represent the cost type parameters. Therefore, the normalized *T*-SFHyS decision matrices are obtained using the normalized formula in Tables 6–10.

Step 3: The proposed *T*-SFHySSWWA operator was applied to the acquired data, and then we obtained the opinions of decision-makers on each alternative in the form of *T*-SFHySNs, $\widetilde{r}_{\widehat{\mathscr{L}}_{ij}}$ $(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$ such as

$$T-SFHySSWWA\left(\widetilde{r}_{\widehat{\mathscr{L}}_{11}}, \widetilde{r}_{\widehat{\mathscr{L}}_{12}}, \cdots, \widetilde{r}_{\widehat{\mathscr{L}}_{44}}\right) = \oplus_{SW\,j=1}^{\;4}\omega_j\left(\oplus_{SW\,i=1}^{\;4}\Omega_i\widetilde{r}_{\widehat{\mathscr{L}}_{ij}}\right)$$

$$= \left\langle \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\frac{\psi}{1+\psi}\mu_{ij}^{\ \tau}\right)^{\Omega_i}\right)^{\omega_j}\right)}, \sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\frac{\psi\delta_{ij}^{\ \tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)},\right.$$

Table 1
T-SFHyS Decision Matrix for A_1.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.8, 0.22, 0.5\rangle$	$\langle 0.7, 0.63, 0.5\rangle$	$\langle 0.6, 0.5, 0.4\rangle$	$\langle 0.7, 0.3, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.6, 0.2, 0.5\rangle$	$\langle 0.9, 0.3, 0.1\rangle$	$\langle 0.7, 0.25, 0.3\rangle$	$\langle 0.4, 0.4, 0.5\rangle$
$\mathscr{D}_3$	$\langle 0.8, 0.6, 0.4\rangle$	$\langle 0.7, 0.34, 0.5\rangle$	$\langle 0.6, 0.3, 0.4\rangle$	$\langle 0.3, 0.6, 0.5\rangle$
$\mathscr{D}_4$	$\langle 0.7, 0.3, 0.3\rangle$	$\langle 0.6, 0.2, 0.5\rangle$	$\langle 0.4, 0.4, 0.5\rangle$	$\langle 0.5, 0.63, 0.7\rangle$

Table 2
T-SFHyS Decision Matrix for A_2.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.7, 0.2, 0.5\rangle$	$\langle 0.8, 0.2, 0.5\rangle$	$\langle 0.6, 0.2, 0.4\rangle$	$\langle 0.8, 0.2, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.6, 0.1, 0.3\rangle$	$\langle 0.9, 0.3, 0.2\rangle$	$\langle 0.8, 0.2, 0.3\rangle$	$\langle 0.7, 0.2, 0.5\rangle$
$\mathscr{D}_3$	$\langle 0.5, 0.2, 0.4\rangle$	$\langle 0.6, 0.05, 0.5\rangle$	$\langle 0.6, 0.3, 0.3\rangle$	$\langle 0.3, 0.1, 0.6\rangle$
$\mathscr{D}_4$	$\langle 0.7, 0.2, 0.4\rangle$	$\langle 0.6, 0.1, 0.4\rangle$	$\langle 0.7, 0.3, 0.5\rangle$	$\langle 0.5, 0.3, 0.7\rangle$

Table 3
T-SFHyS Decision Matrix for A_3.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.7, 0.2, 0.5\rangle$	$\langle 0.7, 0.3, 0.4\rangle$	$\langle 0.6, 0.3, 0.4\rangle$	$\langle 0.8, 0.1, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.6, 0.2, 0.6\rangle$	$\langle 0.9, 0.4, 0.1\rangle$	$\langle 0.6, 0.2, 0.3\rangle$	$\langle 0.4, 0.4, 0.5\rangle$
$\mathscr{D}_3$	$\langle 0.8, 0.1, 0.3\rangle$	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.6, 0.2, 0.5\rangle$	$\langle 0.4, 0.3, 0.5\rangle$
$\mathscr{D}_4$	$\langle 0.7, 0.2, 0.6\rangle$	$\langle 0.3, 0.2, 0.5\rangle$	$\langle 0.4, 0.1, 0.5\rangle$	$\langle 0.5, 0.1, 0.6\rangle$

Table 4
T-SFHyS Decision Matrix for A_4.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.8, 0.3, 0.5\rangle$	$\langle 0.7, 0.2, 0.5\rangle$	$\langle 0.7, 0.2, 0.4\rangle$	$\langle 0.6, 0.1, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.6, 0.2, 0.4\rangle$	$\langle 0.8, 0.2, 0.1\rangle$	$\langle 0.7, 0.1, 0.3\rangle$	$\langle 0.4, 0.2, 0.7\rangle$
$\mathscr{D}_3$	$\langle 0.7, 0.2, 0.4\rangle$	$\langle 0.7, 0.1, 0.5\rangle$	$\langle 0.6, 0.2, 0.4\rangle$	$\langle 0.3, 0.25, 0.5\rangle$
$\mathscr{D}_4$	$\langle 0.6, 0.3, 0.3\rangle$	$\langle 0.6, 0.3, 0.3\rangle$	$\langle 0.8, 0.1, 0.5\rangle$	$\langle 0.5, 0.15, 0.6\rangle$

Table 5
T-SFHyS Decision Matrix for A_5.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.6, 0.1, 0.5\rangle$	$\langle 0.6, 0.05, 0.5\rangle$	$\langle 0.6, 0.17, 0.4\rangle$	$\langle 0.5, 0.2, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.6, 0.2, 0.4\rangle$	$\langle 0.8, 0.1, 0.1\rangle$	$\langle 0.8, 0.4, 0.3\rangle$	$\langle 0.7, 0.1, 0.5\rangle$
$\mathscr{D}_3$	$\langle 0.6, 0.15, 0.4\rangle$	$\langle 0.7, 0.25, 0.3\rangle$	$\langle 0.6, 0.3, 0.4\rangle$	$\langle 0.6, 0.2, 0.5\rangle$
$\mathscr{D}_4$	$\langle 0.7, 0.12, 0.4\rangle$	$\langle 0.7, 0.3, 0.5\rangle$	$\langle 0.4, 0.1, 0.5\rangle$	$\langle 0.5, 0.1, 0.8\rangle$

Table 6
Normalized T-SFHyS Decision Matrix for A_1.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.5, 0.22, 0.8\rangle$	$\langle 0.7, 0.63, 0.5\rangle$	$\langle 0.4, 0.5, 0.6\rangle$	$\langle 0.7, 0.3, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.5, 0.2, 0.6\rangle$	$\langle 0.9, 0.3, 0.1\rangle$	$\langle 0.3, 0.25, 0.7\rangle$	$\langle 0.4, 0.4, 0.5\rangle$
$\mathscr{D}_3$	$\langle 0.4, 0.6, 0.8\rangle$	$\langle 0.7, 0.34, 0.5\rangle$	$\langle 0.4, 0.3, 0.6\rangle$	$\langle 0.3, 0.6, 0.5\rangle$
$\mathscr{D}_4$	$\langle 0.3, 0.3, 0.7\rangle$	$\langle 0.6, 0.2, 0.5\rangle$	$\langle 0.5, 0.4, 0.4\rangle$	$\langle 0.5, 0.63, 0.7\rangle$

Table 7
Normalized q-ROFHyS Decision Matrix for A_2.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.5, 0.2, 0.7\rangle$	$\langle 0.8, 0.2, 0.5\rangle$	$\langle 0.4, 0.2, 0.6\rangle$	$\langle 0.8, 0.2, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.3, 0.1, 0.6\rangle$	$\langle 0.9, 0.3, 0.2\rangle$	$\langle 0.3, 0.2, 0.8\rangle$	$\langle 0.7, 0.2, 0.5\rangle$
$\mathscr{D}_3$	$\langle 0.4, 0.2, 0.5\rangle$	$\langle 0.6, 0.05, 0.5\rangle$	$\langle 0.3, 0.3, 0.6\rangle$	$\langle 0.3, 0.1, 0.6\rangle$
$\mathscr{D}_4$	$\langle 0.4, 0.2, 0.7\rangle$	$\langle 0.6, 0.1, 0.4\rangle$	$\langle 0.5, 0.3, 0.7\rangle$	$\langle 0.5, 0.3, 0.7\rangle$

$$\sqrt[\tau]{\frac{1}{\psi}\left((1+\psi)\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\frac{\psi\nu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}-1\right)}\rangle$$

For $\tau = 3$ and $\psi = 2$, $A_1 = \langle 0.5504, 0.2157, 0.5796\rangle$, $A_2 = \langle 0.5863, 0.1579, 0.5861\rangle$, $A_3 = \langle 0.5649, 0.2378, 0.5455\rangle$, $A_4 = \langle 0.5167, 0.3125, 0.5956\rangle$, $A_5 = \langle 0.5857, 0.2497, 0.5937\rangle$.

Table 8
Normalised q-ROFHyS Decision Matrix for A_3.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.5, 0.2, 0.7\rangle$	$\langle 0.7, 0.3, 0.4\rangle$	$\langle 0.4, 0.3, 0.6\rangle$	$\langle 0.8, 0.1, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.6, 0.2, 0.6\rangle$	$\langle 0.9, 0.4, 0.1\rangle$	$\langle 0.3, 0.2, 0.6\rangle$	$\langle 0.4, 0.4, 0.5\rangle$
$\mathscr{D}_3$	$\langle 0.3, 0.1, 0.8\rangle$	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.5, 0.2, 0.6\rangle$	$\langle 0.4, 0.3, 0.5\rangle$
$\mathscr{D}_4$	$\langle 0.6, 0.2, 0.7\rangle$	$\langle 0.3, 0.2, 0.5\rangle$	$\langle 0.5, 0.1, 0.4\rangle$	$\langle 0.5, 0.1, 0.6\rangle$

Table 9
Normalised q-ROFHyS Decision Matrix for A_4.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.5, 0.3, 0.8\rangle$	$\langle 0.7, 0.2, 0.5\rangle$	$\langle 0.4, 0.2, 0.7\rangle$	$\langle 0.6, 0.1, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.4, 0.2, 0.6\rangle$	$\langle 0.8, 0.2, 0.1\rangle$	$\langle 0.3, 0.1, 0.7\rangle$	$\langle 0.4, 0.2, 0.7\rangle$
$\mathscr{D}_3$	$\langle 0.4, 0.2, 0.7\rangle$	$\langle 0.7, 0.1, 0.5\rangle$	$\langle 0.4, 0.2, 0.6\rangle$	$\langle 0.3, 0.25, 0.5\rangle$
$\mathscr{D}_4$	$\langle 0.3, 0.3, 0.6\rangle$	$\langle 0.6, 0.3, 0.3\rangle$	$\langle 0.5, 0.1, 0.8\rangle$	$\langle 0.5, 0.15, 0.6\rangle$

Table 10
Normalized q-ROFHyS Decision Matrix for A_5.

	$\widehat{\mathscr{L}}_1$	$\widehat{\mathscr{L}}_2$	$\widehat{\mathscr{L}}_3$	$\widehat{\mathscr{L}}_4$
$\mathscr{D}_1$	$\langle 0.5, 0.1, 0.6\rangle$	$\langle 0.6, 0.05, 0.5\rangle$	$\langle 0.4, 0.17, 0.6\rangle$	$\langle 0.5, 0.2, 0.4\rangle$
$\mathscr{D}_2$	$\langle 0.4, 0.2, 0.6\rangle$	$\langle 0.8, 0.1, 0.1\rangle$	$\langle 0.3, 0.4, 0.8\rangle$	$\langle 0.7, 0.1, 0.5\rangle$
$\mathscr{D}_3$	$\langle 0.4, 0.15, 0.6\rangle$	$\langle 0.7, 0.25, 0.3\rangle$	$\langle 0.4, 0.3, 0.6\rangle$	$\langle 0.6, 0.2, 0.5\rangle$
$\mathscr{D}_4$	$\langle 0.4, 0.12, 0.7\rangle$	$\langle 0.7, 0.3, 0.5\rangle$	$\langle 0.5, 0.1, 0.4\rangle$	$\langle 0.5, 0.1, 0.8\rangle$

Step 4: Use the score formula to calculate the score values for all alternatives.
$S(A_1) = -0.03$, $S(A_2) = 0.0002$, $S(A_3) = 0.0222$, $S(A_4) = -0.008$, and $S(A_5) = -0.0095$.
Step 5: After calculation, we get the ranking of alternatives
$S(A_3)\rangle S(A_2)\rangle S(A_4)\rangle S(A_5)\rangle S(A_1)$. So, $A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$.
Hence, the best alternative is A_3.
cusing the T-SFHySSWWG operator.

8.2. By using T-SFHySSWWG operators

Step 1′: This is similar to Step 1.
Step 2′: This is similar to Step 2.
Step 3′: The proposed T-SFHySSWWG operator was applied to the acquired data, and then we obtained the opinions of decision-makers on each alternative in the form of T-SFHySNs $\aleph_{e_{ij}}$ such as

$$T - SFHySSWWG\left(\widetilde{r}_{\hat{\mathscr{L}}_{11}}, \widetilde{r}_{\hat{\mathscr{L}}_{12}}, \cdots, \widetilde{r}_{\hat{\mathscr{L}}_{44}}\right) = \otimes_{SW_{j=1}}^{4}\left(\otimes_{SW_{i=1}}^{4}\widetilde{r}_{\hat{\mathscr{L}}_{ij}}{}^{\Omega_i}\right)^{\omega_j}$$

$$= \left\langle \sqrt[\tau]{\left((1+\psi)\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\frac{\psi\mu_{ij}{}^{\tau}+1}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j} - 1\right)\frac{1}{\psi}}, \sqrt[\tau]{\frac{1+\psi}{\psi}\left(1 - \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1 - \frac{\psi\delta_{ij}{}^{\tau}}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}\right)} \right\rangle$$

Table 11
Ranking results varying SW parameter ψ in the T-SFHySSWWA operator.

Parameter	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	Rankings
$\psi = -0.99$	−0.2773	−0.1543	−0.1367	−0.1879	−0.2006	$A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$
$\psi = -0.1905$	−0.2365	−0.0978	−0.0879	−0.1225	−0.1225	$A_3 \succ A_2 \succ A_4 \approx A_5 \succ A_1$
$\psi = 0.01$	−0.1829	−0.0446	−0.0456	−0.1177	−0.1160	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
$\psi = 3$	−0.1314	−0.0023	−0.0037	−0.0877	−0.0739	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
$\psi = 5$	−0.1162	0.0082	0.0079	−0.0791	−0.0616	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
$\psi = 7$	−0.1059	0.0149	0.0158	−0.0731	−0.0531	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
$\psi = 10$	−0.0950	0.0218	0.0241	−0.0668	−0.0439	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$

$$\sqrt[\tau]{\frac{1+\psi}{\psi}\left(1-\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\frac{\psi {\nu_{ij}}^{\tau}}{1+\psi}\right)^{\Omega_i}\right)^{\omega_j}\right)}\rangle$$

Without loss of generality here, the rung parameter and SW parameter values are considered as $q = 3$ and $\psi = 2$, respectively.

$A_1 = \langle 0.5167, 0.2234, 0.6009\rangle$, $A_2 = \langle 0.5438, 0.1296, 0.6070\rangle$, $A_3 = \langle 0.5262, 0.3215, 0.5655\rangle$, $A_4 = \langle 0.4929, 0.2598, 0.6154\rangle$, $A_5 = \langle 0.5662, 0.3215, 0.6046\rangle$.

Step $4'$: Use the score formula to calculate the score values for all alternatives.

$S(A_1) = -0.1$, $S(A_2) = -0.07$, $S(A_3) = -0.04$, $S(A_4) = -0.1$, $S(A_5) = -0.05$.

Step $5'$: After calculation, we get the ranking of alternatives

$S(A_3)\rangle S(A_5)\rangle S(A_2)\rangle S(A_4)\rangle S(A_1)$. So, the ranking result of the alternatives is found as $A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$.

Hence, the best alternative is A_3.

9. Impact of the Parameters on Decision Results

This section investigates the influence of the WG parameter ψ and rung parameter q on the score values and the rankings of the alternatives.

9.1. Impact of Sugeno-Weber parameter on decision results

In the previous example, Steps 3 and 4 are done more than once, each time with a different value of SW parameter ψ, to show what effect ψ has. For accessibility, the rung parameter is set to $q = 3$ for both the averaging and geometric operator cases. The scores and rankings for the T-SFHySSWWA and T-SFHySSWWG operators are shown in Tables 11 and 12, respectively. From Tables 11 and 12, you can see that different values of the SW parameter ψ have led to different score values, which have led to different rankings.

From Fig. 3, it is significantly noticeable that the alternatives' score values increase with the SW parameter's increasing values based on the T-SFHySSWWA operator. When using the same T-SFHySSWWA AO while maintaining the fixed value $q = 3$, many alternative orderings are discovered to vary the value of ψ from -1 to 10. The alternatives are ranked in the following order: $A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$, when $\psi \in (-1, -0.3034)$. When the value of ψ lies in $(-0.3034, -0.1905)$, the ranking result slightly differs as $A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$, and for $\psi \in (-0.1905, 5.5566)$ ordering is found as $A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$. In the reset range ranking $A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$ is obtained. Therefore, the best alternative is A_2 or A_3.

Again, this can be seen by examining Fig. 4, where it can be seen that the score values of the alternatives decrease as the values of the SW parameter based on the T-SFHySSWWG operator increase; another way of putting this is to say that a gradual decrease in score values can be seen there. In Fig. 4, the variation in the score values of various alternatives is depicted for the case where $q = 3$ is held constant while ψ is varied within the range of $[-1, 10]$. It can be shown from Fig. 4 that the ranking presents itself as $A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$, $A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$, and $A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$ when ψ belongs to $(-1, -0.8475)$, $(-0.8475, 0.2353)$, and $(0.2353, 0.4355)$, respectively. Because of this, utilizing the T-SFHySSWWG operator will result in the best alternative being either A_2 or A_3.

As a result, a DM's outlook can be either negative or positive based on their judgment. Therefore, decision-makers with a pessimistic outlook on a potential alternative based on criteria ought to select a higher value of the SW parameter ψ.

9.2. The influence of the parameter τ, on ranking results

Tables 13 and 14 provide an in-depth analysis of how the rung parameter τ influences the decision made by the T-SFHySSWWA and T-SFHySSWWG operators, respectively. It is clear that by keeping the value of the SW parameter constant at $\psi = 2$, the orderings of the alternatives can be constructed for various τ values in [2,10]. These rankings are based on a variety of different performance factors.

Figs. 5 and 6 show that based on both the operators T-SFHySSWWA and T-SFHySSWWG, several ranking results are found for varying rung parameters τ in $[2, 10]$.

Further, from the visualization of Figs. 5 and 6, it is apparent that the score values of the alternatives increase with increasing the value of the rung parameter τ, based on both averaging and geometric operators.

Table 12
Ranking results varying SW parameter ψ in the T-SFHySSWWG operator.

Parameter	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	Rankings
$\psi = -0.99$	−0.0154	0.0621	0.0955	−0.0097	0.0284	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
$\psi = 0.001$	−0.1833	−0.0450	−0.0459	−0.1180	−0.1164	$A_2 \succ A_3 \succ A_5 \succ A_4 \succ A_1$
$\psi = 3$	−0.2258	−0.0892	−0.0853	−0.1465	−0.1537	$A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$
$\psi = 5$	−0.2357	−0.1007	−0.0950	−0.1539	−0.1628	$A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$
$\psi = 7$	−0.2420	−0.1083	−0.1012	−0.1587	−0.1687	$A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$
$\psi = 10$	−0.2482	−0.1161	−0.1074	−0.1637	−0.1745	$A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$

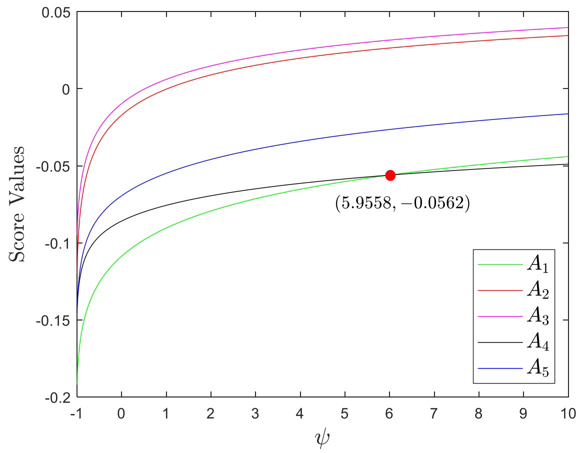

Fig. 3. Effect of SW parameter on score values based on T-SFHySSWWA operator ($q = 3$).

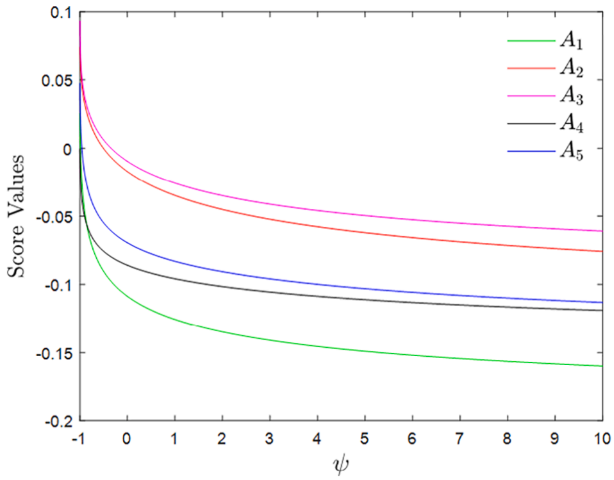

Fig. 4. Effect of SW parameter on score values based on T-SFHySSWWG operator ($q = 3$).

Table 13
Ranking results varying rung parameter τ in T-SFHySSWWA operator.

Parameter	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	Rankings
$\tau = 2$	−0.2450	−0.0529	−0.0757	−0.1279	−0.1624	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$
$\tau = 4$	−0.0793	0.0089	0.0148	−0.0694	−0.0458	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
$\tau = 6$	−0.0184	0.0241	0.0306	−0.0359	−0.0203	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
$\tau = 8$	0.0040	0.0264	0.0301	−0.0182	−0.0130	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
$\tau = 10$	0.0116	0.0243	0.0258	−0.0093	−0.0094	$A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$

Table 14
Ranking results varying rung parameter τ in T-SFHySSWWG operator.

Parameter	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	Rankings
$\tau = 2$	−0.3367	−0.1329	−0.1570	−0.1889	−0.2406	$A_2 \succ A_3 \succ A_4 \succ A_5 \succ A_1$
$\tau = 4$	−0.1348	−0.0450	−0.0348	−0.1015	−0.0906	$A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_1$
$\tau = 6$	−0.0480	−0.0063	0.0021	−0.0495	−0.0401	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
$\tau = 8$	0.0040	0.0264	0.0021	−0.0182	−0.0130	$A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4$
$\tau = 10$	0.0010	0.0134	0.0152	−0.0116	−0.0130	$A_3 \succ A_2 \succ A_1 \succ A_4 \succ A_5$

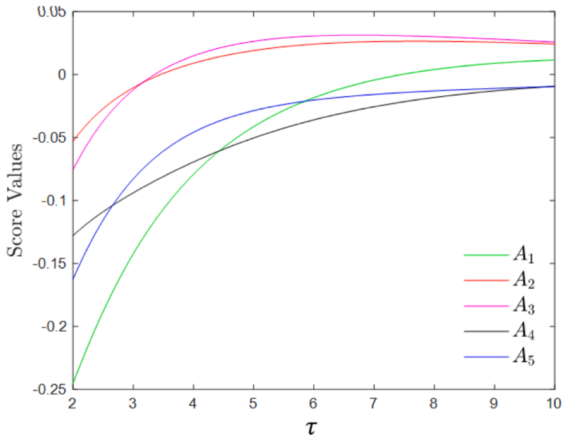

Fig. 5. Effect of rung parameter on score values based on T-SFHySSWWA operator ($\psi = 2$).

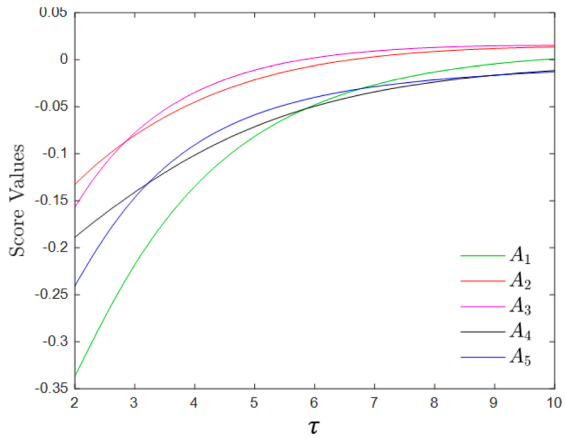

Fig. 6. Effect of rung parameter on score values based on T-SFHySSWWG operator ($\psi = 2$).

Table 15
Characteristic analysis of different approaches.

Operators	Parametrization tool	Sub- parametrization tool	Neutral membership degree	Capturing information by *T*-SF context	AOs considering SW *t*-N&*t*-CNs
IFS AOs [42]	✓	×	×	×	×
PyFS AOs [29]	✓	×	×	×	×
q-ROFS AOs [13]	✓	×	×	×	×
PFS AOs [23]	✓	×	✓	×	×
SFS AOs [8]	✓	×	✓	×	×
T-SFS AOs [7]	✓	×	✓	✓	×
IFHyS AOs [31]	✓	✓	×	×	×
PFHyS AOs [49]	✓	✓	×	×	×
q-ROFHyS AOs [9]	✓	✓	×	×	×
Picture fuzzy HyS AOs [31]	✓	✓	✓	×	×
SFHyS AOs (Yet to introduce)	✓	✓	✓	×	✓
Proposed operators	✓	✓	✓	✓	✓

10. Superiority and advantages of the suggested technique

We have developed a novel technique that can be executed in a T-SFHySS environment by T-SFHySSWWA or T-SFHySSWWG operators. The technique that is anticipated to be utilised is reliable and practical. Our planned approach is superior to existing

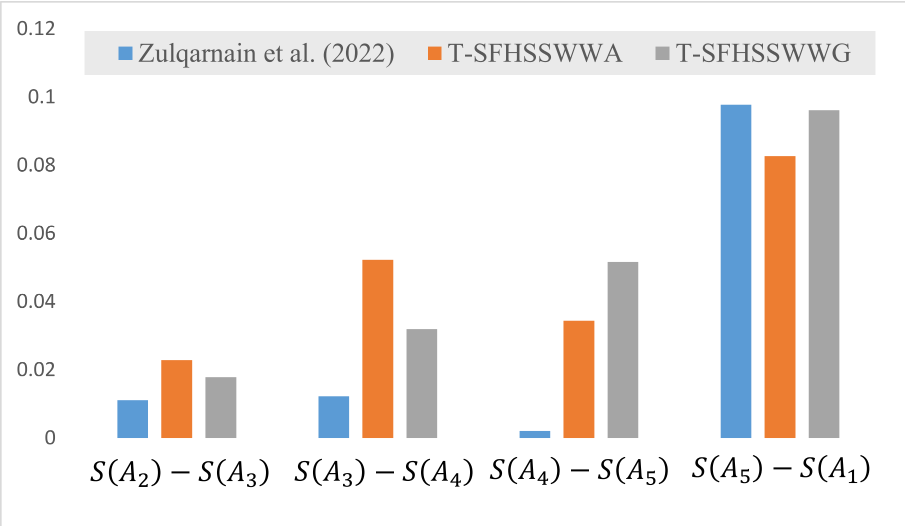

Fig. 7. Comparison with Zulqarnian et al.s' [49] method.

methods and can handle the most delicate MCDM issues. The included model serves multiple functions and is more relaxed to accommodate differences in engagement and output from the development process. There are distinct differences between how the various models rank and judge the prescribed technique to be acceptable. Each evaluation system has its own rating algorithm. As a result of these scientific studies and analyses, we've determined that the conventional approach to building yields less consistent results than the hybrid method. Moreover, numerous hybrid forms of FS, IFHySS, and PFHySS have become uncommon for *T*-SFHySS due to many exceptional circumstances. Therefore, the method we've devised will be significantly more powerful, solid, and excellent than the numerous FS systems, as well as superior. Table 15 contrasts the proposed method with several previous designs based on the characteristics of those frameworks.

The purposeful techniques and associated metrics benefit recent methods in that they prevent conclusions from being drawn based on factors that need to be more helpful. Due to this, this tool can also use the DM process to merge data that needs to be precise or clarified.

- **Compared with Zulqarnian et al.s' (2022) method**

The framework described by Zulqarnian et al. [49] is contrasted with the method that is being proposed here. It is important to note that the ranking of the options stays the same regardless of which method is used. In addition, the suggested method is superior when the differences in the score values of the successive alternatives are determined; the suggested technique comes out on top. As seen in Fig. 7, the differences between the most recent two alternatives, which are consecutive to one another, are significantly greater than those between the conventional methods and each of the alternatives.

11. Conclusions

To conclude, it is necessary to classify potential solutions and select the most feasible options. DM is challenging because it varies from scene to scene. Therefore, it is essential to consider both the advantages and disadvantages of each option. Moreover, DM is more advantageous for your overall well-being and increases the likelihood of revealing the most suitable option. It is crucial to determine precisely how much fundamental information decision-makers require. In DM, the most effective strategy is to pay close attention to and concentrate on your objectives. The investigation's primary objective is to introduce the concept of *T*-SFHySS to the scientific community. Certain newly established operational principles for *T*-SFHySS consider the SW*t*-N &*t*-CNs. Sugeno-Weber AOs, particularly the *T*-SFHySSWWA and *T*-SFHySSWWG operators, have already been designed based on the provided concept. Additionally, some of the essential characteristics of the suggested operators have been discussed. A DM concern is addressed within the context of *T*-SFHySS. The predicted model is the foundation of this challenge. Then, we use several existing methods to demonstrate the validity of the new method, and we conduct a character analysis to determine how much the new method influenced and dominated the initial research. The benefit of the solution is that it allows you to tackle practical problems by utilizing their parameterized characteristics. Consequently, the proven approach, as opposed to any of the existing operators in the *T*-SFHyS context, is the one that can resolve the DM issue. Several hybrid AOs for *T*-SFHySS and associated decision-making processes will be presented in the future. Moreover, applying these frameworks permits the extension of the created AOs to *T*-SFHyS contexts with decision-making strategies.

Although the proposed model has the benefits mentioned above, it is possible to discuss its limitations. The proposed study can only address this situation if the weight of the DMs or criteria is known or partially known. The proposed model needs to account for the interdependencies between criteria. A model of unknown weight under *T*-SFHyS conditions may be developed to overcome such limitations. The following issues may be examined as an extension of the developed method: This work is a foundation for future research. This work will probably lead academics and developers to work in a new direction. In the future, the suggested approaches can be extended to several environments, viz., linguistic information [11], indetermSoft sets and indetermHyperSoft sets, probabilistic linguistic term sets (Han et al., 2022), bipolar fuzzy sets (Mahmood et al., 2023), and other types of fuzzy sets. Also, several decision-making methodologies, viz., the bi-polar preference-based weights allocation method [16], the best-worst method [39], the multiplicative consistency preference relation approach [41], the bi-objective optimization model [3], etc., can be developed in this context.

CRediT authorship contribution statement

Arun Sarkar: Conceptualization, Methodology, Software, Validation, Visualization, Investigation, Writing – original draft, Writing – review & editing. **Tapan Senapati:** Conceptualization, Writing – original draft, Writing – review & editing. **LeSheng Jin:** Writing – original draft, Writing – review & editing. **Radko Mesiar:** Conceptualization, Visualization, Writing – original draft, Writing – review & editing. **Animesh Biswas:** Writing – original draft, Writing – review & editing. **Ronald R. Yager:** Writing – original draft, Writing – review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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ORIGINAL ARTICLE

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Schweizer-Sklar operations based hybrid aggregation operator to dual hesitant q-rung orthopair fuzzy set and its application on MCGDM

Souvik Gayen[1] | Arun Sarkar[2] | Animesh Biswas[1]

[1]Department of Mathematics, University of Kalyani, Kalyani, India
[2]Department of Mathematics, Heramba Chandra College, Kolkata, India

Correspondence
Arun Sarkar, Department of Mathematics, Heramba Chandra College, Kolkata – 700029, India.
Email: asarkarmth@gmail.com

Abstract
Dual hesitant q-rung orthopair fuzzy (DHq-ROF) set appears as a powerful tool in compare to other variants of fuzzy sets to deal with uncertainties associated with available information in various real-life decision-making cases. In order to make DHq-ROF aggregation information process flexible, at first some operations viz., addition, multiplication, scalar multiplication, exponential laws based on Schweizer-Sklar class of t-conorms and t-norms are defined. Subsequently, using these operations, weighted average and geometric operators and ordered weighted average and geometric operators are introduced. But weighted average or geometric operators and ordered weighted average or geometric operators consider only the weight of the opinions and the weight of the ordered position of each given opinion respectively. To resolve weights of the arguments, hybrid aggregation operators viz., DHq-ROF Schweizer-Sklar hybrid averaging, DHq-ROF Schweizer-Sklar hybrid geometric operators are developed and their properties are discussed. Afterwards, a new method to deal with multicriteria group decision making problems under DHq-ROF environment is framed. To illustrate the proposed method a decision making problem related to investment company selection is considered and solved. To show the advantages of the proposed study, a comparative analysis among the developed and existing studies is discussed.

KEYWORDS
dual hesitant fuzzy set, dual hesitant q-rung orthopair fuzzy set, group decision making, hybrid aggregation operators, Schweizer-Sklar t-conorms and t-norms

1 | INTRODUCTION

Multicriteria decision making (MCDM) refers to the problem for sorting alternatives based on numerous criteria and choosing the best one. In actual decision-making circumstances, it is challenging to resolve MCDM problems having vague information due to the fuzziness of human cognition and the complexity of decision-making environments. Zadeh (1965) proposed the theory of fuzzy sets by introducing membership values only. Importing the concept of non-membership, Atanassov (1986) originated intuitionistic fuzzy (IF) sets (IFSs) and Yagar (2013, 2014) expanded the idea of IFS to create the Pythagorean fuzzy (PF) set (PFS), where the sum of the squares of membership and non-membership degrees is not greater than 1. Sometimes PFS fails to describe evaluation information in multicriteria group decision making (MCGDM problems). For example, PFS cannot consider $(0.9,0.7)$ as a pair of membership and non-membership values. Because of $0.9^2+0.7^2>1$. To overcome this situation, Yagar (2017) defined q-rung orthopair fuzzy (q-ROF) set (q-ROFS), which is a generalized version of IFS and PFS satisfying the constraint $0 \le \mu^q+\nu^q \le 1$

where μ is membership degree and ν is non-membership degree and $q \geq 1$. When $q = 1$, q-ROFS reduces to IFS, and for $q = 2$, it reduces to PFS. So q-ROFS can describe vague phenomena more widely than IFS and PFS.

To aggregate q-ROF arguments, Liu and Wang (2018) proposed q-ROF Archimedean t-conorms and t-norms based weighted averaging and geometric operators. Rong et al. (2020) defined Schweizer-Sklar (SS) operations on q-ROF numbers (q-ROFNs) and proposed q-ROF SS weighted average and q-ROF SS weighted geometric operators to solve MCGDM problems by COPRAS method. Later, Utilizing Maclaurin symmetric mean (MSM) operator, Liu, Chen, and Wang (2020) introduced several q-ROF power MSM operators to aggregate q-ROFNs. Furthermore, various MSM operation-based aggregation operators are proposed by several researchers (Liu and Wang (2020), Ali and Mahmood (2020)) under q-ROF context. Next, extending the notion of Einstein norms operation, Akram et al. (2021) developed a series of Einstein geometric operators on q-ROF environment for solving MADM problems. Further, Feng et al. (2022) defined the regular Minkowski distance and a generic class of score functions viz., Minkowski score functions in q-ROF domain. Recently, Gayen et al., (2022) proposed Hamacher aggregation operators for aggregating q-rung orthopair trapezoidal fuzzy numbers. Also, some research works (Mahamood et al., 2019; Liao et al., 2020; Liu et al., 2022; Riaz et al., 2022; Paul et al., 2022, Mahamood et al., 2023) on IF, PF and q-ROF contexts are reviewed.

However, due to the increasing complexity in real-life decision-making, the above-mentioned decision-making methods sometimes fail to capture hesitant characteristics of the problems. In such situations, DMs could experience hesitation among a group of values in choosing the decision values corresponding to some options in view of criteria. By considering such hesitancy, Torra (2010) introduced the idea of hesitant fuzzy set (HFS), where membership degree is represented by a set of possible discrete values in $[0,1]$. Afterwards, the disadvantage of HFS was then brought out by Zhu et al. (2012) from the view point of non considering non-membership degrees. Later, they proposed the idea of dual hesitant fuzzy set (DHFS), which assumes both possible membership and non-membership degrees. Combining the benefits of DHFS and q-ROFS, Xu et al. (2018) introduced the concept of dual hesitant q-ROF (DHq-ROF) set (DHq-ROFS) where sum of q^{th} $(q \geq 1)$ power of the maximum membership and maximum non-membership degree to an element is not greater than 1. So, in this scenario, DMs are given more freedom to express their assessment values of the problems. Thus, DHq-ROFS exhibits more flexibility and usefulness over HFS, IFS, PFS, DHFS, dual hesitant Pythagorean fuzzy set (DHPFS) and q-ROFS, which is presented in Table 1. It is important to mention that almost all the current DHq-ROF information aggregation operations are developed based on Heronian mean (Deb et al., 2022; Xu et al., 2018), Hamacher (Sarkar et al., 2023; Wang, Wei, Wang, et al., 2019), Bonferroni mean (Sarkar & Biswas, 2021), Muirhead mean (Wang, Wei, Wei, & Wei, 2019) algebraic and geometric mean (Hussain et al., 2020), etc.

To capture optimistic or pessimistic behaviour of the DMs, SS t-conorms and t-norms (SSt-CN&t-Ns) (Schweizer & Sklar, 1983) which satisfy the properties of Archimedean t-conorms and t-norms are used. It is also much more adaptable than other operations, viz., algebraic t-conorms and t-norms, Einstein t-conorm and t-norm, Hamacher t-conorm and t-norm, Aczel-Alsina t-conorms and t-norms, etc. Further, weighted average or geometric operators consider only the weight of the opinions but disregard the importance of the ordered position of the opinions, while ordered weighted average or geometric operators consider only the weight of the ordered position of each given opinion but ignore the importance of the individual opinion. To overcome this issue, Xu and Da, (2003) introduced the concept of hybrid aggregation operators, which consider the weight of the argument values as well as its ordered positions. Afterwards, some researchers (Akram & Shazadi, 2021; Wang, Wei, Wang, et al., 2019) paid their attentions to develop hybrid aggregation operators on different fuzzy domains.

Furthermore, Money investment is a hot topic in recent days. It is important to measure risk factors, profitability, growth analysis, environmental impact, etc., for each company before investment. It is more complex to select best profitable company among various companies available in the market based on the above aspect. The MCGDM technique is most suitable for solve the problem. Several researchers (Garg, 2019;

TABLE 1 Comparison of DHq-ROFS model with published models in the literature

Features	Uncertainty	Falsity	Hesitation	Capturing information by q-ROF
FS	■	□	□	□
IFS	■	■	□	□
PFS	■	■	□	□
DHFS	■	■	■	□
DHPFS	■	■	■	□
DHq-ROFS	■	■	■	■

Rahaman & Ali, 2020; Vijayakumar et al., 2022) worked on investment problems in various fuzzy domain. Viewing this, a real-life based money investment MCGDM problem has been considered to solve in this paper under DHq-ROF environment.

The motivations of this research are summarized as follows:

1. q-ROFS is an impressive expansion of IFS and PFS that allows for more comprehensive modelling of scenarios than IFS and PFS. The DMs are encouraged to use DHFSs when hesitating and uncertain situations occur. The advantages of both q-ROFS and DHFS are taken into account to provide the acceptability zone of DHq-ROFS with greater flexibility in the evaluation of information.
2. The operational rules are crucial in processing decision-related data. By using a flexible parameter, SS operations established its efficiency to make reasonable resolutions through aggregation processing. So, use of SS operations in the proposed method manages the risk level of the DMs in decision making processes.
3. Weighted average/geometric and ordered weighted average/geometric operators are particular cases of hybrid averaging/geometric operators. There is no research work on SS operation-based hybrid aggregation operators on DHq-ROFS. So, it would be relevant research to consider the advantages of hybrid aggregation operators and flexible characteristics of SSt-CN&t-Ns in DHq-ROF environment.
4. To overcome the limitation of the current operators in DHq-ROF context.

The following novelty and main contributions of this work are outlined:

1. The use of DHq-ROFNs is implemented so that uncertainty can be captured more effectively. The DHq-ROFNs have greater acceptance as q rises, allowing the DMs more leeway when data are collected.
2. On the basis of SSt-CN&t-Ns, certain fundamental algebraic operational rules—scalar multiplication, exponential, sum, and product are introduced.
3. Based on newly developed operational tools, DHq-ROF SS weighted average (DHq-ROFSSWA), DHq-ROF SS ordered weighted average (DHq-ROFSSOWA), DHq-ROF SS weighted geometric (DHq-ROFSSWG), DHq-ROF SS ordered weighted geometric (DHq-ROFSSOWG), DHq-ROF SS hybrid average (DHq-ROFSSHA), and DHq-ROF SS hybrid geometric (DHq-ROFSSHG) operators are proposed.
4. To diminish the limitation of the current operators, specific instances of the aforementioned proposed operators are presented by changing the associated parameter.
5. This paper suggests a novel MCGDM technique using developed SS operation-based aggregation operators in a DHq-ROF environment. A money investment problem has been resolved using the decision-making approach to demonstrate the efficacy and applicability of the suggested approach. A comparison of the suggested technique with other current methodologies and existing operators shows the method's validity.

The remaining part of the paper is prepared as: in Section 2, some basic definitions and properties are discussed. The developed aggregation operators for DHq-ROF information, viz., DHq-ROFSSWA, DHq-ROFSSOWA, DHq-ROFSSWG, DHq-ROFSSOWG, DHq-ROFSSHA, DHq-ROFSSHG based on newly defined operational laws are presented in Section 3. Based on the developed operators, a unique MCGDM method discourses in Section 4. In Section 5, An illustrative example is discussed and solved. Further, sensitivity analysis, and comparative studies are also presented, and in Section 6, conclusions and some future directions are given.

2 | PRELIMINARIES

In the present section, some basic definitions connected to q-ROFSs (Yager, 2017), DHq-ROFSs (Xu et al., 2018) and operations on them are reviewed to develop the proposed methodology.

Definition 2.1 (Yager, 2017). Let $\mathcal{X}$ be a universal set, a q-ROFS $\tilde{\mathcal{P}}$ on $\mathcal{X}$ is presented as

$$\tilde{\mathcal{P}} = \left\{ \left(x, \mu_{\tilde{\mathcal{P}}}(x), \nu_{\tilde{\mathcal{P}}}(x)\right) \middle| x \in \mathcal{X} \right\},$$

where $\mu_{\tilde{\mathcal{P}}} : \mathcal{X} \to [0,1]$ and $\nu_{\tilde{\mathcal{P}}} : \mathcal{X} \to [0,1]$ indicate the degree of membership and non-membership, respectively, of an element $x \in \mathcal{X}$ to the set $\tilde{\mathcal{P}}$, and satisfies $0 \le \left(\mu_{\tilde{\mathcal{P}}}(x)\right)^q + \left(\nu_{\tilde{\mathcal{P}}}(x)\right)^q \le 1$, $q \ge 1$. The hesitancy degree is given by $\pi_{\tilde{\mathcal{P}}}(x) = \sqrt[q]{1 - \left(\mu_{\tilde{\mathcal{P}}}(x)\right)^q - \left(\nu_{\tilde{\mathcal{P}}}(x)\right)^q}$.

For convenience, $\left(\mu_{\tilde{\mathcal{P}}}(x), \nu_{\tilde{\mathcal{P}}}(x)\right)$ is known as a q-ROFN (Yager, 2017) and is symbolized by $\tilde{\mathfrak{p}} = (\mu, \nu)$. For comparing q-ROFNs, Liu and Wang (2018) and Wei et al. (2018) introduced score and accuracy functions in the following manners.

Definition 2.2. (Liu & Wang, 2018). For any q-ROFN $\tilde{\mathfrak{p}} = (\mu, \nu)$, score function of $\tilde{\mathfrak{p}}$ is described as $S(\tilde{\mathfrak{p}}) = \frac{1}{2}(1 + \mu^q - \nu^q)$. The accuracy function of $\tilde{\mathfrak{p}}$ is described as $A(\tilde{\mathfrak{p}}) = \mu^q + \nu^q$.

Liu and Wang (2018) suggested an ordering technique of q-ROFNs as follows.

Definition 2.3. (Liu & Wang, 2018). For any two q-ROFNs $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$, the ranking of $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$ is done by the following rules.

(i) If $S(\tilde{\mathfrak{p}}_1) > S(\tilde{\mathfrak{p}}_2)$, then $\tilde{\mathfrak{p}}_1 \succ \tilde{\mathfrak{p}}_2$;

(ii) If $S(\tilde{\mathfrak{p}}_1) = S(\tilde{\mathfrak{p}}_2)$, then

- If $A(\tilde{\mathfrak{p}}_1) > A(\tilde{\mathfrak{p}}_2)$, then $\tilde{\mathfrak{p}}_1 \succ \tilde{\mathfrak{p}}_2$;
- If $A(\tilde{\mathfrak{p}}_1) = A(\tilde{\mathfrak{p}}_2)$, then $\tilde{\mathfrak{p}}_1 \approx \tilde{\mathfrak{p}}_2$.

Based on q-ROFS (Liu & Wang, 2018; Yager, 2017) and DHFS (Zhu et al., 2012), Xu et al. (2018) introduced the idea and basic operational rules on DHq-ROFS.

Definition 2.4. (Xu et al., 2018). Let $\mathcal{X}$ be a universal set. A DHq-ROFS $\tilde{\mathcal{K}}$ on $\mathcal{X}$ is described as:

$\tilde{\mathcal{K}} = \left(\left\langle x, \tilde{h}_{\tilde{\mathcal{K}}}(x), \tilde{\mathfrak{g}}_{\tilde{\mathcal{K}}}(x) \right\rangle \mid x \in \mathcal{X}\right)$,

where $\tilde{h}_{\tilde{\mathcal{K}}}(x) = \bigcup_{\gamma \in \tilde{h}_{\tilde{\mathcal{K}}}(x)} \{\gamma\}$ and $\tilde{\mathfrak{g}}_{\tilde{\mathcal{K}}}(x) = \bigcup_{\eta \in \tilde{\mathfrak{g}}_{\tilde{\mathcal{K}}}(x)} \{\eta\}$ are two discrete sets of real numbers in $[0,1]$, denoting the possible membership and non-membership degrees of an element $x \in \mathcal{X}$ to the set $\tilde{\mathcal{K}}$ satisfying the conditions:

$0 \le \gamma, \eta \le 1$ and $0 \le \left(\max_{\gamma \in \tilde{h}_{\tilde{\mathcal{K}}}(x)} \{\gamma\}\right)^q + \left(\max_{\eta \in \tilde{\mathfrak{g}}_{\tilde{\mathcal{K}}}(x)} \{\eta\}\right)^q \le 1$.

For convenience, pair $\tilde{\mathcal{K}} = \left\langle \tilde{h}_{\tilde{\mathcal{K}}}(x), \tilde{\mathfrak{g}}_{\tilde{\mathcal{K}}}(x) \right\rangle$ is known as a DHq-ROF number (DHq-ROFN) which is denoted by $\tilde{\kappa} = \left\langle \tilde{h}, \tilde{\mathfrak{g}} \right\rangle$.

Definition 2.5. (Xu et al., 2018). The score function, $S(\tilde{\kappa})$ of a DHq-ROFN $\tilde{\kappa} = \left\langle \tilde{h}, \tilde{\mathfrak{g}} \right\rangle$, is defined by

$$S(\tilde{\kappa}) = \frac{1}{2}\left(1 + \frac{1}{l_{\tilde{h}}} \sum_{\gamma \in \tilde{h}} \gamma^q - \frac{1}{l_{\tilde{h}}} \sum_{\eta \in \tilde{\mathfrak{g}}} \eta^q\right), \tag{1}$$

and accuracy function of $\tilde{\kappa}$, indicated by $A(\tilde{\kappa})$, is given by

$$A(\tilde{\kappa}) = \frac{1}{l_{\tilde{h}}} \sum_{\gamma \in \tilde{h}} \gamma^q + \frac{1}{l_{\tilde{h}}} \sum_{\eta \in \tilde{\mathfrak{g}}} \eta^q, \tag{2}$$

where $l_{\tilde{h}}$ and $l_{\tilde{\mathfrak{g}}}$ respectively, denoting the number of elements in $\tilde{h}$ and $\tilde{\mathfrak{g}}$.

The following is the procedure for ordering DHq-ROFNs:

Let $\tilde{\kappa}_1 = \left(\tilde{h}_1, \tilde{\mathfrak{g}}_1\right)$ and $\tilde{\kappa}_2 = \left(\tilde{h}_2, \tilde{\mathfrak{g}}_2\right)$ be any two DHq-ROFNs,

(i) If $S(\tilde{\kappa}_1) > S(\tilde{\kappa}_2)$, then $\tilde{\kappa}_1$ is superior to $\tilde{\kappa}_2$, denoted by $\tilde{\kappa}_1 \succ \tilde{\kappa}_2$;

(ii) If $S(\tilde{\kappa}_1) = S(\tilde{\kappa}_2)$, then

- If $A(\tilde{\kappa}_1) > A(\tilde{\kappa}_2)$, then $\tilde{\kappa}_1 \succ \tilde{\kappa}_2$;
- If $A(\tilde{\kappa}_1) = A(\tilde{\kappa}_2)$, then $\tilde{\kappa}_1$ is equivalent to $\tilde{\kappa}_2$, denoted by $\tilde{\kappa}_1 \approx \tilde{\kappa}_2$.

Definition 2.6. (Xu et al., 2018). Let $\tilde{\kappa}_1 = \left(\tilde{h}_1, \tilde{\mathfrak{g}}_1\right)$, $\tilde{\kappa}_2 = \left(\tilde{h}_2, \tilde{\mathfrak{g}}_2\right)$ and $\tilde{\kappa} = \left(\tilde{h}, \tilde{\mathfrak{g}}\right)$ be any three DHq-ROFNs and $\lambda > 0$. Then,

(1) $\tilde{\kappa}_1 \oplus \tilde{\kappa}_2 = \left\langle \bigcup_{\substack{\gamma_i \in \tilde{h}_i \\ i=1,2}} \left\{(\gamma_1{}^q + \gamma_2{}^q - \gamma_1{}^q \gamma_2{}^q)^{\frac{1}{q}}\right\}, \bigcup_{\substack{\eta_i \in \tilde{k}_i \\ i=1,2}} \{\eta_1, \eta_2\} \right\rangle$,

(2) $\tilde{\kappa}_1 \otimes \tilde{\kappa}_2 = \left\langle \bigcup_{\substack{\gamma_i \in \tilde{h}_i \\ i=1,2}} \{\gamma_1, \gamma_2\}, \bigcup_{\substack{\eta_i \in \tilde{k}_i \\ i=1,2}} \left\{(\eta_1{}^q + \eta_2{}^q - \eta_1{}^q \eta_2{}^q)^{\frac{1}{q}}\right\} \right\rangle$,

(3) $\lambda\tilde{\kappa} = \left\langle \bigcup_{\gamma \in \tilde{h}} \left\{\left(1 - (1-\gamma^q)^\lambda\right)^{\frac{1}{q}}\right\}, \bigcup_{\eta \in \tilde{k}} \{\eta^\lambda\} \right\rangle$,

(4) $\tilde{\kappa}^\lambda = \left\langle \bigcup_{\gamma \in \tilde{h}} \{\gamma^\lambda\}, \bigcup_{\eta \in \tilde{k}} \left\{\left(1 - (1-\eta^q)^\lambda\right)^{\frac{1}{q}}\right\} \right\rangle$.

Definition 2.7. (Schweizer & Sklar, 1983). Assume α and β are any two real numbers in $[0,1]$. Then, SSt-CN, $U_{SS}(\alpha,\beta)$ and SSt-N, $I_{SS}(\alpha,\beta)$ are defined with the subsequent expressions

$U_{SS}(\alpha,\beta)=(\alpha^\tau+\beta^\tau-\alpha^\tau\beta^\tau)^{\frac{1}{\tau}}$,
and $I_{SS}(\alpha,\beta)=1-((1-\alpha)^\tau+(1-\beta)^\tau-(1-\alpha)^\tau(1-\beta)^\tau)^{\frac{1}{\tau}}$,
where $(\alpha,\beta)\in[0,1]\times[0,1]$, and $\tau>0$.

3 | DEVELOPMENT OF SS*t*-CNS&*t*-NS-BASED DH*q*-ROF HYBRID AGGREGATION OPERATORS

In this section, weighted average, ordered weighted average, weighted geometric, ordered weighted geometric, hybrid average and hybrid geometric aggregation operators based on SS*t*-CN&*t*-Ns are developed to aggregate DH*q*-ROFNs.

3.1 | Basic operations on DH*q*-ROFNs based on SS*t*-CNs&*t*-Ns

According to SS*t*-CN&*t*-Ns, the mathematical operations of DH*q*-ROFNs are defined as follows.

Suppose $\tilde{\kappa}_1=\left(\tilde{h}_1,\tilde{g}_1\right)$, $\tilde{\kappa}_2=\left(\tilde{h}_2,\tilde{g}_2\right)$ and $\tilde{\kappa}=\left(\tilde{h},\tilde{g}\right)$ be any three DH*q*-ROFNs and $\tau>0$ be a Schweizer & Sklar parameter, then the operational rules, viz., addition "$\oplus_{SS}$", multiplication "$\otimes_{SS}$", scalar multiplication and exponent are defined as follows:

(1) $\tilde{\kappa}_1\oplus_{SS}\tilde{\kappa}_2=$

$$\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\ i=1,2}}\left\{(\gamma_1{}^{\tau q}+\gamma_2{}^{\tau q}-\gamma_1{}^{\tau q}\gamma_2{}^{\tau q})^{\frac{1}{\tau q}}\right\}\bigcup_{\substack{\eta\in\tilde{g}_i\\ i=1,2}}\left\{\left(1-((1-\eta_1{}^q)^\tau+(1-\eta_2{}^q)^\tau-(1-\eta_1{}^q)^\tau(1-\eta_2{}^q)^\tau)^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\}\right\rangle;$$

$$(2)\tilde{\kappa}_1\otimes_{SS}\tilde{\kappa}_2=\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\ i=1,2}}\left\{\left(1-((1-\gamma_1{}^q)^\tau+(1-\gamma_2{}^q)^\tau-(1-\gamma_1{}^q)^\tau(1-\gamma_2{}^q)^\tau)^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\},\bigcup_{\substack{\eta_i\in\tilde{g}_i\\ i=1,2}}\left\{(\eta_1{}^{\tau q}+\eta_2{}^{\tau q}-\eta_1{}^{\tau q}\eta_2{}^{\tau q})^{\frac{1}{\tau q}}\right\}\right\rangle;$$

$$(3)\lambda\tilde{\kappa}=\left\langle\bigcup_{\gamma\in\tilde{h}}\left\{\left(1-(1-\gamma^{q\tau})^\lambda\right)^{\frac{1}{q\tau}}\right\},\bigcup_{\eta\in\tilde{g}}\left\{\left(1-\left(1-(1-(1-\eta^q)^\tau)^\lambda\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\}\right\rangle,\lambda>0;$$

$$(4)\tilde{\kappa}^\lambda=\left\langle\bigcup_{\gamma\in\tilde{h}}\left\{\left(1-\left(1-(1-(1-\gamma^q)^\tau)^\lambda\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\},\bigcup_{\vartheta\in\tilde{g}}\left\{\left(1-(1-\vartheta^{q\tau})^\lambda\right)^{\frac{1}{q\tau}}\right\}\right\rangle,\lambda>0.$$

The following theorem illustrates different characteristics of the above operations.

Theorem 3.1. *Let $\tilde{\kappa}_1=\left(\tilde{h}_1,\tilde{g}_1\right)$ and $\tilde{\kappa}_2=\left(\tilde{h}_2,\tilde{g}_2\right)$ be any two DHq-ROFNs and $\tau>0$. Then the following properties hold.*

1. $\tilde{\kappa}_1\oplus_{SS}\tilde{\kappa}_2=\tilde{\kappa}_2\oplus_{SS}\tilde{\kappa}_1$
2. $\tilde{\kappa}_1\otimes_{SS}\tilde{\kappa}_2=\tilde{\kappa}_2\otimes_{SS}\tilde{\kappa}_1$
3. $n(\tilde{\kappa}_1\oplus_{SS}\tilde{\kappa}_2)=n\tilde{\kappa}_1\oplus_{SS}n\tilde{\kappa}_2$, $n\geq0$.
4. $n_1\tilde{\kappa}_1\oplus_{SS}n_2\tilde{\kappa}_1=(n_1+n_2)\tilde{\kappa}_1$, $n_1,n_2\geq0$.
5. $\tilde{\kappa}_1^n\otimes_{SS}\tilde{\kappa}_2^n=(\tilde{\kappa}_1\otimes_{SS}\tilde{\kappa}_2)^n$, $n\geq0$.
6. $\tilde{\kappa}_1^{n_1}\otimes_{SS}\tilde{\kappa}_1^{n_2}=\tilde{\kappa}_1{}^{n_1+n_2}$.

Proof. Obvious.

3.2 | Development of aggregation operators using SS*t*-CN&*t*-N based operations on DH*q*-ROFNs

Now, based on the above-defined operations on DH*q*-ROFNs, several aggregation operators, viz., DH*q*-ROFSSWA, DH*q*-ROFSSOWA, DH*q*-ROFSSWG, DH*q*-ROFSSOWG, DH*q*-ROFSSHA and DH*q*-ROFSSHG are developed in the subsection.

3.2.1 | DH*q*-ROFSSWA operator

Definition 3.1. Let $\tilde{\kappa}_i=\left\langle\tilde{h}_i,\tilde{g}_i\right\rangle$ $(i=1,2,...,n)$ be a set of n DH*q*-ROFNs. If $DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,...,\tilde{\kappa}_n)=(\oplus_{SS})_{i=1}^n(\omega_i\tilde{\kappa}_i)$. Then $DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,...,\tilde{\kappa}_n)$ is called DH*q*-ROFSSWA operator, where $\omega=(\omega_1,\omega_2,...,\omega_n)$ be the weighted vector of $\tilde{\kappa}_i$ $(i=1,2,...,n)$, $\omega_i>0$, and $\sum_{i=1}^n\omega_i=1$.

The properties of the developed DHq-ROFSSWA operator are described as follows.

Theorem 3.2. *Let $\tilde{\kappa}_i = \langle \tilde{h}_i, \tilde{g}_i \rangle$ $(i=1,2,\ldots,n)$ be a set of n DHq-ROFNs and $\tau > 0$, then the aggregated value using DHq-ROFSSWA is also a DHq-ROFN and can be given as*

$$DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n) = \oplus_{SS_{i=1}^{n}}(\omega_i\tilde{\kappa}_i)$$

$$= \left\langle \bigcup_{\substack{\gamma_i \in \tilde{h}_i \\ (i=1,2,\ldots,n)}} \left\{ \left(1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i}\right)^{\frac{1}{\tau q}} \right\}, \bigcup_{\substack{\eta_i \in \tilde{g}_i \\ (i=1,2,\ldots,n)}} \left\{ \left(1-\left(1-\prod_{i=1}^{n}(1-(1-\eta_i^{q})^{\tau})^{\omega_i}\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\} \right\rangle, \tag{3}$$

where $\omega = (\omega_1,\omega_2,\ldots,\omega_n)$ is the weighted vector of $\tilde{\kappa}_i$ $(i=1,2,\ldots,n)$ and $\omega_i > 0$, $\sum_{i=1}^{n}\omega_i = 1$.

Proof. This Theorem is executed using the mathematical induction method.

$$\omega_1\tilde{\kappa}_1 = \left\langle \bigcup_{\gamma_1 \in \tilde{h}_1} \left\{ (1-(1-\gamma_1^{q\tau})^{\omega_1})^{\frac{1}{q\tau}} \right\}, \bigcup_{\eta_1 \in \tilde{g}_1} \left\{ \left(1-(1-(1-(1-\eta_1^{q})^{\tau})^{\omega_1})^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\} \right\rangle,$$

$$\omega_2\tilde{\kappa}_2 = \left\langle \bigcup_{\gamma \in \tilde{h}_2} \left\{ (1-(1-\gamma_2^{q\tau})^{\omega_2})^{\frac{1}{q\tau}} \right\}, \bigcup_{\eta_2 \in \tilde{g}_2} \left\{ \left(1-(1-(1-(1-\eta_2^{q})^{\tau})^{\omega_2})^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\} \right\rangle;$$

Now,

$$\oplus_{SS_{i=1}^{2}}(\omega_i\tilde{\kappa}_i) = \left\langle \cup_{\substack{\gamma_1 \in \tilde{h}_1, \\ \gamma_2 \in \tilde{h}_2}} \{((1-(1-\gamma_1^{q\tau})^{\omega_1}) + (1-(1-\gamma_2^{q\tau})^{\omega_2}) - (1-(1-\gamma_1^{q\tau})^{\omega_1})(1-(1-\gamma_2^{q\tau}) \right.$$

$$\left.))^{\frac{1}{q\tau}}\}, \cup_{\substack{\eta_1 \in \tilde{g}_1, \\ \eta_2 \in \tilde{g}_2}} \left\{ \left(1-((1-(1-(1-\eta_1^{q})^{\tau})^{\omega_1}) + (1-(1-(1-\eta_2^{q})^{\tau})^{\omega_2}) - ((1-(1-(1-\eta_1^{q})^{\tau})^{\omega_1})(1-(1-(1-\eta_2^{q})^{\tau})^{\omega_2})))^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\} \right\rangle$$

$$= \left\langle \cup_{\substack{\gamma_1 \in \tilde{h}_1, \\ \gamma_2 \in \tilde{h}_2}} \{1-(1-\gamma_1^{q\tau})^{\omega_1}(1-\gamma_2^{q\tau})^{\omega_2}\}, \cup_{\substack{\eta_1 \in \tilde{g}_1, \\ \eta_2 \in \tilde{g}_2}} \left\{ \left(1-(1-(1-(1-\eta_1^{q})^{\tau})^{\omega_1}(1-(1-\eta_2^{q})^{\tau})^{\omega_2})^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\} \right\rangle$$

$$= \left\langle \bigcup_{\substack{\gamma_1 \in \tilde{h}_1, \\ \gamma_2 \in \tilde{h}_2}} \left\{ \left(1-\prod_{i=1}^{2}(1-\gamma_i^{q\tau})^{\omega_i}\right)^{\frac{1}{\tau q}} \right\}, \bigcup_{\substack{\eta_1 \in \tilde{g}_1, \\ \eta_2 \in \tilde{g}_2}} \left\{ \left(1-\left(1-\prod_{i=1}^{2}(1-(1-\eta_i^{q})^{\tau})^{\omega_i}\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\} \right\rangle.$$

Therefore, it is valid for $n=2$.
Now, let it is valid for $n=v$,
i.e., $DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_v) = \oplus_{SS_{i=1}^{v}}(\omega_i\tilde{\kappa}_i)$

$$= \left\langle \bigcup_{\substack{\gamma_i \in \tilde{h}_i \\ (i=1,\ldots,v)}} \left\{ (1-\prod_{i=1}^{v}(1-\gamma_i^{q\tau})^{\omega_i})^{\frac{1}{\tau q}} \right\}, \bigcup_{\substack{\eta_i \in \tilde{g}_i \\ i=1,\ldots,v}} \left\{ \left(1-(1-\prod_{i=1}^{v}(1-(1-\eta_i^{q})^{\tau})^{\omega_i})^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\} \right\rangle.$$

Then, $DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_v,\tilde{\kappa}_{v+1}) = \oplus_{SS_{i=1}^{v}}(\omega_i\tilde{\kappa}_i)\oplus_{SS}(\omega_{v+1}\tilde{\kappa}_{v+1})$

$$= \left\langle \bigcup_{\substack{\gamma_i \in \tilde{h}_i \\ (i=1,2,\ldots,v+1)}} \left\{ \left(1-\prod_{i=1}^{v+1}(1-\gamma_i^{q\tau})^{\omega_i}\right)^{\frac{1}{\tau q}} \right\}, \bigcup_{\substack{\eta_i \in \tilde{g}_i \\ (i=1,2,\ldots,v+1)}} \left\{ \left(1-\left(1-\prod_{i=1}^{v+1}(1-(1-\eta_i^{q})^{\tau})^{\omega_i}\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\} \right\rangle;$$

Therefore, it is valid for $n=v+1$, and hence true for all n.
Now, it is to be shown that the aggregated value represents a DHq-ROFN.

i.e., it is to be proved that

$$0 \le \left(\left(1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i}\right)^{\frac{1}{\tau}}\right)+\left(1-\left(1-\prod_{i=1}^{n}(1-(1-\eta_i^{q})^{\tau})^{\omega_i}\right)^{\frac{1}{\tau}}\right)\le 1,$$

now $0\le\gamma_i^q+\eta_i^q\le1$ So, $\gamma_i^q\le1-\eta_i^q$ for all $i=(1,2,\ldots,n)$,

$\Leftrightarrow (1-(1-\eta_i^q)^{\tau})^{\omega_i}\le(1-\gamma_i^{q\tau})^{\omega_i}$

$\Leftrightarrow 1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i}\le 1-\prod_{i=1}^{n}(1-(1-\eta_i^q)^{\tau})^{\omega_i}$

$\Leftrightarrow (1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i})^{\frac{1}{\tau}}-(1-\prod_{i=1}^{n}(1-(1-\eta_i^q)^{\tau})^{\omega_i})^{\frac{1}{\tau}}\le 0$

$\Leftrightarrow \left((1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i})^{\frac{1}{\tau}}\right)+\left(1-(1-\prod_{i=1}^{n}(1-(1-\eta_i^q)^{\tau})^{\omega_i})^{\frac{1}{\tau}}\right)\le 1$

Also, $\left((1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i})^{\frac{1}{\tau}}\right)\ge 0$, and $\left(1-(1-\prod_{i=1}^{n}(1-(1-\eta_i^q)^{\tau})^{\omega_i})^{\frac{1}{\tau}}\right)\ge 0$,

So, $\left((1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i})^{\frac{1}{\tau}}\right)+\left(1-(1-\prod_{i=1}^{n}(1-(1-\eta_i^q)^{\tau})^{\omega_i})^{\frac{1}{\tau}}\right)\ge 0.$

Thus, $0\le\left((1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i})^{\frac{1}{\tau}}\right)+\left(1-(1-\prod_{i=1}^{n}(1-(1-\eta_i^q)^{\tau})^{\omega_i})^{\frac{1}{\tau}}\right)\le 1.$

So, this is a DH*q*-ROFN.

This concludes the theorem's proof.

Theorem 3.3. (Idempotency) *Let* $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)\mid i=1,2,\ldots,n\right\}$ *be a collection of DHq-ROFNs. If* $\tilde{\kappa}_i=\tilde{\kappa}=\left(\tilde{h},\tilde{g}\right)$ $\forall i$, *then* $DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\tilde{\kappa}$.

Proof. $DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\oplus_{SS_{i=1}^{n}}\omega_i\tilde{\kappa}_i$

$$=\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\(i=1,2,\ldots,n)}}\left\{\left(1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i}\right)^{\frac{1}{\tau q}}\right\},\bigcup_{\substack{\eta_i\in\tilde{g}_i\\(i=1,2,\ldots,n)}}\left\{\left(1-\left(1-\prod_{i=1}^{n}(1-(1-\eta_i^q)^{\tau})^{\omega_i}\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\}\right\rangle$$

Since $\tilde{\kappa}_i=\tilde{\kappa}=\left(\tilde{h},\tilde{g}\right)$ $\forall i=1,2,\ldots,n$,

$$DHq-ROFSSWA(\tilde{\kappa},\tilde{\kappa},\ldots,\tilde{\kappa})=\oplus_{SS_{i=1}^{n}}(\omega_i\tilde{\kappa})$$

$$=\left\langle\bigcup_{\gamma\in\tilde{h}}\left\{\left(1-\prod_{i=1}^{n}(1-\gamma^{q\tau})^{\omega_i}\right)^{\frac{1}{\tau q}}\right\},\bigcup_{\eta\in\tilde{g}}\left\{\left(1-\left(1-\prod_{i=1}^{n}(1-(1-\eta^q)^{\tau})^{\omega_i}\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\}\right\rangle$$

$=\left\langle\tilde{h},\tilde{g}\right\rangle=\tilde{\kappa}$

Theorem 3.4. (Monotonicity) *Let* $\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)$ *and* $\tilde{\kappa}'_i=\left(\tilde{h}'_i,\tilde{g}'_i\right)$ $(i=1,2,\ldots,n)$ *be two collections of DHq-ROFNs and if* $\gamma_i\le\gamma'_i$ *and* $\vartheta_i\ge\vartheta'_i$ *for all i, then*

$$DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)\le DHq-ROFSSWA\left(\tilde{\kappa}'_1,\tilde{\kappa}'_2,\ldots,\tilde{\kappa}'_n\right)$$

Proof. *Here,* $DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$

$$=\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\(i=1,2,\ldots,n)}}\left\{(1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i})^{\frac{1}{\tau q}}\right\},\bigcup_{\substack{\eta_i\in\tilde{g}_i\\(i=1,2,\ldots,n)}}\left\{\left(1-(1-\prod_{i=1}^{n}(1-(1-\eta_i^q)^{\tau})^{\omega_i})^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\}\right\rangle,$$

and

$$-ROFSSWA\left(\tilde{\kappa}'_1,\tilde{\kappa}'_2,\ldots,\tilde{\kappa}'_n\right)=\left\langle\bigcup_{\substack{\gamma'_i\in\tilde{h}'_i\\(i=1,2,\ldots,n)}}\left\{(1-\prod_{i=1}^{n}(1-\gamma_i'^{q\tau})^{\omega_i})^{\frac{1}{\tau q}}\right\},\bigcup_{\substack{\eta'_i\in\tilde{g}'_i\\(i=1,2,\ldots,n)}}\left\{\left(1-(1-\prod_{i=1}^{n}(1-(1-\eta_i'^{q})^{\tau})^{\omega_i})^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\}\right\rangle.$$

Now, for $i=1,2,\ldots,n$,

$\gamma_i\le\gamma'_i$

$\Leftrightarrow (1-\gamma_i'^{q\tau})^{\omega_i}\le(1-\gamma_i^{q\tau})^{\omega_i}$,

$\Leftrightarrow \prod_{i=1}^{n}(1-\gamma_i'^{q\tau})^{\omega_i}\le\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i}$

Hence, $(1-\prod_{i=1}^{n}(1-\gamma_i^{q\tau})^{\omega_i})^{\frac{1}{\tau q}}\le(1-\prod_{i=1}^{n}(1-\gamma_i'^{q\tau})^{\omega_i})^{\frac{1}{\tau q}}$.

Similarly, it can be shown that if $\eta_i\ge\eta'_i$ *then*

$\left(1-(1-\prod_{i=1}^{n}(1-(1-\eta_i^q)^{\tau})^{\omega_i})^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\ge\left(1-(1-\prod_{i=1}^{n}(1-(1-\eta_i'^{q})^{\tau})^{\omega_i})^{\frac{1}{\tau}}\right)^{\frac{1}{q}}$.

Then by Definition 2.5.

$DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n) \le DHq-ROFSSWA(\tilde{\kappa}'_1,\tilde{\kappa}'_2,\ldots,\tilde{\kappa}'_n)$

Theorem 3.5. *(Boundedness) Let* $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)\mid i=1,2,\ldots,n\right\}$ *be a set of DHq-ROFNs. If* $\tilde{\kappa}^+=\cup_{\gamma_i\in\tilde{h}_i,\eta_i\in\tilde{g}_i}\left\{\left\{\max_j\gamma_j\right\},\left\{\min_j\eta_j\right\}\right\}$ *and* $\tilde{\kappa}^-=\cup_{\gamma_i\in\tilde{h}_i,\eta_i\in\tilde{g}_i}\left\{\left\{\min_j\gamma_j\right\},\left\{\max_j\eta_j\right\}\right\}$ *then*

$\tilde{\kappa}^- \le DHq-ROFSSWA(\kappa_1,\kappa_2,\ldots,\kappa_n)\le\tilde{\kappa}^+$.

Proof. *Since* $\min_j\gamma_j\le\gamma_j\le\max_j\gamma_j$ *and* $\min_j\eta_j\le\eta_j\le\max_j\eta_j$ *then*

$\tilde{\kappa}^-\le\tilde{\kappa}_i$ *for* $i=1,2,\ldots,n$.

Then by monotonicity, $DHq-ROFSSWA(\tilde{\kappa}^-,\tilde{\kappa}^-,\ldots,\tilde{\kappa}^-)\le DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$.

By idempotency, $\tilde{\kappa}^-\le DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$.

Similarly, $DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)\le\tilde{\kappa}^+$.

So, $\tilde{\kappa}^-\le DHq-ROFSSWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)\le\tilde{\kappa}^+$.

Note 1. If $\tau=1$ then DH*q*-ROFSSWA operator reduces to DH*q*-ROF weighted average (DH*q*-ROFWA (Wang, Wei, Wang, et al., 2019)) as

$$DHq-ROFWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\(i=1,2,\ldots,n)}}\left\{\left(1-\prod_{i=1}^n\left(1-\gamma_i{}^q\right)^{\omega_i}\right)^{\frac{1}{q}}\right\},\bigcup_{\substack{\eta_i\in\tilde{g}_i\\(i=1,2,\ldots,n)}}\left\{\prod_{i=1}^n\eta_i{}^{\omega_i}\right\}\right\rangle.$$

3.2.2 | DH*q*-ROFSSOWA operator

In this subsection *DHq – ROFSSOWA* operator is developed.

Definition 3.2. Let $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)\mid i=1,2,\ldots,n\right\}$ be a set of DH*q*-ROFNs. If $DHq-ROFSSOWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\oplus_{SS_{i=1}^n}\left(\omega_i\tilde{\kappa}_{\sigma(i)}\right)$, then $DHq-ROFSSOWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$ is called DH*q*-ROFSSOWA operator, where σ is a permutation on $\{1,2,\ldots,n\}$ in such a way that $\tilde{\kappa}_{\sigma(i-1)}\ge\tilde{\kappa}_{\sigma(i)}$ $\forall i=2,3,\ldots,n$. Here, $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ be the weighted vector corresponding to the DH*q*-ROFNs such that $\omega_i>0$, and $\sum_{i=1}^n\omega_i=1$.

Theorem 3.6. *Let* $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)\mid i=1,2,\ldots,n\right\}$ *be a set of DHq-ROFNs and* $\tau>0$, *then the aggregated value using DHq-ROFSSOWA is also a DHq-ROFN and can be presented as follows:*

$$DHq-ROFSSOWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\oplus_{SS_{i=1}^n}\left(\omega_i\tilde{\kappa}_{\sigma(i)}\right)$$

$$=\left\langle\bigcup_{\substack{\gamma_{\sigma(i)}\in\tilde{h}_{\sigma(i)}\\(i=1,2,\ldots,n)}}\left\{\left(1-\prod_{i=1}^n\left(1-\gamma_{\sigma(i)}{}^{q\tau}\right)^{\omega_i}\right)^{\frac{1}{\tau q}}\right\},\right.$$

$$\left.\bigcup_{\substack{\eta_{\sigma(i)}\in\tilde{g}_{\sigma(i)}\\(i=1,2,\ldots,n)}}\left\{\left(1-\left(1-\prod_{i=1}^n\left(1-\left(1-\eta_{\sigma(i)}{}^q\right)^{\tau}\right)^{\omega_i}\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\}\right\rangle \tag{4}$$

where σ *is a permutation on* $\{1,2,\ldots,n\}$ *in such a way that* $\tilde{\kappa}_{\sigma(i-1)}\ge\tilde{\kappa}_{\sigma(i)}$ $\forall i=2,3,\ldots,n$, *and* $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ *is the associated weight vector such that* $\omega_i>0$ *and* $\sum_{i=1}^n\omega_i=1$.

Proof. This proof is analogous to Theorem 3.2.

Note 2. The idempotency, monotonicity, and boundedness properties are also satisfied for the DH*q*-ROFSSOWA operator, which can be proved similarly.

Note 3. If $\tau=1$ then DH*q*-ROFSSOWA operator is reduced to DH*q*-ROF weighted ordered average (DH*q*-ROFOWA (Wang, Wei, Wang, et al., 2019)) operator.

3.2.3 | DH*q*-ROFSSWG operator

In this subsection $DHq-ROFSSWG$ operator is developed.

Definition 3.3. Let $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right) \mid i=1,2,\ldots,n\right\}$ be a set of DH*q*-ROFNs. If

$$DHq-ROFSSWG(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\otimes_{SS}{}_{i=1}^{n}(\tilde{\kappa}_i)^{\omega_i},$$

then $DHq-ROFSSWG(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$ is called DH*q*-ROFSSWG operator, where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ be the weighted vector of $(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$ and $\omega_i>0$, and $\sum_{i=1}^{n}\omega_i=1$.

Theorem 3.7. *Let* $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right) \mid i=1,2,\ldots,n\right\}$ *be a set of DHq-ROFNs and* $\tau>0$, *then aggregated value using DHq-ROFSSWG operator is also a DHq-ROFN and can be presented as follows:*

$$DHq-ROFSSWG(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\otimes_{SS}{}_{i=1}^{n}(\tilde{\kappa}_i)^{\omega_i}$$

$$=\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\(i=1,2,\ldots,n)}}\left\{\left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\gamma_i{}^{q}\right)^{\tau}\right)^{\omega_i}\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}}\right\},\bigcup_{\substack{\eta_i\in\tilde{g}_i\\(i=1,2,\ldots,n)}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\eta_i{}^{q\tau}\right)^{\omega_i}\right)^{\frac{1}{\tau q}}\right\}\right\rangle, \tag{5}$$

where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ *be the weighted vector of* $\tilde{\kappa}_i$ $(i=1,2,\ldots,n)$ *and* $\omega_i>0$, *and* $\sum_{i=1}^{n}\omega_i=1$.

Proof. This proof is analogous to Theorem 3.2.

Note 4. The idempotency, monotonicity, and boundedness properties are also satisfied for the DH*q*-ROFSSWG operator, which can be proved in a similar way.

Note 5. If $\tau=1$ then DH*q*-ROFSSWG operator is reduced to DH*q*-ROF weighted geometric (DH*q*-ROFWG (Wang, Wei, Wang, et al., 2019)) operator.

3.2.4 | DH*q*-ROFSSOWG operator

In this subsection $DHq-ROFSSOWG$ operator is developed.

Definition 3.4. Let $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right) \mid i=1,2,\ldots,n\right\}$ be a set of DH*q*-ROFNs. If

$DHq-ROFSSOWG(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\otimes_{SS}{}_{i=1}^{n}\left(\tilde{\kappa}_{\sigma(i)}\right)^{\omega_i}$,

then $DHq-ROFSSOWG(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$ is called DH*q*-ROFSSOWG operator, where σ is a permutation on $\{1,2,\ldots n\}$ such that $\tilde{\kappa}_{\sigma(i-1)}\geq\tilde{\kappa}_{\sigma(i)}$ $\forall i=2,3,\ldots,n$, and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ be the associate weighted vector such that $\omega_i>0$, and $\sum_{i=1}^{n}\omega_i=1$.

Theorem 3.8. *Let* $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right) \mid i=1,2,\ldots,n\right\}$ *be a set of DHq-ROFNs and* $\tau>0$, *then the aggregated value using DHq-ROFSSOWG is also a DHq-ROFN and can be presented as follows:*

$$DHq-ROFSSOWG(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\otimes_{SS}{}_{i=1}^{n}\left(\tilde{\kappa}_{\sigma(i)}\right)^{\omega_i}$$

$$= \left\langle \bigcup_{\substack{\gamma_{\sigma(i)} \in \tilde{h}_{\sigma(i)} \\ (i=1,2,\ldots,n)}} \left\{ \left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\gamma_{\sigma(i)}{}^{q}\right)^{\tau}\right)^{\omega_i}\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\}, \bigcup_{\substack{\eta_{\sigma(i)} \in \tilde{g}_{\sigma(i)} \\ (i=1,2,\ldots,n)}} \left\{ \left(1-\prod_{i=1}^{n}\left(1-\eta_{\sigma(i)}{}^{q\tau}\right)^{\omega_i}\right)^{\frac{1}{\tau q}} \right\} \right\rangle,$$

where σ is a permutation on $\{1,2,\ldots,n\}$ such that $\tilde{\kappa}_{\sigma(i-1)} \geq \tilde{\kappa}_{\sigma(i)}$ $\forall i=2,3,\ldots,n$, and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ be weighted vector such that $\omega_i>0$, and $\sum_{i=1}^{n}\omega_i=1$.

Proof. This proof is analogous to Theorem 3.2.

Note 6. The idempotency, monotonicity, and boundness properties are also satisfied for the DH*q*-ROFSSOWG operator, which can be proved in a similar manner.

Note 7. If $\tau=1$ then the DH*q*-ROFSSOWG operator is reduced to DH*q*-ROF ordered weighted geometric (DH*q*-ROFOWG (Wang, Wei, Wang, et al., 2019)) operator.

3.2.5 | DH*q*-ROFSSHA operator

In this subsection $DHq-ROFSSHA$ operator is developed.

Definition 3.5. Let $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right) \mid i=1,2,\ldots,n\right\}$ be a set of DH*q*-ROFNs. If

$$DHq-ROFSSHA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\oplus_{SS}{}_{i=1}^{n}\left(\omega_i\tilde{\dot{\kappa}}_{\sigma(i)}\right),$$

Then $DHq-ROFSSHA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$ is called DH*q*-ROFSSHA operator, where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ be associated weight vector such that $\omega_i>0$, and $\sum_{i=1}^{n}\omega_i=1$. $\tilde{\dot{\kappa}}_{\sigma(i)}$ is the i^{th} largest elements of the DH*q*-ROF arguments $\tilde{\dot{\kappa}}_i$ $\left(\tilde{\dot{\kappa}}_i=n\Omega_i\tilde{\kappa}_i=\left(n\Omega_i\tilde{h}_i,n\Omega_i\tilde{g}_i\right),\ i=1,2,\ldots,n\right)$, $\Omega=(\Omega_1,\Omega_2,\ldots,\Omega_n)$ is the weighting vector of DH*q*-ROF arguments $\tilde{\kappa}_i$ with $\Omega_i>0$, and $\sum_{i=1}^{n}\Omega_i=1$ and n is the balancing co-efficient.

Moreover, if $\omega=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$ (or $\Omega=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$) then DH*q*-ROFSSHA operator is reduced to DH*q*-ROFSSWA (or DH*q*-ROFSSOWA) operator, respectively.

Theorem 3.9. *Let $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right) \mid i=1,2,\ldots,n\right\}$ be a set of DHq-ROFNs and $\tau>0$, then the aggregated value using DHq-ROFSSHA is also a DHq-ROFN and can be presented as follows:*

$$DHq-ROFSSHA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\oplus_{SS}{}_{i=1}^{n}\left(\omega_i\tilde{\dot{\kappa}}_{\sigma(i)}\right)$$

$$= \left\langle \bigcup_{\substack{\dot{\gamma}_i \in \tilde{\dot{h}}_i \\ (i=1,2,\ldots,n)}} \left\{ \left(1-\prod_{i=1}^{n}\left(1-\dot{\gamma}_{\sigma(i)}^{q\tau}\right)^{\omega_i}\right)^{\frac{1}{q\tau}} \right\},\right.$$

$$\left. \bigcup_{\substack{\dot{\eta}_i \in \tilde{\dot{g}}_i \\ (i=1,2,\ldots,n)}} \left\{ \left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\dot{\eta}_{\sigma(i)}^{q}\right)^{\tau}\right)^{\omega_i}\right)^{\frac{1}{\tau}}\right)^{\frac{1}{q}} \right\} \right\rangle, \tag{6}$$

where $\tilde{\dot{\kappa}}_i=n\Omega_i\tilde{\kappa}_i=\left(n\Omega_i\tilde{h}_i,n\Omega_i\tilde{g}_i\right)=\left(\tilde{\dot{h}},\tilde{\dot{g}}\right)$.

Proof. The Theorem can be proved in an analogous way to Theorem 3.2.

Note 8. The idempotency, monotonicity, and boundness properties are also satisfied for the DHq-ROFSSHA operator, which can be proved in a similar way.

Note 9. If $\tau = 1$ then the DHq-ROFSSHA operator is reduced to DHq-ROF hybrid average (DHq-ROFHA) operator.

3.2.6 | DHq-ROFSSHG operator

In this subsection $DHq-ROFSSHG$ operator is developed.

Definition 3.6. Let $\left\{\tilde{\kappa}_i = \left(\tilde{h}_i, \tilde{g}_i\right) \mid i = 1,2,...,n\right\}$ be a set of DHq-ROFNs. If

$$DHq-ROFSSHG(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n) = \otimes_{SS}{}_{i=1}^{n}\left(\tilde{\dot{\kappa}}_{\sigma(i)}\right)^{\omega_i},$$

Then $DHq-ROFSSHG(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n)$ is called DHq-ROFSSHG operator, where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ be the associate weighted vector such that $\omega_i > 0$, and $\sum_{i=1}^{n} \omega_i = 1$. $\tilde{\dot{\kappa}}_{\sigma(i)}$ is the i^{th} largest elements of the DHq-ROF arguments $\tilde{\dot{\kappa}}_i$ $\left(\tilde{\dot{\kappa}}_i = \tilde{\kappa}_i^{n\Omega_i} = \left(\tilde{h}_i^{n\Omega_i}, \tilde{g}_i^{n\Omega_i}\right)\right)$, $\Omega = (\Omega_1, \Omega_2, ..., \Omega_n)$ is the weighting vector of DHq-ROF arguments $\tilde{\kappa}_i$ with $\Omega_i > 0$, and $\sum_{i=1}^{n} \Omega_i = 1$ and n is the balancing co-efficient.

Moreover, if $\omega = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ then DHq-ROFSSHG operator is reduced to DHq-ROFSSWG operator. Again, DHq-ROFSSOWG operator can be generated from the proposed operator DHq-ROFSSHG if $\Omega = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$.

Theorem 3.10. *Let* $\left\{\tilde{\kappa}_i = \left(\tilde{h}_i, \tilde{g}_i\right) \mid i = 1,2,...,n\right\}$ *be a set of DHq-ROFNs,* $\tau > 0$, *then the aggregated value using DHq-ROFSSHG is also a DHq-ROFN and can be given as follows:*

$$DHq-ROFSSHG(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n) = \otimes_{SS}{}_{i=1}^{n}\left(\omega_i \tilde{\dot{\kappa}}_{\sigma(i)}\right)$$

$$= \left\langle \bigcup_{\substack{\dot{\gamma}_i \in \tilde{\dot{h}}_i \\ (i=1,2,...,n)}} \left\{ \left(1 - \left(1 - \prod_{i=1}^{n}\left(1 - \left(1 - \dot{\gamma}_{\sigma(i)}^{q}\right)^{t}\right)^{\omega_i}\right)^{\frac{1}{t}}\right)^{\frac{1}{q}} \right\}, \right.$$

$$\left. \bigcup_{\substack{\dot{\eta}_i \in \tilde{\dot{g}} \\ (i=1,2,...,n)}} \left\{ \left(1 - \prod_{i=1}^{n}\left(1 - \dot{\eta}_{\sigma(i)}^{qt}\right)^{\omega_i}\right)^{\frac{1}{qt}} \right\} \right\rangle, \tag{7}$$

where $\tilde{\dot{\kappa}}_i = \tilde{\kappa}_i^{n\Omega_i} = \left(\tilde{h}_i^{n\Omega_i}, \tilde{g}_i^{n\Omega_i}\right) = \left(\tilde{\dot{h}}, \tilde{\dot{g}}\right)$.

Proof. The Theorem can be proved in an analogous way to Theorem 3.2.

Note 8. The idempotency, monotonicity, and boundness properties are also satisfied for the DHq-ROFSSHA operator, which can be proved in a similar way.

Note 9. If $\tau = 1$ then the DHq-ROFSSHA operator is reduced to DHq-ROF hybrid geometric (DHq-ROFHG) operator.

4 | METHODOLOGICAL DEVELOPMENT OF MCGDM USING DHq-ROF HYBRID AGGREGATION OPERATOR BASED ON SSt-CN&t-NS OPERATION

Let $A = \{A_1, A_2, ..., A_m\}$ be a finite set of alternatives, $C = \{C_1, C_2, ..., C_n\}$ be a set of criteria and $D = \left\{D^{(1)}, D^{(1)}, ..., D^{(p)}\right\}$ be p number of DMs. Let $w = (w_1, w_2, ..., w_n)^T$ be the weight vector of criteria such that $\sum_{i=1}^{n} w_i = 1$ where $w_i \in [0,1]$ and $W = (W_1, W_1, ..., W_n)^T$ represents the corresponding associated weighted vector. And $\varpi = \{\varpi_1, \varpi_2, ..., \varpi_p\}$ be the weight vector of the DMs such that $\sum_{j=1}^{p} \varpi_j = 1$ where $\varpi_j \in [0,1]$. To reduced DMs' hesitation and uncertainity for judgement on each alternative w.r.t each criterion is suggested for giving their opinion in DHq-ROFNs. Let $\mathcal{K}^{(l)} = \left[\tilde{\kappa}_{ij}^{(l)}\right]_{m\times n} = \left(\tilde{h}_{ij}^{(l)}, \tilde{g}_{ij}^{(l)}\right)_{m\times n}$ is a DHq-ROF decision matrix (DHq-ROFDM). Where $\tilde{h}_{ij}^{(l)}$ and $\tilde{g}_{ij}^{(l)}$ indicate, respectively, set of possible membership and non-membership values of i^{th} alternative for j^{th} criterion by the l^{th} DM. The main objective is to select the best alternative(s) using the proposed method. The following is a step-by-step summary of the computing process. Also a flowchart of the developed methodology is shown in Figure 1.

Step 1. Construct the decision matrices by figuring out the criteria and alternatives, $\mathcal{K}^{(l)} = \left[\tilde{\kappa}_{ij}^{(l)}\right]_{m\times n}$, where $\tilde{\kappa}_{ij}^{(l)} = \left(\tilde{h}_{ij}^{(l)}, \tilde{g}_{ij}^{(l)}\right)_{m\times n}$ $(l = 1,2,...p; i = 1,2,...,m; j = 1,2,...,n)$.

Step 2. In the decision-making problem, the criteria are two types cost type and benefit type. In cost type, the smaller value is better, and for benefit type, the bigger value is better. So normalized the decision matrix in the following manner.

$$R^{(l)} = \tilde{r}_{ij}^{(l)} = \begin{cases} \tilde{\kappa}_{ij}^{(l)} & \text{for benefit type } C_j, \\ \left(\tilde{\kappa}_{ij}^{(l)}\right)^c & \text{for cost types } C_j \end{cases} \tag{8}$$

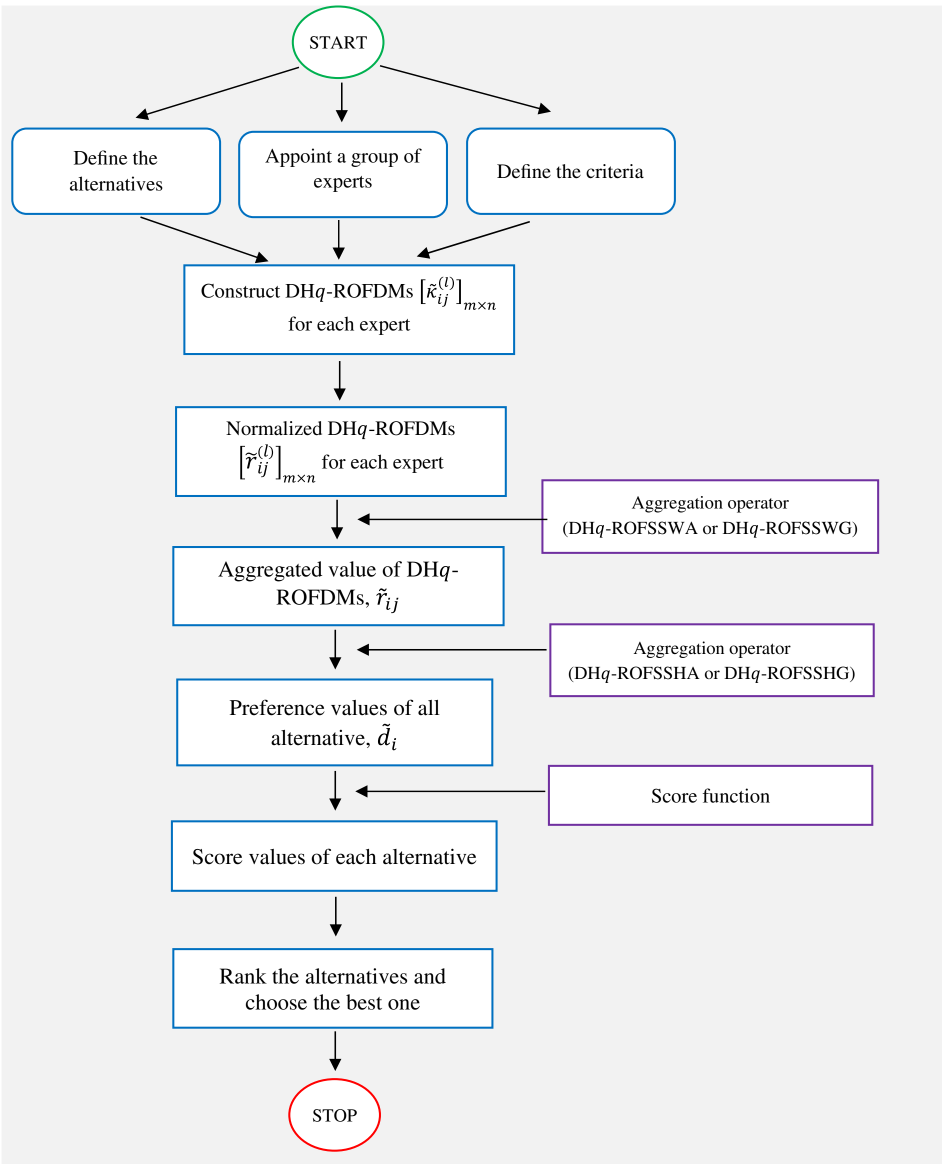

FIGURE 1 Flowchart of proposed method.

where $l=1,2,\ldots p$; $i=1,2,\ldots,n$; $j=1,2,\ldots,m$, and $\left(\tilde{\kappa}_{ij}^{(l)}\right)^c$ is the complement of $\tilde{\kappa}_{ij}^{(l)}$, i.e. $\left(\tilde{\kappa}_{ij}^{(l)}\right)^c=\left(\tilde{g}_{ij}^{(l)},\tilde{h}_{ij}^{(l)}\right)$.

Step 3. Utilize DH*q*-ROFSSWA (or DH*q*-ROFSSWG) operator to aggregate all individual DH*q*-ROFDM $R^{(l)}=\left(\tilde{r}_{ij}^{(l)}\right)_{n\times m}$ into a single DH*q*-ROFDM, $R=[\tilde{r}_{ij}]_{n\times m}$.

$$\tilde{r}_{ij}=DHq-ROFSSWA\left(\tilde{r}_{ij}^{(1)},\tilde{r}_{ij}^{(2)},\ldots,\tilde{r}_{ij}^{(p)}\right), \tag{9}$$

or

$$\tilde{r}_{ij}=DHq-ROFSSWG\left(\tilde{r}_{ij}^{(1)},\tilde{r}_{ij}^{(2)},\ldots,\tilde{r}_{ij}^{(p)}\right). \tag{10}$$

Step 4. To get overall $\tilde{d}_i$, utilize the DH*q*-ROFSSHA (or DH*q*-ROFSSHG) operator to aggregate all the attributes values $\tilde{r}_{ij}$ $(j=1,2,\ldots,n)$ of the alternative A_i $(i=1,2,\ldots,m)$.

$$\tilde{d}_i=DHq-ROFSSHA(\tilde{r}_{i1},\tilde{r}_{i2},\ldots,\tilde{r}_{in}), \tag{11}$$

or

$$\tilde{d}_i=DHq-ROFSSHG(\tilde{r}_{i1},\tilde{r}_{i2},\ldots,\tilde{r}_{in}) \tag{12}$$

Step 5. Using Equation (1), compute the scores $S\left(\tilde{d}_i\right)$ $(i=1,2,\ldots,m)$ of the overall DH*q*-ROFN $\tilde{d}_i$ $(i=1,2,\ldots,n)$.

Step 6. Utilizing Definition 2.5, rank all the alternatives A_i $(i=1,2,\ldots.,m)$ and select the best ones(s) in accordance with the scores $\left(\tilde{d}_i\right)$ $(i=1,2,\ldots,m)$.

5 | ILLUSTRATIVE EXAMPLE

In this section, an illustrative example (Rahman & Ali, 2020) regarding selection of best suitable company for investment from a set of five available companies is adopted in DH*q*-ROF context and solved by the proposed methodology. The vision and mission of the companies are defined as follows:

Based on some preliminary knowledge an investor finds out five profitable companies available in the market, viz., A_1: Furniture Company, A_2: Mobile Company, A_3: Chemical Company, A_4: Computer Company, A_5: Medicine Company. Further four DMs $D^{(l)}$ $(l=1,2,3,4)$, who are experts on various types of marketing fields are assigned. The weight vector of DMs is $\varpi=(0.1,0.2,0.3,0.4)^T$. According to board of experts, four important and unavoidable criteria, viz., C_1: Risk analysis, C_2: Growth analysis, C_3: Environmental impact, C_4: Expected Benefit with weight vector $w=(0.1,0.2,0.3,0.4)^T$ are taken into account. To overcome hesitancy / inaccuracy of DMs, DH*q*-ROFN are used to express their judgement values to each company with respect to each criterion. Here risk analysis and environmental impact are considered as cost type criteria and growth analysis and expected benefit are benefit type criteria. Based on experts' suggestion, the investor will invest his/ her wealth to the selected company. To identify the best company for investing, the following steps are executed.

Step 1. The individual evaluation values of DMs are presented in Tables 2-5.

Step 2. Since risk analysis and environmental impact are cost-type criteria, normalization of the decision matrix by Equation (8) is required and are given following Tables 6-9.

Step 3. Considering $q=3$, $\tau=2$ and utilize DH*q*-ROFSSWA operator to aggregate all the individual DH*q*-ROF decision matrices $R^{(l)}=\left(\tilde{r}_{ij}^{(l)}\right)_{n\times m}$ into a collective DH*q*-ROFDM, $R=[\tilde{r}_{ij}]_{n\times m}$, shown in Table 10.

Step 4. Utilize the DH*q*-ROFSSHA operator to aggregate each company's collective evaluation values, $\tilde{r}_{ij}$ $(i=1,2,\ldots,5;j=1,2,\ldots,4)$, with respect to given criteria to get the comprehensive decision values, $\tilde{d}_i$ $(i=1,2,3,4,5)$, are as follows. It is also noted that the associated weight vector Ω is the same as w.

$$\tilde{d}_1=\left\langle \begin{Bmatrix} 0.5537,0.5588,0.5559,0.5609,0.5593,0.5641,0.5614,0.5662,0.5588,0.5637, \\ 0.5609,0.5657,0.5642,0.5688,0.5662,0.5707 \end{Bmatrix}, \begin{Bmatrix} 0.6001,0.6087,0.6022,0.6109,0.6083,0.6171,0.6105,0.6193,0.6071,0.6159 \\ 0.6093,0.6181,0.6154,0.6244,0.6177,0.6267,0.6099,0.6187,0.6121,0.6209 \\ 0.6182,0.6273,0.6205,0.6296,0.6171,0.6261,0.6193,0.6283,0.6256,0.6348 \\ 0.6279,0.6371,0.6085,0.6173,0.6107,0.6195,0.6168,0.6258,0.6191,0.6281 \\ 0.6157,0.6246,0.6179,0.6269,0.6242,0.6333,0.6264,0.6357,0.6185,0.6275 \\ 0.6207,0.6298,0.6270,0.6363,0.6293,0.6386,0.6258,0.6350,0.6281,0.6374 \\ 0.6346,0.6440,0.6369,0.6463 \end{Bmatrix} \right\rangle$$

TABLE 2 DHq-ROFDM of $D^{(1)}$

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.5\},\{0.8\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.4\},\{0.7\}\rangle$	$\langle\{0.3,0.5\},\{0.7\}\rangle$
A_2	$\langle\{0.4\},\{0.8,0.9\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.5,0.6\},\{0.8\}\rangle$	$\langle\{0.3\},\{0.8\}\rangle$
A_3	$\langle\{0.6\},\{0.5\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.2,0.5\},\{0.7,0.9\}\rangle$	$\langle\{0.3,0.8\},\{0.8\}\rangle$
A_4	$\langle\{0.5\},\{0.6,0.7\}\rangle$	$\langle\{0.4\},\{0.6\}\rangle$	$\langle\{0.3,0.4\},\{0.6,0.9\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$
A_5	$\langle\{0.8\},\{0.6\}\rangle$	$\langle\{0.6\},\{0.6,0.8\}\rangle$	$\langle\{0.3\},\{0.7\}\rangle$	$\langle\{0.2,0.5\},\{0.6,0.8\}\rangle$

TABLE 3 DHq-ROFDM of $D^{(2)}$

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.4,0.7\},\{0.6\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_2	$\langle\{0.2,0.3\},\{0.7\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_3	$\langle\{0.6\},\{0.6,0.8\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.3,0.6\},\{0.6,0.9\}\rangle$	$\langle\{0.4,0.8\},\{0.5,0.7\}\rangle$
A_4	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.6,0.8\},\{0.6\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.5\},\{0.8\}\rangle$
A_5	$\langle\{0.4\},\{0.6\}\rangle$	$\langle\{0.2\},\{0.7\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$

TABLE 4 DHq-ROFDM of $D^{(3)}$

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.3,0.5\},\{0.6,0.7\}\rangle$	$\langle\{0.4\},\{0.7\}\rangle$	$\langle\{0.5,0.8\},\{0.3,0.6\}\rangle$	$\langle\{0.4,0.5\},\{0.6\}\rangle$
A_2	$\langle\{0.3\},\{0.8\}\rangle$	$\langle\{0.3,0.6\},\{0.7\}\rangle$	$\langle\{0.1,0.3\},\{0.8\}\rangle$	$\langle\{0.2\},\{0.9\}\rangle$
A_3	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.6\},\{0.6\}\rangle$	$\langle\{0.2,0.4\},\{0.7,0.9\}\rangle$	$\langle\{0.3,0.7\},\{0.4,0.6,0.8\}\rangle$
A_4	$\langle\{0.5\},\{0.7,0.8\}\rangle$	$\langle\{0.5,0.6,0.7\},\{0.8\}\rangle$	$\langle\{0.1\},\{0.9\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_5	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.2,0.4\},\{0.8\}\rangle$	$\langle\{0.3\},\{0.7,0.9\}\rangle$

TABLE 5 DHq-ROFDM of $D^{(4)}$

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.3\},\{0.8\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.4,0.5\},\{0.7\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$
A_2	$\langle\{0.2,0.3\},\{0.7,0.8\}\rangle$	$\langle\{0.3\},\{0.8\}\rangle$	$\langle\{0.1,0.3\},\{0.8,0.9\}\rangle$	$\langle\{0.2,0.5\},\{0.8\}\rangle$
A_3	$\langle\{0.6\},\{0.6\}\rangle$	$\langle\{0.6,0.8\},\{0.8\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.3,0.7\},\{0.6,0.8\}\rangle$
A_4	$\langle\{0.4\},\{0.7\}\rangle$	$\langle\{0.6,0.8\},\{0.8\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.5,0.9\},\{0.3,0.7\}\rangle$
A_5	$\langle\{0.4,0.6\},\{0.6\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.2,0.5\},\{0.5,0.8\}\rangle$	$\langle\{0.2,0.3\},\{0.8,0.9\}\rangle$

$$\tilde{d}_2 = \langle\{0.6031, 0.6117, 0.6089, \ldots (64\,values)\}, \{0.5648, 0.5729, 0.5689, \ldots (32\,values)\}\rangle$$

$\tilde{d}_3 = \left\langle \begin{array}{l} \{0.5557, 0.5783, 0.6130, 0.6270, \ldots (1024\,values)\}, \\ \{0.5264, 0.5552, 0.5434, 0.5734, \ldots (192\,values)\} \end{array} \right\rangle;$

$\tilde{d}_4 = \langle\{0.6454, 0.6838, 0.6496, \ldots (384\,values)\}, \{0.5169, 0.5776, 0.5188, 0.5798\}\rangle;$

$\tilde{d}_5 = \langle\{0.5505, 0.551, 0.5529, \ldots (8\,values)\}, \{0.6116, 0.6168, 0.6205, \ldots (128\,values)\}\rangle;$

Step 5. The score value of each company is calculated from each $\tilde{d}_i$ $(i = 1,2,3,4,5)$ using Equation (1) and are giving as follows.

$S(\tilde{d}_1) = 0.4682$, $S(\tilde{d}_2) = 0.5189$, $S(\tilde{d}_3) = 0.5502$, $S(\tilde{d}_4) = 0.6071$, $S(\tilde{d}_5) = 0.4472$.

Step 6. Based on the score value, the ranking of each company for investment is given as $A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$.

From the above ranking, the most suitable company for investment is found as A_4 i.e., computer company.

TABLE 6 DHq-ROF normalized decision matrix $R^{(1)}$

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.8\},\{0.5\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.7\},\{0.4\}\rangle$	$\langle\{0.3,0.5\},\{0.7\}\rangle$
A_2	$\langle\{0.8,0.9\},\{0.4\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.8\},\{0.5,0.6\}\rangle$	$\langle\{0.3\},\{0.8\}\rangle$
A_3	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.7,0.9\},\{0.2,0.5\}\rangle$	$\langle\{0.3,0.8\},\{0.8\}\rangle$
A_4	$\langle\{0.6,0.7\},\{0.5\}\rangle$	$\langle\{0.4\},\{0.6\}\rangle$	$\langle\{0.6,0.9\},\{0.3,0.4\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$
A_5	$\langle\{0.6\},\{0.8\}\rangle$	$\langle\{0.6\},\{0.6,0.8\}\rangle$	$\langle\{0.7\},\{0.3\}\rangle$	$\langle\{0.2,0.5\},\{0.6,0.8\}\rangle$

TABLE 7 DHq-ROF normalized decision matrix $R^{(2)}$

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.6\},\{0.5\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.6\},\{0.4,0.7\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_2	$\langle\{0.7\},\{0.2,0.3\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.7\},\{0.5\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_3	$\langle\{0.6,0.8\},\{0.6\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.6,0.9\},\{0.3,0.6\}\rangle$	$\langle\{0.4,0.8\},\{0.5,0.7\}\rangle$
A_4	$\langle\{0.7\},\{0.5\}\rangle$	$\langle\{0.6,0.8\},\{0.6\}\rangle$	$\langle\{0.7,0.8\},\{0.4\}\rangle$	$\langle\{0.5\},\{0.8\}\rangle$
A_5	$\langle\{0.6\},\{0.4\}\rangle$	$\langle\{0.2\},\{0.7\}\rangle$	$\langle\{0.8\},\{0.4\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$

TABLE 8 DHq-ROF normalized decision matrix $R^{(3)}$

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.6,0.7\},\{0.3,0.5\}\rangle$	$\langle\{0.4\},\{0.7\}\rangle$	$\langle\{0.3,0.6\},\{0.5,0.8\}\rangle$	$\langle\{0.4,0.5\},\{0.6\}\rangle$
A_2	$\langle\{0.8\},\{0.3\}\rangle$	$\langle\{0.3,0.6\},\{0.7\}\rangle$	$\langle\{0.8\},\{0.1,0.3\}\rangle$	$\langle\{0.2\},\{0.9\}\rangle$
A_3	$\langle\{0.6\},\{0.5\}\rangle$	$\langle\{0.6\},\{0.6\}\rangle$	$\langle\{0.7,0.9\},\{0.2,0.4\}\rangle$	$\langle\{0.3,0.7\},\{0.4,0.6,0.8\}\rangle$
A_4	$\langle\{0.7,0.8\},\{0.5\}\rangle$	$\langle\{0.5,0.6,0.7\},\{0.8\}\rangle$	$\langle\{0.9\},\{0.1\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_5	$\langle\{0.7\},\{0.5\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.8\},\{0.2,0.4\}\rangle$	$\langle\{0.3\},\{0.7,0.9\}\rangle$

TABLE 9 DHq-ROF normalized decision matrix $R^{(4)}$

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.8\},\{0.3\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.7\},\{0.4,0.5\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$
A_2	$\langle\{0.7,0.8\},\{0.2,0.3\}\rangle$	$\langle\{0.3\},\{0.8\}\rangle$	$\langle\{0.8,0.9\},\{0.1,0.3\}\rangle$	$\langle\{0.2,0.5\},\{0.8\}\rangle$
A_3	$\langle\{0.6\},\{0.6\}\rangle$	$\langle\{0.6,0.8\},\{0.8\}\rangle$	$\langle\{0.7,0.8\},\{0.4\}\rangle$	$\langle\{0.3,0.7\},\{0.6,0.8\}\rangle$
A_4	$\langle\{0.7\},\{0.4\}\rangle$	$\langle\{0.6,0.8\},\{0.8\}\rangle$	$\langle\{0.8\},\{0.2\}\rangle$	$\langle\{0.5,0.9\},\{0.3,0.7\}\rangle$
A_5	$\langle\{0.6\},\{0.4,0.6\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.5,0.8\},\{0.2,0.5\}\rangle$	$\langle\{0.2,0.3\},\{0.8,0.9\}\rangle$

5.1 | Results and discussions

The method is associated with rung parameter and SS parameter. Now the variation of results with the change of those parameters will be discussed in the next subsection. Further, the results are provided by DHq-ROFSSHG operator are also discussed here. a brief comparative analysis with the existing methods is furnished in the next subsequent subsection.

5.1.1 | The impact of several parameters on the result of decision-making

Now, the impact of rung parameter, q and SS parameter, τ on decision making results are discussed. Those parameters play crucial roles in determining the decision making results. Different score values are obtained for each company by varying the values of the parameters. Varying the

TABLE 10 Aggregated DHq-ROFDM R

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.7378,0.7518\},\{0.3488,0.4062\}\rangle$	$\langle\{0.4000\},\{0.7177,0.7561,0.7269,0.7664\}\rangle$	$\langle\{0.6417,0.6602\},\{0.4274,0.4672,0.4838,0.5301,0.4755,0.5208,0.5397,0.5932\}\rangle$	$\langle\{0.4694,0.4918,0.4797,0.5000\},\{0.6468\}\rangle$
A_2	$\langle\{0.7504,0.7857,0.7740,0.8036\},\{0.2419,0.2845,0.2623,0.3087\}\rangle$	$\langle\{0.3777,0.5065\},\{0.7561\}\rangle$	$\langle\{0.7857,0.8443\},\{0.1611,0.2501,0.2240,0.3488,0.1638,0.2543,0.2278,0.3548\}\rangle$	$\langle\{0.3129,0.4393,0.3851,0.4602\},\{0.8000\}\rangle$
A_3	$\langle\{0.5933,0.6711\},\{0.5674\}\rangle$	$\langle\{0.5784,0.7128\},\{0.5078,0.6370\}\rangle$	$\langle\{0.6856,0.7412,0.7983,0.8227,0.7785,0.8070,0.8423,0.8589,0.7359,0.7751,0.8201,0.8404,0.8041,0.8273,0.8571,0.8715\},\{0.2857,0.3519,0.3266,0.4030,0.3128,0.3858,0.3579,0.4424\}\rangle$	$\langle\{0.3346,0.6067,0.5798,0.6627,0.6249,0.6871,0.6743,0.7189,0.5612,0.6540,0.6369,0.6943,0.6665,0.7135,0.7033,0.7397\},\{0.5231,0.5781,0.5929,0.6590,0.6415,0.7173,0.5580,0.6182,0.6346,0.7089,0.6889,0.7772\}\rangle$
A_4	$\langle\{0.6930,0.7347,0.7000,0.7397\},\{0.4569\}\rangle$	$\langle\{0.5670,0.7094,0.5907,0.7169,0.6332,0.7334,0.6585,0.7450,0.6698,0.7507,0.6934,0.7633\},\{0.7271\}\rangle$	$\langle\{0.8250,0.8352,0.8443,0.8528\},\{0.1939,0.1995\}\rangle$	$\langle\{0.4937,0.8028\},\{0.4843,0.6917\}\rangle$
A_5	$\langle\{0.6397\},\{0.4553,0.5353\}\rangle$	$\langle\{0.4111\},\{0.7528,0.7772\}\rangle$	$\langle\{0.7345,0.7931\},\{0.2389,0.3435,0.2939,0.4242\}\rangle$	$\langle\{0.3199,0.3319,0.3717,0.3777\},\{0.7428,0.7701,0.7962,0.8297,0.7664,0.7961,0.8251,0.8639\}\rangle$

rung parameter $q \in [3,12]$ and SS parameter $\tau \in [1,10]$ and using proposed operators the results are presented in Tables 11-14 and Figures 2-5. A brief description of each situation is given below.

Firstly, if DHq-ROFSSWA and DHq-ROFSSHA operators are used in the proposed methodology by fixing SS parameter at $\tau = 3$, the associated score values achieved through the developed method with respect to the companies under consideration are shown in Table 11 and Figure 2 with the change of rung parameter $q \in [3,12]$. From Figure 2, it is clear that the score values of A_4 decrease when the values of the parameter q are increased from 3 to 12. Whereas the score values of other companies initially increase but decrease in the later. Additionally, it is found that when $q \in [3,8.2]$ the ordering of the five companies is $A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$ and when $q \in [8.2,12]$ the ordering is found as $A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$. So, for both situation the computer sector is the best choice for investing the money.

Further, when fixed rung parameter $q = 3$ is considered and varying the SS parameter $\tau \in [1,10]$, the derived score values for the various options are shown in Table 12 and Figure 3. From Figure 3, it is observed that the score values of all alternatives are increasing. Additionally, it is noticed that when $\tau \in [1,3.32]$ the ordering is found as $A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$. Again, the ordering is archived as $A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$ for $\tau \in [5.43,10]$ and elsewhere found as $A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$. So, considering all cases company A_4 is the best choice for investing the money.

Secondly, if DHq-ROFSSWG and DHq-ROFSSHG operators are used with the consideration of the SS parameter $\tau = 2$ and varying rung parameter $q \in [3,12]$, the associated score values for the various companies are shown in Table 13 and Figure 4. From Figure 4, it is noticed that the score values are increasing. Additionally, it is found that when $q \in [3,3.24]$ the ordering became $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$. Again for $q \in [3.24,6.21]$, the ranking appeared as $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$. Also, the ordering is archived as $A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$ for $q \in [6.21,10.22]$.

TABLE 11 The effect of the parameter q (fixed SS parameter $\tau = 3$) utilizing DHq-ROFSSHA operator

Parameter	$S(\tilde{d}_1)$	$S(\tilde{d}_2)$	$S(\tilde{d}_3)$	$S(\tilde{d}_4)$	$S(\tilde{d}_5)$	Ordering
$q=3$	0.4920	0.5616	0.5791	0.6375	0.4883	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
$q=4$	0.5061	0.5673	0.5827	0.6345	0.5018	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
$q=5$	0.5141	0.5677	0.5812	0.6268	0.5096	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
$q=6$	0.5167	0.5650	0.5768	0.6171	0.5135	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
$q=7$	0.5165	0.5608	0.5712	0.6069	0.5150	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
$q=8$	0.5152	0.5559	0.5651	0.5969	0.5150	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
$q=9$	0.5135	0.5508	0.5591	0.5875	0.5141	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$q=10$	0.5116	0.5458	0.5533	0.5788	0.5128	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$q=11$	0.5098	0.5412	0.5480	0.5710	0.5113	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$q=12$	0.5082	0.5369	0.5432	0.5638	0.5097	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$

TABLE 12 The effect of the SS parameter τ for fixed $q = 3$ utilizing DHq-ROFSSHA operator

Parameter	$S(\tilde{d}_1)$	$S(\tilde{d}_2)$	$S(\tilde{d}_3)$	$S(\tilde{d}_4)$	$S(\tilde{d}_5)$	Ordering
$\tau=1$	0.4351	0.4640	0.5100	0.5648	0.3958	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
$\tau=2$	0.4682	0.5189	0.5502	0.6071	0.4472	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
$\tau=3$	0.4920	0.5616	0.5791	0.6375	0.4883	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
$\tau=4$	0.5142	0.5943	0.6024	0.6616	0.5212	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$\tau=5$	0.5317	0.6200	0.6219	0.6811	0.5464	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$\tau=6$	0.5454	0.6405	0.6385	0.6973	0.5668	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau=7$	0.5576	0.6575	0.6528	0.7110	0.5835	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau=8$	0.5687	0.6717	0.6653	0.7226	0.5973	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau=9$	0.5786	0.6839	0.6763	0.7327	0.6089	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau=10$	0.5876	0.6944	0.6860	0.7415	0.6188	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$

TABLE 13 The effect of the parameter q (fixed SS parameter $\tau = 2$) utilizing DHq-ROFSSHG operator

Parameter	$S(\tilde{d}_1)$	$S(\tilde{d}_2)$	$S(\tilde{d}_3)$	$S(\tilde{d}_4)$	$S(\tilde{d}_5)$	Ordering
$q=3$	0.3746	0.2606	0.4134	0.4152	0.2489	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q=4$	0.3977	0.2913	0.4229	0.4187	0.2839	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q=5$	0.4180	0.3211	0.4340	0.4269	0.3160	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q=6$	0.4355	0.3468	0.4448	0.4366	0.3431	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q=7$	0.4496	0.3683	0.4544	0.4464	0.3656	$A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$
$q=8$	0.4606	0.3862	0.4627	0.4553	0.3842	$A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$
$q=9$	0.4688	0.4010	0.4697	0.4632	0.3997	$A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$
$q=10$	0.4753	0.4134	0.4754	0.4699	0.4127	$A_3 \succ A_1 \succ A_4 \succ A_2 \succ A_5$
$q=11$	0.4805	0.4239	0.4802	0.4755	0.4237	$A_1 \succ A_3 \succ A_4 \succ A_2 \succ A_5$
$q=12$	0.4845	0.4329	0.4840	0.4802	0.4330	$A_1 \succ A_3 \succ A_4 \succ A_5 \succ A_2$

Moreover, when $q \in [10.22, 11.64]$ the ordering is obtained as $A_1 \succ A_3 \succ A_4 \succ A_2 \succ A_5$ and the ordering is found as $A_1 \succ A_3 \succ A_4 \succ A_5 \succ A_2$ for $q \in [11.64, 12]$.

Further, when considering fixed rung parameter $q = 3$ and varying the parameter $\tau \in [1, 10]$, the derived score values for the various companies are shown in Table 14 and Figure 5. From Figure 5, it is noticed that the score values are decreasing. Additionally, it is noticed that when $\tau \in [1, 2.47]$ the ordering is found as $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$. Again for $\tau \in [2.47, 5.15]$ the ordering is achieved $A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$. Moreover, the ranking is $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$ when $\tau \in [5.15, 8.3]$ and for $\tau \in [8.3, 10]$ the ranking is $A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$.

TABLE 14 The effect of the SS parameter τ for fixed $q=3$ over DHqROFSSHG operator

Parameter	$S(\tilde{d}_1)$	$S(\tilde{d}_2)$	$S(\tilde{d}_3)$	$S(\tilde{d}_4)$	$S(\tilde{d}_5)$	Ordering
$\tau=1$	0.3747	0.2707	0.4249	0.4352	0.2573	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau=2$	0.3746	0.2606	0.4134	0.4152	0.2489	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau=3$	0.3694	0.2511	0.3992	0.3986	0.2409	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
$\tau=4$	0.3619	0.2426	0.3859	0.3849	0.2342	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
$\tau=5$	0.3551	0.2349	0.3744	0.3742	0.2285	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
$\tau=6$	0.3489	0.2279	0.3645	0.3657	0.2235	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau=7$	0.3434	0.2215	0.3561	0.3588	0.2191	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau=8$	0.3384	0.2158	0.3489	0.3531	0.2153	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau=9$	0.3340	0.2107	0.3427	0.3484	0.2119	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$
$\tau=10$	0.3301	0.2061	0.3373	0.3444	0.2089	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$

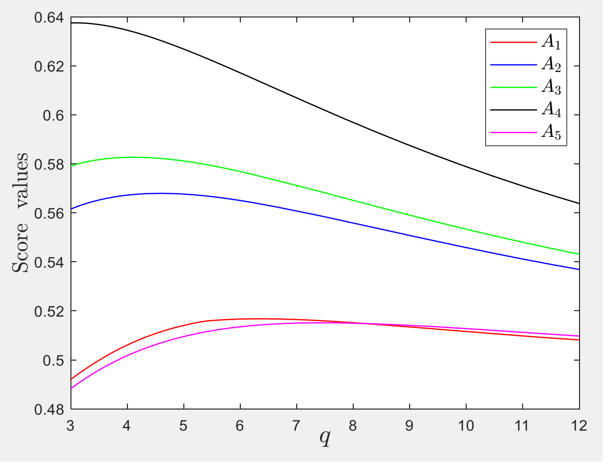

FIGURE 2 Score value using variation of rung parameter q for fixed SS parameter $\tau=3$ (DHq-ROFSSHA).

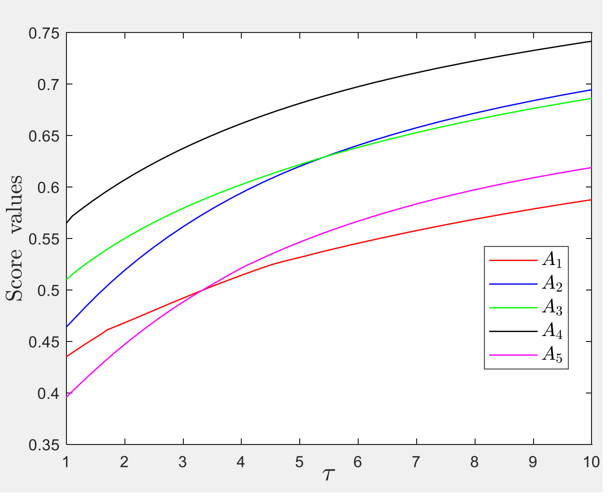

FIGURE 3 Score value using variation of SS parameter τ for fixed $q=3$ (DHq-ROFSSHA).

So, from above situation, company A_3 can be considered as a best company if $\tau \in [2.47, 5.15]$ and company A_4 can be considered as a best company if $\tau \notin [2.47, 5.15]$.

So, considering all cases it can be said that company A_4 is the most profitable company according to all DMs.

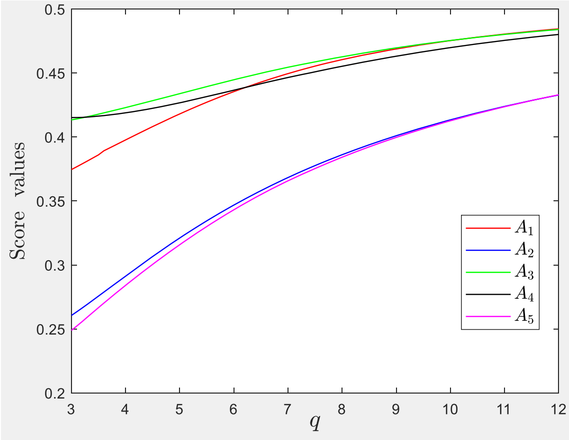

FIGURE 4 Score value using variation of rung parameter q for fixed SS parameter $\tau = 2$ (DHq-ROFSSHG).

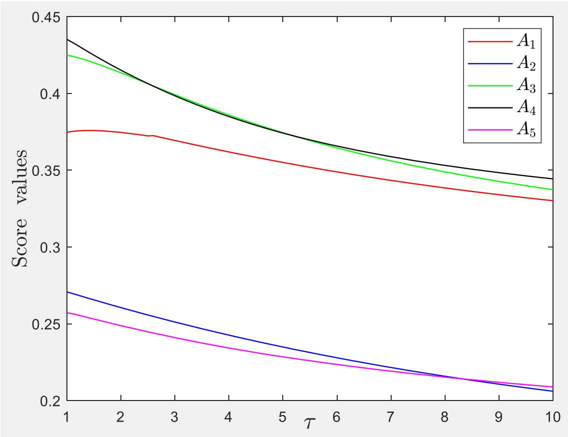

FIGURE 5 Score value using variation of SS parameter τ for fixed $q = 3$ (DHq-ROFSSHG).

5.2 | Comparative analysis

In this section, the developed method is compared with Rahman and Ali's (2020) method and some other existing aggregation operators (Wang, Wei, Wang, et al., 2019; Wang, Wei, Wei, & Wei, 2019; Xu et al., 2018).

Firstly, the result is compared with the methodology developed by Rahman and Ali (2020) using Pythagorean fuzzy Einstein hybrid geometric operator. It is well known that Pythagorean fuzzy set is a particular case with the consideration of the rung parameter $q = 2$. Table 15 describes the score values and rankings of the companies using Rahman and Ali's (2020) method and the proposed method. In both methods, ranking remains the same, i.e., $A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$. But the difference of score values between two consecutive companies (rank-wise) by the proposed method is larger than Rahman and Ali's (2020) method. As a result, the proposed method can distinguish the rank of the alternatives more effectively. So, the proposed method is better than the existing Rahman and Ali's (2020) method. A graphical representation of the differences of score value is given in Figure 6.

To demonstrate validity and efficacy of the DHq-ROFSSHA and DHq-ROFSSHG operators, the results are compared with the existing DHq-ROF Hamacher hybrid weighted average (DHq-ROFHHWA) (Wang, Wei, Wang, et al., 2019), DHq-ROF Hamacher hybrid weighted geometric (DHq-ROFHHWG) (Wang, Wei, Wang, et al., 2019), DHq-ROF weighted Muirhead mean (DHq-ROFWMM) (Wang, Wei, Wei, & Wei, 2019) DHq-ROF weighted dual Muirhead mean (DHq-ROFWDMM) (Wang, Wei, Wei, & Wei, 2019), q-rung dual hesitant fuzzy weighted Heronian mean (q-RDHFWHM) (Xu et al., 2018) and q-rung dual hesitant fuzzy weighted geometric Heronian mean (q-RDHFWGHM) (Xu et al., 2018) operators. For comparison, $q = 3$ is considered in all operators, viz., DHq-ROFSSHA, DHq-ROFSSHG, DHq-ROFHHWA and DHq-ROFHHWG operators. Under this condition, the example is solved by the proposed method using these operators. The score value and ranking of the companies are presented in Table 16.

TABLE 15 Comparison with Rahaman and Ali's (2020) method

Method	Score values					Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	
Rahman and Ali (2020)	−0.221	−0.177	−0.154	−0.099	−0.225	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
Proposed	0.4682	0.5189	0.5502	0.6071	0.4472	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$

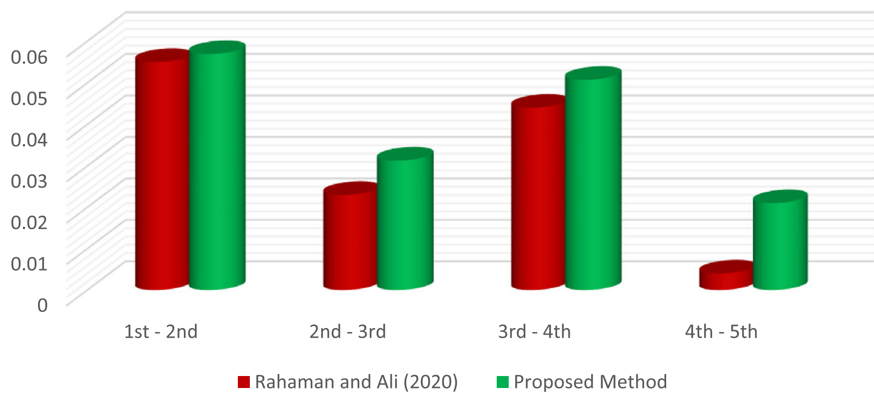

FIGURE 6 Bar diagram of differences of the alternatives' score values.

TABLE 16 Comparison with existing operators

Operators	Score values					Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	
DH*q*-ROFHHWA (Wang, Wei, Wang, et al., 2019)	0.3250	0.3601	0.3789	0.4320	0.3161	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
DH*q*-ROFHHWG (Wang, Wei, Wang, et al., 2019)	0.5231	0.3851	0.5526	0.5591	0.3628	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
DH*q*-ROFWMM (Wang, Wei, Wei, & Wei, 2019)	0.4065	0.3955	0.4656	0.4907	0.3277	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
DH*q*-ROFWDMM (Wang, Wei, Wei, & Wei, 2019)	0.6203	0.7062	0.6692	0.7363	0.6026	$A_4 \succ A_2 \succ A_3 \succ A_1 \succ A_5$
q-RDHFWHM (Xu et al., 2018)	0.4804	0.5447	0.5767	0.6285	0.4717	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
q-RDHFWGHM (Xu et al., 2018)	0.4535	0.4530	0.5322	0.5579	0.3908	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
DH*q*-ROFSSHA	0.4682	0.5189	0.5502	0.6071	0.4472	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
DH*q*-ROFSSHG	0.3746	0.2606	0.4134	0.4152	0.2489	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$

From the above results, it is observed that the ranking of the companies by averaging operators viz., DH*q*-ROFSSHA, DH*q*-ROFHHWA (Wang, Wei, Wang, et al., 2019), *q*-RDHFWHM (Xu et al., 2018) operators are the same. Also, the ranking of the companies by geometric operators, viz., DH*q*-ROFSSHG and DH*q*-ROFHHWG (Wang, Wei, Wang, et al., 2019) and DH*q*-ROFWMM (Wang, Wei, Wei, & Wei, 2019) operators are same. Also, DH*q*-ROFWMM (Wang, Wei, Wei, & Wei, 2019) and *q*-RDHFWGHM (Xu et al., 2018) operators give the same ranking. But in all above cases the computer company, A_4 is the best choice for investing. So, it can be said that the proposed operators fused the data correctly.

Figure 7 represents the graphical representation of the difference of the score values between two consecutive companies (rank-wise). The difference of score value between A_4 and A_3 is highest in DH*q*-ROFSSHA than other operators. As a result, the proposed method can distinguish the rank of the companies more effectively. So, proposed operators are better than existing operators. Finally, it can be said that DH*q*-ROFSSHA operator gives the best results as compared to other operators in this example.

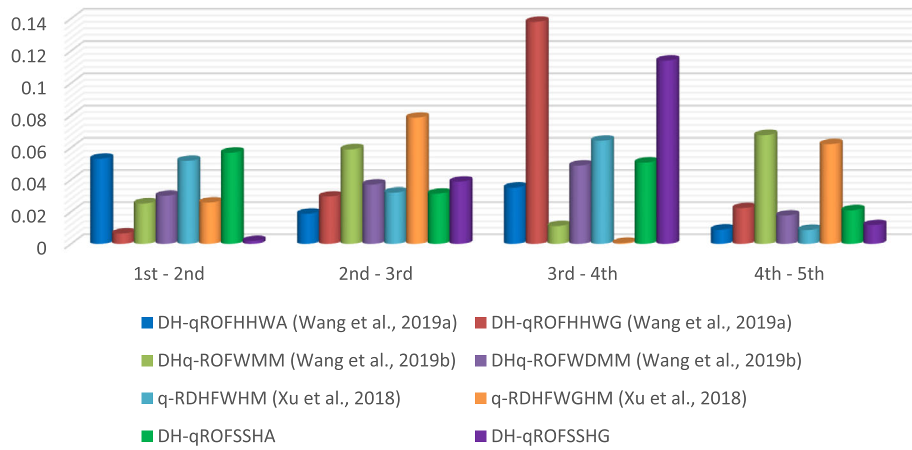

FIGURE 7 Graphical representation of score value difference of alternatives.

6 | CONCLUSION AND SCOPE OF FUTURE DIRECTIONS

It is well known that the notion of a DHq-ROFS generalizes the concept of fuzzy set to process complex uncertain information more accurately. Thus, the developed operators, viz., DHq-ROFSSWA, DHq-ROFSSWG, DHq-ROFSSOWA, DHq-ROFSSOWG operators based on SSt-CN&t-Ns will must add extra miles in the process of decision making. The main advantage of using these operators is that those operators possesses the capability of controlling the optimistic and pessimistic nature of a DM through the flexible SS parameter. However, weighted average or geometric operators consider only the weight of the opinions but disregard the importance of the ordered position of the opinions, while ordered weighted average or geometric operators consider only the weight of the ordered position of each given opinion but ignore the importance of the individual opinion. To overcome this drawback and also importance of these operators in mind, the DHq-ROFSSHA and DHq-ROFSSHG operators is also proposed. The benefit of these operators is that those consider weight of DHq-ROF arguments and ordered positions of the DHq-ROF arguments simultaneously. Also, using SSt-CN&t-Ns, all generalized cases are considered by varying SS parameter, τ. Therefore, these operators are more trustworthy than other existing operators on such sets and can be employed more effectively to handle decision-making problems. It is obvious that proposed operators can compensate for human hesitancy and the connection between fused arguments; moreover, the newly developed techniques may dynamically change to the parameter depending on the risk attitude of the DM. An investment problem is considered and solved using the developed methods to validate the applicability of the proposed methodology. A sensitivity analysis is also performed to capture sensitive nature of the solutions with the change of SS and rung parameters. Four DMs' evaluation values in the form of DHq-ROFNs in order to select the best company among five companies. It establishes that the proposed methodology is suitable for handling MCGDM problem effectively. The proposed method and the suggested approach are compared, and it is demonstrated that the new method is more consistent and useful than the existing method.

A limitation of the proposed study is that it is unable to handle such situation when the weight of the DMs or criteria are unknown. In the future an unknown weight model under DHq-ROF environment may be developed to overcome the limitation. As an extension of the developed method, the following issues may be studied in future. SS operation-based aggregation operators would be developed on probabilistic HFS (Batool et al., 2020), picture fuzzy set (Ullah, 2021), complex q-ROFS, T-spherical fuzzy sets, etc. Linguistic DHq-ROFS power aggregation operators, Aczel-Alsina Aggregation operators on DHq-ROFS would also be developed. Some correlation coefficients, similarity measures on DHq-ROFSs may also be considered. However, it is hoped that the proposed method would add an extra dimension in the field of making decision under uncertain complex decision making situations.

CONFLICT OF INTEREST STATEMENT

All the authors declare that the article does not have any conflict of interest.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

ORCID

Souvik Gayen https://orcid.org/0000-0001-8432-5918
Arun Sarkar https://orcid.org/0000-0002-9263-4400
Animesh Biswas https://orcid.org/0000-0002-5813-1076

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APPENDIX A

Symbol	Description
$\mathcal{X}$	Universal set
$\tilde{\mathcal{P}}$	q-ROFS
q	Rung parameter
$\mu_{\tilde{\mathcal{P}}}$	Membership degree of $\tilde{\mathcal{P}}$
$\nu_{\tilde{\mathcal{P}}}$	Non membership degree of $\tilde{\mathcal{P}}$
$\tilde{\mathfrak{p}}$	q-ROFN
$\tilde{\mathcal{K}}$	DHq-ROFS
$\tilde{h}_{\tilde{\mathcal{K}}}$	Possible membership degrees
$\tilde{g}_{\tilde{\mathcal{K}}}$	Possible non-membership degrees
$\tilde{\kappa}$	DHq-ROFN
$\tilde{h}$	Membership value of DHq-ROFN $\tilde{\kappa}$
$\tilde{g}$	Non-membership value of DHq-ROFN $\tilde{\kappa}$
$S(\tilde{\kappa})$	Score function of DHq-ROFN $\tilde{\kappa}$
$A(\tilde{\kappa})$	Accuracy function of DHq-ROFN $\tilde{\kappa}$
τ	Schweizer & Sklar parameter
I_{SS}	Schweizer & Sklarr t-norm
U_{SS}	Schweizer & Sklar t-conorm
λ	A scaler
ω	Weight vector

APPENDIX B

Abbreviations	Explanation
MCDM	Multicriteria decision making
MCGDM	Multicriteria group decision-making
DM	Decision maker
IFS	Intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
q-ROF	q-rung orthopair fuzzy
q-ROFS	q-rung orthopair fuzzy set
q-ROFN	q-ROF number
MSM	Maclaurin symmetric mean
SS	Schweizer-Sklar
HFS	hesitant fuzzy set HFS
DHFS	dual hesitant fuzzy set DHFS
DHq-ROFS	dual hesitant q-ROF set
DHPFS	dual hesitant Pythagorean fuzzy set
SSt-CN&t-Ns	Schweizer-Sklar t-conorms and t-norms
DHq-ROFSSWA	DHq-ROF SS weighted average
DHq-ROFSSOWA	DHq-ROF SS ordered weighted average
DHq-ROFSSWG	DHq-ROF SS weighted geometric
DHq-ROFSSOWG	DHq-ROF SS ordered weighted geometric
DHq-ROFSSHA	DHq-ROF SS hybrid average
DHq-ROFSSHG	DHq-ROF SS hybrid geometric
DHq-ROFHA	DHq-ROF hybrid average
DHq-ROFHG	DHq-ROF hybrid geometric
DHq-ROFDM	DHq-ROF decision matrix
DHq-ROFHHWA	DHq-ROF Hamacher hybrid weighted average
DHq-ROFHHWG	DHq-ROF Hamacher hybrid weighted geometric
DHq-ROFWMM	DHq-ROF weighted Muirhead mean
DHq-ROFWDMM	DHq-ROF weighted dual Muirhead mean
q-RDHFWHM	q-rung dual hesitant fuzzy weighted Heronian mean
q-RDHFWGHM	q-rung dual hesitant fuzzy weighted geometric Heronian mean

AUTHOR BIOGRAPHIES

Souvik Gayen is a research scholar in the Department of Mathematics, University of Kalyani, India. He received M.Sc. degree in Mathematics from IIT Kanpur, India in 2017. He has published some research articles in leading international journals and attended several international and national conferences. The area of his research includes Multicriteria decision making, Data clustering, Pattern recognition, etc, in various fuzzy environment. Gayen is also a member of the Operational Research Society of India.

Dr. Arun Sarkar received M.Sc. degree in Applied Mathematics from the University of Kalyani, India, in 2009. He is now working as an Assistant Professor in the Department of Mathematics at Heramba Chandra College, Kolkata 700029, India, since 2017. He is also awarded the Ph.D. degree in the Department of Mathematics, University of Kalyani, India. His research interests include aggregation operators in different variants of fuzzy environments, fuzzy decision making and their applications, etc. He has produced several research articles in leading International Journals and Conferences. He is a member of Operational Research Society of India and Indian Statistical Institute, Kolkata.

Dr. Animesh Biswas is rendering his service as a Professor in the Department of Mathematics, University of Kalyani, India. Formerly, he served Sikkim Manipal University of Health, Medical and Technological Sciences, India and RCC Institute of Information Technology, India as a Lecturer in Mathematics. The area of his research includes Mathematical Modelling, Optimization Techniques, Decision Making under uncertainties, Fuzzy Logic, Fuzzy Control, Soft Computing, etc. Dr. Biswas produced research articles in different reputed international journals. He attended various international conferences within India and abroad, delivered invited talks, presented papers. Dr. Biswas is a member of Indian Statistical Institute, Kolkata, India, Operational Research Society of India, Soft Computing Research Society, India, Tripura Mathematical Society, India, Computational Intelligence Society, IEEE, World Academy of Science, Engineering and Technology (WASET), International Association of Engineers (IAENG), International Economics Development Research Center (IEDRC), Multicriteria Decision Making Society, Germany, etc.

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RESEARCH

Novel Correlation Measure for Generalized Orthopair Fuzzy Sets and Its Decision-Making Applications

Paul Augustine Ejegwa[1] **· Arun Sarkar**[2]

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Abstract
A generalized orthopair fuzzy set (GOFS), also known as a q-rung orthopair fuzzy set (q-ROFS), is a higher variant of ordinary fuzzy sets by relaxing restrictions on the degrees of membership and non-membership. In fact, GOFSs generalize intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (PFSs), and Fermatean fuzzy sets (FFSs) with an improved ability to tackle vagueness. On the other hand, correlation analysis measures the statistical relationships between two samples or variables. Certain approaches for measuring the correlation coefficient of GOFSs have been studied, however, with some setbacks. In this paper, we propose a new correlation coefficient that measures the interrelation between any two arbitrary GOFSs with a better rating. Some properties of the novel generalized orthopair correlation coefficient are presented to validate its appropriateness. In addition, the novel correlation coefficient is validated with some numerical examples and adjudged to outperform some existing approaches via comparative analysis. Finally, we discuss the applications of the novel approach in problems involving pattern recognition and medical diagnosis based on simulated data presented as generalized orthopair fuzzy values.

Keywords Correlation measure · Generalized orthopair fuzzy set · Generalized orthopair fuzzy value · Pattern recognition · Disease diagnosis

1 Introduction

Among the variants of fuzzy sets, viz., intuitionistic fuzzy set (IFS) [1], Pythagorean fuzzy set (PFS) [2–4], Fermatean fuzzy set (FFS) [5], etc., the q-rung orthopair fuzzy set (q-ROFS) presented by Yager [6] is an indispensable and meaningful tool for dealing with uncertain information. In q-ROFS, membership degree, ξ, and non-membership

✉ Paul Augustine Ejegwa
ejegwa.augustine@uam.edu.ng

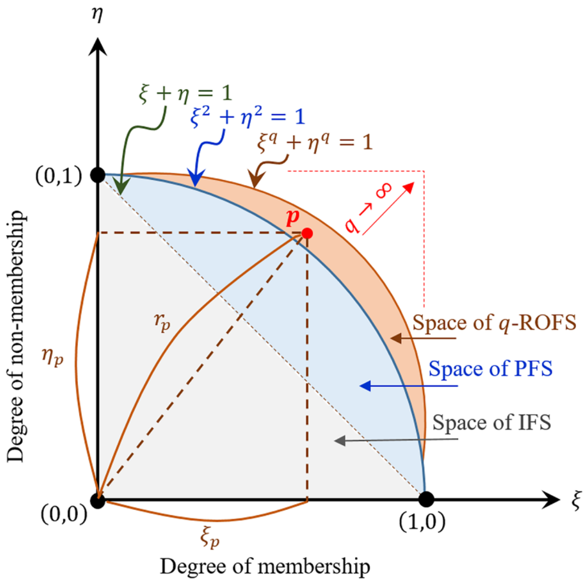
1 Department of Mathematics, University of Agriculture, P.M.B. 2373, Makurdi, Nigeria

2 Department of Mathematics, Heramba Chandra College, Kolkata 700029, India

degree, η, satisfy the condition that $\xi^q + \eta^q \leq 1$ for all $q \geq 1$. q-ROFS is a generalized form of IFSs, PFSs, and FFSs with certain constraints as a more generic fuzzy set. By substituting the values of the rung parameter, $q = 1, 2, 3$, q-ROFS reduces to IFS, PFS, and FFS, respectively. As a result, q-ROFS is the most valuable and refined extension of fuzzy sets, in which decision-makers (DMs) can change the range of their judgment values by altering the rung parameter q depending on various indeterminate degrees. A comparison among the domains of IFSs, PFSs, and q-ROFSs is portrayed in Fig. 1. Many scholars have conducted in-depth exploration and research on the q-ROFS because of its excellent characteristics and have produced a large number of results, including q-ROF multicriteria decision-making (MCDM) [7–10], measure theory [11–14], graph theory [15–17], multiattribute decision-making (MADM) [18], medical diagnosis [19, 20], and so on.

In decision-making, pattern recognition, data analysis, machine learning, and other fields, correlation is one of the most widely used indices. It assesses how effectively two samples or qualities move in a straight line. The idea of a correlation coefficient has been studied under IFSs with various applications [21–27]. Furthermore, the notions of correlation and coefficient of correlation were applied to PFSs. Garg [28] proposed a novel correlation coefficient and a weighted correlation coefficient formulation between two PFSs. Using these, some numerical examples of pattern recognition and medical diagnosis were solved. Thao [29] proposed a new correlation coefficient between PFSs and applied it to the pattern recognition problem. Considering membership and non-membership degrees, strength, and direction of commitments of a PFS, Lin et al. [30] developed some novel directional correlation coefficients between two PFSs. Singh and Ganie [31] introduced some novel correlation coefficients of PFSs, satisfying the condition that the correlation coefficient of two PFSs is one if and only if the two PFSs are equal. Other correlation coefficients of PFSs have been developed and applied in various areas [32–34].

Fig. 1 Graphical representation of q-ROFS



Nevertheless, whereas the q-ROFS theory has been successfully implemented in numerous decision-making processes, real-world situations frequently have many attribute values that are complicated to comprehend quantitatively. In such circumstances, it appears appropriate to convey things qualitatively. Under the q-ROF context, Du [35] defined a correlation coefficient rule and investigated some of its basic properties. The correlation coefficient rule was applied in cluster analysis under q-ROF environments. After that, Singh and Ganie [36] introduced another kind of correlation coefficient for q-ROFSs. Using a correlation-based closeness coefficient, they also proposed a novel approach for solving an MCDM problem in a q-ROF environment. In the q-ROF context, Bashir et al. [37] formulated some correlation coefficient methods as generalizations of correlation coefficients of IFSs and PFSs. Mahmood and Ali [38] studied entropy measures and the TOPSIS approach using the correlation coefficient with application to MADM based on complex q-rung fuzzy information. Li et al. [39] developed two new q-rung orthopair fuzzy correlation coefficients and their applications in clustering analysis.

Although the outputs from the approach in ref. [35] seem to be accurate, it exhibits two weaknesses, namely, that its output violates the conditions of the correlation coefficient as q increase and that it does not capture the hesitation margin, yielding outputs that are reliable to error due to omission. In the same vein, in the course of the numerical verifications of the approaches in ref. [36], it is observed that the first approach can only measure the correlation of dissimilar q-ROFSs. Precisely, the first approach in ref. [36] is not a good approach to measuring correlation coefficients. Though the second strategy is superior to the first, it is inconsistent in terms of satisfying the correlation measure's requirements. The techniques of the correlation coefficient for GOFSs in refs. [35–37, 39] lack consistency with a better performance index. In addition to the techniques in ref. [36] that incorporate the complete parametric features of GOFSs, the techniques in refs. [35, 37, 39] discarded the grades of hesitancy or hesitation margins of the GOFSs. This exclusion renders the outputs from these techniques unreliable because the nature of the hesitation margin of GOFSs is the chief factor that clearly differentiates GOFSs from IFSs, PFSs, and FFSs and positions it as a more competent soft computing tool. In fact, without hesitation margin, a GOFS can be reduced to a fuzzy set, and thus relinquishes its ability to reasonably curb imprecision.

However, in a quest to remedy the setbacks in refs. [35–37, 39], a novel correlation coefficient approach based on GOFSs/q-ROFSs is proposed with a better rating. In fact, the novel approach resolves the aforementioned drawbacks in refs. [35–37, 39]. Succinctly speaking, the contributions of the works are as follows:

1) Appraisal of the correlation measuring approaches based on GOFSs in refs. [35–37, 39] to pinpoint their limitations for the justification of a new correlation measuring approach,
2) Propose a novel correlation measuring approach, which resolves the limitations in the hitherto approaches with better performance ratings,
3) Ascertain the advantages of the novel approach over the existing approaches via comparative analysis based on numerical illustrations,

4) Characterizations of the novel approach to demonstrate its alignment with the properties of the correlation coefficient, and
5) Demonstrations of decision-making problems pertaining to pattern recognition and disease diagnosis based on the novel approach under generalized orthopair (GOF) information.

The remainder of this work is laid out as follows: We go through the essential concepts of generalized orthopair fuzzy sets in Sect. 2. In Sect. 3, we reiterate some existing approaches to computing correlation coefficients under the GOF domain. Section 4 discusses the novel approach to estimating the correlation coefficient between GOFSs and obtaining some of its properties. In Sect. 5, we discuss the applications of the novel approach pertaining to pattern recognition and disease diagnosis. The work is concluded in Sect. 6 with recommendations for future research.

2 Preliminaries

With the inclusion of a non-membership degree, the idea of fuzzy sets was extended to IFS. We denote X as a universe of discourse (i.e., a non-empty set) throughout the work.

Definition 2.1 [1] An IFS $\widetilde{N}$ in X is given by the structure

$$\widetilde{N} = \left\{ \langle x, \mu_{\widetilde{N}}(x), \nu_{\widetilde{N}}(x) \rangle | x \in X \right\}, \tag{1}$$

where $\mu_{\widetilde{N}}, \nu_{\widetilde{N}}: X \to [0,1]$ denote the grade of membership and non-membership for $x \in X$ to the set $\widetilde{N}$, such that.

$$\mu_{\widetilde{N}}(x), \nu_{\widetilde{N}}(x) \in [0,1], \text{ and } 0 \le \mu_{\widetilde{N}}(x) + \nu_{\widetilde{N}}(x) \le 1 \tag{2}$$

The indeterminacy degree for IFS is presented by $\pi_{\widetilde{N}}(x) = 1 - \mu_{\widetilde{N}}(x) - \nu_{\widetilde{N}}(x)$. For usefulness, $\left(\mu_{\widetilde{N}}(x), \nu_{\widetilde{N}}(x)\right)$ is taken as an intuitionistic fuzzy number (IFN) and is denoted by $\widetilde{N} = (\mu, \nu)$.

Definition 2.2 [40] Let $\widetilde{N}$, $\widetilde{N}_1$, and $\widetilde{N}_2$ represent any three IFSs in X, then some fundamental operational rules are defined below:

(i) $\overline{\widetilde{N}} = \left\{ \langle x, \nu_{\widetilde{N}}(x), \mu_{\widetilde{N}}(x) \rangle | x \in X \right\}$;
(ii) $\widetilde{N}_1 = \widetilde{N}_2$ if $\mu_{\widetilde{N}_1}(x) = \mu_{\widetilde{N}_2}(x)$ and $\nu_{\widetilde{N}_1}(x) = \nu_{\widetilde{N}_2}(x)$ $\forall x \in X$;
(iii) $\widetilde{N}_1 \subseteq \widetilde{N}_2$ if $\mu_{\widetilde{N}_1}(x) \le \mu_{\widetilde{N}_2}(x)$ and $\nu_{\widetilde{N}_1}(x) \ge \nu_{\widetilde{N}_2}(x)$ $\forall x \in X$;
(iv) $\widetilde{N}_1 \cap \widetilde{N}_2 = \left\{ \langle x, \min\left(\mu_{\widetilde{N}_1}(x), \mu_{\widetilde{N}_2}(x)\right), \max\left(\nu_{\widetilde{N}_1}(x), \nu_{\widetilde{N}_2}(x)\right) \rangle | x \in X \right\}$;
(v) $\widetilde{N}_1 \cup \widetilde{N}_2 = \left\{ \langle x, \max\left(\mu_{\widetilde{N}_1}(x), \mu_{\widetilde{N}_2}(x)\right), \min\left(\nu_{\widetilde{N}_1}(x), \nu_{\widetilde{N}_2}(x)\right) \rangle | x \in X \right\}$.

By extending IFS [1], Yager [2] presented a novel set called PFS, as defined in the following manner.

Definition 2.3 [2] A PFS symbolized by $\widetilde{P}$ in X is the structure

$$\widetilde{P} = \{\langle x, \mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x)\rangle | x \in X\}, \tag{3}$$

where $\mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x) \in [0, 1]$ denote the grade of membership and non-membership for $x \in X$ to the set $\widetilde{P}$, such that

$$0 \leq \left(\mu_{\widetilde{P}}(x)\right)^2 + \left(\nu_{\widetilde{P}}(x)\right)^2 \leq 1. \tag{4}$$

The indeterminacy degree for PFS is given by $\pi_{\widetilde{P}}(x) = \sqrt{1 - \left(\mu_{\widetilde{P}}(x)\right)^2 - \left(\nu_{\widetilde{P}}(x)\right)^2}$. For convenience, Zhang and Xu [41] named $\left(\mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x)\right)$ as a Pythagorean fuzzy number (PFN) and is represented by $\widetilde{p} = (\mu, \nu)$.

With the introduction of PFS, it is clearly realized that the spaces for considering membership and non-membership values have been extended. Yager and Abbasov [4] proposed score and accuracy functions for PFNs.

Definition 2.4 [4] Score and accuracy functions of any PFN, $\widetilde{p} = (\mu, \nu)$ are denoted as $S(\widetilde{p})$ and $A(\widetilde{p})$, respectively, presented by

$$\left.\begin{array}{l} S(\widetilde{p}) = \mu^2 - \nu^2 \\ A(\widetilde{p}) = \mu^2 + \nu^2 \end{array}\right\}, \tag{5}$$

where $-1 \leq S(\widetilde{p}) \leq 1$ and $A(\widetilde{p}) \in [0, 1]$.

Yager and Abbasov [4] defined a method for ranking of PFNs as follows.

Definition 2.5 [4] The following principles are used for ordering between any two PFNs, $\widetilde{p}_1$ and $\widetilde{p}_2$.

(i) If $S(\widetilde{p}_1) > S(\widetilde{p}_2)$, then $\widetilde{p}_1 \succ \widetilde{p}_2$;
(ii) If $S(\widetilde{p}_1) = S(\widetilde{p}_2)$, then
(iii) $\widetilde{p}_1 \succ \widetilde{p}_2$ when $A(\widetilde{p}_1) > A(\widetilde{p}_2)$, if $A(\widetilde{p}_1) = A(\widetilde{p}_2)$, then $\widetilde{p}_1 \approx \widetilde{p}_2$.

Basic operations on PFNs as presented by Yager [2, 3] and Yager and Abbasov [4] are presented as follows:

Definition 2.6 [4] If any three PFNs are represented by $\widetilde{p} = (\mu, \nu)$, $\widetilde{p}_1 = (\mu_1, \nu_1)$, and $\widetilde{p}_2 = (\mu_2, \nu_2)$, then some basic operating principles of them are defined below ($\lambda > 0$):

(i) $\widetilde{p}_1 \oplus \widetilde{p}_2 = \left(\sqrt{{\mu_1}^2 + {\mu_2}^2 - {\mu_1}^2{\mu_2}^2}, \nu_1\nu_2\right)$;
(ii) $\widetilde{p}_1 \oplus \widetilde{p}_2 = \left(\mu_1\mu_2, \sqrt{{\nu_1}^2 + {\nu_2}^2 - {\nu_1}^2{\nu_2}^2}\right)$
(iii) $\lambda\widetilde{p} = \left(\sqrt{1 - \left(1 - \mu^2\right)^\lambda}, \nu^\lambda\right)$;
(iv) $\widetilde{p}^\lambda = \left(\mu^\lambda, \sqrt{1 - \left(1 - \nu^2\right)^\lambda}\right)$.

Definition 2.7 [4] Let $\widetilde{P}$, $\widetilde{P}_1$, and $\widetilde{P}_2$ represent any three PFSs in X, then some fundamental operational rules are defined below:

(i) $\overline{\overline{\widetilde{P}}} = \{\langle x, \nu_{\widetilde{P}}(x), \mu_{\widetilde{P}}(x)\rangle | x \in X\}$
(ii) $\widetilde{P}_1 = \widetilde{P}_2$ iff $\mu_{\widetilde{P}_1}(x) = \mu_{\widetilde{P}_2}(x)$ and $\nu_{\widetilde{P}_1}(x) = \nu_{\widetilde{P}_2}(x)\ \forall x \in X$.
(iii) (ii) $\widetilde{P}_1 \subseteq \widetilde{P}_2$ iff $\mu_{\widetilde{P}_1}(x) \le \mu_{\widetilde{P}_2}(x)$ and $\nu_{\widetilde{P}_1}(x) \ge \nu_{\widetilde{P}_2}(x)\ \forall x \in X$.
(iv) $\widetilde{P}_1 \cup \widetilde{P}_2 = \left\{\langle x, \max\left(\mu_{\widetilde{P}_1}(x), \mu_{\widetilde{P}_2}(x)\right), \min\left(\nu_{\widetilde{P}_1}(x), \nu_{\widetilde{P}_2}(x)\right)\rangle | x \in X\right\}$.
(v) $\widetilde{P}_1 \cap \widetilde{P}_2 = \left\{\langle x, \min\left(\mu_{\widetilde{P}_1}(x), \mu_{\widetilde{P}_2}(x)\right), \max\left(\nu_{\widetilde{P}_1}(x), \nu_{\widetilde{P}_2}(x)\right)\rangle | x \in X\right\}$.

Definition 2.8 [6] Let X be a universe of discourse. A GOFS symbolized by $\widetilde{\wp}$ in X is represented by

$$\widetilde{\wp} = \left\{\langle x, \xi_{\widetilde{\wp}}(x), \eta_{\widetilde{\wp}}(x)\rangle | x \in X\right\}, \tag{6}$$

where $\xi_{\widetilde{\wp}}(x) \in [0, 1]$ and $\eta_{\widetilde{\wp}}(x) \in [0, 1]$ denote the degree of membership and non-membership, respectively, of the element $x \in X$ to the set $\widetilde{\wp}$ satisfying the condition that.

$$0 \le \left(\xi_{\widetilde{\wp}}(x)\right)^q + \left(\eta_{\widetilde{\wp}}(x)\right)^q \le 1, q \ge 1 \tag{7}$$

The degree of indeterminacy $\pi_{\widetilde{\wp}}(x)$ of x in $\widetilde{\wp}$ is given as

$$\pi_{\widetilde{\wp}}(x) = \left[1 - \left(\xi_{\widetilde{\wp}}(x)\right)^q - \left(\eta_{\widetilde{\wp}}(x)\right)^q\right]^{\frac{1}{q}} \tag{8}$$

For simplicity, $\left(\xi_{\widetilde{\wp}}(x), \eta_{\widetilde{\wp}}(x)\right)$ is called a generalized orthopair fuzzy number (GOFN) and is symbolized by $\widetilde{\wp} = (\xi, \eta)$.

Liu and Wang [18] proposed the following score and accuracy functions for q-ROFNs.

Definition 2.9 [18] For any GOFN $\widetilde{\wp} = (\xi, \eta)$, the score function, $S\left(\widetilde{\wp}\right)$ of $\widetilde{\wp}$ is defined as follows:

$$S\left(\widetilde{\wp}\right) = \frac{1}{2}(1 + \xi^q - \eta^q)$$

where $S\left(\widetilde{\wp}\right) \in [0, 1]$.

For a q-ROFN $\widetilde{\wp} = (\xi, \eta)$, the accuracy function, $A\left(\widetilde{\wp}\right)$ of p is defined as

$$A\left(\widetilde{\wp}\right) = \xi^q + \eta^q$$

Liu and Wang [18] developed the following technique for ordering the GOFNs:

Definition 2.10 [18] Let $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ be any two GOFNs, then the ordering of these GOFNs are as follows:

- $\widetilde{\wp}_1 \succ \widetilde{\wp}_2$ when $S\left(\widetilde{\wp}_1\right) > S\left(\widetilde{\wp}_2\right)$;
- For $S\left(\widetilde{\wp}_1\right) = S\left(\widetilde{\wp}_2\right)$

(i) $\widetilde{\wp}_1 \succ \widetilde{\wp}_2$ when $A\left(\widetilde{\wp}_1\right) > A\left(\widetilde{\wp}_2\right)$;
(ii) $\widetilde{\wp}_1 \approx \widetilde{\wp}_2$ for $A\left(\widetilde{\wp}_1\right) = A\left(\widetilde{\wp}_2\right)$.

Four basic operations on q-ROFNs were formulated by Liu and Wang [18], which we recall as follows:

Definition 2.11 [18] Let $\widetilde{\wp} = (\xi, \eta)$, $\widetilde{\wp}_1 = \left(\xi_1, \eta_1\right)$, and $\widetilde{\wp}_2 = \left(\xi_2, \eta_2\right)$ be three q-ROFNs, and $\lambda > 0$, then some fundamental operations are defined as follows:

(1) $\widetilde{\wp}_1 \oplus \widetilde{\wp}_2 = \left(\sqrt[q]{{\xi_1}^q + {\xi_2}^q - {\xi_1}^q{\xi_2}^q}, \eta_1\eta_2\right)$;
(2) $\widetilde{\wp}_1 \otimes \widetilde{\wp}_2 = \left(\xi_1\xi_2, \sqrt[q]{{\eta_1}^q + {\eta_2}^q - {\eta_1}^q{\eta_2}^q}\right)$;
(3) $\lambda\widetilde{\wp} = \left(\sqrt[q]{1 - (1 - \xi^q)^\lambda}, \eta^\lambda\right)$;
(4) $\widetilde{\wp}^\lambda = \left(\xi^\lambda, \sqrt[q]{1 - (1 - \eta^q)^\lambda}\right)$.

Here, definitions of the correlation coefficient for GOFSs in the domains of $[0, 1]$ and $[-1, 1]$ are provided as follows:

Definition 2.12 If $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ are GOFSs in $X = \left\{x_1, x_2, \ldots, x_n\right\}$, then the correlation coefficient from a statistical viewpoint for $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ denoted by $\Omega_*(\widetilde{\wp}_1, \widetilde{\wp}_2)$, is a function, $\rho_* : \widetilde{\wp}_1 \times \widetilde{\wp}_2 \rightarrow [-1, 1]$, which satisfies.

(i) $\rho_*(\widetilde{\wp}_1, \widetilde{\wp}_2) \in [-1, 1]$,
(ii) $\rho_*(\widetilde{\wp}_1, \widetilde{\wp}_2) = \rho_*(\widetilde{\wp}_2, \widetilde{\wp}_1)$,
(iii) $\rho_*(\widetilde{\wp}_1, \widetilde{\wp}_2) = 1\widetilde{\wp}_1 = \widetilde{\wp}_2$

As $\rho_*(\widetilde{\wp}_1, \widetilde{\wp}_2)$ moves closer to 1, it shows that there is a strong positive correlation. On the other hand, as $\rho_*(\widetilde{\wp}_1, \widetilde{\wp}_2)$ moves closer to -1, it shows that there is a weak negative correlation. Whereas, $\rho_*(\widetilde{\wp}_1, \widetilde{\wp}_2) = 1$ and $\rho_*(\widetilde{\wp}_1, \widetilde{\wp}_2) = -1$ indicate a perfect positive correlation and a perfect negative correlation, respectively.

Definition 2.13 If $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ are GOFSs in $X = \left\{x_1, x_2, \ldots, x_n\right\}$, then the correlation coefficient for $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ denoted by $\Omega(\widetilde{\wp}_1, \widetilde{\wp}_2)$, is a function, $\rho :: \widetilde{\wp}_1 \times \widetilde{\wp}_2 \rightarrow [0, 1]$, which satisfies.

(i) $\rho(\widetilde{\wp}_1, \widetilde{\wp}_2) \in [0, 1]$,
(ii) $\rho(\widetilde{\wp}_1, \widetilde{\wp}_2) = \rho(\widetilde{\wp}_2, \widetilde{\wp}_1)$,
(iii) if and only if $\widetilde{\wp}_1 = \widetilde{\wp}_2$.

As $\rho(\widetilde{\wp}_1, \widetilde{\wp}_2)$ moves closer to 1, it shows that the correlation is strong. On the other hand, as $\rho(\widetilde{\wp}_1, \widetilde{\wp}_2)$ moves closer to 0, it shows that the correlation is very weak. Whereas, $\rho(\widetilde{\wp}_1, \widetilde{\wp}_2) = 1$ and $\rho(\widetilde{\wp}_1, \widetilde{\wp}_2) = 0$ indicate a perfect correlation and no correlation, respectively.

3 Some Existing Approaches of Measuring Correlation in Generalized Orthopair Fuzzy Domain

Here, we recall some approaches to measuring correlation coefficients represented in the domain of GOFSs.

3.1 Du's Approach

Let $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ be two GOFSs in $X = \{x_1, x_2, \ldots, x_n\}$, where X is a finite universe of discourse, then the correlation coefficient in ref. [35] between $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ is given by.

$$\rho_1(\widetilde{\wp}_1, \widetilde{\wp}_2) = \left(\frac{\left(\sum_{i=1}^{n} \left(\xi^q_{\widetilde{\wp}_1}(x_i)\xi^q_{\widetilde{\wp}_2}(x_i) + \eta^q_{\widetilde{\wp}_1}(x_i)\eta^q_{\widetilde{\wp}_2}(x_i) \right) \right)^2}{\sum_{i=1}^{n} \left(\xi^{2q}_{\widetilde{\wp}_1}(x_i) + \eta^{2q}_{\widetilde{\wp}_1}(x_i) \right) \sum_{i=1}^{n} \left(\xi^{2q}_{\widetilde{\wp}_2}(x_i) + \eta^{2q}_{\widetilde{\wp}_2}(x_i) \right)} \right)^{\frac{1}{2q}}, \text{ for } q \geq 1 \tag{9}$$

3.2 Singh and Ganie's Approaches

Two approaches to computing the correlation coefficient between GOFSs were introduced in ref. [36]. Assume $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ are two GOFSs in $X = \{x_1, x_2, \ldots, x_n\}$, where X is a finite universe of discourse, then the correlation coefficients between $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ are given as follows:

First approach;

$$\rho_2(\widetilde{\wp}_1, \widetilde{\wp}_2) = \frac{\mathcal{C}(\widetilde{\wp}_1, \widetilde{\wp}_2)}{\sqrt{\mathcal{T}(\widetilde{\wp}_1)\mathcal{T}(\widetilde{\wp}_2)}}, \tag{10}$$

Where

$$\begin{aligned} C(\widetilde{\wp}_1, \widetilde{\wp}_2) = & \frac{1}{n-1} \sum_{i=1}^{n} \Big(\Big(\xi^q_{\widetilde{\wp}_1}(x_i) - \big(\overline{\xi_{\widetilde{\wp}_1}}\big)^q \Big) \Big(\xi^q_{\widetilde{\wp}_2}(x_i) - \big(\overline{\xi_{\widetilde{\wp}_2}}\big)^q \Big) + \Big(\eta^q_{\widetilde{\wp}_1}(x_i) \\ & - \big(\overline{\eta_{\widetilde{\wp}_1}}\big)^q \Big) \Big(\eta^q_{\widetilde{\wp}_2}(x_i) - \big(\overline{\eta_{\widetilde{\wp}_2}}\big)^q \Big) + \Big(\pi^q_{\widetilde{\wp}_1}(x_i) - \big(\overline{\pi_{\widetilde{\wp}_1}}\big)^q \Big) \Big(\pi^q_{\widetilde{\wp}_2}(x_i) - \big(\overline{\pi_{\widetilde{\wp}_2}}\big)^q \Big) \Big), \end{aligned}$$

$$T(\widetilde{\wp}_1) = \frac{1}{n-1} \sum_{i=1}^{n} \left(\left(\xi^q_{\widetilde{\wp}_1}(x_i) - \big(\overline{\xi_{\widetilde{\wp}_1}}\big)^q \right)^2 + \left(\eta^q_{\widetilde{\wp}_1}(x_i) - \big(\overline{\eta_{\widetilde{\wp}_1}}\big)^q \right)^2 + \left(\pi^q_{\widetilde{\wp}_1}(x_i) - \big(\overline{\pi_{\widetilde{\wp}_1}}\big)^q \right)^2 \right),$$

$$T(\widetilde{\wp}_2) = \frac{1}{n-1} \sum_{i=1}^{n} \left(\left(\xi^q_{\widetilde{\wp}_2}(x_i) - \big(\overline{\xi_{\widetilde{\wp}_2}}\big)^q \right)^2 + \left(\eta^q_{\widetilde{\wp}_2}(x_i) - \big(\overline{\eta_{\widetilde{\wp}_2}}\big)^q \right)^2 + \left(\pi^q_{\widetilde{\wp}_2}(x_i) - \big(\overline{\pi_{\widetilde{\wp}_2}}\big)^q \right)^2 \right),$$

$$\overline{\xi_{\widetilde{\wp}_1}} = \frac{\sum_{i=1}^{n} \xi_{\widetilde{\wp}_1}(x_i)}{n}, \overline{\eta_{\widetilde{\wp}_1}} = \frac{\sum_{i=1}^{n} \eta_{\widetilde{\wp}_1}(x_i)}{n}, \overline{\pi_{\widetilde{\wp}_1}} = \frac{\sum_{i=1}^{n} \pi_{\widetilde{\wp}_1}(x_i)}{n},$$

$$\overline{\xi_{\widetilde{\wp}_2}} = \frac{\sum_{i=1}^{n} \xi_{\widetilde{\wp}_2}(x_i)}{n}, \overline{\eta_{\widetilde{\wp}_2}} = \frac{\sum_{i=1}^{n} \eta_{\widetilde{\wp}_2}(x_i)}{n}, \overline{\pi_{\widetilde{\wp}_2}} = \frac{\sum_{i=1}^{n} \pi_{\widetilde{\wp}_2}(x_i)}{n}, \text{ for } q \geq 1.$$

Second approach;

$$\rho_3(\widetilde{\wp}_1, \widetilde{\wp}_2) = \frac{1}{3}(\theta_1 + \theta_2 + \theta_3), \tag{11}$$

Where

$$\theta_1 = \frac{\sum_{i=1}^{n}\left[\left(\xi^q_{\widetilde{\wp}_1}(x_i) - \left(\overline{\xi_{\widetilde{\wp}_1}}\right)^q\right)\left(\xi^q_{\widetilde{\wp}_2}(x_i) - \left(\overline{\xi_{\widetilde{\wp}_2}}\right)^q\right)\right]}{\left[\sqrt{\sum_{i=1}^{n}\left(\xi^q_{\widetilde{\wp}_1}(x_i) - \left(\overline{\xi_{\widetilde{\wp}_1}}\right)^q\right)^2}\sqrt{\sum_{i=1}^{n}\left(\xi^q_{\widetilde{\wp}_2}(x_i) - \left(\overline{\xi_{\widetilde{\wp}_2}}\right)^q\right)^2}\right]},$$

$$\theta_2 = \frac{\sum_{i=1}^{n}\left[\left(\eta^q_{\widetilde{\wp}_1}(x_i) - \left(\overline{\eta_{\widetilde{\wp}_1}}\right)^q\right)\left(\eta^q_{\widetilde{\wp}_2}(x_i) - \left(\overline{\eta_{\widetilde{\wp}_2}}\right)^q\right)\right]}{\left[\sqrt{\sum_{i=1}^{n}\left(\eta^q_{\widetilde{\wp}_1}(x_i) - \left(\overline{\eta_{\widetilde{\wp}_1}}\right)^q\right)^2}\sqrt{\sum_{i=1}^{n}\left(\eta^q_{\widetilde{\wp}_2}(x_i) - \left(\overline{\eta_M}\right)^q\right)^2}\right]},$$

$$\theta_3 = \frac{\sum_{i=1}^{n}\left[\left(\pi^q_{\widetilde{\wp}_1}(x_i) - \left(\overline{\pi_{\widetilde{\wp}_1}}\right)^q\right)\left(\pi^q_{\widetilde{\wp}_2}(x_i) - \left(\overline{\pi_{\widetilde{\wp}_2}}\right)^q\right)\right]}{\left[\sqrt{\sum_{i=1}^{n}\left(\pi^q_{\widetilde{\wp}_1}(x_i) - \left(\overline{\pi_{\widetilde{\wp}_1}}\right)^q\right)^2}\sqrt{\sum_{i=1}^{n}\left(\pi^q_{\widetilde{\wp}_2}(x_i) - \left(\overline{\pi_{\widetilde{\wp}_2}}\right)^q\right)^2}\right]},$$

$$\overline{\xi_{\widetilde{\wp}_1}} = \frac{\sum_{i=1}^{n} \xi_{\widetilde{\wp}_1}(x_i)}{n}, \overline{\eta_{\widetilde{\wp}_1}} = \frac{\sum_{i=1}^{n} \eta_{\widetilde{\wp}_1}(x_i)}{n}, \overline{\pi_{\widetilde{\wp}_1}} = \frac{\sum_{i=1}^{n} \pi_{\widetilde{\wp}_1}(x_i)}{n},$$

$$\overline{\xi_{\widetilde{\wp}_2}} = \frac{\sum_{i=1}^{n} \xi_{\widetilde{\wp}_2}(x_i)}{n}, \overline{\eta_{\widetilde{\wp}_2}} = \frac{\sum_{i=1}^{n} \eta_{\widetilde{\wp}_2}(x_i)}{n}, \overline{\pi_{\widetilde{\wp}_2}} = \frac{\sum_{i=1}^{n} \pi_{\widetilde{\wp}_2}(x_i)}{n} \text{ for } q \geq 1.$$

3.3 Bashir et al.'s Techniques

Bashir et al. [37] introduced new techniques of CC q-ROFSs, which are presented as;

$$\rho_4(\widetilde{\wp}_1, \widetilde{\wp}_2) = \frac{\sum_{i=1}^{n}\left(\xi^q_{\widetilde{\wp}_1}(x_i)\xi^q_{\widetilde{\wp}_2}(x_i) + \eta^q_{\widetilde{\wp}_1}(x_i)\eta^q_{\widetilde{\wp}_2}(x_i)\right)}{\max\left(\sum_{i=1}^{n}\left(\xi^{2q}_{\widetilde{\wp}_1}(x_i) + \eta^{2q}_{\widetilde{\wp}_1}(x_i)\right), \sum_{i=1}^{n}\left(\eta^{2q}_{\widetilde{\wp}_2}(x_i) + \eta^{2q}_{\widetilde{\wp}_2}(x_i)\right)\right)}, \tag{12}$$

$$\rho_5\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\frac{\sum_{i=1}^{n}\left(\xi^q_{\widetilde{\wp}_1}\left(x_i\right)\xi^q_{\widetilde{\wp}_2}\left(x_i\right)+\eta^q_{\widetilde{\wp}_1}\left(x_i\right)\eta^q_{\widetilde{\wp}_2}\left(x_i\right)\right)}{\sqrt{\sum_{i=1}^{n}\left(\xi^{2q}_{\widetilde{\wp}_1}\left(x_i\right)+\eta^{2q}_{\widetilde{\wp}_1}\left(x_i\right)\right)\sum_{i=1}^{n}\left(\eta^{2q}_{\widetilde{\wp}_2}\left(x_i\right)+\eta^{2q}_{\widetilde{\wp}_2}\left(x_i\right)\right)}}, \quad (13)$$

for $q \geq 1$. Equation (9) equals Eq. (13) whenever $q = 1$.

3.4 Li et al.'s Techniques

In ref. [39], some novel techniques of computing CC q-ROFSs were introduced, which are presented as follows;

$$\rho_6\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=(1-\lambda)\theta_\xi+\lambda\theta_\eta, \quad (14)$$

Where

$$\theta_\xi=\frac{\sum_{i=1}^{n}\left(\left(\xi^q_{\widetilde{\wp}_1}\left(x_i\right)-\overline{\xi_{\widetilde{\wp}_1}}\right)\left(\xi^q_{\widetilde{\wp}_2}\left(x_i\right)-\overline{\xi_{\widetilde{\wp}_2}}\right)\right)}{\sqrt{\sum_{i=1}^{n}\left(\xi^q_{\widetilde{\wp}_1}\left(x_i\right)-\overline{\xi_{\widetilde{\wp}_1}}\right)^2\sum_{i=1}^{n}\left(\xi^q_{\widetilde{\wp}_2}\left(x_i\right)-\overline{\xi_{\widetilde{\wp}_2}}\right)^2}},$$

$$\theta_\eta=\frac{\sum_{i=1}^{n}\left(\left(\eta^q_{\widetilde{\wp}_1}\left(x_i\right)-\overline{\eta_{\widetilde{\wp}_1}}\right)\left(\eta^q_{\widetilde{\wp}_2}\left(x_i\right)-\overline{\eta_{\widetilde{\wp}_2}}\right)\right)}{\sqrt{\sum_{i=1}^{n}\left(\eta^q_{\widetilde{\wp}_1}\left(x_i\right)-\overline{\eta_{\widetilde{\wp}_1}}\right)^2\sum_{i=1}^{n}\left(\eta^q_{\widetilde{\wp}_2}\left(x_i\right)-\overline{\eta_{\widetilde{\wp}_2}}\right)^2}},$$

for

$$\overline{\xi_{\widetilde{\wp}_1}}=\frac{\sum_{i=1}^{n}\xi_{\widetilde{\wp}_1}\left(x_i\right)}{n},\overline{\eta_{\widetilde{\wp}_1}}=\frac{\sum_{i=1}^{n}\eta_{\widetilde{\wp}_1}\left(x_i\right)}{n},$$

$$\overline{\xi_{\widetilde{\wp}_2}}=\frac{\sum_{i=1}^{n}\xi_{\widetilde{\wp}_2}\left(x_i\right)}{n},\overline{\eta_{\widetilde{\wp}_2}}=\frac{\sum_{i=1}^{n}\eta_{\widetilde{\wp}_2}\left(x_i\right)}{n},$$

wherein $q \geq 1$ and $\lambda \in [0, 1]$.

Similarly,

$$\rho_7\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=(1-\lambda)\theta^*_\xi+\lambda\theta^*_\eta, \quad (15)$$

Where

$$\theta^*_\xi=\frac{\sum_{i=1}^{n}\left(\left(\xi^q_{\widetilde{\wp}_1}\left(x_i\right)-\overline{\xi_{\widetilde{\wp}_1}}\right)\left(\xi^q_{\widetilde{\wp}_2}\left(x_i\right)-\overline{\xi_{\widetilde{\wp}_2}}\right)\right)}{max\left(\sum_{i=1}^{n}\left(\xi^q_{\widetilde{\wp}_1}\left(x_i\right)-\overline{\xi_{\widetilde{\wp}_1}}\right)^2,\sum_{i=1}^{n}\left(\xi^q_{\widetilde{\wp}_2}\left(x_i\right)-\overline{\xi_{\widetilde{\wp}_2}}\right)^2\right)},$$

$$\theta_{\eta}^{*} = \frac{\sum_{i=1}^{n}\left(\left(\eta_{\widetilde{\wp}_1}^{q}\left(x_i\right) - \overline{\eta_{\widetilde{\wp}_1}}\right)\left(\eta_{\widetilde{\wp}_2}^{q}\left(x_i\right) - \overline{\eta_{\widetilde{\wp}_2}}\right)\right)}{\max\left(\sum_{i=1}^{n}\left(\eta_{\widetilde{\wp}_1}^{q}\left(x_i\right) - \overline{\eta_{\widetilde{\wp}_1}}\right)^2, \sum_{i=1}^{n}\left(\eta_{\widetilde{\wp}_2}^{q}\left(x_i\right) - \overline{\eta_{\widetilde{\wp}_2}}\right)^2\right)},$$

for

$$\overline{\xi_{\widetilde{\wp}_1}} = \frac{\sum_{i=1}^{n}\xi_{\widetilde{\wp}_1}\left(x_i\right)}{n}, \overline{\eta_{\widetilde{\wp}_1}} = \frac{\sum_{i=1}^{n}\eta_{\widetilde{\wp}_1}\left(x_i\right)}{n},$$

$$\overline{\xi_{\widetilde{\wp}_2}} = \frac{\sum_{i=1}^{n}\xi_{\widetilde{\wp}_2}\left(x_i\right)}{n}, \overline{\eta_{\widetilde{\wp}_2}} = \frac{\sum_{i=1}^{n}\eta_{\widetilde{\wp}_2}\left(x_i\right)}{n},$$

wherein $q \geq 1$ and $\lambda \in [0, 1]$.

4 New Approach of Computing Correlation Coefficient in Generalized Orthopair Fuzzy Domain

Here, we introduce an efficient approach to calculating the correlation coefficient for GOFSs. The development of the new correlation coefficient for GOFSs is achieved by studying the approach in ref. [24] under the environment of GOFSs through the incorporation of the hesitation margins and the removal of the restriction of $c > 2$. For GOFSs $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ in $X = \{x_1, x_2, \dots, x_n\}$ where $n < \infty$, the correlation coefficient between the GOFSs can be measured by

$$\overline{\rho}\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{1}{3n}\sum_{i=1}^{n}\left(\mu_i\left(1 - \Delta\xi_i\right) + \nu_i\left(1 - \Delta\eta_i\right) + \varphi_i\left(1 - \Delta\pi_i\right)\right), \quad (16)$$

Where

$$\mu_i = \frac{q - \Delta\xi_i - \Delta\xi_{\max}}{q - \Delta\xi_{\min} - \Delta\xi_{\max}}, \nu_i = \frac{q - \Delta\eta_i - \Delta\eta_{\max}}{q - \Delta\eta_{\min} - \Delta\eta_{\max}}, \varphi_i = \frac{q - \Delta\pi_i - \Delta\pi_{\max}}{q - \Delta\pi_{\min} - \Delta\pi_{\max}},$$

$$\Delta\xi_{\min} = \min_i\left\{\left|\xi_{\widetilde{\wp}_1}^{q}\left(x_i\right) - \xi_{\widetilde{\wp}_2}^{q}\left(x_i\right)\right|\right\}, \ \Delta\eta_{\min} = \min_i\left\{\left|\eta_{\widetilde{\wp}_1}^{q}\left(x_i\right) - \eta_{\widetilde{\wp}_2}^{q}\left(x_i\right)\right|\right\},$$

$$\Delta\pi_{\min} = \min_i\left\{\left|\pi_{\widetilde{\wp}_1}^{q}\left(x_i\right) - \pi_{\widetilde{\wp}_2}^{q}\left(x_i\right)\right|\right\},$$

$$\Delta\xi_{\max} = \max_i\left\{\left|\xi_{\widetilde{\wp}_1}^{q}\left(x_i\right) - \xi_{\widetilde{\wp}_2}^{q}\left(x_i\right)\right|\right\}, \ \Delta\eta_{\max} = \max_i\left\{\left|\eta_{\widetilde{\wp}_1}^{q}\left(x_i\right) - \eta_{\widetilde{\wp}_2}^{q}\left(x_i\right)\right|\right\}$$

$$\Delta\pi_{\max} = \max_i\left\{\left|\pi_{\widetilde{\wp}_1}^{q}\left(x_i\right) - \pi_{\widetilde{\wp}_2}^{q}\left(x_i\right)\right|\right\},$$

$$\Delta\xi_i = \left|\xi^q_{\widetilde{\wp}_1}(x_i) - \xi^q_{\widetilde{\wp}_2}(x_i)\right|,\ \Delta\eta_i = \left|\eta^q_{\widetilde{\wp}_1}(x_i) - \eta^q_{\widetilde{\wp}_2}(x_i)\right|,$$

$$\Delta\pi_i = \left|\pi^q_{\widetilde{\wp}_1}(x_i) - \pi^q_{\widetilde{\wp}_2}(x_i)\right| \text{ For } q \geq 1,\ i = 1, 2, \dots, n.$$

The instance for which $\Delta\xi_{\min} + \Delta\xi_{\max} = 1$, $\Delta\eta_{\min} + \Delta\eta_{\max} = 1$, and $\Delta\pi_{\min} - \Delta\pi_{\max} = 1$ is very uncommon for $q = 1$. Howbeit, when it happens, $q > 1$ should be used. Therefore, there is no fear of the possibility of either μ_i, ν_i, or φ_i becoming undefined. Although GOFSs generalize IFSs (i.e., GOFSs for $q = 1$), PFSs (i.e., GOFSs for $q = 2$), and FFSs (i.e., GOFSs for $q = 3$), the beauty and distinctiveness of GOFSs are expressed for $q > 3$ ($q \in \mathbb{N}$).

In many cases, the weight of each element $x_i \in X$ is necessary to be taken into consideration while computing the correlation coefficient. For instance, in multi-attribute decision-making cases, every attribute has a different significance and needs to be assigned a different weight. By putting that into consideration, Eq. (16) becomes

$$\overline{\rho}_\omega(\widetilde{\wp}_1, \widetilde{\wp}_2) = \frac{1}{3}\sum\nolimits_{i=1}^{n}\omega_i\left(\mu_i(1 - \Delta\xi_i) + \nu_i(1 - \Delta\eta_i) + \varphi_i(1 - \Delta\pi_i)\right), \quad (17)$$

where the parameters are the same as in Eq. (16), and $\omega_i \geq 0$ for $\sum\nolimits_{i=1}^{n}\omega_i = 1$. If $\omega = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then Eqs. (16) and (17) are equivalent. The value of ω_i can be decided by expert opinion, statistical distribution, coefficient of variation method, or analytic hierarchy process.

Proposition 4.1. *For GOFSs $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ in X, $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2)$ satisfies.*

(i) $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) = \overline{\rho}(\widetilde{\wp}_2, \widetilde{\wp}_1)$,
(ii) $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) = 1$ iff $\widetilde{\wp}_1 = \widetilde{\wp}_2$.

Proof. Firstly, we establish (i). Recall that
$\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) = \frac{1}{3n}\sum\nolimits_{i=1}^{n}\left(\mu_i(1 - \Delta\xi_i) + \nu_i(1 - \Delta\eta_i) + \varphi_i(1 - \Delta\pi_i)\right)$, and so

$$\begin{aligned}\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) =& \frac{1}{3n}\sum\nolimits_{i=1}^{n}\left(\mu_i\left(1 - \left|\xi^q_{\widetilde{\wp}_1}(x_i) - \xi^q_{\widetilde{\wp}_2}(x_i)\right|\right) + \nu_i\left(1 - \left|\eta^q_{\widetilde{\wp}_1}(x_i) - \eta^q_{\widetilde{\wp}_2}(x_i)\right|\right)\right.\\ &\left. + \varphi_i\left(1 - \left|\pi^q_{\widetilde{\wp}_1}(x_i) - \pi^q_{\widetilde{\wp}_2}(x_i)\right|\right)\right)\end{aligned}$$

$$\begin{aligned}=& \frac{1}{3n}\sum\nolimits_{i=1}^{n}\left(\mu_i\left(1 - \left|\xi^q_{\widetilde{\wp}_2}(x_i) - \xi^q_{\widetilde{\wp}_1}(x_i)\right|\right) + \nu_i\left(1 - \left|\eta^q_{\widetilde{\wp}_2}(x_i) - \eta^q_{\widetilde{\wp}_1}(x_i)\right|\right)\right.\\ &\left. + \varphi_i\left(1 - \left|\pi^q_{\widetilde{\wp}_2}(x_i) - \pi^q_{\widetilde{\wp}_1}(x_i)\right|\right)\right)\end{aligned}$$

$$\overline{\rho}(\widetilde{\wp}_2, \widetilde{\wp}_1),$$

which proves (i).

Again, suppose $\widetilde{\wp}_1 = \widetilde{\wp}_2$. Then,

$$\left|\xi^q_{\widetilde{\wp}_1}(x_i) - \xi^q_{\widetilde{\wp}_2}(x_i)\right| = 0, \left|\eta^q_{\widetilde{\wp}_1}(x_i) - \eta^q_{\widetilde{\wp}_2}(x_i)\right| = 0,$$

$$\left|\pi^q_{\widetilde{\wp}_1}(x_i) - \pi^q_{\widetilde{\wp}_2}(x_i)\right| = 0.$$

Consequently, $\Delta\xi_i = \Delta\eta_i = \Delta\pi_i = 0$, $\Delta\xi_{\min} = \Delta\eta_{\min} = \Delta\pi_{\min} = 0$, and $\Delta\xi_{\max} = \Delta\eta_{\max} = \Delta\pi_{\max} = 0$. Thus, $\mu_i = \nu_i = \varphi_i = 1$, and so $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) = 1$.

Conversely, assume $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) = 1$, then it is straightforward that $\widetilde{\wp}_1 = \widetilde{\wp}_2$. Hence, (ii) holds.

Proposition 4.2. GOFSs $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ in X, $\overline{\rho}_\omega(\widetilde{\wp}_1, \widetilde{\wp}_2)$ satisfies.

(i) $\overline{\rho}_\omega(\widetilde{\wp}_1, \widetilde{\wp}_2) = \overline{\rho}_\omega(\widetilde{\wp}_2, \widetilde{\wp}_1)$,
(ii) $\overline{\rho}_\omega(\widetilde{\wp}_1, \widetilde{\wp}_2) = 1$ iff $\widetilde{\wp}_1 = \widetilde{\wp}_2$.

Proof. Straightforward

Theorem 4.3. If $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2)$ and $\overline{\rho}_\omega(\widetilde{\wp}_1, \widetilde{\wp}_2)$ are correlation coefficients between GOFSs $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ in X, then $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2), \overline{\rho}_\omega(\widetilde{\wp}_1, \widetilde{\wp}_2) \in [0, 1]$.

Proof. In this case, we need to first and foremost prove that $0 \le \overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) \le 1$, i.e., $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) \ge 0$ and $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) \le 1$. The fact that $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) \ge 0$ is clear. So, it suffices to show that $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) \le 1$. To establish this, let us assume that $\sum_{i=1}^n \mu_i(1 - \Delta\xi_i) = \mathcal{A}$, $\sum_{i=1}^n \nu_i(1 - \Delta\eta_i) = \mathcal{B}$, $\sum_{i=1}^n \varphi_i(1 - \Delta\pi_i) = \mathcal{C}$.

We then apply the principle of Cauchy–Schwarz inequality, and so

$$\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) = \frac{1}{3n}\sum_{i=1}^n \left(\mu_i(1 - \Delta\xi_i) + \nu_i(1 - \Delta\eta_i) + \varphi_i(1 - \Delta\pi_i)\right)$$

$$\le \frac{\sum_{i=1}^n \mu_i(1 - \Delta\xi_i) + \sum_{i=1}^n \nu_i(1 - \Delta\eta_i) + \sum_{i=1}^n \varphi_i(1 - \Delta\pi_i)}{3n}$$

$$= \frac{\mathcal{A} + \mathcal{B} + \mathcal{C}}{3n}.$$

Thus,

$$\begin{aligned} \overline{c}\rho(\widetilde{\wp}_1, \widetilde{\wp}_2) - 1 &= \frac{\mathcal{A} + \mathcal{B} + \mathcal{C}}{3n} - 1 \\ &= \frac{\mathcal{A} + \mathcal{B} + \mathcal{C} - 3n}{3n} \\ &= -\frac{(3n - \mathcal{A} - \mathcal{B} - \mathcal{C})}{3n} \le 0, \end{aligned}$$

which implies that $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) \le 1$. Hence, $\overline{\rho}(\widetilde{\wp}_1, \widetilde{\wp}_2) \in [0, 1]$. The proof of $\overline{\rho}_\omega(\widetilde{\wp}_1, \widetilde{\wp}_2) \in [0,1]$ is similar.

4.1 Numerical Verification

Some numerical examples of GOFSs are provided to verify the new approach to computing the correlation coefficient.

Example 1. The following are GOFSs defined in $X = \{x_1, x_2, x_3\}$:

$$\widetilde{\wp}_1 = \{\langle x_1, 0.1, 0.2\rangle, \langle x_2, 0.2, 0.1\rangle, \langle x_3, 0.29, 0.0\rangle\},$$

$$\widetilde{\wp}_2 = \{\langle x_1, 0.1, 0.3\rangle, \langle x_2, 0.2, 0.2\rangle, \langle x_3, 0.29, 0.1\rangle\}.$$

The GOFSs are very similar, so the correlation coefficient is expected to approach 1. Now, we find the correlation coefficient using Eq. (16) as follows:

From Table 1, we have $\Delta\xi_{\min} = \Delta\xi_{\max} = 0$, $\Delta\eta_{\min} = \Delta\eta_{\max} = 0.1$, and $\Delta\pi_{\min} = \Delta\pi_{\max} = 0.1$. Clearly, $\mu_1 = \nu_1 = \varphi_1 = 1$, $\mu_2 = \nu_2 = \varphi_2 = 1$, and $\mu_3 = \nu_3 = \varphi_3 = 1$. Thus, the correlation coefficient is.

$$\begin{aligned}\overline{\rho}\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) =& \frac{1}{9}(1(1-0) + 1(1-0.1) + 1(1-0.1) + 1(1-0) + 1(1-0.1) \\ &+ 1(1-0.1) + 1(1-0) + 1(1-0.1) + 1(1-0.1)) = 0.9333.\end{aligned}$$

From Table 2, we have $\Delta\ \xi_{\min} = \Delta\ \xi_{\max} = 0$, $\Delta\ \eta_{\min} = 0.01$, $\Delta\ \eta_{\max} = 0.05$, $\Delta\ \pi_{\min} = 0.0099$, and $\Delta\ \pi_{\max} = 0.05$. Thus, $\mu_1 = 1$, $\nu_1 = 0.9794$, $\varphi_1 = 0.9793$, $\mu_2 = 1$, $\nu_2 = 0.9897$, $\varphi_2 = 0.9896$, and $\mu_3 = \nu_3 = \varphi_3 = 1$. Hence, $\overline{\rho}\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 0.9869$.

From Table 3, we have $\Delta\xi_{\min} = \Delta\xi_{\max} = 0$, $\Delta\eta_{\min} = 0.01$, $\Delta\eta_{\max} = 0.019$, $\Delta\pi_{\min} = 0.0009$, and $\Delta\pi_{\max} = 0.019$. Thus, $\mu_1 = 1$, $\nu_1 = 0.994$, $\varphi_1 = 0.9939$, $\mu_2 = 1$, $\nu_2 = 0.998$, $\varphi_2 = 0.9979$, and $\mu_3 = \nu_3 = \varphi_3 = 1$. Hence, $\overline{\rho}\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 0.9962$.

From Table 4, we have $\Delta\xi_{\min} = \Delta\xi_{\max} = 0$, $\Delta\eta_{\min} = 0.0001$, $\Delta\eta_{\max} = 0.0065$, $\Delta\pi_{\min} = 0$, and $\Delta\pi_{\max} = 0.0068$. Thus, $\mu_1 = 1$, $\nu_1 = 0.9984$, $\varphi_1 = 0.9983$, $\mu_2 = 1$, $\nu_2 = 0.9996$, $\varphi_2 = 0.9996$, and $\mu_3 = \nu_3 = \varphi_3 = 1$. Hence, $\overline{\rho}\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 0.9988$.

The results prove the similarity between $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$; however, the correlation coefficient is better as q increases.

Example 2. Let $\widetilde{\wp}_3$ and $\widetilde{\wp}_4$ be GOFSs in $X = \{x_1, x_2\}$ defined as:

$$\widetilde{\wp}_3 = \{\langle x_1, 0.4, 0.3\rangle, \langle x_2, 0.3, 0.2\rangle\},$$

$$\widetilde{\wp}_4 = \{\langle x_1, 0.3, 0.2\rangle, \langle x_2, 0.2, 0.1\rangle\},$$

Table 1 Computation for $q = 1$

X	$\Delta\xi_i$	$\Delta\eta_i$	$\Delta\pi_i$
x_1	0	0.1	0.1
x_2	0	0.1	0.1
x_3	0	0.1	0.1

Table 2 Computation for $q = 2$

X	$\Delta\ \xi_i$	$\Delta\eta_i$	$\Delta\pi_i$
x_1	0	0.05	0.05
x_2	0	0.03	0.03
x_3	0	0.01	0.0099

Similarly, these GOFSs are very similar to each other, and it is expected that their correlation coefficient will be close to one but certainly not one since $\widetilde{\wp}_3 \neq \widetilde{\wp}_4$. Now, we find the correlation coefficient using Eqs. (9)-(15) and (16) for both of the examples and obtain the results in Table 5.

In Table 5, we observe that the new method of the correlation coefficient between GOFSs is more consistent with better performance indices. Though the results of the method in ref. [35] seem comparable to the new method, the results cannot be trusted because the method does not include the hesitation margins of the considered GOFSs. The results of the methods in refs. [36, 39] at $q = 1$ yield a perfect positive correlation coefficient, though the GOFSs are not equal in antithesis to the condition of the correlation coefficient. In addition, the methods in refs. [37, 39] also excluded the hesitation margins, so their outputs will not be reliable due to errors of omission.

In a nutshell, the new method upholds all the conditions of correlation measure and possesses accuracy and reliability. It resolves all the drawbacks in refs. [35–37, 39]. This present work is the extension and generalization of the intuitionistic fuzzy correlation coefficient in ref. [24] by (i) taking into cognizance the significance of hesitation margin to enhance reliable decision, (ii) removing the restriction in ref. [24], and (iii) generalizing the approach in ref. [24] to capture the properties of GOFSs.

5 Applications

In what follows, we discuss two applications of the novel approach, involving pattern recognition and medical diagnosis, to demonstrate the practicability of the proposed approach. Pattern recognition and medical diagnosis are decision-making problems that are engrossed with uncertainties, and so the better way to resolve these uncertainties is by deploying the idea of GOFSs, which is proven to have the capacity to tackle uncertainties in decision-making. To achieve more reliable output, we restricted the GOFSs to $q \geq 3$.

Table 3 Computation for $q = 3$

X	$\Delta\xi_i$	$\Delta\eta_i$	$\Delta\pi_i$
x_1	0	0.19	0.019
x_2	0	0.007	0.0071
x_3	0	0.001	0.0009

Table 4 Computation for $q = 4$

X	$\Delta\xi_i$	$\Delta\eta_i$	$\Delta\pi_i$
x_1	0	0.1065	0.0068
x_2	0	0.0015	0.0068
x_3	0	0.0001	0

5.1 Problem of Pattern Recognition

Here, we consider the case of pattern recognition discussed in refs. [42, 43]. There are three patterns, C_1, C_2, and C_3, which are represented by the following GOFSs in the given finite universe, $X = \{x_1, x_2, x_3\}$, respectively:

$$C_1 = \{(x_1, 1.0, 0.0), (x_2, 0.8, 0.0), (x_3, 0.7, 0.1)\}$$

$$C_2 = \{(x_1, 0.8, 0.1), (x_2, 1.0, 0.0), (x_3, 0.9, 0.0)\},$$

$$C_3 = \{(x_1, 0.6, 0.2), (x_2, 0.8, 0.0), (x_3, 1.0, 0.0)\}.$$

Given an unknown pattern Q, which is represented by the GOFS;

$$Q = \{(x_1, 0.5, 0.3), (x_2, 0.6, 0.2), (x_3, 0.8, 0.1)\},$$

the task is to classify pattern Q into one of the classes: C_1, C_2,and C_3. According to the recognition principle of the maximum degree of the correlation coefficient between GOFSs, the process of assigning the pattern Q to C_m is described by

$$m = \arg\max_{1 \le i \le 3} \{\overline{\rho}(C_i, Q)\} \tag{19}$$

By Eq. (16), we compute the correlation coefficient between C_i $(i = 1, 2, 3)$ and Q, as shown in Table 6.

We take $\overline{\rho}(C_1, Q) = \overline{\rho}_1$, $\overline{\rho}(C_2, Q) = \overline{\rho}_2$, and $\overline{\rho}(C_3, Q) = \overline{\rho}_3$ for simplicity's sake. According to the recognition rule given by Eq. (19), the pattern Q should be classified with C_3. This result is consistent with that found in refs. [42, 43]. Figure 2 shows the graphic picture of Table 6.

Similarly, from Fig. 2, we see that pattern Q is classified with C_3.

5.2 Problem of Disease Diagnosis

Medical diagnosis (Dx) is the procedure of determining which ailment or disease describes a patient's signs and symptoms. It is oftentimes called a "diagnosis'" with the medical context being understood. This procedure is often enmeshed with vagueness, which necessitates the deployment of GOFS technique.

Table 5 Correlation coefficient values

Methods	Example 1	Example 2
$\overline{\rho}$		
$q=1$	0.9333	0.8667
$q=2$	0.9869	0.9402
$q=3$	0.9962	0.9760
$q=4$	0.9988	0.9907
ρ_1		
$q=1$	0.9531	0.9941
$q=2$	0.9651	0.9944
$q=3$	0.9666	0.9957
$q=4$	0.9683	0.9968
ρ_2		
$q=1$	0.7811	0.6842
$q=2$	0.6877	0.5069
$q=3$	0.6079	0.3859
$q=4$	0.5441	0.2958
ρ_3		
$q=1$	0.9531	0.9941
$q=2$	0.9158	0.9891
$q=3$	0.8610	0.9875
$q=4$	0.8051	0.9880
ρ_4		
$q=1$	1.0000	1.0000
$q=2$	0.8685	0.9986
$q=3$	0.7587	0.9962
$q=4$	0.6826	0.9946
ρ_5		
$q=1$	1.0000	1.0000
$q=2$	0.7355	0.9991
$q=3$	0.7626	0.9973
$q=4$	0.7697	0.9955
ρ_6		
$q=1$	1.0000	1.0000
$q=2$	0.9881	0.9987
$q=3$	0.9889	0.9959
$q=4$	0.9936	0.9932
ρ_7		
$q=1$	1.0000	1.0000
$q=2$	0.7647	0.6598
$q=3$	0.6707	0.4473
$q=4$	0.6112	0.3093

Table 6 Ranking results by varying q

Varying q	$\bar{\rho}(C_1,Q)$	$\bar{\rho}(C_2,Q)$	$\bar{\rho}(C_3,Q)$	Ranking order
$q=3$	0.8628	0.8005	0.8744	$\bar{\rho}_3 \succ \bar{\rho}_1 \succ \bar{\rho}_2$
$q=4$	0.8074	0.7552	0.8485	$\bar{\rho}_3 \succ \bar{\rho}_2 \succ \bar{\rho}_1$
$q=5$	0.7821	0.7362	0.8405	$\bar{\rho}_3 \succ \bar{\rho}_1 \succ \bar{\rho}_2$
$q=6$	0.7741	0.7296	0.8351	$\bar{\rho}_3 \succ \bar{\rho}_1 \succ \bar{\rho}_2$
$q=7$	0.7736	0.7291	0.8309	$\bar{\rho}_3 \succ \bar{\rho}_1 \succ \bar{\rho}_2$
$q=8$	0.7763	0.7315	0.8273	$\bar{\rho}_3 \succ \bar{\rho}_1 \succ \bar{\rho}_2$
$q=9$	0.7800	0.7353	0.8241	$\bar{\rho}_3 \succ \bar{\rho}_1 \succ \bar{\rho}_2$
$q=10$	0.7838	0.7397	0.8211	$\bar{\rho}_3 \succ \bar{\rho}_1 \succ \bar{\rho}_2$

We consider the case of disease diagnosis discussed in ref. [44]. Let us consider a set of diagnoses.

$$Q = \{Q_1(\text{viral fever}),\ Q_2(\text{malaria}),\ Q_3(\text{typhoid}),\ Q_4(\text{stomach problem}),\ Q_5(\text{chest problem})\},$$

and a set of symptoms

$$S = \{s_1(\text{temperature}),\ s_2(\text{headache}),\ s_3(\text{stomach pain}),\ s_4(\text{cough}),'\ s_5(\text{chest pain})\}.$$

Each diagnosis Q_i $(i = 1, 2, 3, 4, 5)$ is represented by generalized orthopair fuzzy values in Table 7.

Assume that the following GOFS is used to represent a patient P called Bob in terms of all symptoms presented in Table 7.

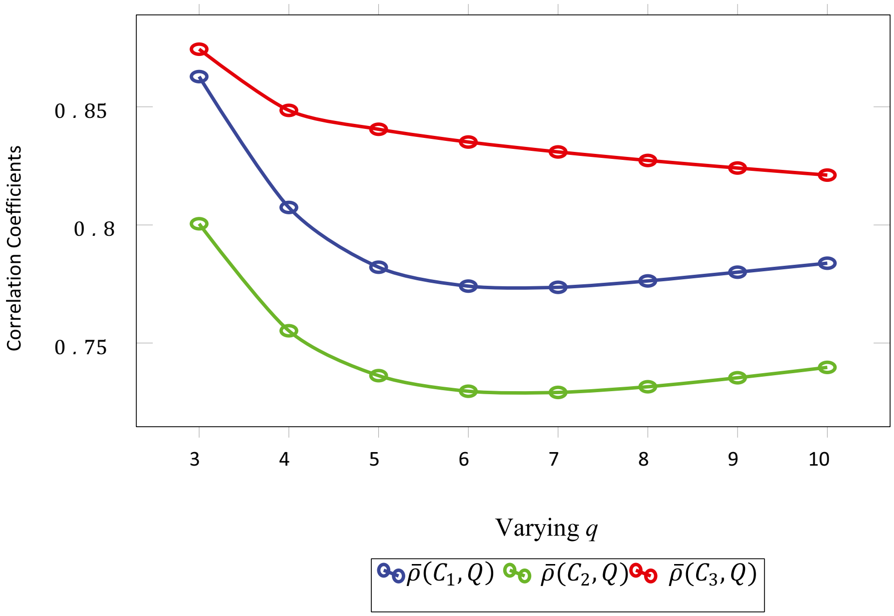


Fig. 2 Plot of Table 6

Table 7 Disease presentations in generalized orthopair fuzzy values

	s_1	s_2	s_3	s_4	s_5
Q_1	(0.4,0.0)	(0.3,0.5)	(0.1,0.7)	(0.4,0.3)	(0.1,0.7)
Q_2	(0.7,0.0)	(0.2,0.6)	(0.0,0.9)	(0.7,0.0)	(0.1,0.8)
Q_3	(0.3,0.3)	(0.6,0.1)	(0.2,0.7)	(0.2,0.6)	(0.1,0.9)
Q_4	(0.1,0.7)	(0.2,0.4)	(0.8,0.0)	(0.2,0.7)	(0.2,0.7)
Q_5	(0.1,0.8)	(0.0,0.8)	(0.2,0.8)	(0.2,0.8)	(0.8,0.1)

$$P = \{(s_1, 0.0, 0.8), (s_2, 0.4, 0.4), (s_3, 0.6, 0.1), (s_4, 0.1, 0.7), (s_5, 0.1, 0.8)\}$$

The goal is to figure out which of the diseases Q_i $(i = 1, 2, 3, 4, 5)$ is afflicting patient P. We use the recognition rule, $m = \arg\max_{1 \le i \le 5} \{\overline{\rho}(Q_i, P)\}$ because the medical diagnosis problem is basically a pattern recognition problem. By Eq. (16), we obtain the results in Table 8.

From the results in Table 8, patient P can thus be assigned to the disease Q_4 (stomach problem), which agrees with the conclusion reached in ref. [44]. Figure 3 shows the graphic picture of Table 8.

Similarly, from Fig. 3, we see that patient P should be treated for the disease Q_4 (stomach problem).

5.3 Comparative Analysis

Here, we present the comparative analysis of the new method of correlation coefficient under GOFSs with the existing methods of correlation coefficient under GOFSs in terms of the application examples in Sects. 5.1 and 5.2, respectively, to authenticate the present method. For comparison purposes, we consider the GOFSs for $q = 8$. For the application example in Sect. 5.1, we apply the new method of correlation coefficient under GOFSs with the existing methods of correlation coefficient under GOFSs in refs. [35–37, 39] and obtain the results in Table 9.

All the methods in Table 9 show that the pattern Q should be classed with C_3. Among the existing methods of the correlation coefficient for GOFSs, the methods in [36] are the only ones that incorporate the hesitation margins into the computations, besides the newly developed method. Between the methods in [36] and the new method, it is certain

Table 8 Ranking results by varying q

Varying q	$\overline{\rho}(Q_1, P)$	$\overline{\rho}(Q_2, P)$	$\overline{\rho}(Q_3, P)$	$\overline{\rho}(Q_4, P)$	$\overline{\rho}(Q_5, P)$
$q=3$	0.8969	0.9140	0.9084	0.9702	0.8943
$q=4$	0.8990	0.9022	0.8968	0.9582	0.8788
$q=5$	0.9143	0.9094	0.9032	0.9537	0.8866
$q=6$	0.9302	0.9198	0.9145	0.9541	0.9005
$q=7$	0.9439	0.9298	0.9295	0.9572	0.9154
$q=8$	0.9552	0.9387	0.9411	0.9617	0.9293
$q=9$	0.9643	0.9464	0.9500	0.9666	0.9415
$q=10$	0.9714	0.9529	0.9571	0.9716	0.9520

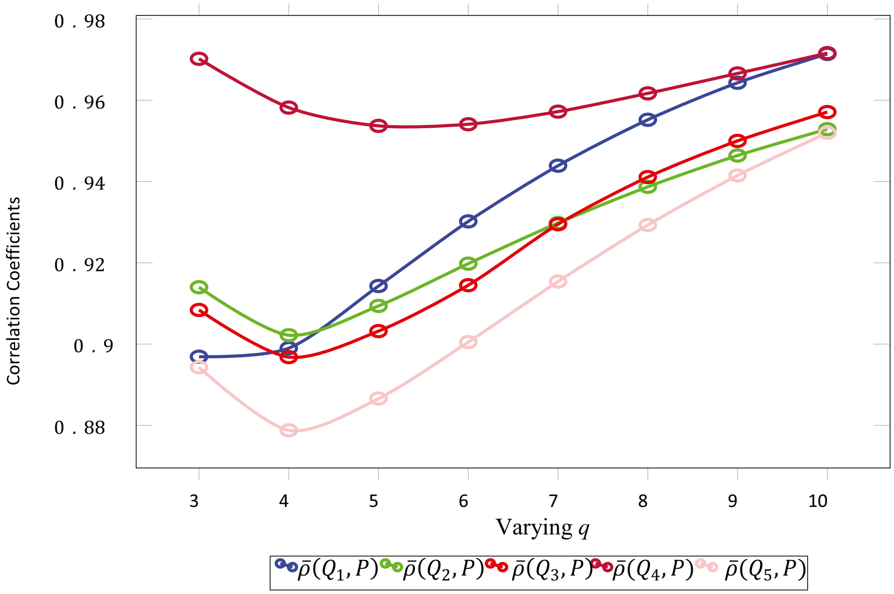


Fig. 3 Plot of Table 8

that the new method yields the most consistent, accurate, and reliable results. Though the method in [35] stands out, it is obvious due to the omission of hesitation margins.

Similarly, for the application example in Sect. 5.2, we apply the new method of correlation coefficient under GOFSs with the existing methods of correlation coefficient under GOFSs in [35–37, 39] and obtain the results in Table 10.

From the results in Table 10, we see that all the methods indicate that patient P is suffering from a disease Q_4 (stomach problem). It is only one of the methods in [37], i.e., ρ_5 that indicates that patient P is suffering from a disease Q_3 (typhoid). Again, we see that the newly developed method of correlation coefficient under GOFS yields the most consistent, accurate, and reliable results compare to the other methods. It is worth noting that the accuracy of the method in [35] is dependent on the nature of the data set because, while it yields somewhat better results in the first

Table 9 Comparative results for Sect. 5.1

Methods	(C_1, Q)	(C_2, Q)	(C_3, Q)
$\bar{\rho}$	0.7763	0.7315	0.8273
ρ_1	0.9438	0.9829	0.9993
ρ_2	−0.3484	−0.0297	0.7668
ρ_3	−0.2540	0.3160	0.8760
ρ_4	0.0159	0.0739	0.1659
ρ_5	0.0957	0.4827	0.9978
ρ_6	−0.2011	0.5029	0.9984
ρ_7	−0.0329	0.0008	0.1051

Table 10 Comparative results for Sect. 5.2

Methods	(Q_1, P)	(Q_2, P)	(Q_3, P)	(Q_4, P)	(Q_5, P)
$\overline{\rho}$	0.9552	0.9387	0.9411	0.9617	0.9293
ρ_1	0.5517	0.4102	0.7877	0.9838	0.6363
ρ_2	−0.2178	−0.1728	0.0440	0.2118	0.3840
ρ_3	0.1691	−0.0031	0.4139	0.5960	−0.0651
ρ_4	0.1616	0.1278	0.3870	0.4255	0.2695
ρ_5	0.4844	0.2451	0.6877	0.5333	0.4131
ρ_6	0.2093	0.1006	0.3567	0.9595	0.1555
ρ_7	0.0760	0.0506	0.1991	0.2372	0.1342

application example, it produces less accurate results in this example. This again justifies the advantage of the newly developed method of correlation coefficient under GOFSs in comparison to the existing methods.

6 Conclusion

The construct of generalized orthopair fuzzy correlation coefficient is a competent computational intelligence device applicable in cases of decision-making. Some approaches for measuring the correlation coefficient of GOFSs have been discussed in the literature, however, with certain drawbacks. In a quest to resolve these drawbacks, we developed a novel generalized orthopair fuzzy correlation coefficient method, which was proven to be more reliable compared to the existing approaches of measuring the generalized orthopair fuzzy correlation coefficient. In fact, the novel approach resolves all the drawbacks observed with the existing approaches. In order to validate the novel approach, some properties of the correlation coefficient were discussed based on the new approach. The novel generalized orthopair fuzzy correlation coefficient method was applied to decision-making problems involving pattern recognition and medical diagnosis to demonstrate its usefulness in computational intelligence based on the recognition principle. The superiority of the novel approach was substantiated via comparative analysis. For further research, the novel generalized orthopair fuzzy correlation coefficient approach could be investigated in other higher fuzzy variants and applied to real-life problems.

Author Contribution Paul Augustine Ejegwa contributed to the conception of the work, the drafting of the work, the computation of the results, and the interpretations. Arun Sarkar contributed to the conception, design, and drafting of the introduction and preliminaries of the article.

Availability of Data and Material Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Code availability Available on request.

Declarations

Competing interests The authors declare no competing interests.

Ethics Approval Not applicable.

Consent to Participate Not applicable.

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Conflict of Interest The authors declare no conflict of interest.

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Development of Archimedean power Heronian mean operators for aggregating linguistic *q*-rung orthopair fuzzy information and its application to financial strategy making

Nayana Deb[1] · Arun Sarkar[2] · Animesh Biswas[1]

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Abstract

The linguistic *q*-rung orthopair fuzzy (L*q*-ROF) set (L*q*-ROFS) is an important implement in the research area in modelling vague decision information by incorporating the advantages of *q*-rung orthopair fuzzy sets with linguistic variables. To effectively fuse the L*q*-ROF information, this paper introduces a series of L*q*-ROF power Heronian mean operators with their weighted variants based on Archimedean operations. The defined operators meet certain important characteristics such as they can provide generality and flexibility in the aggregation process, model uncertainty of decision-making with interrelated attributes, and the influence of unreasonable attribute values can also be reduced by them. A generalized distance measure and an entropy measure were defined for L*q*-ROFSs, subsequently. Afterwards, a multi-attribute group decision-making approach under L*q*-ROF context, where the information about decision-makers' and attribute weights might be known or unknown, is designed utilizing individual expressions of the proposed operators. The developed approach includes a TOPSIS-based algorithm and an entropy-based algorithm in order to compute the weight of decision-makers' and attributes, respectively. Finally, the proposed method's feasibility is validated by solving a financial strategy-making problem, and its superiority is demonstrated via some detailed comparisons.

Keywords Linguistic *q*-rung orthopair fuzzy sets · Power Heronian mean · Entropy measure · Distance measure · Multi-attribute group decision-making · Unknown weight evaluation

1 Introduction

In decision theory, multi-attribute decision-making (MADM) is one of the important branches. It is a process to find an optimal solution from a set of feasible alternatives satisfying multiple attributes. Due to increasing complexities in daily life, it becomes hard for a single decision-maker (DM) to make a proper decision by considering all relevant aspects of a certain problem. Consequently, the assessment values provided by a single DM may be inaccurate. While the multi-attribute group decision-making (MAGDM) method can provide more accurate assessment results under ambiguous and complex situations considering the cognition of a group of DMs instead of individual DM. MAGDM is used to organize and resolve planning and decision-making problems as well as to evaluate the most advantageous alternative based on expert(s) decision-supporting given attributes. Nevertheless, due to the imprecise cognition of human beings and the presence of ambiguities in decision problems, it is often difficult to handle decision-making processes using crisp numbers. To deal with such circumstances, fuzzy set theory (Zadeh 1965) in decision-making has been appeared as an efficient technique and has been successfully applied in various fields. Moreover, several extensions of fuzzy sets have been developed in the literature, viz. intuitionistic fuzzy set (IFS) (Atanassov 1986), Pythagorean fuzzy set (PFS) (Yager et al. 2013, 2014), and *q*-rung orthopair fuzzy set

✉ Arun Sarkar
asarkarmth@gmail.com

Animesh Biswas
abiswaskln@rediffmail.com

1 Department of Mathematics, University of Kalyani, Kalyani 741235, India

2 Department of Mathematics, Heramba Chandra College, Kolkata 700029, India

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(q-ROFSs) (Yager 2017). All the above-mentioned variants of fuzzy sets can deal with problems using quantitative information. Sometimes, it seems insufficient and inadequate to express evaluation values using quantitative forms when decision-making problems become more complex. To deal with such situations, Zadeh (1975) introduced the linguistic concept in fuzzy sets. It can reflect the DMs' perception more rationally and comprehensively under various vague situations. The notion of linguistic term set (LTS) was first introduced by Herrera et al. (1996) for the assessment of qualitative information. Several applications of LTS are found in various fields (Teng et al. 2022; Luo et al. 2022). Xu (2005) proposed the idea of continuous LTS (CLTS) to reduce the loss of information in the computation procedure. With the advancement of CLTS, many researchers paid their attention to develop several theories combining CLTS and variants of fuzzy sets. Zhang (2014) developed linguistic intuitionistic fuzzy (LIF) sets (LIFSs) imposing linguistic concepts on IFS. In LIFSs, each element's membership and non-membership degrees satisfy the condition that the sum of their subscripts needs to be less than or equal to the carnality of the LTS under consideration. Kumar and Chen (Kumar and Chen 2022) developed an approach to solve MAGDM problems under LIF context. Garg and Kumar (Garg and Kumar 2018a) utilized Einstein t-norm (t-N) and t-conorm (t-CN) operations to define a series of LIF aggregation operators for developing MAGDM approach based on possibility degree measures. LIFS theory has been further applied in a variety of decision-making processes (Garg and Kumar 2018b; Liu and Liu 2018a, 2020; Liu et al. 2018) successfully.

Apart from the advantages of LIFSs, there are some limitations. For example, LIFS fails to handle situations when a DM provides his/her decision information as (s_3, s_4) under a CLTS, $\{s_\alpha | \alpha \in [0,6].\}$, as $3+4 \nleq 6$. To overcome this issue, Garg (2018) proposed linguistic PFS (LPFS) by integrating PFS with LTS. In LPFS, the square sum of subscripts of membership and non-membership degrees is not beyond the square of the cardinality of corresponding LTS. After the inception of LPFS, several research works are going on. Lin et al. (2018) extended the partition BM operator to LPFSs and developed LPF-weighted interaction PBM operator along with its geometric form. Based on the correlation coefficient and entropy measure, Lin et al. (2019a) further proposed a novel TOPSIS method to solve MADM problems under LPF environment. Liu et al. (2019a) introduced a series of generalized LPF aggregation operators. Liu et al. (Liu et al. 2020a) introduced Pythagorean fuzzy linguistic-Muirhead mean and dual Muirhead mean aggregation operators to deal with MADM problems. Recently, Sarkar and Biswas (Sarkar and Biswas 2021a) defined some distance and entropy measures for LPFSs and introduced a TOPSIS method to solve MCGDM problems. Within a short span, the LPFS theory has attracted researchers to perform research works using this concept.

Sometimes, the condition of LIFS or LPFS may be violated under some linguistic decision-making situations. To overcome that drawback, Liu and Liu (2018b) presented linguistic q-ROFS (Lq-ROFS). Using Lq-ROFS, the scope of selection of linguistic membership and non-membership grades is increased than LIFS and LPFS. Liu and Liu (2018b) further proposed generalized power BM operators and Hamming distance measures under Lq-ROF environment. Moreover, incorporating Muirhead mean operator, Liu and Liu (Liu and Liu 2019) developed several aggregation operators on Lq-ROF environment. Considering the cosine similarity measure, Peng et al. (2016) constructed some new similarity measures of Lq-ROFSs. Lin et al. (2019b) investigated MADM problems by developing partitioned HM-based Lq-ROF operators.

Since the complexity in real decision-making situations is constantly increasing, the following perspectives are needed to be considered for generating an efficient Lq-ROF information aggregation tool aiming to solve MAGDM problems: by reducing the effect of unduly low and unduly high arguments

(1) When a biased DM is involved, the aggregate result is adversely affected by certain extreme assessment values that he/ she provided. To resolve that issue, Yager (2001) introduced power average (PA) aggregation operator, which has the ability to reduce the effect of unduly low and unduly high arguments. PA reinforces the unreasonable assessment values by calculating the support measures and assigning them to produce different power weights. Therefore, PA operators can be utilized as an effective tool to relieve such biasness in the evaluation computation processes under various fuzzy contexts (Biswas and Deb 2021).

(2) In practical MAGDM problems, the attributes are not always independent, i.e. interrelationships between attributes are often found. The aggregation operators having an assumption that the aggregated arguments are independent fail to produce accurate decision results. Meanwhile, there exist some novel aggregation operations, viz., Bonferroni mean (BM) and Heronian mean (HM), which can deal with interrelated input arguments. However, due to the characteristics that it neglects the calculation redundancy and considers the association between an attribute and itself, HM is more beneficial than BM (Deb and Biswas 2021; Sarkar and Biswas 2021b).

(3) To deal with MAGDM issues, the prerequisite aggregation operators must be general and

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adaptable enough to account for all DM preferences while aggregating the evaluation values. The aggregation operators combined with Archimedean t-N and t-CN (Klir and Yuan 1995; Nguyen and Walker 1997) (At-N&t-CNs) are more flexible and more versatile for aggregating fuzzy information (Sarkar and Biswas 2019, 2021c; Sarkar et al. 2021). However, most of the existing aggregation operators under Lq-ROF environment are based on algebraic t-N and t-CN, which are only a class of At-N&t-CNs. So, involving Archimedean operations in the aggregation function will result in the development of a variety of flexible aggregation operators (Beliakov et al. 2007) using algebraic, Einstein, Hamacher, Dombi, Frank and more operations for the Lq-ROFNs. Therefore, the aggregation operators incorporating such operational rules will generate flexible and reasonable results in MAGDM context.

Therefore, with the view of the above discussions, this paper is aimed to define some At-N&t-CN based Lq-ROF PA aggregation operators, viz. Lq-ROF Archimedean PA (Lq-ROFAPA) and Lq-ROF Archimedean power-weighted average (Lq-ROFAPWA) operators. Also, combining PA with HM operator, a series of aggregation operators, viz. Lq-ROF Archimedean power HM (Lq-ROFAPHM) and Lq-ROF Archimedean power-weighted HM (Lq-ROFAPWHM) operators are introduced in order to develop a MAGDM approach.

In the process of MAGDM, experts' weights and attributes' weights play important roles according to different abilities, interests, and situations. As per the authors' knowledge, there are few researches on MAGDM with unknown weights for both experts and attributes under Lq-ROF environments that exist in the literature. To cope with situations like this, a TOPSIS-based algorithm has been presented in this paper to derive the DMs' weights, and to fix the attributes' weights, an entropy-based algorithm is also proposed. It is to be mentioned here that a generalized distance measure and an entropy measure have been presented for generating DMs' weights and attributes' weights, respectively. Hence, the proposed MAGDM methodology can help DMs to make a robust decision despite the unavailability in weights of both experts and attributes.

In order to acquire the above objectives, this paper is organized as follows: Section 2 consists of some basic concepts in connection with Lq-ROFSs. In Sect. 3, firstly, At-N&t-CN-based operational laws of the Lq-ROFNs are defined, and then, generalized distance measure and entropy measure of Lq-ROFSs are established. Based on the new operational rules of Lq-ROFNs and PA and HM operators, several aggregation operators, viz. Lq-ROFAPA, Lq-ROFAPWA, Lq-ROFAPHM and Lq-ROFAPWHM operators, are introduced followed by discussing their properties and special cases in Sect. 4. Section 5 is devoted to construct MAGDM approach under Lq-ROF environment. For calculating the DMs' weights and attributes' weights, algorithm 1 and algorithm 2, respectively, are also provided in this section. Consequently, a practical example is provided in Sect. 6 to reveal the effectiveness and advantages of the proposed method. In Sect. 7, a comparative analysis with the existing approaches is presented, and some conclusions of this study are made in Sect. 8.

2 Preliminaries

In this section, some basic concepts of LTS, q-ROFS, At-N&t-CNs, PA and HM operators are briefly reviewed.

2.1 LTS

Definition 1 Herrera et al. (Herrera et al. 1996) Let $\mathcal{S} = \{\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell\}$ be a finite-ordered discrete set with odd cardinality. The set $\mathcal{S}$ would represent LTS if the following conditions are satisfied:

(i) The set is ordered: if $i > j$, then $\mathcal{S}_i > \mathcal{S}_j$, which means $\mathcal{S}_i$ is superior to $\mathcal{S}_j$.
(ii) Negation operator: $neg(\mathcal{S}_i) = \mathcal{S}_j$, where $j = \ell - i$.
(iii) Min operator: if $i \le j$, i.e. $\mathcal{S}_i \le \mathcal{S}_j$, then $\min(\mathcal{S}_i, \mathcal{S}_j) = \mathcal{S}_i$.
(iv) Max operator: if $i \ge j$, i.e. $\mathcal{S}_i \ge \mathcal{S}_j$ then $\max(\mathcal{S}_i, \mathcal{S}_j) = \mathcal{S}_i$.

The semantic of $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_\ell$ depend on real situations under consideration, for example, in the context of quality assessment of the mobile phone, the LTS that may be taken as

$$\begin{aligned}\mathcal{S} &= \{\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6\} \\ &= \{extreme\ low, very\ low, low, medium, high, veryhigh, \\ &\quad extreme\ high\}\end{aligned}$$

Xu (2005) extended the concept of discrete LTS, $\mathcal{S}$, and introduced CLTS, $\mathcal{S}_{[0,\ell]}$, in the form of $\mathcal{S}_{[0,\ell]} = \{\mathcal{S}_h | \mathcal{S}_0 \le \mathcal{S}_h \le \mathcal{S}_\ell, h \in [0, \ell]\}$ satisfying all the conditions of discrete LTS.

Combining the concept of CLTS with q-ROFS, Liu and Liu (2018b) introduced the notion of Lq-ROFSs as follows:

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2.2 Lq-ROFS

Definition 2 Liu and Liu (2018b) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse. An Lq-ROFS $\widetilde{A}$ defined on X is represented by:

$$\tilde{A} = \left\{ \left\langle x, \mathcal{S}_{\xi_{\tilde{A}}}(x), \mathcal{S}_{\eta_{\tilde{A}}}(x) \right\rangle | x \in X \right\}, \quad (1)$$

where $\mathcal{S}_{\xi_{\tilde{A}_i}}, \mathcal{S}_{\eta_{\tilde{A}_i}} \in \mathcal{S}_{[0,\ell]}$ express the linguistic membership degree and linguistic non-membership degree of $x \in X$, respectively, satisfying the condition $0 \le (\xi_{\tilde{A}_i})^q + (\eta_{\tilde{A}_i})^q \le \ell^q$ $(q \ge 1)$. $\tilde{\alpha} = \langle \mathcal{S}_\xi, \mathcal{S}_\eta \rangle$ is called as the Lq-ROFN and $\mathcal{S}_{\pi_{\tilde{\alpha}}}(x) = \mathcal{S}_{(\ell^q - \xi^q - \eta^q)^{\frac{1}{q}}}$ is known as the linguistic indeterminacy degree of x to $\tilde{\alpha}$. The family of all Lq-ROFSs in X is denoted by Lq-ROFS(X). The graphical representation of Lq-ROFS is depicted in Fig. 1.

In Fig. 1, the triangular region joining the points $(0,0)$, $(0,\ell)$ and $(\ell,0)$ represents the satisfying zone of LIFNs, whereas the circular region in the first quadrant having centred at the origin and radius ℓ designates the satisfying region of LPFNs. Similarly, the elliptic region in first quadrant having centre at $(0,0)$ represents the satisfying region of Lq-ROFNs. From the figure, it is clear that the satisfying zone of Lq-ROFNs fully covers the satisfying zone of LIFNs as well as LPFNs, which is the advantage of choosing Lq-ROFNs.

Liu and Liu (2018b) also defined the score function LS($\tilde{\alpha}$) and the accuracy function LA($\tilde{\alpha}$) as follows.

Definition 3 Liu and Liu (2018b) Let $\tilde{\alpha} = \langle \mathcal{S}_\xi, \mathcal{S}_\eta \rangle$ be an Lq-ROFN, the score function $LS(\tilde{\alpha})$ and accuracy function $LA(\tilde{\alpha})$ of the Lq-ROFN can be defined as:

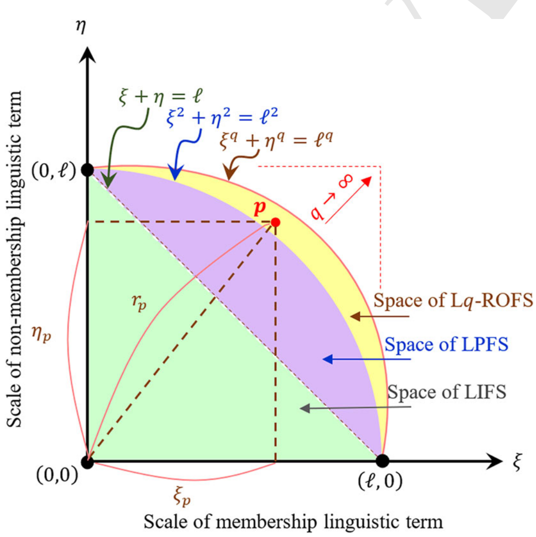

Fig. 1 Graphical representation of Lq-ROFS

$$\mathrm{LS}(\tilde{\alpha}) = \left(\frac{\ell^q + \xi^q - \eta^q}{2} \right)^{\frac{1}{q}}, \quad (2)$$

and

$$\mathrm{LA}(\tilde{\alpha}) = (\xi^q + \eta^q)^{\frac{1}{q}}. \quad (3)$$

To compare any two Lq-ROFNs, the following comparison method is presented based on the above-defined functions.

Definition 4 Liu and Liu (2018b) Let $\tilde{\alpha}_1 = \langle \mathcal{S}_{\xi_1}, \mathcal{S}_{\eta_1} \rangle$, $\tilde{\alpha}_2 = \langle \mathcal{S}_{\xi_2}, \mathcal{S}_{\eta_2} \rangle$ be any two Lq-ROFNs.

(i) If $\mathrm{LS}(\tilde{\alpha}_1) < \mathrm{LS}(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \prec \tilde{\alpha}_2$
(ii) If $\mathrm{LS}(\tilde{\alpha}_1) = \mathrm{LS}(\tilde{\alpha}_2)$, then

- if $\mathrm{LA}(\tilde{\alpha}_1) < \mathrm{LA}(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \prec \tilde{\alpha}_2$ which means $\tilde{\alpha}_2$ is better than $\tilde{\alpha}_1$;
- if $LA(\tilde{\alpha}_1) = LA(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \approx \tilde{\alpha}_2$, which means $\tilde{\alpha}_1$ is equal to $\tilde{\alpha}_2$.

Example 1 An expert assesses the quality of steering system compliance for three cars and provides the evaluation information as $\tilde{\alpha}_1 = \langle \mathcal{S}_5, \mathcal{S}_4 \rangle$, $\tilde{\alpha}_2 = \langle \mathcal{S}_6, \mathcal{S}_4 \rangle$, $\tilde{\alpha}_3 = \langle \mathcal{S}_4, \mathcal{S}_3 \rangle \in \mathcal{S}_{[0,8]}$, respectively. Let $q = 3$, then, the score functions are calculated as:

$$\mathrm{LS}(\tilde{\alpha}_1) = 6.5924, \mathrm{LS}(\tilde{\alpha}_2) = 6.9244, \mathrm{LS}(\tilde{\alpha}_3) = 6.4990$$

and then it follows that $\tilde{\alpha}_2 \succ \tilde{\alpha}_1 \succ \tilde{\alpha}_3$.

2.3 At-N&t-CNs

Definition 5 Klir and Yuan (1995), Nguyen and Walker (1997) The Archimedean t-Ns (At-Ns) and Archimedean t-CNs (At-CNs) can be produced by the relative additive generators (Klement and Mesiar 2005). The At-N is characterized by $\mathcal{I}(a,b) = f^{-1}(f(a) + f(b))$, where f is a monotonically decreasing function and satisfies $f(a) : [0,m] \to \mathbb{R}^+$, $f^{-1}(a) : \mathbb{R}^+ \to [0,m]$, $\lim_{a\to\infty} f^{-1}(a) = 0$ and $f^{-1}(0) = 1$. The At-CN is formed as $\mathcal{U}(a,b) = g^{-1}(g(a) + g(b))$, where g is a monotonically increasing function and satisfies $g(a) : [0,m] \to \mathbb{R}^+$, $g^{-1}(a) : \mathbb{R}^+ \to [0,m]$, $\lim_{a\to\infty} g^{-1}(a) = 0$ and $g^{-1}(0) = 0$ with $g(x) = f(1-x)$.

2.4 HM operator

Definition 6 Sykora (2009) Let a_i $(i = 1, 2, \ldots, n)$ be a collection of non-negative numbers. If

$$\mathrm{HM}(a_1, a_2, \ldots, a_n) = \frac{2}{n(n+1)} \sum_{\substack{i,j=1 \\ i \le j}}^{n} \sqrt{a_i a_j}$$

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, then $\mathrm{HM}(a_1, a_2, \ldots, a_n)$ is called the HM.

Sykora (2009) extended basic HM to a generalized form by introducing two parameters ϕ and ψ as follows:

$$\mathrm{HM}^{\phi,\psi}(a_1, a_2, \ldots, a_n) = \left(\frac{2}{n(n+1)} \sum_{\substack{i,j=1 \\ i \le j}}^{n} (a_i)^{\phi} (a_j)^{\psi} \right)^{\frac{1}{\phi+\psi}}. \tag{4}$$

It is well known that HM has the advantage of considering the interrelationships between any two input arguments.

2.5 The power average operator

The PA operator (Yager 2001) is an important aggregation tool that can reduce the negative impacts of unreasonable high or low input values given by any biased DM. It is defined as follows:

Definition 7 Yager (2001) Let $(a_1, a_2, \ldots, a_n)$ be a set of evaluated values; then, the PA operator is the mapping defined by

$$\mathrm{PA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} \frac{(1 + T(a_i))}{\sum_{k=1}^{n} (1 + T(a_k))} a_i, \tag{5}$$

where $T(a_i) = \sum_{\substack{j=1 \\ i \ne j}}^{n} Sup(a_i, a_j)$, $\mathrm{Sup}(a_i, a_j) = 1 - d(a_i, a_j)$ and $\mathrm{Sup}(a_i, a_j)$ is the support degree for a_i from a_j, which satisfies the following three properties:

(1) $\mathrm{Sup}(a_i, a_j) \in [0, 1]$;
(2) $\mathrm{Sup}(a_i, a_j) = \mathrm{Sup}(a_j, a_i)$;
(3) if $d(a_i, a_j) \le d(a_l, a_r)$, then $Sup\ (a_i, a_j) \ge Sup\ (a_l, a_r)$, where (a_i, a_j) represents the distance between a_i and a_j.

3 At-N&t-CNs-based operational laws and measures for Lq-ROFNs

In this section, some basic operational rules, distance and entropy measures, are defined successively.

3.1 Fundamental operational laws for Lq-ROFNs based on At-N&t-CNs

Following the concepts of At-N&t-CNs, the following operational laws on Lq-ROFNs are defined.

Definition 8 Let $\overline{\mathcal{S}} = \{\mathcal{S}_\sigma : \sigma \in [0, \ell]\}$ be a CLTS, $\tilde{\alpha}_1 = \langle \mathcal{S}_{\xi_1}, \mathcal{S}_{\eta_1} \rangle$, $\tilde{\alpha}_2 = \langle \mathcal{S}_{\xi_2}, \mathcal{S}_{\eta_2} \rangle$ and $\tilde{\alpha} = \langle \mathcal{S}_{\xi}, \mathcal{S}_{\eta} \rangle$ be three Lq-ROFNs, $\lambda > 0$. Then, the Archimedean operational laws are defined as

(i)

$$\tilde{\alpha}_1 \oplus_A \tilde{\alpha}_2 = \mathcal{S}_{\mathcal{U}(\xi_1,\xi_2)}, \mathcal{S}_{\mathcal{I}(\eta_1,\eta_2)} = \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(g\left(\left(\frac{\xi_1}{\ell}\right)^q\right)+g\left(\left(\frac{\xi_2}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(f^{-1}\left(f\left(\left(\frac{\eta_1}{\ell}\right)^q\right)+f\left(\left(\frac{\eta_2}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}} \right\rangle;$$

(ii)

$$\tilde{\alpha}_1 \otimes_A \tilde{\alpha}_2 = \mathcal{S}_{\mathcal{I}(\xi_1,\xi_2)}, \mathcal{S}_{\mathcal{U}(\eta_1,\eta_2)} = \left\langle \mathcal{S}_{\ell\left(f^{-1}\left(f\left(\left(\frac{\xi_1}{\ell}\right)^q\right)+f\left(\left(\frac{\xi_2}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(g^{-1}\left(g\left(\left(\frac{\eta_1}{\ell}\right)^q\right)+g\left(\left(\frac{\eta_2}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}} \right\rangle$$

(iii) $\lambda \odot_A \tilde{\alpha} = \mathcal{S}_{\ell\left(g^{-1}\left(\lambda g\left(\left(\frac{\xi}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(f^{-1}\left(\lambda f\left(\left(\frac{\eta}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}$;

(iv) $\tilde{\alpha}^{\lambda} = \mathcal{S}_{\ell\left(f^{-1}\left(\lambda f\left(\left(\frac{\xi}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(g^{-1}\left(\lambda g\left(\left(\frac{\eta}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}$.

In the following subsections, two information measures, viz. distance and entropy measures, for Lq-ROFSs are defined, and their necessary properties are examined.

3.2 Generalized distance measure of Lq-ROFSs

Distance measures are extensively used to calculate the closeness indices of various arguments. In this section, the distance between any two Lq-ROFNs is defined by utilizing linguistic membership, non-membership, and indeterminacy degrees.

Definition 9 Let $\{\tilde{\alpha}_i = \langle \mathcal{S}_{\xi_i}, \mathcal{S}_{\eta_i} \rangle | i = 1, 2, 3\}$ be any three Lq-ROFNs and $\varepsilon \ge 1$. The generalized distance between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, denoted by $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2)$, is defined as

$$d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{\ell^q} \left(\frac{1}{2} \left(\left|\xi_1^q - \xi_2^q\right|^{\varepsilon} + \left|\eta_1^q - \eta_2^q\right|^{\varepsilon} + \left|\pi_{\tilde{\alpha}_1}^q - \pi_{\tilde{\alpha}_2}^q\right|^{\varepsilon} \right) \right)^{\frac{1}{\varepsilon}}, \tag{6}$$

satisfying the properties:

(i) $0 \le d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) \le 1$;
(ii) If $\tilde{\alpha}_1 = \tilde{\alpha}_2$, then $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) = 0$;
(iii) $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) = d_{\mathcal{G}}(\tilde{\alpha}_2, \tilde{\alpha}_1)$;
(iv) $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) + d_{\mathcal{G}}(\tilde{\alpha}_2, \tilde{\alpha}_3) \ge d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_3)$.

Proof (i) $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{\ell^q}\left(\frac{1}{2}\left(|{\xi_1}^q - {\xi_2}^q|^{\varepsilon} + |{\eta_1}^q - {\eta_2}^q|^{\varepsilon} + |{\pi_{\tilde{\alpha}_1}}^q - {\pi_{\tilde{\alpha}_2}}^q|^{\varepsilon}\right)\right)^{\frac{1}{\varepsilon}}$ i.e.

$$d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) \le \frac{1}{\ell^q} \left(\frac{1}{2} \left(\max\left|\xi_1^q - \xi_2^q\right|^{\varepsilon} + \max\left|\eta_1^q - \eta_2^q\right|^{\varepsilon} + \left|\pi_{\tilde{\alpha}_1}^q - \pi_{\tilde{\alpha}_2}^q\right|^{\varepsilon} \right) \right)^{\frac{1}{\varepsilon}} \tag{7}$$

The expression in the R.H.S. is maximum when $\xi_1 = \ell$ $(\eta_1 = 0)$ and $\xi_2 = 0$ $(\eta_2 = \ell)$ or, $\xi_1 = 0$ $(\eta_1 = \ell)$ and $\xi_2 =$

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ℓ $(\eta_2 = 0)$, then $\pi_{\tilde{\alpha}_1} = 0$, $\pi_{\tilde{\alpha}_2} = 0$, i.e. $0 \le d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) \le \frac{1}{\ell^q}\left(\frac{1}{2}\left(\ell^{q\varepsilon} + \ell^{q\varepsilon}\right)\right)^{\frac{1}{\varepsilon}} = 1.$

The proof of conditions (ii) and (iii) can be easily derived from the definition of distance measure.

$$(iv)\, d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) + d_{\mathcal{G}}(\tilde{\alpha}_2, \tilde{\alpha}_3)$$
$$= \frac{1}{\ell^q}\left(\frac{1}{2}\left(\left|\xi_1^q - \xi_2^q\right|^{\varepsilon} + \left|\eta_1^q - \eta_2^q\right|^{\varepsilon} + \left|\pi_{\tilde{\alpha}_1}^q - \pi_{\tilde{\alpha}_2}^q\right|^{\varepsilon}\right)\right)^{\frac{1}{\varepsilon}}$$
$$+ \frac{1}{\ell^q}\left(\frac{1}{2}\left(\left|\xi_2^q - \xi_3^q\right|^{\varepsilon} + \left|\eta_2^q - \eta_3^q\right|^{\varepsilon} + \left|\pi_{\tilde{\alpha}_2}^q - \pi_{\tilde{\alpha}_3}^q\right|^{\varepsilon}\right)\right)^{\frac{1}{\varepsilon}}$$
$$\ge \frac{1}{\ell^q 2^{\frac{1}{\varepsilon}}}\left(\left(\left|\xi_1^q - \xi_2^q + \xi_2^q - \xi_3^q\right|^{\varepsilon} + \left|\eta_1^q - \eta_2^q + \eta_2^q - \eta_3^q\right|^{\varepsilon} + \left|\pi_{\tilde{\alpha}_1}^q - \pi_{\tilde{\alpha}_2}^q + \pi_{\tilde{\alpha}_2}^q - \pi_{\tilde{\alpha}_3}^q\right|^{\varepsilon}\right)^{\frac{1}{\varepsilon}}\right)$$

(Using Minkowski's inequality)

$$= \frac{1}{\ell^q}\left(\frac{1}{2}\left(\left|\xi_1^q - \xi_3^q\right|^{\varepsilon} + \left|\eta_1^q - \eta_3^q\right|^{\varepsilon} + \left|\pi_{\tilde{\alpha}_1}^q - \pi_{\tilde{\alpha}_3}^q\right|^{\varepsilon}\right)\right)^{\frac{1}{\varepsilon}}$$
$$= d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_3)$$

Therefore, $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) + d_{\mathcal{G}}(\tilde{\alpha}_2, \tilde{\alpha}_3) \ge d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_3)$.

Now, three special cases of $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2)$ are discussed for Lq-ROFNs:

1. If $\varepsilon = 1$, $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2)$ reduces to a Lq-ROF Hamming distance

$$d_H(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{\ell^q}\left(\frac{1}{2}\left(\left|\xi_1^q - \xi_2^q\right| + \left|\eta_1^q - \eta_2^q\right| + \left|\pi_{\tilde{\alpha}_1}^q - \pi_{\tilde{\alpha}_2}^q\right|\right)\right)$$

2. If $\varepsilon = 2$, then $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2)$ is reduced to a Lq-ROF Euclidean distance

$$d_E(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{\ell^q}\left(\frac{1}{2}\left(\left|\xi_1^q - \xi_2^q\right|^2 + \left|\eta_1^q - \eta_2^q\right|^2 + \left|\pi_{\tilde{\alpha}_1}^q - \pi_{\tilde{\alpha}_2}^q\right|^2\right)\right)^{\frac{1}{2}}$$

3. If $\varepsilon \to +\infty$, then $d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2)$ is reduced to a Lq-ROF Chebyshev distance

$$d_{+\infty}(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{\ell^q}\max\left(\left|\xi_1^q - \xi_2^q\right|, \left|\eta_1^q - \eta_2^q\right|, \left|\pi_{\tilde{\alpha}_1}^q - \pi_{\tilde{\alpha}_2}^q\right|\right)$$

Example 2 Consider two Lq-ROFNs, $\tilde{\alpha}_1 = \langle \mathcal{S}_7, \mathcal{S}_6 \rangle$ and $\tilde{\alpha}_2 = \langle \mathcal{S}_3, \mathcal{S}_5 \rangle$, where $\mathcal{S}_i(x) \in \mathcal{S}_{[0,8]}$. The generalized distance between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ (assigning $q = 3$, $\varepsilon = 4$) is obtained by using Definition 9. Here, $\pi_{\tilde{\alpha}_1}{}^3 = 8^3 - 7^3 - 6^3 = -47$ and $\pi_{\tilde{\alpha}_2}{}^3 = 8^3 - 3^3 - 5^3 = 360$, then

$$d_{\mathcal{G}}(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{8^3}\left(\frac{1}{2}\left(\left|7^3 - 3^3\right|^4 + \left|6^3 - 5^3\right|^4 + \left|-47 - 360\right|^4\right)\right)^{\frac{1}{4}} = 0.7226.$$

On the basis of the concept of generalized distance between Lq-ROFNs, in this subsection, the distance measure among two Lq-ROFSs is defined as follows:

Definition 10 Let $\widetilde{A}$ and $\widetilde{B}$ be two Lq-ROFSs in X representing the collection of Lq-ROFNs $\tilde{\alpha}_i = \langle \mathcal{S}_{\xi_{\tilde{\alpha}_i}}, \mathcal{S}_{\eta_{\tilde{\alpha}_i}} \rangle$ and $\tilde{\beta}_i = \langle \mathcal{S}_{\xi_{\tilde{\beta}_i}}, \mathcal{S}_{\eta_{\tilde{\beta}_i}} \rangle$, respectively, and $\widetilde{\alpha}_i, \widetilde{\beta}_i \in \chi_{[0,\ell]}$, $(i = 1, 2, \ldots, n)$. Then, the generalized distance measure $d_{\mathcal{G}}\left(\widetilde{A}, \widetilde{B}\right)$ between two Lq-ROFSs $\widetilde{A}$ and $\widetilde{B}$ in X, is a mapping $d_{\mathcal{G}} : Lq - ROFS(X) \times Lq - ROFS(X) \to [0, 1]$ is defined as follows:

$$d_{\mathcal{G}}(\tilde{A}, \tilde{B}) = \frac{1}{n}\sum_{i=1}^{n} d_{\mathcal{G}}\left(\tilde{\alpha}_i, \tilde{\beta}_i\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\ell^q}\left(\frac{1}{2}\left(\left|\xi_{\tilde{\alpha}_i}^q - \xi_{\tilde{\beta}_i}^q\right|^{\varepsilon} + \left|\eta_{\tilde{\alpha}_i}^q - \eta_{\tilde{\beta}_i}^q\right|^{\varepsilon} + \left|\pi_{\tilde{\alpha}_i}^q - \pi_{\tilde{\beta}_i}^q\right|^{\varepsilon}\right)\right)^{\frac{1}{\varepsilon}},$$
$$\text{where } \varepsilon \ge 1 \tag{8}$$

3.3 Entropy measure of Lq-ROFSs

Entropy is an effective tool to measure the uncertainties in a fuzzy set. In this subsection, an entropy measure for Lq-ROFSs is developed, which is presented as follows:

Definition 11 Let $\widetilde{A} = \{\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\}$, and $\widetilde{B} = \left\{\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right\}$ be any two Lq-ROFSs. A real-valued function $E_{\mathrm{L}q-\mathrm{ROFS}} : Lq - ROFS(X) \to [0, 1]$ is called an entropy measure for Lq-ROFSs if it satisfies the following axiomatic requirements:

(i) $E_{\mathrm{L}q-\mathrm{ROFS}}\left(\widetilde{A}\right) = 0$ if and only if $\widetilde{A}$ is a crisp set, i.e. $\mathcal{S}_{\xi_{\tilde{\alpha}_i}} = \mathcal{S}_{\ell}$ and $\mathcal{S}_{\eta_{\tilde{\alpha}_i}} = \mathcal{S}_0$, or $\mathcal{S}_{\xi_{\tilde{\alpha}_i}} = \mathcal{S}_0$ and $\mathcal{S}_{\eta_{\tilde{\alpha}_i}} = \mathcal{S}_{\ell}$, for all $i = 1, 2, \ldots, n$;

(ii) $E_{\mathrm{L}q-\mathrm{ROFS}}\left(\widetilde{A}\right) = 1$ if and only if $\pi_{\tilde{\alpha}_i} = \ell$ for all $i = 1, 2, \ldots, n$;

(iii) $E_{\mathrm{L}q-\mathrm{ROFS}}\left(\widetilde{A}\right) = E_{\mathrm{L}q-\mathrm{ROFS}}\left(\widetilde{A}^C\right)$; where $\widetilde{A}^C = \left\{\widetilde{\alpha}_1{}^C, \widetilde{\alpha}_2{}^C, \ldots, \widetilde{\alpha}_n{}^C\right\}$ and $\widetilde{\alpha}_i{}^C$ denotes the complement of $\widetilde{\alpha}_i$ i.e. $\widetilde{\alpha}_i{}^C = \langle \mathcal{S}_{\xi_{\tilde{\alpha}_i}}, \mathcal{S}_{\eta_{\tilde{\alpha}_i}} \rangle$.

(iv) $E_{\mathrm{L}q-\mathrm{ROFS}}\left(\widetilde{A}\right) \le E_{\mathrm{L}q-\mathrm{ROFS}}\left(\widetilde{B}\right)$ if $\widetilde{A}$ is less fuzzy than $\widetilde{B}$, i.e.

$\mathcal{S}_{\xi_{\tilde{\alpha}_i}} \le \mathcal{S}_{\xi_{\tilde{\beta}_i}}$ and $\mathcal{S}_{\eta_{\tilde{\alpha}_i}} \ge \mathcal{S}_{\eta_{\tilde{\beta}_i}}$ for $\mathcal{S}_{\xi_{\tilde{\beta}_i}} \le \mathcal{S}_{\eta_{\tilde{\beta}_i}}$,or $\mathcal{S}_{\xi_{\tilde{\alpha}_i}} \ge \mathcal{S}_{\xi_{\tilde{\beta}_i}}$ and $\mathcal{S}_{\eta_{\tilde{\alpha}_i}} \le \mathcal{S}_{\eta_{\tilde{\beta}_i}}$ for $\mathcal{S}_{\xi_{\tilde{\beta}_i}} \ge \mathcal{S}_{\eta_{\tilde{\beta}_i}}$, for all $i = 1, 2, \ldots, n$.

In this subsection, a new entropy measure for Lq-ROFSs is introduced. For each $\widetilde{A} \in \mathrm{L}q - \mathrm{ROFS(X)}$, the entropy measure $E_{\mathrm{L}q-\mathrm{ROFS}}\left(\widetilde{A}\right)$ for $\widetilde{A}$ is defined by

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$$E_{Lq-ROFS}(\tilde{A}) = \frac{1}{2n}\sum_{i=1}^{n}\left(1 - d_{\mathcal{G}}\left(\tilde{\alpha}_i, \tilde{\alpha}_i^C\right)\right)\left(1 + \left(\frac{\pi_{\tilde{\alpha}_i}}{\ell}\right)^q\right)$$
$$= \frac{1}{2n}\sum_{i=1}^{n}\left(1 - \frac{1}{\ell^q}\left|\xi_{\tilde{\alpha}_i}^q - \eta_{\tilde{\alpha}_i}^q\right|\right)\left(2 - \frac{\xi_{\tilde{\alpha}_i}^q + \eta_{\tilde{\alpha}_i}^q}{\ell^q}\right). \quad (9)$$

Theorem 1 *Let $\widetilde{A} = \{\widetilde{\alpha}_i | i = 1, 2, \ldots, n\}$ be a Lq-ROFS. A real-valued function $E_{Lq-ROFS}\left(\widetilde{A}\right)$ on Lq-ROFS(X) defined by Eq. (9) is an entropy for Lq-ROFSs.*

Proof Definitely, as a meaningful entropy of Lq-ROFSs, it should satisfy the axioms (i)–(iv) in Definition 11.

(i) Let $\widetilde{A}$ be a crisp set, then either $\mathcal{S}_{\xi_{\tilde{\alpha}_i}} = \mathcal{S}_\ell$ and $\mathcal{S}_{\eta_{\tilde{\alpha}_i}} = \mathcal{S}_0$ or $\mathcal{S}_{\xi_{\tilde{\alpha}_i}} = \mathcal{S}_0$ and $\mathcal{S}_{\eta_{\tilde{\alpha}_i}} = \mathcal{S}_\ell$ for $i = 1, 2, \ldots, n$. So from Eq. (9), it can be easily found that $E_{Lq-ROFS}\left(\widetilde{A}\right) = 0$.

On the other hand, suppose $E_{Lq-ROFS}\left(\widetilde{A}\right) = 0$.

$$E_{Lq-ROFS}(\tilde{B}) - E_{Lq-ROFS}(\tilde{A})$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\left(2 - \frac{\xi_{\tilde{\beta}_i}^q + \eta_{\tilde{\beta}_i}^q}{\ell^q}\right)\left(1 - \frac{1}{\ell^q}\left|\xi_{\tilde{\beta}_i}^q - \eta_{\tilde{\beta}_i}^q\right|\right)\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\left(2 - \frac{\xi_{\tilde{\beta}_i}^q + \eta_{\tilde{\beta}_i}^q}{\ell^q}\right)\left(1 - \frac{1}{\ell^q}\left|\xi_{\tilde{\beta}_i}^q - \eta_{\tilde{\beta}_i}^q\right|\right) - \frac{1}{2}\left(2 - \frac{\xi_{\tilde{\alpha}_i}^q + \eta_{\tilde{\alpha}_i}^q}{\ell^q}\right)\left(1 - \frac{1}{\ell^q}\left|\xi_{\tilde{\alpha}_i}^q - \eta_{\tilde{\alpha}_i}^q\right|\right)\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(\left(1 + \frac{\xi_{\tilde{\beta}_i}^q}{\ell^q} - \frac{\eta_{\tilde{\beta}_i}^q}{\ell^q}\right)\left(1 - \frac{\xi_{\tilde{\beta}_i}^q}{2\ell^q} - \frac{\eta_{\tilde{\beta}_i}^q}{2\ell^q}\right) - \left(1 + \frac{\xi_{\tilde{\alpha}_i}^q}{\ell^q} - \frac{\eta_{\tilde{\alpha}_i}^q}{\ell^q}\right)\left(1 - \frac{\xi_{\tilde{\alpha}_i}^q}{2\ell^q} - \frac{\eta_{\tilde{\alpha}_i}^q}{2\ell^q}\right)\right) \quad (10)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(\left(1 + \frac{\xi_{\tilde{\beta}_i}^q}{2\ell^q} - \frac{3\eta_{\tilde{\beta}_i}^q}{2\ell^q} - \frac{\xi_{\tilde{\beta}_i}^{2q}}{2\ell^{2q}} + \frac{\eta_{\tilde{\beta}_i}^{2q}}{2\ell^{2q}}\right) - \left(1 + \frac{\xi_{\tilde{\alpha}_i}^q}{2\ell^q} - \frac{3\eta_{\tilde{\alpha}_i}^q}{2\ell^q} - \frac{\xi_{\tilde{\alpha}_i}^{2q}}{2\ell^{2q}} + \frac{\eta_{\tilde{\alpha}_i}^{2q}}{2\ell^{2q}}\right)\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\left(\frac{\xi_{\tilde{\beta}_i}^q}{\ell^q} - \frac{\xi_{\tilde{\alpha}_i}^q}{\ell^q}\right) - \frac{3}{2}\left(\frac{\eta_{\tilde{\beta}_i}^q}{\ell^q} - \frac{\eta_{\tilde{\alpha}_i}^q}{\ell^q}\right) - \frac{1}{2}\left(\left(\frac{\xi_{\tilde{\beta}_i}^q}{\ell^q}\right)^2 - \left(\frac{\xi_{\tilde{\alpha}_i}^q}{\ell^q}\right)^2\right) + \frac{1}{2}\left(\left(\frac{\eta_{\tilde{\beta}_i}^q}{\ell^q}\right)^2 - \left(\frac{\eta_{\tilde{\alpha}_i}^q}{\ell^q}\right)^2\right)\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\left(\frac{\xi_{\tilde{\beta}_i}^q}{\ell^q} - \frac{\xi_{\tilde{\alpha}_i}^q}{\ell^q}\right)\left(1 - \frac{\xi_{\tilde{\beta}_i}^q}{\ell^q} - \frac{\xi_{\tilde{\alpha}_i}^q}{\ell^q}\right) + \frac{1}{2}\left(\frac{\eta_{\tilde{\beta}_i}^q}{\ell^q} - \frac{\eta_{\tilde{\alpha}_i}^q}{\ell^q}\right)\left(\frac{\eta_{\tilde{\beta}_i}^q}{\ell^q} + \frac{\eta_{\tilde{\alpha}_i}^q}{\ell^q} - 3\right)\right)$$

Since $1 + \left(\frac{\pi_{\tilde{\alpha}_i}}{\ell}\right)^q \neq 0$, it is clear from Eq. (9) that $1 - \frac{1}{\ell^q}\left|\xi_{\tilde{\alpha}_i}{}^q - \eta_{\tilde{\alpha}_i}{}^q\right| = 0$ for all $i = 1, 2, \ldots, n$, which implies $\mathcal{S}_{\xi_{\tilde{\alpha}_i}} = \mathcal{S}_\ell$ and $\mathcal{S}_{\eta_{\tilde{\alpha}_i}} = \mathcal{S}_0$ or $\mathcal{S}_{\xi_{\tilde{\alpha}_i}} = \mathcal{S}_0$ and $\mathcal{S}_{\eta_{\tilde{\alpha}_i}} = \mathcal{S}_\ell$. Hence, $\widetilde{A}$ is a crisp set.

(ii) Let $\pi_{\tilde{\alpha}_i} = \ell$ for all $i = 1, 2, \ldots, n$. From Eq. (9), it is obvious that $E_{Lq-ROFS}\left(\widetilde{A}\right) = 1$.

Again suppose $E_{Lq-ROFS}\left(\widetilde{A}\right) = 1$. Then from Eq. (9),

$$\frac{1}{2}\left(1 - \frac{1}{\ell^q}\left|\xi_{\tilde{\alpha}_i}^q - \eta_{\tilde{\alpha}_i}^q\right|\right)\left(1 + \left(\frac{\pi_{\tilde{\alpha}_i}}{\ell}\right)^q\right) = 1$$
$$\Leftrightarrow \left(1 - \frac{1}{\ell^q}\left|\xi_{\tilde{\alpha}_i}^q - \eta_{\tilde{\alpha}_i}^q\right|\right)\left(1 + \left(\frac{\pi_{\tilde{\alpha}_i}}{\ell}\right)^q\right) = 2$$
$$\Leftrightarrow 1 + \left(\frac{\pi_{\tilde{\alpha}_i}}{\ell}\right)^q - \frac{1}{\ell^q}\left|\xi_{\tilde{\alpha}_i}^q - \eta_{\tilde{\alpha}_i}^q\right|\left(1 + \left(\frac{\pi_{\tilde{\alpha}_i}}{\ell}\right)^q\right) = 2$$
$$\Leftrightarrow \frac{1}{\ell^q}\left|\xi_{\tilde{\alpha}_i}^q - \eta_{\tilde{\alpha}_i}^q\right|\left(1 + \left(\frac{\pi_{\tilde{\alpha}_i}}{\ell}\right)^q\right) = -\left(1 - \left(\frac{\pi_{\tilde{\alpha}_i}}{\ell}\right)^q\right).$$

Since $1 + \left(\frac{\pi_{\tilde{\alpha}_i}}{\ell}\right)^q > 0$ and $\left|\xi_{\tilde{\alpha}_i}{}^q - \eta_{\tilde{\alpha}_i}{}^q\right|$ is a non-negative number, the above equality holds only if both sides are zero. Thus, it is easily found that $\pi_{\tilde{\alpha}_i} = \ell$.

(iii) The proof is trivial from the definitions of $\widetilde{A}$ and $\widetilde{A}^C$.

(iv) Let $\widetilde{A}, \widetilde{B} \in \mathrm{L}q - \mathrm{ROFS(X)}$ be any two Lq-ROFSs with $\widetilde{A} \subseteq \widetilde{B}$, i.e. $\xi_{\tilde{\alpha}_i} \le \xi_{\tilde{\beta}_i}$ and $\eta_{\tilde{\alpha}_i} \ge \eta_{\tilde{\beta}_i}$ so $\xi_{\tilde{\alpha}_i} \le \xi_{\tilde{\beta}_i} \le \eta_{\tilde{\beta}_i} \le \eta_{\tilde{\alpha}_i}$ for $\xi_{\tilde{\beta}_i} \le \eta_{\tilde{\beta}_i}$.

Now,

Since $\xi_{\tilde{\alpha}_i} \le \xi_{\tilde{\beta}_i} \le \eta_{\tilde{\beta}_i} \le \eta_{\tilde{\alpha}_i}$,

$$\xi_{\tilde{\alpha}_i} \le \eta_{\tilde{\alpha}_i} \wedge \xi_{\tilde{\beta}_i} \le \eta_{\tilde{\beta}_i} \Rightarrow \left(\xi_{\tilde{\alpha}_i}\right)^q \le \frac{\ell^q}{2} \wedge \left(\xi_{\tilde{\beta}_i}\right)^q \le \frac{\ell^q}{2},$$

i.e. $\left(\xi_{\tilde{\alpha}_i}\right)^q + \left(\xi_{\tilde{\beta}_i}\right)^q \le \ell^q$ for $i = 1, 2, \ldots, n$.

Thus, the expression (10) is non-negative, i.e. $E_{Lq-ROFS}\left(\widetilde{A}\right) \le E_{Lq-ROFS}\left(\widetilde{B}\right)$ holds.

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Similarly, as for $\widetilde{A} \supseteq \widetilde{B}$ for $\xi_{\tilde{\beta}_i} \geq \eta_{\tilde{\beta}_i}$, it is obtained that $E_{Lq-ROFS}\left(\widetilde{A}\right) \leq E_{Lq-ROFS}\left(\widetilde{B}\right)$.

Equation (9) expresses the degree of fuzziness for Lq-ROFSs, which includes many single points (Lq-ROFNs). For a single point (Lq-ROFN) belongs to the Lq-ROFS, the entropy of Lq-ROFN $\tilde{\alpha}$ based on Eq. (9) can be calculated as

$$E(\tilde{\alpha}) = \frac{1}{2}\left(2 - \frac{1}{\ell^q}(\xi^q + \eta^q)\right)\left(1 - \frac{1}{\ell^q}|\xi^q - \eta^q|\right) \tag{11}$$

4 Linguistic q-rung orthopair fuzzy aggregation operators

4.1 Lq-ROF Archimedean PA operators

The PA operator is extended under Lq-ROF context to propose Lq-ROFAPA operator and its weighted form Lq-ROFAPWA operator as follows:

Definition 12 Let $\{\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n\}$ be a set of Lq-ROFNs, where $\tilde{\alpha}_i = \langle \mathcal{S}_{\xi_i}, \mathcal{S}_{\eta_i} \rangle$ $(i = 1, 2, \ldots, n)$ and $q \geq 1$. The Lq-ROFAPA operator is defined as

$$\mathrm{L}q - \mathrm{ROFAPA}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = \oplus_{A_{i=1}}^{n}\left(\frac{1 + T(\tilde{\alpha}_i)}{\sum_{k=1}^{n}(1 + T(\tilde{\alpha}_k))}\tilde{\alpha}_i\right), \tag{12}$$

where $T(\tilde{\alpha}_i) = \sum_{j=1, j \neq i}^{n} Sup(\tilde{\alpha}_i, \tilde{\alpha}_j)$, $Sup(\tilde{\alpha}_i, \tilde{\alpha}_j)$ is the support for $\tilde{\alpha}_i$ from $\tilde{\alpha}_j$.

To express Eq. (12) in a simple way, put $\mathfrak{T}_i = \frac{(1+T(\tilde{\alpha}_i))}{\sum_{k=1}^{n}(1+T(\tilde{\alpha}_k))}$, then, Eq. (12) can be expressed as

$$\mathrm{L}q - \mathrm{ROFAPA}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = \oplus_{Ai=1}^{n}\mathfrak{T}_i\tilde{\alpha}_i,$$

where $\mathfrak{T}_i$ can be regarded as the weight of the Lq-ROFNs $\tilde{\alpha}_i$, and satisfies $0 \leq \mathfrak{T}_i \leq 1$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n}\mathfrak{T}_i = 1$.

Theorem 2 *Let $\tilde{\alpha}_i = \langle \mathcal{S}_{\xi_i}, \mathcal{S}_{\eta_i} \rangle$ $(i = 1, 2, \ldots, n)$ be a set of Lq-ROFNs. Then, the aggregated value using Lq-ROFAPA operator is also a Lq-ROFN, where*

$$\mathrm{L}q - \mathrm{ROFAPA}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = \oplus_{A_{i=1}}^{n}\mathfrak{T}_i\tilde{\alpha}_i = \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\sum_{i=1}^{n}\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)^{1/q}}, \mathcal{S}_{\ell\left(f^{-1}\left(\sum_{i=1}^{n}\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)^{1/q}} \right\rangle \tag{13}$$

Proof Based on Definition 8,

$$\mathfrak{T}_i\tilde{\alpha}_i = \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(f^{-1}\left(\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}} \right\rangle.$$

Then, it is obtained that

$$\mathfrak{T}_1\tilde{\alpha}_1 \oplus_A \mathfrak{T}_2\tilde{\alpha}_2 = \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\mathfrak{T}_1 g\left(\left(\frac{\xi_1}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(f^{-1}\left(\mathfrak{T}_1 f\left(\left(\frac{\eta_1}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}} \right\rangle \oplus_A \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\mathfrak{T}_2 g\left(\left(\frac{\xi_2}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(f^{-1}\left(\mathfrak{T}_2 f\left(\left(\frac{\eta_2}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}} \right\rangle = \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\sum_{i=1}^{2}\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)^{1/q}}, \mathcal{S}_{\ell\left(f^{-1}\left(\sum_{i=1}^{2}\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)^{1/q}} \right\rangle$$

So, the Theorem is true for $n = 2$.

Now let the Theorem is true for $n = m$, i.e., $\mathrm{L}q - \mathrm{ROFAPA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_m) = \oplus_{A_{i=1}}^{m}\mathfrak{T}_i\tilde{\alpha}_i$

$$= \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\sum_{i=1}^{m}\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)^{1/q}}, \mathcal{S}_{\ell\left(f^{-1}\left(\sum_{i=1}^{m}\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)^{1/q}} \right\rangle.$$

Now it would be shown that it is true for $n = m + 1$,

$$\mathrm{L}q - \mathrm{ROFAPA}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_m, \tilde{\alpha}_{m+1}) = \left(\oplus_{A_{i=1}}^{m}\mathfrak{T}_i\tilde{\alpha}_i\right) \oplus_A \left(\mathfrak{T}_{m+1}\tilde{\alpha}_{m+1}\right)$$
$$= \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\sum_{i=1}^{m}\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)^{1/q}}, \mathcal{S}_{\ell\left(f^{-1}\left(\sum_{i=1}^{m}\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)^{1/q}} \right\rangle \oplus_A \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\mathfrak{T}_{m+1} g\left(\left(\frac{\xi_{m+1}}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(f^{-1}\left(\mathfrak{T}_{m+1} f\left(\left(\frac{\eta_{m+1}}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}} \right\rangle$$
$$= \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\sum_{i=1}^{m+1}\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)^{1/q}}, \mathcal{S}_{\ell\left(f^{-1}\left(\sum_{i=1}^{m+1}\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)^{1/q}} \right\rangle$$

Since it is true for $n = m + 1$, the Theorem is proved for all n.

In reality, it is not so logical to treat all the aspects of attributes that influence the decision-making process equally while dealing with a MAGDM situation. This implies that the attribute weights should be varied. However, in the process of aggregating DM's perspective information, the Lq-ROFAPA operator does not evaluate the influence of weight on the result. Therefore, the weight component must be integrated further into Lq-ROFAPA, as shown in Definition 12.

Definition 13 Let $\{\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n\}$ be a set of Lq-ROFNs, and $q \geq 1$. $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n)$, which satisfies $\sum_{i=1}^{n}\omega_i = 1$ and $\omega_i \in [0, 1]$. The Lq-ROFAPWA operator of the Lq-ROFNs is defined as

$$\mathrm{L}q - \mathrm{ROFAPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = \oplus_{A_{i=1}}^{n}\left(\frac{\omega_i(1 + T(\tilde{\alpha}_i))}{\sum_{k=1}^{n}\omega_k(1 + T(\tilde{\alpha}_k))}\tilde{\alpha}_i\right), \tag{14}$$

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where $T(\widetilde{\alpha}_i) = \sum_{j=1, j \neq i}^{n} Sup(\widetilde{\alpha}_i, \widetilde{\alpha}_j)$.,

For ease of expression, Eq. (14) may be written as.

$$\mathrm{L}q - \mathrm{ROFAPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \oplus_{A}{}_{i=1}^{n} \Omega_i \tilde{\alpha}_i,$$

where Ω_i is the weight of the Lq-ROFNs $\widetilde{\alpha}_i$, that satisfies $\sum_{i=1}^{n} \Omega_i = 1$ and for $\Omega_i = \frac{\omega_i(1+T(\widetilde{\alpha}_i))}{\sum_{k=1}^{n} \omega_k(1+T(\widetilde{\alpha}_k))}$, $\Omega_i \in [0, 1]$.

Theorem 3 *Let $\widetilde{\alpha}_i = \langle \mathcal{S}_{\xi_i}, \mathcal{S}_{\eta_i} \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of Lq-ROFNs. Then, the aggregated result by using Lq-ROF Archimedean power-weighted averaging (Lq-ROFAPWA) operator is still a Lq-ROFN and*

$$\begin{aligned}
&\mathrm{L}q - \mathrm{ROFAPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \\
&= \oplus_{A}{}_{i=1}^{n} \left(\frac{\omega_i(1+T(\tilde{\alpha}_i))}{\sum_{k=1}^{n} \omega_k(1+T(\tilde{\alpha}_k))} \tilde{\alpha}_i \right) \\
&= \oplus_{A}{}_{i=1}^{n} (\Omega_i \tilde{\alpha}_i) = \\
&\left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\sum_{i=1}^{n} \Omega_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)^{1/q}}, \mathcal{S}_{\ell\left(f^{-1}\left(\sum_{i=1}^{n} \Omega_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)^{1/q}} \right\rangle
\end{aligned} \tag{15}$$

where $\Omega_i = \frac{\omega_i(1+T(\widetilde{\alpha}_i))}{\sum_{k=1}^{n} \omega_k(1+T(\widetilde{\alpha}_k))}$, and $T(\widetilde{\alpha}_i) = \sum_{j=1, j \neq i}^{n} Sup(\widetilde{\alpha}_i, \widetilde{\alpha}_j)$.

Proof The proof is the same as Theorem 2, so it is omitted here.

The specific expressions of the Lq-ROFAPWA operator are constructed as follows:

If Hamacher t-N&t-CNs are applied, i.e. $f(t) = \log\left(\frac{\rho}{t} + 1 - \rho\right)$ is used in Eq. (15), Lq-ROFAPWA operator converted to Lq-ROF Hamacher power WA (Lq-ROFHPWA) operator and is expressed as

$$\begin{aligned}
&Lq - ROFHPWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \\
&= \left\langle \mathcal{S}_{\ell\left(1 - \frac{\rho}{\prod_{i=1}^{n}\left(\frac{\rho}{1-\left(\frac{\xi_i}{\ell}\right)^q} + 1 - \rho\right)^{\Omega_i} + \rho - 1}\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(\frac{\rho}{\prod_{i=1}^{n}\left(\frac{\rho}{\left(\frac{\eta_i}{\ell}\right)^q} + 1 - \rho\right)^{\Omega_i} + \rho - 1}\right)^{\frac{1}{q}}} \right\rangle
\end{aligned} \tag{16}$$

If Dombi t-N&t-CNs are applied, i.e. $f(t) = \left(\frac{1}{t} - 1\right)^{\tau}$, is put in Eq. (15), L$q$-ROFAPWA operator converted to Lq-ROF Dombi power WA (Lq-ROFDPWA) operator, which can be expressed as:

$$\begin{aligned}
&Lq - ROFDPWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \\
&= \left\langle \mathcal{S}_{\ell\left(1 - \frac{1}{1 + \left(\sum_{i=1}^{n} \Omega_i \left(\frac{1}{1-\left(\frac{\xi_i}{\ell}\right)^q} - 1\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{1/q}}, \mathcal{S}_{\ell\left(\frac{1}{1 + \left(\sum_{i=1}^{n} \Omega_i \left(\frac{1}{\left(\frac{\eta_i}{\ell}\right)^q} - 1\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{1/q}} \right\rangle.
\end{aligned} \tag{17}$$

If Frank t-N&t-CNs are applied, i.e. put $f(t) = \log\left(\frac{\zeta - 1}{\zeta^t - 1}\right)$, $\zeta > 1$ to Eq. (15), Lq-ROFAPWA operator converted to Lq-ROF Frank power WA (Lq-ROFFPWA) operator can be obtained as:

$$\begin{aligned}
&Lq - ROFFPWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \\
&= \left\langle \mathcal{S}_{\ell\left(1 - \frac{\log\left(\frac{\zeta - 1}{\left(\prod_{i=1}^{n}\left(\frac{\zeta - 1}{\zeta^{1-\left(\frac{\xi_i}{\ell}\right)^q} - 1}\right)^{\Omega_i}\right)} + 1\right)}{\log \zeta}\right)^{1/q}}, \right\rangle.
\end{aligned} \tag{18}$$

$$\mathcal{S}_{\ell\left(\frac{\log\left(\frac{\zeta - 1}{\left(\prod_{i=1}^{n}\left(\frac{\zeta - 1}{\zeta^{\left(\frac{\eta_i}{\ell}\right)^q} - 1}\right)^{\Omega_i}\right)} + 1\right)}{\log \zeta}\right)^{\frac{1}{q}}}$$

4.2 Archimedean HM-based power aggregation operators on Lq-ROF environment

In this section, Lq-ROFAPHM and Lq-ROFAPWHM aggregation operators are developed based on power operational laws, At-N&t-CNs and HM operator under Lq-ROF context.

Definition 14 Let $\phi, \psi > 0$ and $\widetilde{\alpha}_i = \langle \mathcal{S}_{\xi_i}, \mathcal{S}_{\eta_i} \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of Lq-ROFNs, then the Lq-ROFAPHM aggregation operator is defined as follows:where $T(\widetilde{\alpha}_i) = \sum_{j=1, j \neq i}^{n} Sup(\widetilde{\alpha}_i, \widetilde{\alpha}_j)$;

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$$\mathrm{L}q-\mathrm{ROFAPHM}^{\phi,\psi}(\tilde{\alpha}_1,\ \tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\left(\frac{2}{n(n+1)}\oplus A_{\substack{i,j=1\\ i\leq j}}^{n}\left(\left(\frac{n(1+T(\tilde{\alpha}_i))}{\sum_{k=1}^{n}(1+T(\tilde{\alpha}_k))}\tilde{\alpha}_i\right)^{\phi}\otimes_A\left(\frac{n\left(1+T(\tilde{\alpha}_j)\right)}{\sum_{k=1}^{n}(1+T(\tilde{\alpha}_k))}\tilde{\alpha}_j\right)^{\psi}\right)\right)^{\frac{1}{\phi+\psi}}, \tag{19}$$

$Sup\left(\widetilde{\alpha}_i,\widetilde{\alpha}_j\right)=1-d\left(\widetilde{\alpha}_i,\widetilde{\alpha}_j\right)$. To simplify Eq. (19), let $\mathfrak{T}_i=\frac{\left(1+T\left(\widetilde{\alpha}_i\right)\right)}{\sum_{k=1}^{n}\left(1+T\left(\widetilde{\alpha}_k\right)\right)}$, then Eq. (19) can be denoted as:

$$\mathrm{L}q-\mathrm{ROFAPWHM}_{\omega}^{\phi,\psi}(\tilde{\alpha}_1,\ \tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)$$
$$=\left(\frac{2}{n(n+1)}\oplus A_{\substack{i,j=1\\ i\leq j}}^{n}\left((n\Omega_i\tilde{\alpha}_i)^{\phi}\otimes_A\left(n\Omega_j\tilde{\alpha}_j\right)^{\psi}\right)\right)^{\frac{1}{\phi+\psi}},$$

where $\mathfrak{T}_i$ means the power weights of $\widetilde{\alpha}_i$.

Theorem 4 *Let $\phi,\psi>0$ and $\widetilde{\alpha}_i=\langle\mathcal{S}_{\xi_i},\mathcal{S}_{\eta_i}\rangle$ $(i=1,2,\ldots,n)$ be a group of Lq-ROFNs, then their fused results by utilizing the Lq-ROFAPHM operator are also a Lq-ROFN, and*

$$Lq-ROFAPHM^{\phi,\psi}(\tilde{\alpha}_1,\ \tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\left(\frac{2}{n(n+1)}\oplus A_{\substack{i,j=1\\ i\leq j}}^{n}\left((n\mathfrak{T}_i\tilde{\alpha}_i)^{\phi}\otimes_A\left(n\mathfrak{T}_j\tilde{\alpha}_j\right)^{\psi}\right)\right)^{\frac{1}{\phi+\psi}}$$
$$=\left\langle\mathcal{S}_{\ell\left(f^{-1}\left(\frac{1}{\phi+\psi}f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n}g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)+\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}},\right. \tag{20}$$
$$\left.\mathcal{S}_{\ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n}f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}\right\rangle,$$

where $\mathfrak{T}_i=\frac{\left(1+T\left(\widetilde{\alpha}_i\right)\right)}{\sum_{k=1}^{n}\left(1+T\left(\widetilde{\alpha}_k\right)\right)}$.

Proof On the basis of the operational laws as defined in Definition 8,

$$n\mathfrak{T}_i\tilde{\alpha}_i=\left\langle\mathcal{S}_{\ell\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}},\mathcal{S}_{\ell\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}\right\rangle;$$

$$n\mathfrak{T}_j\tilde{\alpha}_j=\left\langle\mathcal{S}_{\ell\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}},\mathcal{S}_{\ell\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}\right\rangle.$$

Further, it is obtained

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$$(n\mathfrak{T}_i\tilde{\alpha}_i)^{\phi}\otimes_A\left(n\mathfrak{T}_j\tilde{\alpha}_j\right)^{\psi}=$$

$$\left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)\right)\right)^{\frac{1}{q}}} \right\rangle \otimes_A$$

$$\left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(g^{-1}\left(\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)\right)\right)^{\frac{1}{q}}} \right\rangle$$

$$= \left\langle \mathcal{S}_{\ell\left(f^{-1}\left(f\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)\right)\right)+f\left(f^{-1}\left(\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}, quad\right.$$

$$\left. \mathcal{S}_{\ell\left(g^{-1}\left(g\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)\right)\right)+g\left(g^{-1}\left(\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}} \right\rangle$$

$$= \left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)+\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)^{\frac{1}{q}}},\right.$$

$$\left.\left(\mathcal{S}_{\ell\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)\right)\right)^{\frac{1}{q}}} \right)\right\rangle$$

.

By mathematical induction method, it can be shown that

$$\oplus^{n}{}_{\substack{A_{i,j=1}\\ i\le j}} \left((n\mathfrak{T}_i\tilde{\alpha}_i)^{\phi}\otimes_A\left(n\mathfrak{T}_j\tilde{\alpha}_j\right)^{\psi}\right)$$

$$= \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\sum_{\substack{i,j=1\\ i\le j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)+\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}},\right.$$

$$\left. \mathcal{S}_{\ell\left(f^{-1}\left(\sum_{\substack{i,j=1\\ i\le j}}^{n} f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}} \right\rangle$$

Now,

$$\frac{2}{n(n+1)} \oplus A^{n}_{\substack{i,j=1\\ i\le j}} \left((n\mathfrak{T}_i\tilde{\alpha}_i)^{\phi}\otimes_A\left(n\mathfrak{T}_j\tilde{\alpha}_j\right)^{\psi}\right) = \left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)+\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}},\right.$$

$$\left. \mathcal{S}_{\ell\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}} \right\rangle$$

Finally,

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$$\left(\frac{2}{n(n+1)}\frac{2}{n(n+1)}\oplus A_{\substack{i,j=1\\ i\le j}}^{n}\left(\left(n\mathfrak{T}_i\tilde{\alpha}_i\right)^{\phi}\otimes_A\left(n\mathfrak{T}_j\tilde{\alpha}_j\right)^{\psi}\right)\right.$$

$$=\left\langle \mathcal{S}_{\ell\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)+\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}\right\rangle^{\left(\left(n\mathfrak{T}_i\tilde{\alpha}_i\right)^{\phi}\otimes_A\left(n\mathfrak{T}_j\tilde{\alpha}_j\right)^{\psi}\right)^{\frac{1}{\phi+\psi}}}$$

$$=\left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\frac{1}{\phi+\psi}f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)+\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}},\right.$$

$$\left.\mathcal{S}_{\ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}\right\rangle.$$

Hence, the Theorem is proved.

Theorem 5 (*Idempotency*). *Let* $\tilde{\alpha}_i=\langle\mathcal{S}_{\xi_i},\mathcal{S}_{\eta_i}\rangle$ $(i=1,2,\ldots,n)$ *be a collection of n Lq-ROFNs. If* $\tilde{\alpha}_i=\tilde{\alpha}=\langle\mathcal{S}_{\xi},\mathcal{S}_{\eta}\rangle$ *for all* $i=1,2,\ldots,n$, *then*

$$Lq-ROFAPHM^{\phi,\psi}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\tilde{\alpha}.$$

Proof Since $\tilde{\alpha}_i=\langle\mathcal{S}_{\xi_i},\mathcal{S}_{\eta_i}\rangle=\langle\mathcal{S}_{\xi},\mathcal{S}_{\eta}\rangle=\tilde{\alpha}$ for all $i=1,2,\ldots,n$;

$Sup(\tilde{\alpha}_i,\tilde{\alpha}_j)=1,\quad [d(\tilde{\alpha}_i,\tilde{\alpha}_j)=0 for \tilde{\alpha}_i=\tilde{\alpha}_j]$ for all $i,j=1,2,\ldots,n$;

So $\mathfrak{T}_i=\frac{1}{n}$ for all $i=1,2,\ldots,n$;

Then,

$$Lq-ROFAPHM^{\phi,\psi}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=Lq-ROFAPHM^{\phi,\psi}(\tilde{\alpha},\tilde{\alpha},\ldots,\tilde{\alpha})$$

$$=\left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\frac{1}{\phi+\psi}f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} g\left(f^{-1}\left((\phi+\psi)f\left(g^{-1}\left(g\left(\left(\frac{\xi}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} f\left(g^{-1}\left((\phi+\psi)g\left(f^{-1}\left(f\left(\left(\frac{\eta}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}\right\rangle$$

$$=\left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\frac{1}{\phi+\psi}f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} g\left(f^{-1}\left((\phi+\psi)f\left(\left(\frac{\xi}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} f\left(g^{-1}\left((\phi+\psi)g\left(\left(\frac{\eta}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}\right\rangle$$

$$=\left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\frac{1}{\phi+\psi}f\left(g^{-1}\left(\frac{2}{n(n+1)}\frac{n(n+1)}{2}g\left(f^{-1}\left((\phi+\psi)f\left(\left(\frac{\xi}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\frac{n(n+1)}{2}f\left(g^{-1}\left((\phi+\psi)g\left(\left(\frac{\eta}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}\right\rangle$$

$$=\left\langle \mathcal{S}_{\ell\left(f^{-1}\left(f\left(\left(\frac{\xi}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(g^{-1}\left(g\left(\left(\frac{\eta}{\ell}\right)^q\right)\right)\right)^{\frac{1}{q}}}\right\rangle=\left\langle \mathcal{S}_{\ell\left(\left(\frac{\xi}{\ell}\right)^q\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(\left(\frac{\eta}{\ell}\right)^q\right)^{\frac{1}{q}}}\right\rangle=\langle\mathcal{S}_{\xi},\mathcal{S}_{\eta}\rangle$$

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This completes the proof of the Theorem.

Theorem 6 (*Boundedness*). *Let* $\tilde{\alpha}_i = \langle \mathcal{S}_{\xi_i}, \mathcal{S}_{\eta_i} \rangle$ $(i = 1, 2, \ldots, n)$ *be a collection of Lq-ROFNs, and* $\xi^- = \min_i\{\xi_i\}$, $\xi^+ = \max_i\{\xi_i\}$, $\eta^- = \min_i\{\eta_i\}$, $\eta^+ = \max_i\{\eta_i\}$ *then*

$$\tilde{\alpha}^- \leq \mathrm{L}q-\mathrm{ROFAPHM}^{\phi,\psi}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leq \tilde{\alpha}^+,$$

where

$$\tilde{\alpha}^- = \left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\frac{1}{\phi+\psi}f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right)+\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}, \right.$$
$$\left. \mathcal{S}_{\ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta^+}{\ell}\right)^q\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta^+}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}} \right\rangle$$

And

$$\tilde{\alpha}^+ = \left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\frac{1}{\phi+\psi}f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi^+}{\ell}\right)^q\right)\right)\right)+\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi^+}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}, \right.$$
$$\left. \mathcal{S}_{\ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta^-}{\ell}\right)^q\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta^-}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}} \right\rangle.$$

Proof Since g is an increasing function for all $i = 1, 2, \ldots, n$;

$$g\left(\left(\frac{\xi^-}{\ell}\right)^q\right) \leq g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)$$

i.e. $g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right) \leq g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)$.

Again since f is a decreasing function, it is observed that

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$\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right) \geq \phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)$.

Similarly, $\psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right) \geq \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)$.

Then, $f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right)\right) \leq f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)$ i.e.

$$\sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right)\right)\right)$$

$$\leq \sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)$$

i.e.
$$g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right)\right)\right)\right)$$

$$\leq g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)$$

So,

$$\frac{1}{\phi+\psi} f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)$$
$$\geq \frac{1}{\phi+\psi} f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)$$

i.e.

$$\ell\left(f^{-1}\left(\frac{1}{\phi+\psi} f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi^-}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}$$
$$\leq \ell\left(f^{-1}\left(\frac{1}{\phi+\psi} f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n} g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\mathfrak{T}_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right) + \psi f\left(g^{-1}\left(n\mathfrak{T}_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}} \quad (21)$$

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Similarly, it can be shown that

$$\ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta^{+}}{\ell}\right)^{q}\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta^{+}}{\ell}\right)^{q}\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}$$

$$\geq \ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\mathfrak{T}_i f\left(\left(\frac{\eta_i}{\ell}\right)^{q}\right)\right)\right)+\psi g\left(f^{-1}\left(n\mathfrak{T}_j f\left(\left(\frac{\eta_j}{\ell}\right)^{q}\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}. \tag{22}$$

Therefore from Eqs. (21) and (22), it is found that $\tilde{\alpha}^{-} \leq \mathrm{Lq-ROFAPHM}^{\phi,\psi}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$.

In a similar way, it can be shown that $\mathrm{L}q-\mathrm{ROFAPHM}^{\phi,\psi}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \leq \tilde{\alpha}^{+}$.

Hence, Theorem is proved.

Definition 15 Let $\phi, \psi > 0$ and $\widetilde{\alpha}_i = \langle \mathcal{S}_{\xi_i}, \mathcal{S}_{\eta_i} \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of Lq-ROFNs whose weighted vectors are $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, which satisfies $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n}\omega_i = 1$, and then, the L$q$-ROFAPWHM

$$\mathrm{L}q-\mathrm{ROFAPWHM}_{\omega}^{\phi,\psi}(\tilde{\alpha}_1,\ \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\frac{2}{n(n+1)} \oplus A_{\substack{i,j=1\\ i\le j}}^{n} \left(\left(\frac{n\omega_i(1+T(\tilde{\alpha}_i))}{\sum_{k=1}^{n}\omega_k(1+T(\tilde{\alpha}_k))}\tilde{\alpha}_i\right)^{\phi} \otimes_A \left(\frac{n\omega_j(1+T(\tilde{\alpha}_j))}{\sum_{k=1}^{n}\omega_k(1+T(\tilde{\alpha}_k))}\tilde{\alpha}_j\right)^{\psi}\right)\right)^{\frac{1}{\phi+\psi}}, \tag{23}$$

aggregation operator is defined as follows:where $T(\widetilde{\alpha}_i) = \sum_{\substack{j=1\\ j\neq i}}^{n} Sup(\widetilde{\alpha}_i, \widetilde{\alpha}_j)$; $Sup(\widetilde{\alpha}_i, \widetilde{\alpha}_j) = 1 - d(\widetilde{\alpha}_i, \widetilde{\alpha}_j)$.

To simplify Eq. (23), let $\Omega_i = \frac{\omega_j(1+T(\widetilde{\alpha}_j))}{\sum_{k=1}^{n}\omega_k(1+T(\widetilde{\alpha}_k))}$; then, Eq. (23) can be denoted as:

$$\mathrm{L}q-\mathrm{ROFAPHM}^{\phi,\psi}(\tilde{\alpha}_1,\ \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\frac{2}{n(n+1)} \oplus A_{\substack{i,j=1\\ i\le j}}^{n} \left((n\mathfrak{T}_i\tilde{\alpha}_i)^{\phi} \otimes_A (n\mathfrak{T}_j\tilde{\alpha}_j)^{\psi}\right)\right)^{\frac{1}{\phi+\psi}},$$

where $\boldsymbol{\Omega}_i$ means the power weights of $\widetilde{\alpha}_i$.

Theorem 7 *Let $\widetilde{\alpha}_i = \langle \mathcal{S}_{\xi_i}, \mathcal{S}_{\eta_i} \rangle$ $(i = 1, 2, \ldots, n)$ be a collection of Lq-ROFNs, whose weighted vectors are $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, satisfies $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n}\omega_i = 1$. Let $\phi, \psi > 0$ be any numbers. Then, the aggregated value using Lq-ROFAPWHM is also a Lq-ROFN and*

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$$\mathrm{L}q-\mathrm{ROFAPWHM}_{\omega}^{\phi,\psi}(\tilde{\alpha}_1,\ \tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\left(\frac{2}{n(n+1)}\oplus A_{\substack{i,j=1\\ i\leq j}}^{n}\left(\left(\frac{n\omega_i(1+T(\tilde{\alpha}_i))}{\sum_{k=1}^{n}\omega_k(1+T(\tilde{\alpha}_k))}\tilde{\alpha}_i\right)^{\phi}\otimes_A\left(\frac{n\omega_j(1+T(\tilde{\alpha}_j))}{\sum_{k=1}^{n}\omega_k(1+T(\tilde{\alpha}_k))}\tilde{\alpha}_j\right)^{\psi}\right)\right)^{\frac{1}{\phi+\psi}}$$

$$=\left\langle \mathcal{S}_{\ell\left(f^{-1}\left(\frac{1}{\phi+\psi}f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n}g\left(f^{-1}\left(\phi f\left(g^{-1}\left(n\Omega_i g\left(\left(\frac{\xi_i}{\ell}\right)^q\right)\right)\right)+\psi f\left(g^{-1}\left(n\Omega_j g\left(\left(\frac{\xi_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}},\ \mathcal{S}_{\ell\left(g^{-1}\left(\frac{1}{\phi+\psi}g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\leq j}}^{n}f\left(g^{-1}\left(\phi g\left(f^{-1}\left(n\Omega_i f\left(\left(\frac{\eta_i}{\ell}\right)^q\right)\right)\right)+\psi g\left(f^{-1}\left(n\Omega_j f\left(\left(\frac{\eta_j}{\ell}\right)^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}}\right\rangle \tag{24}$$

where $\Omega_i=\omega_j\left(1+T\left(\widetilde{\alpha}_j\right)\right)\sum_{k=1}^{n}\omega_k\left(1+T\left(\widetilde{\alpha}_k\right)\right)$.

Proof The proof is similar to Theorem 4.

Case 1: If the additive generators of algebraic t-N&t-CNs are, respectively, assigned to f and g, i.e. $f(t)=-\log t$ and $g(t)=-\log(1-t)$, then a Lq-ROF power-weighted HM (Lq-ROFPWHM) operator is constructed:

$$Lq-ROFPWHM_{\omega}^{\phi,\psi}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\left\langle \mathcal{S}_{\ell\left(1-\left(\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(1-\left(1-\left(1-\left(\frac{\xi_i}{\ell}\right)^q\right)^{n\Omega_i}\right)^{\phi}\left(1-\left(1-\left(\frac{\xi_j}{\ell}\right)^q\right)^{n\Omega_j}\right)^{\psi}\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(\phi+\psi)}}},\ \mathcal{S}_{\ell\left(1-\left(\prod_{\substack{i,j=1\\ i\leq j}}^{n}\left(1-\left(1-\left(1-\left(\frac{\xi_i}{\ell}\right)^q\right)^{n\Omega_i}\right)^{\phi}\left(1-\left(1-\left(\frac{\xi_j}{\ell}\right)^q\right)^{n\Omega_j}\right)^{\psi}\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{q(\phi+\psi)}}}\right\rangle \tag{25}$$

Theorem 7 presents a generalized form of the Lq-ROF aggregation operator. If specific functions are assigned to f and g in Eq. (24), then particular operators can be constructed.

Case 2: If the additive generators of Einstein t-N&t-CNs are, respectively, assigned to f and g, i.e. $f(t)=\log\left(\frac{2-t}{t}\right)$ and $g(t)=\log\left(\frac{1+t}{1-t}\right)$, then a L$q$-ROF Einstein

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power-weighted HM (Lq-ROFEPWHM) operator is constructed:

$$Lq-ROFEPWHM^{\phi,\psi}_{\omega}(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n) = \left\langle \mathcal{S}_{\ell\left(\frac{2}{\left(\frac{\prod_{i,j=1, i\le j}^{n}(A_{ij}+3)^{\frac{2}{n(n+1)}}+3\prod_{i,j=1, i\le j}^{n}(A_{ij}-1)^{\frac{2}{n(n+1)}}}{\prod_{i,j=1, i\le j}^{n}(A_{ij}+3)^{\frac{2}{n(n+1)}}-\prod_{i,j=1, i\le j}^{n}(A_{ij}-1)^{\frac{2}{n(n+1)}}}+1\right)^{\frac{1}{\phi+\psi}}+1}\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(1-\frac{2}{\left(\frac{\left(\prod_{i,j=1, i\le j}^{n}\left(\frac{4}{B_{ij}-1}+1\right)\right)^{\frac{2}{n(n+1)}}+3}{\left(\prod_{i,j=1, i\le j}^{n}\left(\frac{4}{B_{ij}-1}+1\right)\right)^{\frac{2}{n(n+1)}}-1}\right)^{\frac{1}{\phi+\psi}}+1}\right)^{\frac{1}{q}}} \right\rangle. \tag{26}$$

where $A_{ij} = \left(\frac{\left(1+\left(\frac{\xi_i}{\ell}\right)^q\right)^{n\Omega_i}+3\left(1-\left(\frac{\xi_i}{\ell}\right)^q\right)^{n\Omega_i}}{\left(1+\left(\frac{\xi_i}{\ell}\right)^q\right)^{n\Omega_i}-\left(1-\left(\frac{\xi_i}{\ell}\right)^q\right)^{n\Omega_i}}\right)^{\phi} \left(\frac{\left(1+\left(\frac{\xi_j}{\ell}\right)^q\right)^{n\Omega_j}+3\left(1-\left(\frac{\xi_j}{\ell}\right)^q\right)^{n\Omega_j}}{\left(1+\left(\frac{\xi_j}{\ell}\right)^q\right)^{n\Omega_j}-\left(1-\left(\frac{\xi_j}{\ell}\right)^q\right)^{n\Omega_j}}\right)^{\psi}$

$$B_{ij} = \left(\frac{\left(\frac{2}{\left(\frac{\eta_i}{\ell}\right)^q}-1\right)^{n\Omega_i}+3}{\left(\frac{2}{\left(\frac{\eta_i}{\ell}\right)^q}-1\right)^{n\Omega_i}-1}\right)^{\phi}\left(\frac{\left(\frac{2}{\left(\frac{\eta_j}{\ell}\right)^q}-1\right)^{n\Omega_j}+3}{\left(\frac{2}{\left(\frac{\eta_j}{\ell}\right)^q}-1\right)^{n\Omega_j}-1}\right)^{\psi}.$$

Case 3: If the additive generators of Hamacher t-N&t-CNs are, respectively, assigned to f and g, i.e. $f(t) = \log\left(\frac{\rho}{t}+1-\rho\right)$ and $g(t) = \log\left(\frac{\rho}{1-t}+1-\rho\right)$, then a L$q$-ROF Hamacher power-weighted HM (Lq-ROFHPWHM) operator is constructed:

$$Lq-ROFHPWHM^{\phi,\psi}_{\omega}(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n) = \left\langle \mathcal{S}_{\ell\left(\frac{\rho}{\left(\frac{\rho^2}{\left(\prod_{i,j=1, i\le j}^{n}\left(\frac{\rho^2}{A_{ij}-1}+1\right)\right)^{\frac{2}{n(n+1)}}-1}+1\right)^{\frac{1}{\phi+\psi}}+\rho-1}\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(1-\frac{\rho}{\left(\frac{\rho^2}{\left(\prod_{i,j=1, i\le j}^{n}\left(\frac{\rho^2}{B_{ij}-1}+1\right)\right)^{\frac{2}{n(n+1)}}-1}+1\right)^{\frac{1}{\phi+\psi}}+\rho-1}\right)^{\frac{1}{q}}} \right\rangle. \tag{27}$$

where $A_{ij} = \left(\frac{\rho^2}{\left(\frac{\rho}{1-\left(\frac{\xi_i}{\ell}\right)^q}+1-\rho\right)^{n\Omega_i}-1}+1\right)^{\phi}\left(\frac{\rho^2}{\left(\frac{\rho}{1-\left(\frac{\xi_j}{\ell}\right)^q}+1-\rho\right)^{n\Omega_j}-1}+1\right)^{\psi}$, and $B_{ij} =$

$$\left(\frac{\rho^2}{\left(\frac{\rho}{\left(\frac{\eta_i}{\ell}\right)^q}+1-\rho\right)^{n\Omega_i}-1}+1\right)^{\phi}\left(\frac{\rho^2}{\left(\frac{\rho}{\left(\frac{\eta_j}{\ell}\right)^q}+1-\rho\right)^{n\Omega_j}-1}+1\right)^{\psi}.$$

Case 4: If the additive generators of Dombi t-N&t-CNs are, respectively, assigned to f and g, i.e. $f(t) = \left(\frac{1}{t}-1\right)^{\tau}$ and $g(t) = \left(\frac{1}{1-t}-1\right)^{\tau}$, then a L$q$-ROF Dombi power-

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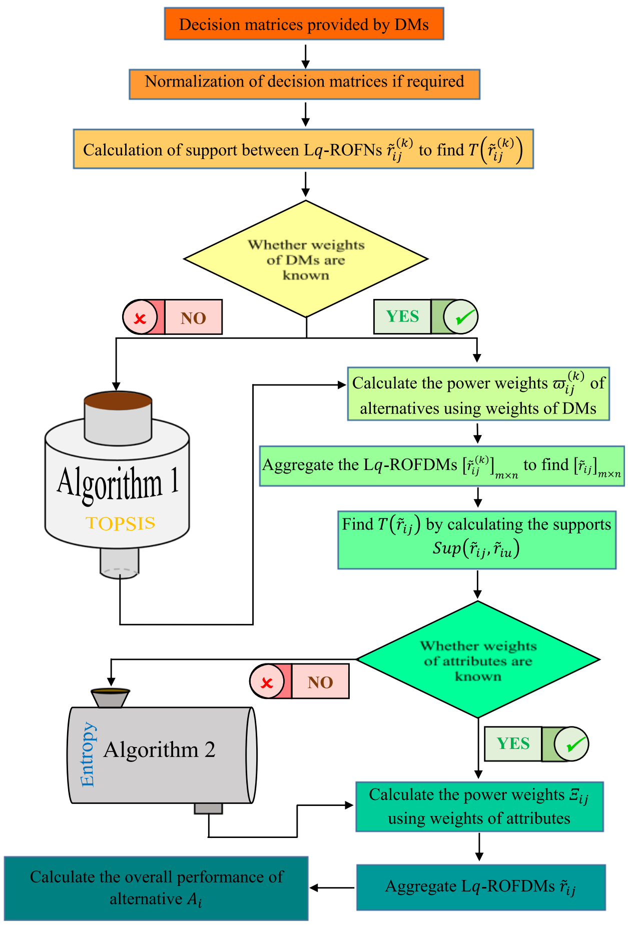

Fig. 2 Flowchart of the proposed methodology

Table 1 The Lq-ROFDM $\widetilde{\mathcal{X}}^{(1)}$ provided by the DM $\mathfrak{D}^{(1)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$(\mathcal{S}_6, \mathcal{S}_1)$	$(\mathcal{S}_3, \mathcal{S}_1)$	$(\mathcal{S}_3, \mathcal{S}_3)$	$(\mathcal{S}_1, \mathcal{S}_6)$
A_2	$(\mathcal{S}_3, \mathcal{S}_4)$	$(\mathcal{S}_3, \mathcal{S}_4)$	$(\mathcal{S}_2, \mathcal{S}_5)$	$(\mathcal{S}_2, \mathcal{S}_4)$
A_3	$(\mathcal{S}_1, \mathcal{S}_3)$	$(\mathcal{S}_2, \mathcal{S}_3)$	$(\mathcal{S}_3, \mathcal{S}_2)$	$(\mathcal{S}_6, \mathcal{S}_1)$
A_4	$(\mathcal{S}_6, \mathcal{S}_2)$	$(\mathcal{S}_4, \mathcal{S}_3)$	$(\mathcal{S}_5, \mathcal{S}_1)$	$(\mathcal{S}_7, \mathcal{S}_1)$

Table 2 The Lq-ROFDM $\widetilde{\mathcal{X}}^{(2)}$ provided by the DM $\mathfrak{D}^{(2)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$(\mathcal{S}_3, \mathcal{S}_2)$	$(\mathcal{S}_4, \mathcal{S}_1)$	$(\mathcal{S}_3, \mathcal{S}_4)$	$(\mathcal{S}_2, \mathcal{S}_3)$
A_2	$(\mathcal{S}_5, \mathcal{S}_2)$	$(\mathcal{S}_2, \mathcal{S}_1)$	$(\mathcal{S}_3, \mathcal{S}_4)$	$(\mathcal{S}_2, \mathcal{S}_5)$
A_3	$(\mathcal{S}_2, \mathcal{S}_3)$	$(\mathcal{S}_3, \mathcal{S}_3)$	$(\mathcal{S}_1, \mathcal{S}_2)$	$(\mathcal{S}_3, \mathcal{S}_3)$
A_4	$(\mathcal{S}_5, \mathcal{S}_2)$	$(\mathcal{S}_3, \mathcal{S}_3)$	$(\mathcal{S}_5, \mathcal{S}_2)$	$(\mathcal{S}_4, \mathcal{S}_1)$

Table 3 The Lq-ROFDM $\widetilde{\mathcal{X}}^{(3)}$ provided by the DM $\mathfrak{D}^{(3)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$(\mathcal{S}_3, \mathcal{S}_3)$	$(\mathcal{S}_3, \mathcal{S}_5)$	$(\mathcal{S}_6, \mathcal{S}_1)$	$(\mathcal{S}_2, \mathcal{S}_6)$
A_2	$(\mathcal{S}_3, \mathcal{S}_2)$	$(\mathcal{S}_2, \mathcal{S}_4)$	$(\mathcal{S}_2, \mathcal{S}_1)$	$(\mathcal{S}_3, \mathcal{S}_4)$
A_3	$(\mathcal{S}_6, \mathcal{S}_1)$	$(\mathcal{S}_2, \mathcal{S}_5)$	$(\mathcal{S}_3, \mathcal{S}_4)$	$(\mathcal{S}_1, \mathcal{S}_3)$
A_4	$(\mathcal{S}_5, \mathcal{S}_1)$	$(\mathcal{S}_4, \mathcal{S}_4)$	$(\mathcal{S}_6, \mathcal{S}_2)$	$(\mathcal{S}_5, \mathcal{S}_2)$

weighted HM (Lq-ROFDPWHM) operator is constructed:

where $C_{ij} = \left(\frac{\phi}{n\Omega_i \left(\frac{1}{1-\left(\frac{\xi_i}{\ell}\right)^q} - 1 \right)^{\tau}} + \frac{\psi}{n\Omega_j \left(\frac{1}{1-\left(\frac{\xi_j}{\ell}\right)^q} - 1 \right)^{\tau}} \right)$, $D_{ij} =$

$$\left(\frac{\phi}{n\Omega_i \left(\frac{1}{\left(\frac{\eta_i}{\ell}\right)^q} - 1 \right)^{\tau}} + \frac{\psi}{n\Omega_j \left(\frac{1}{\left(\frac{\eta_j}{\ell}\right)^q} - 1 \right)^{\tau}} \right)$$

Case 5: If the additive generators of Frank t-N&t-CNs are, respectively, assigned to f and g, i.e. $f(t) = \log\left(\frac{\zeta-1}{\zeta^t-1}\right)$ and $g(t) = \log\left(\frac{\zeta-1}{\zeta^{1-t}-1}\right)$, $\zeta > 1$, then a Lq-ROF Frank power-weighted HM (Lq-ROFFPWHM) operator is constructed:

where $F_{ij} =$

$$\left(\frac{\zeta-1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{\left(\frac{\zeta-1}{\zeta^{1-\left(\frac{\xi_i}{\ell}\right)^q}-1} \right)^{n\Omega_i}} + 1 \right)}{\log\zeta}}} - 1 \right)^{\phi}$$

$$\left(\frac{\zeta-1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{\left(\frac{\zeta-1}{\zeta^{1-\left(\frac{\xi_j}{\ell}\right)^q}-1} \right)^{n\Omega_j}} + 1 \right)}{\log\zeta}} - 1} \right)^{\psi}$$

and

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$$Lq - ROFDPWHM_{\omega}^{\phi,\psi}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left\langle \mathcal{S}_{\ell\left(\frac{1}{1+\left(\frac{n(n+1)}{2(\phi+\psi)\sum_{\substack{i,j=1\\ i\leq j}}^{n}\frac{1}{C_{ij}}}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}}, \mathcal{S}_{\ell\left(1-\frac{1}{1+\left(\frac{n(n+1)}{2(\phi+\psi)\sum_{\substack{i,j=1\\ i\leq j}}^{n}\frac{1}{D_{ij}}}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}} \right\rangle, \tag{28}$$

$$G_{ij} = \left(\frac{\zeta - 1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{\left(\frac{\zeta-1}{\zeta^{\left(\frac{\eta_i}{\ell}\right)^q}-1}\right)^{n\Omega_i}}+1\right)}{\log\zeta}} - 1}\right)^{\phi} \left(\frac{\zeta - 1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{\left(\frac{\zeta-1}{\zeta^{\left(\frac{\eta_j}{\ell}\right)^q}-1}\right)^{n\Omega_j}}+1\right)}{\log\zeta}} - 1}\right)^{\psi}.$$

For each of the above cases $\Omega_i = \frac{\omega_j\left(1+T\left(\tilde{\alpha}_j\right)\right)}{\sum_{k=1}^{n}\omega_k(1+T(\tilde{\alpha}_k))}$ is considered.

5 An MAGDM method based on Lq-ROF information with unknown attributes and expert weights

In this section, an effective approach for solving MAGDM problems under Lq-ROF environments is designed with unknown attributes and unknown expert weights. The problem is described as follows.

Consider $\mathfrak{D} = \left\{\mathfrak{D}^{(1)}, \mathfrak{D}^{(2)}, \ldots, \mathfrak{D}^{(t)}\right\}$ as the set of DMs, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ as a finite set of alternatives, and let $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\}$ be the set of n attributes. Assume that the DM $\mathfrak{D}^{(k)}$ $(k = 1, 2, \ldots, t)$ provides his/her judgement information for the alternative A_i $(i = 1, 2, \ldots, m)$ in regard to the attribute $\mathcal{C}_j$ $(j = 1, 2, \ldots, n)$, in the form of Lq-ROFNs, which are presented in a Lq-ROF decision matrix (Lq-ROFDM) $\widetilde{\mathcal{X}}^{(k)} = \left[\widetilde{\alpha}_{ij}^{(k)}\right]_{m\times n}$, where $\widetilde{\alpha}_{ij}^{(k)} = \langle \mathcal{S}_{\xi_{\tilde{\alpha}_{ij}}^{(k)}}, \mathcal{S}_{\eta_{\tilde{\alpha}_{ij}}^{(k)}} \rangle$ are Lq-ROFNs. Here, for the DM $\mathfrak{D}^{(k)}$; $\mathcal{S}_{\xi_{\tilde{\alpha}_{ij}}^{(k)}}$ indicates the degree that the alternative $\mathcal{A}_i$ satisfies the attribute $\mathcal{C}_j$ and $\mathcal{S}_{\eta_{\tilde{\alpha}_{ij}}^{(k)}}$ indicates the degree that the alternative $\mathcal{A}_i$ fails to satisfy the attribute $\mathcal{C}_j$. It is to be mentioned here that the weights of DMs and attributes are unknown. In this method, the unknown weight vector of the DMs $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_t)^T$ and weight vector of the attributes $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_n)^T$ are determined first. If those weight vectors are known, the steps relating to these weight evaluations may be skipped.

The aim of this MAGDM method is to select the most desirable alternative(s) on the basis of given Lq-ROFDMs. An algorithm of the MAGDM with Lq-ROF information is provided as follows.

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$$
\begin{aligned}
& Lq-ROFFPWHM_{\omega}^{\phi,\psi}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n) \\
& = \left\langle \mathcal{S}\left(\ell\left(\left(\log\left(\left((\zeta-1)/\left(\frac{\zeta-1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{\left(\prod_{i,j=1,\, i\le j}^{n}\left(\frac{\zeta-1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{F_{ij}}+1\right)}{\log\zeta}}-1}\right)\right)^{\frac{2}{n(n+1)}}}+1\right)}{\log\zeta}}-1}\right)\right)^{\frac{1}{\phi+\psi}}\right)+1\right)\right)/(\log\zeta)\right)\right)^{\frac{1}{q}}, \right. \\
& \left. \mathcal{S}\left(\ell\left(1-\left(\log\left(\left((\zeta-1)/\left(\frac{\zeta-1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{\left(\prod_{i,j=1,\, i\le j}^{n}\left(\frac{\zeta-1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{G_{ij}}+1\right)}{\log\zeta}}-1}\right)\right)^{\frac{2}{n(n+1)}}}+1\right)}{\log\zeta}}-1}\right)\right)^{\frac{1}{\phi+\psi}}\right)+1\right)/\log\zeta\right)\right)\right)^{\frac{1}{q}}\right\rangle,
\end{aligned}
\tag{29}
$$

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Step 1 Convert the individual Lq-ROFDM $\widetilde{\mathcal{X}}^{(k)} = \left[\widetilde{\alpha}_{ij}^{(k)}\right]_{m\times n}$ into normalized Lq-ROFDM $\widetilde{R}^{(k)} = \left[\widetilde{r}_{ij}^{(k)}\right]_{m\times n} = \left[\langle \mathcal{S}_{\xi_{\tilde{r}_{ij}}^{(k)}}, \mathcal{S}_{\eta_{\tilde{r}_{ij}}^{(k)}} \rangle\right]_{m\times n}$. The normalization process is done as follows:

$$\widetilde{r}_{ij}^{(k)} = \begin{cases} \langle \mathcal{S}_{\xi_{\tilde{\alpha}_{ij}}^{(k)}}, \mathcal{S}_{\eta_{\tilde{\alpha}_{ij}}^{(k)}} \rangle & \text{for benefit attributes;} \\ \langle \mathcal{S}_{\eta_{\tilde{\alpha}_{ij}}^{(k)}}, \mathcal{S}_{\xi_{\tilde{\alpha}_{ij}}^{(k)}} \rangle & \text{for cost attributes.} \end{cases} \tag{30}$$

Step 2: On the basis of Eq. (6), calculate the support degrees between the Lq-ROFN $\widetilde{r}_{ij}^{(k)}$ with other Lq-ROFNs $\widetilde{r}_{ij}^{(l)}$ $(k, l = 1, 2, \ldots, t; k \neq l)$, shown as follows:

$$Sup\left(\widetilde{r}_{ij}^{(k)}, \widetilde{r}_{ij}^{(l)}\right) = 1 - d\left(\widetilde{r}_{ij}^{(k)}, \widetilde{r}_{ij}^{(l)}\right), (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n). \tag{31}$$

Calculated support degrees, $Sup\left(\widetilde{r}_{ij}^{(k)}, \widetilde{r}_{ij}^{(l)}\right)$, are arranged in a matrix form, i.e. $\left[Sup\left(\widetilde{r}_{ij}^{(k)}, \widetilde{r}_{ij}^{(l)}\right)\right]_{m\times n}$ and for simplicity, the support degree matrices $\left[Sup\left(\widetilde{r}_{ij}^{(k)}, \widetilde{r}_{ij}^{(l)}\right)\right]_{m\times n}$ are represented by $\widehat{\text{Sup}}^{kl}$ $(k, l = 1, 2, \ldots, t; k \neq l)$.

Step 3: Determine the sum of supports, $T\left(\widetilde{r}_{ij}^{(k)}\right)$ corresponding to $\widetilde{r}_{ij}^{(k)}$, as

$$T\left(\widetilde{r}_{ij}^{(k)}\right) = \sum_{l=1, l\neq k}^{t} Sup\left(\widetilde{r}_{ij}^{(k)}, \widetilde{r}_{ij}^{(l)}\right), \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; k \neq l), \tag{32}$$

and for convenience, the formulated matrix $\left[T\left(\widetilde{r}_{ij}^{(k)}\right)\right]_{m\times n}$ is denoted as $T^{(k)}$.

Step 4: In this step, the unknown weights Ω_k of DMs are determined using Algorithm 1.

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Algorithm 1: Determination of weights of DMs.

Considering equal weight $\omega = (\omega_1, \omega_2, \dots, \omega_t)^T = \left(\frac{1}{t}, \frac{1}{t}, \dots, \frac{1}{t}\right)^T$ for all DMs, calculate intuitive power weight $\mathfrak{w}_{ij}^{(k)}$ of each Lq-ROFN $\tilde{r}_{ij}^{(k)}$ as follows:

$$\mathfrak{w}_{ij}^{(k)} = \frac{\omega_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)}{\sum_{k=1}^{t}\omega_k\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)} = \frac{\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)}{\sum_{k=1}^{t}\left(1+T\left(\tilde{r}_{ij}^{(k)}\right)\right)}, \text{ for } k = 1,2,\dots,t.$$

For reducing complexity of representation, $\left[\mathfrak{w}_{ij}^{(k)}\right]_{m\times n}$ is denoted by $\mathfrak{w}^{(k)}$.

For the alternative A_i, aggregate the decision values corresponding to the attributes C_j provided by DM $\mathfrak{D}^{(k)}$ $(k = 1,2,\dots,t)$ based on Lq-ROFAPWA operator and obtain the comprehensive decision matrix as an ideal solution $\tilde{R}^* = [\tilde{r}_{ij}^*]_{m\times n} = \left[\left\langle s_{\xi_{\tilde{r}_{ij}^*}}, s_{\eta_{\tilde{r}_{ij}^*}}\right\rangle\right]_{m\times n}$,

where, $$\tilde{r}_{ij}^* = Lq-ROFAPWA\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \dots, \tilde{r}_{ij}^{(t)}\right) = \oplus_{A\,k=1}^{\;t}\left(\mathfrak{w}_{ij}^{(k)} \odot_A \tilde{r}_{ij}^{(k)}\right)$$

Determine the left ideal solution $\tilde{R}^- = [\tilde{r}_{ij}^-]_{m\times n}$ and right ideal solution $\tilde{R}^+ = [\tilde{r}_{ij}^+]_{m\times n}$ as follows

$$\tilde{r}_{ij}^- = \left\{\tilde{r}_{ij}^{(k)} \middle| S\left(\tilde{r}_{ij}^{(k)}\right) = \min_{1\le k\le t}\left\{S\left(\tilde{r}_{ij}^{(k)}\right)\right\}\right\},$$
$$\tilde{r}_{ij}^+ = \left\{\tilde{r}_{ij}^{(k)} \middle| S\left(\tilde{r}_{ij}^{(k)}\right) = \max_{1\le k\le t}\left\{S\left(\tilde{r}_{ij}^{(k)}\right)\right\}\right\},$$

where $i = 1,2,\dots,m; j = 1,2,\dots,n$, and t is the total number of DMs.

On the basis of Eq. (6), calculate the distance measures, $\mathfrak{S}_k^*$, $\mathfrak{S}_k^-$ and $\mathfrak{S}_k^+$, of individual matrix $\tilde{R}^{(k)}$ from ideal solution matrix $\tilde{R}^*$, left ideal solution matrix $\tilde{R}^-$ and right ideal solution matrix $\tilde{R}^+$, respectively, as shown as follows:

$$\mathfrak{S}_k^* = d\left(\tilde{R}^{(k)}, \tilde{R}^*\right) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} d\left(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^*\right)$$

$$= \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\ell^q}\left(\frac{1}{2}\left(\left|\left(\xi_{\tilde{r}_{ij}^{(k)}}\right)^q - \left(\xi_{\tilde{r}_{ij}^*}\right)^q\right|^\varepsilon + \left|\left(\eta_{\tilde{r}_{ij}^{(k)}}\right)^q - \left(\eta_{\tilde{r}_{ij}^*}\right)^q\right|^\varepsilon + \left|\left(\pi_{\tilde{r}_{ij}^{(k)}}\right)^q - \left(\pi_{\tilde{r}_{ij}^*}\right)^q\right|^\varepsilon\right)\right)^{\frac{1}{\varepsilon}};$$

$$\mathfrak{S}_k^- = d\left(\tilde{R}^{(k)}, \tilde{R}^-\right) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} d\left(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^-\right);$$

and $$\mathfrak{S}_k^+ = d\left(\tilde{R}^{(k)}, \tilde{R}^+\right) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} d\left(\tilde{r}_{ij}^{(k)}, \tilde{r}_{ij}^+\right), \text{ where } k = 1,2,\dots,t.$$

Calculate the relative closeness index CI_k of the DMs by using the following formula:

$$CI_k = \frac{\mathfrak{S}_k^- + \mathfrak{S}_k^+}{\mathfrak{S}_k^* + \mathfrak{S}_k^- + \mathfrak{S}_k^+}.$$

The weight of the k^{th} DM is defined as

$$\Omega_k = \frac{CI_k}{\sum_{k=1}^{t} CI_k}.$$

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Table 4 The matrix for ideal solution $\left(\widetilde{\boldsymbol{R}}^{*}\right)$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$(\mathcal{S}_{4.3739}, \mathcal{S}_{1.8693})$	$(\mathcal{S}_{3.4044}, \mathcal{S}_{1.7033})$	$(\mathcal{S}_{4.3902}, \mathcal{S}_{2.3774})$	$(\mathcal{S}_{1.7698}, \mathcal{S}_{4.9756})$
A_2	$(\mathcal{S}_{3.8767}, \mathcal{S}_{2.5402})$	$(\mathcal{S}_{2.4281}, \mathcal{S}_{2.5780})$	$(\mathcal{S}_{2.4358}, \mathcal{S}_{2.7963})$	$(\mathcal{S}_{2.4294}, \mathcal{S}_{4.3124})$
A_3	$(\mathcal{S}_{4.0351}, \mathcal{S}_{2.1675})$	$(\mathcal{S}_{2.4328}, \mathcal{S}_{3.5600})$	$(\mathcal{S}_{2.6294}, \mathcal{S}_{2.5217})$	$(\mathcal{S}_{4.1963}, \mathcal{S}_{2.1557})$
A_4	$(\mathcal{S}_{5.3667}, \mathcal{S}_{1.5816})$	$(\mathcal{S}_{3.7221}, \mathcal{S}_{3.3051})$	$(\mathcal{S}_{5.3667}, \mathcal{S}_{1.5816})$	$(\mathcal{S}_{5.6160}, \mathcal{S}_{1.2842})$

Table 5 The left ideal solution matrix $\left(\widetilde{\boldsymbol{R}}^{-}\right)$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$(\mathcal{S}_3, \mathcal{S}_3)$	$(\mathcal{S}_3, \mathcal{S}_5)$	$(\mathcal{S}_3, \mathcal{S}_4)$	$(\mathcal{S}_1, \mathcal{S}_6)$
A_2	$(\mathcal{S}_3, \mathcal{S}_4)$	$(\mathcal{S}_2, \mathcal{S}_4)$	$(\mathcal{S}_2, \mathcal{S}_5)$	$(\mathcal{S}_2, \mathcal{S}_5)$
A_3	$(\mathcal{S}_1, \mathcal{S}_3)$	$(\mathcal{S}_2, \mathcal{S}_5)$	$(\mathcal{S}_1, \mathcal{S}_2)$	$(\mathcal{S}_1, \mathcal{S}_3)$
A_4	$(\mathcal{S}_5, \mathcal{S}_2)$	$(\mathcal{S}_3, \mathcal{S}_3)$	$(\mathcal{S}_5, \mathcal{S}_2)$	$(\mathcal{S}_4, \mathcal{S}_1)$

Table 6 The right ideal solution matrix $\left(\widetilde{\boldsymbol{R}}^{+}\right)$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$(\mathcal{S}_6, \mathcal{S}_1)$	$(\mathcal{S}_4, \mathcal{S}_1)$	$(\mathcal{S}_6, \mathcal{S}_1)$	$(\mathcal{S}_2, \mathcal{S}_3)$
A_2	$(\mathcal{S}_5, \mathcal{S}_2)$	$(\mathcal{S}_2, \mathcal{S}_1)$	$(\mathcal{S}_2, \mathcal{S}_1)$	$(\mathcal{S}_3, \mathcal{S}_4)$
A_3	$(\mathcal{S}_6, \mathcal{S}_1)$	$(\mathcal{S}_3, \mathcal{S}_3)$	$(\mathcal{S}_3, \mathcal{S}_2)$	$(\mathcal{S}_6, \mathcal{S}_1)$
A_4	$(\mathcal{S}_6, \mathcal{S}_2)$	$(\mathcal{S}_4, \mathcal{S}_3)$	$(\mathcal{S}_6, \mathcal{S}_2)$	$(\mathcal{S}_7, \mathcal{S}_1)$

It is to be noted here that Step 4 is to be skipped if the weights of DMs are known.

Step 5: Calculate the power weights $\varpi_{ij}^{(k)}$ corresponding to the Lq-ROFN $\tilde{r}_{ij}^{(k)}$ $(k = 1, 2, \ldots, t)$ as.

$$\varpi_{ij}^{(k)} = \frac{\Omega_k\left(1 + T\left(\tilde{r}_{ij}^{(k)}\right)\right)}{\sum_{k=1}^{t}\Omega_k\left(1 + T\left(\tilde{r}_{ij}^{(k)}\right)\right)}, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n). \tag{33}$$

For simplicity, the constructed power weight matrix $\left[\varpi_{ij}^{(k)}\right]_{m\times n}$ is denoted as $\varpi^{(k)}$.

Step 6: For the alternative A_i, aggregate the decision values of attributes $\mathcal{C}_j$ provided by DM $\mathfrak{D}^{(k)}$ $(k = 1, 2, \ldots, t)$ based on Lq-ROFAPWA operator as

$$\begin{aligned}\tilde{r}_{ij} &= Lq - \text{ROFAPWA}\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \ldots, \tilde{r}_{ij}^{(t)}\right)\\ &= \oplus_{Ak=1}^{t}\left(\varpi_{ij}^{(k)} \odot_A \tilde{r}_{ij}^{(k)}\right)\end{aligned} \tag{34}$$

and obtain the comprehensive decision matrix $\widetilde{R} = \left[\widetilde{r}_{ij}\right]_{m\times n}$.

Step 7: Calculate the support degree $Sup\left(\widetilde{r}_{ij}, \widetilde{r}_{iu}\right)$ $(j, u = 1, 2, \ldots, n; j \neq u)$ of each element of the matrix $\widetilde{R}$ as

$$\text{Sup}\left(\tilde{r}_{ij}, \tilde{r}_{iu}\right) = 1 - d\left(\tilde{r}_{ij}, \tilde{r}_{iu}\right), (i = 1, 2, \ldots, m). \tag{35}$$

For convenience, the column matrix $\left[Sup\left(\widetilde{r}_{ij}, \widetilde{r}_{iu}\right)\right]_{m\times 1}$ is denoted by $\widehat{\text{Sup}}_{ju}$.

Step 8: Calculate the sum of supports corresponding to $\widetilde{r}_{ij}$, $T\left(\widetilde{r}_{ij}\right)$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ as

$$\left[T\left(\tilde{r}_{ij}\right)\right]_{m\times 1} = \sum_{u=1, j\neq u}^{n}\widehat{\text{Sup}}_{ju}, \text{for } j = 1, 2, \ldots, n, \tag{36}$$

and form a matrix as $T = \left[T\left(\widetilde{r}_{ij}\right)\right]_{m\times n}$.

Step 9: In this step, unknown weights of attributes Θ_j are calculated using Algorithm 2.

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Algorithm 2: Calculation of weights of attributes

Calculate entropy measure matrix E of the comprehensive decision matrix $\tilde{R}$ by

$$E = \begin{pmatrix} E_{11}(\tilde{r}_{11}) & E_{12}(\tilde{r}_{12}) & \cdots & E_{1n}(\tilde{r}_{1n}) \\ E_{21}(\tilde{r}_{21}) & E_{22}(\tilde{r}_{22}) & \cdots & E_{2n}(\tilde{r}_{2n}) \\ \vdots & & \ddots & \vdots \\ E_{m1}(\tilde{r}_{m1}) & E_{m2}(\tilde{r}_{m2}) & \cdots & E_{mn}(\tilde{r}_{mn}) \end{pmatrix}$$

where $E_{ij}(\tilde{r}_{ij}) = \frac{1}{2}\left(2 - \frac{1}{\varrho^q}\left({\xi_{ij}}^q + {\eta_{ij}}^q\right)\right)\left(1 - \frac{1}{\varrho^q}\left|{\xi_{ij}}^q - {\eta_{ij}}^q\right|\right)$ using the newly defined entropy measure presented in Eq. (9).

Compute weight of attributes as

$$\Theta_j = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} E_{ij}(\tilde{r}_{ij})}{\sum_{j=1}^{n}\left(1 - \frac{1}{m}\sum_{i=1}^{m} E_{ij}(\tilde{r}_{ij})\right)}, j = 1,2,\ldots,n.$$

It is to be mentioned here that Step 9 is to be skipped if the weights of attributes are known.

Step 10: Calculate the power weights as:

$$\Xi_{ij} = \frac{\Theta_j\left(1 + T(\tilde{r}_{ij})\right)}{\sum_{j=1}^{n}\Theta_j\left(1 + T(\tilde{r}_{ij})\right)}, (i = 1,2,\ldots,m; j = 1,2,\ldots,n) \tag{37}$$

and form the power weight matrix $\Xi = [\Xi_{ij}]_{m\times n}$.

Step 11: Utilizing power weights Ξ_{ij}, calculate the overall performance value of the alternative A_i $(i = 1,2,\ldots,m)$ as

$$\tilde{r}_i = Lq - ROFAPWHM(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in})$$

$$= \left(\frac{2}{n(n+1)} \oplus^{n}_{A_{u,v=1} \atop u\le v}\left((n\Xi_{iu} \odot_A \tilde{r}_{iv})^{\phi} \otimes_A (n\Xi_{iu} \odot_A \tilde{r}_{iv})^{\psi}\right)\right)^{\frac{1}{\phi+\psi}}. \tag{38}$$

Step 12: Calculate the score function of alternatives and rank the alternatives based on the comparison rule presented in Sect. 2.

The flowchart of the above methodology is presented in Fig. 2.

6 Case example

To illustrate the use of the newly proposed MAGDM method, a case example (Garg 2018) is provided in this section. Further, the effectiveness and superiority of the proposed method are demonstrated by comparing it with different existing methods.

Example 3 Assume that a multinational company wanted to adopt a strategy among four available strategies A_i $(i = 1,2,3,4)$ for the planning of financial matters of next year and invites three experts $\mathfrak{D}^{(1)}$, $\mathfrak{D}^{(2)}$ and $\mathfrak{D}^{(3)}$ to provide their preferences on the strategies based on four general characteristics $\mathcal{C}_j$ $(j = 1,2,2,4)$. The experts evaluated each alternative under each attribute using the LTS $\mathcal{S} = \{\mathcal{S}_0 = \text{extremely poor}, \mathcal{S}_1 = \text{very poor}, \mathcal{S}_2 = \text{poor}, \mathcal{S}_3 = \text{slightly poor}, \mathcal{S}_4 = \text{fair}, \mathcal{S}_5 = \text{slightly good}, \mathcal{S}_6 = \text{good}, \mathcal{S}_7 = \text{very good}, \mathcal{S}_8 = \text{extremely good}\}$.

The Lq-ROFDMs $\widetilde{\mathcal{X}}^{(k)} = \left[\widetilde{\alpha}_{ij}^{(k)}\right]_{m\times n}$ $(k = 1,2,3)$ of three experts $\mathfrak{D}^{(k)}$ $(k = 1,2,3)$ are presented in Tables 1, 2, 3, respectively. Then, the developed methodology is applied to obtain the most eligible strategy.

Table 7 Aggregated Lq-ROFDM $\widetilde{R}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$(\mathcal{S}_{4.3491}, \mathcal{S}_{1.8728})$	$(\mathcal{S}_{3.4205}, \mathcal{S}_{1.6830})$	$(\mathcal{S}_{4.3667}, \mathcal{S}_{2.4069})$	$(\mathcal{S}_{1.7754}, \mathcal{S}_{4.9281})$
A_2	$(\mathcal{S}_{3.9088}, \mathcal{S}_{2.5263})$	$(\mathcal{S}_{2.4199}, \mathcal{S}_{2.5252})$	$(\mathcal{S}_{2.4520}, \mathcal{S}_{2.8193})$	$(\mathcal{S}_{2.4216}, \mathcal{S}_{4.3271})$
A_3	$(\mathcal{S}_{4.0067}, \mathcal{S}_{2.1840})$	$(\mathcal{S}_{2.4490}, \mathcal{S}_{3.5465})$	$(\mathcal{S}_{2.6101}, \mathcal{S}_{2.5089})$	$(\mathcal{S}_{4.1735}, \mathcal{S}_{2.1734})$
A_4	$(\mathcal{S}_{5.3585}, \mathcal{S}_{1.5900})$	$(\mathcal{S}_{3.7085}, \mathcal{S}_{3.2981})$	$(\mathcal{S}_{5.3590}, \mathcal{S}_{1.5905})$	$(\mathcal{S}_{5.5877}, \mathcal{S}_{1.2769})$

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Step 1 Since all the attributes are benefit types, normalization is not required. So, $\widetilde{\mathcal{X}}^{(k)} = \widetilde{R}^{(k)} = \left[\widetilde{r}_{ij}^{(k)}\right]_{4\times 4}$ $(k = 1, 2, 3)$.

Step 2 Calculate the support degrees of each element $\widetilde{r}_{ij}^{(k)}$, $(i = 1, 2, 3, 4; j = 1, 2, 3, 4; k = 1, 2, 3)$ on the basis of Eq. (31) (for convenience, take $q = 3$). Then, the support degree matrices $\widehat{\text{Sup}}^{kl}$ $(k, l = 1, 2, 3; k \neq l)$ are formulated as:

$$\widehat{\text{Sup}}^{12} = \widehat{\text{Sup}}^{21} = \begin{bmatrix} 0.6376 & 0.9277 & 0.9277 & 0.6376 \\ 0.8357 & 0.8556 & 0.8953 & 0.8809 \\ 0.9863 & 0.9629 & 0.9492 & 0.6542 \\ 0.8223 & 0.9277 & 0.9863 & 0.4551 \end{bmatrix};$$

$$\widehat{\text{Sup}}^{13} = \widehat{\text{Sup}}^{31} = \begin{bmatrix} 0.6542 & 0.7578 & 0.6542 & 0.9863 \\ 0.8906 & 0.9629 & 0.7578 & 0.9629 \\ 0.6037 & 0.8086 & 0.8906 & 0.6037 \\ 0.8152 & 0.9277 & 0.8152 & 0.5809 \end{bmatrix};$$

$$\widehat{\text{Sup}}^{23} = \widehat{\text{Sup}}^{32} = \begin{bmatrix} 0.9629 & 0.7864 & 0.6775 & 0.6309 \\ 0.8086 & 0.8770 & 0.8556 & 0.8953 \\ 0.6173 & 0.8249 & 0.8595 & 0.9492 \\ 0.9863 & 0.8764 & 0.8223 & 0.8736 \end{bmatrix}.$$

Step 3: Calculate the sum of supports corresponding to $\widetilde{r}_{ij}^{(k)}$, $T\left(\widetilde{r}_{ij}^{(k)}\right)$ $(i = 1, 2, 3, 4; j = 1, 2, 3, 4; k = 1, 2, 3)$ based on Eq. (32). Then, the matrices $T^{(k)} = \left[T\left(\widetilde{r}_{ij}^{(k)}\right)\right]_{4\times 4}$ are determined as.

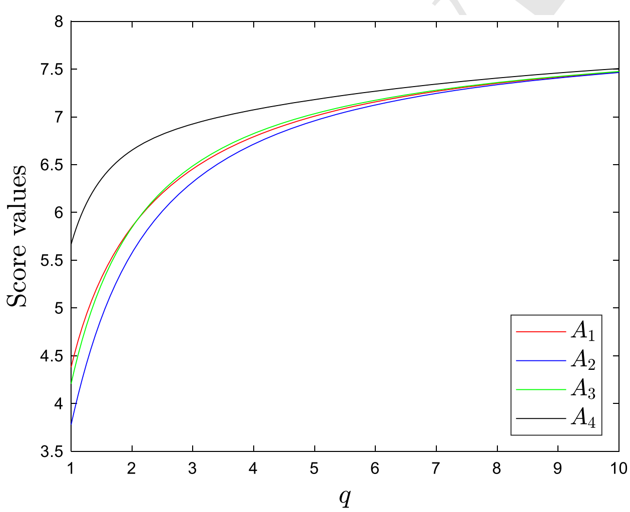

Fig. 3 Score values of alternatives using L*q*-ROFHPWHM operator varying *q*

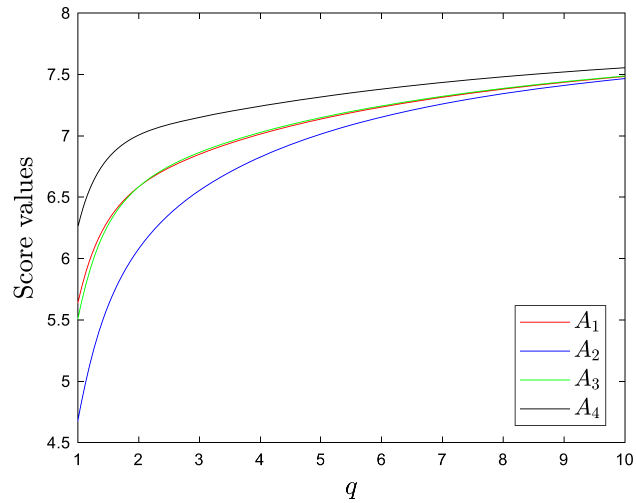

Fig. 4 Score values of alternatives using L*q*-ROFDPWHM operator varying *q*

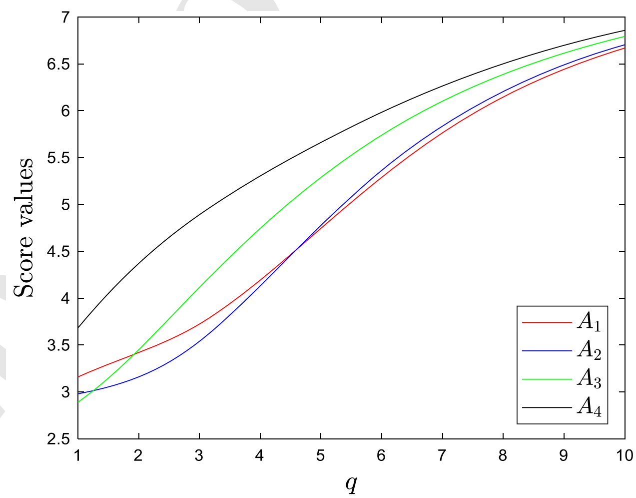

Fig. 5 Score values of alternatives using L*q*-ROFFPWHM operator varying *q*

$$T^{(1)} = \begin{bmatrix} 1.2918 & 1.6855 & 1.5819 & 1.6239 \\ 1.7263 & 1.8185 & 1.6531 & 1.8438 \\ 1.5900 & 1.7715 & 1.8398 & 1.2579 \\ 1.6374 & 1.8555 & 1.8015 & 1.0360 \end{bmatrix},$$

$$T^{(2)} = \begin{bmatrix} 1.6005 & 1.7141 & 1.6053 & 1.2684 \\ 1.6443 & 1.7326 & 1.7509 & 1.7762 \\ 1.6036 & 1.7878 & 1.8087 & 1.6034 \\ 1.8086 & 1.8042 & 1.8086 & 1.3287 \end{bmatrix},$$

and

$$T^{(3)} = \begin{bmatrix} 1.6171 & 1.5442 & 1.3317 & 1.6172 \\ 1.6992 & 1.8398 & 1.6134 & 1.8582 \\ 1.2210 & 1.6335 & 1.7501 & 1.5529 \\ 1.8015 & 1.8042 & 1.6374 & 1.4546 \end{bmatrix}.$$

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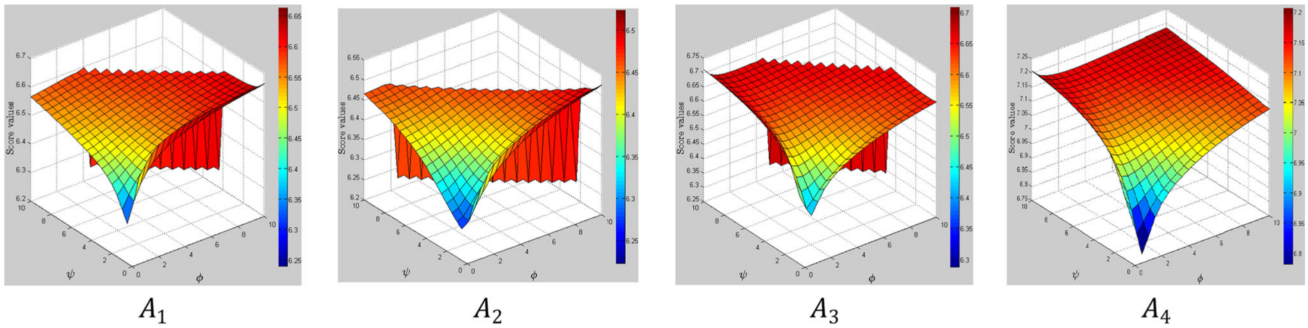

Fig. 6 Scores of A_i based on Lq-ROFHPWHM operator varying ϕ, ψ in $(0, 10]$

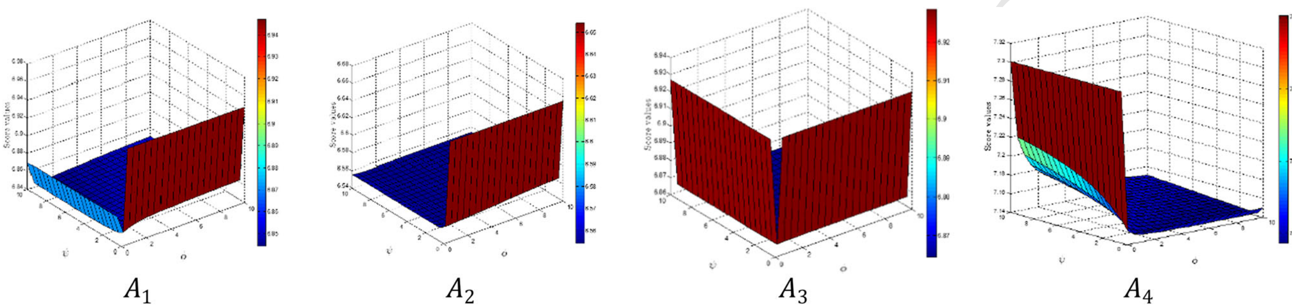

Fig. 7 Scores of A_i based on Lq-ROFDPWHM operator varying ϕ, ψ in $(0, 10]$

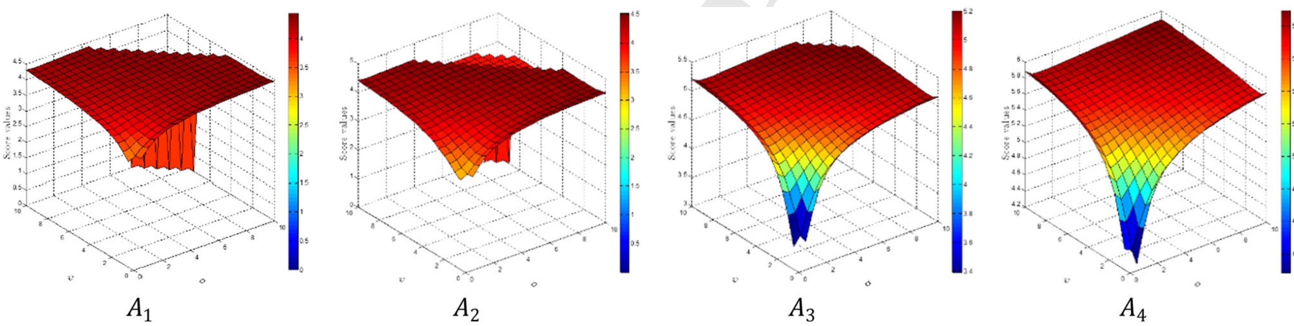

Fig. 8 Scores of A_i based on Lq-ROFFPWHM operator varying ϕ, ψ in $(0, 10]$

Step 4 Since the weights of DMs are completely unknown, Algorithm 1 is followed in order to compute weight $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ of DMs.

Initially assume that weights of three DMs are equal, i.e. $\omega = (\omega_1, \omega_2, \omega_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Utilizing Eq. (33), calculate the intuitive power weights $\mathfrak{w}_{ij}^{(k)}$ corresponding to Lq-ROFN $\widetilde{r}_{ij}^{(k)}$ and $\mathfrak{w}^{(k)}$ $(i = 1, 2, 3, 4; j = 1, 2, 3, 4; k = 1, 2, 3)$ are obtained as

$$\mathfrak{w}^{(1)} = \begin{bmatrix} 0.3052 & 0.3381 & 0.3434 & 0.3494 \\ 0.3378 & 0.3359 & 0.3309 & 0.3354 \\ 0.3493 & 0.3383 & 0.3381 & 0.3045 \\ 0.3198 & 0.3374 & 0.3397 & 0.2986 \end{bmatrix},$$

$$\mathfrak{w}^{(2)} = \begin{bmatrix} 0.3463 & 0.3417 & 0.3465 & 0.3021 \\ 0.3277 & 0.3257 & 0.3431 & 0.3275 \\ 0.3511 & 0.3403 & 0.3344 & 0.3511 \\ 0.3405 & 0.3313 & 0.3405 & 0.3415 \end{bmatrix},$$

and

$$\mathfrak{w}^{(3)} = \begin{bmatrix} 0.3485 & 0.3203 & 0.3101 & 0.3485 \\ 0.3345 & 0.3384 & 0.3260 & 0.3371 \\ 0.2995 & 0.3214 & 0.3274 & 0.3443 \\ 0.3397 & 0.3313 & 0.3198 & 0.3599 \end{bmatrix}.$$

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Table 8 Influence of HM parameters ϕ, ψ on decision results

Varying ϕ and ψ		Score values of alternatives				Ranking order
		$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	
Lq-ROFHPWHM	$\phi = 2, \psi = 10$	6.5789	6.4648	6.6666	7.1622	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\phi = 5, \psi = 5$	6.5754	6.4561	6.6358	7.1216	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\phi = 10, \psi = 2.5$	6.6197	6.2414	6.6528	7.1415	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\phi = 10, \psi = 0$	6.6639	6.5244	6.6503	7.1262	$A_4 \succ A_1 \succ A_3 \succ A_2$
	$\phi = 10, \psi = 10$	6.2615	6.2481	6.2964	7.1817	$A_4 \succ A_3 \succ A_1 \succ A_2$
Lq-ROFDPWHM	$\phi = 0, \psi = 0.5$	6.8703	6.5559	6.9270	7.3009	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\phi = 2, \psi = 10$	6.8450	6.5543	6.8644	7.1787	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\phi = 5, \psi = 5$	6.8501	6.5542	6.8646	7.1511	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\phi = 10, \psi = 2.5$	6.8567	6.5557	6.8649	7.1443	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\phi = 10, \psi = 0$	6.9471	6.6544	6.9285	7.1499	$A_4 \succ A_1 \succ A_3 \succ A_2$
	$\phi = 10, \psi = 10$	6.8501	6.5542	6.8646	7.1511	$A_4 \succ A_3 \succ A_1 \succ A_2$
Lq-ROFFPWHM	$\phi = 0, \psi = 0.5$	3.3368	3.1234	3.3952	4.4615	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\phi = 2, \psi = 10$	4.3175	4.4122	5.1374	5.8166	$A_4 \succ A_3 \succ A_2 \succ A_1$
	$\phi = 5, \psi = 5$	4.2876	4.3612	5.0725	5.7332	$A_4 \succ A_3 \succ A_2 \succ A_1$
	$\phi = 10, \psi = 2.5$	4.3622	4.4266	5.1300	5.7913	$A_4 \succ A_3 \succ A_2 \succ A_1$
	$\phi = 10, \psi = 0$	4.4310	4.5229	5.1693	5.8036	$A_4 \succ A_3 \succ A_2 \succ A_1$
	$\phi = 10, \psi = 10$	0.0000	0.0000	4.483	5.899	$A_4 \succ A_3 \succ A_1 \approx A_2$

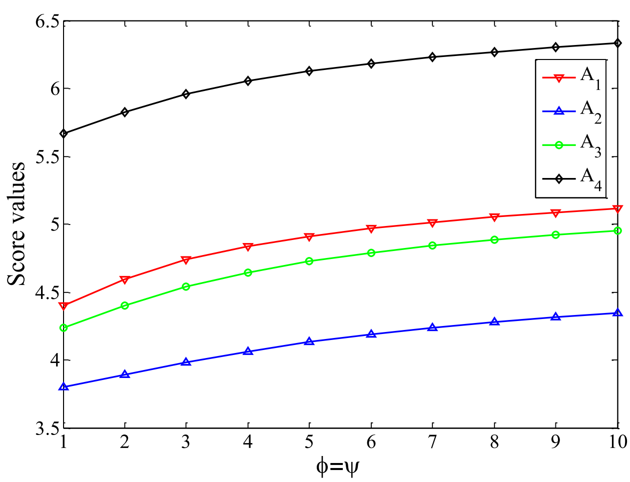

Fig. 9 Score values of alternatives using Lq-ROFHPWHM operator varying HM parameters (Taking $\phi = \psi; q = 1; \rho = 2$)

Then, utilizing the calculated power weights and Lq-ROFAPWA operator, the matrices corresponding to ideal solutions, left ideal solutions, and right ideal solutions are calculated and presented in Tables 4, 5, 6, respectively.

On the basis of Eq. (6), calculate the distance measures of individual matrix $\widetilde{R}^{(k)}$ from ideal solution matrix, left ideal solution matrix and right ideal solution matrix, $\mathfrak{S}_k^*$, $\mathfrak{S}_k^-$ and $\mathfrak{S}_k^+$, $(k = 1, 2, 3)$, respectively, as follows:

$$\mathfrak{S}_1^* = d\left(\tilde{R}^{(1)}, \tilde{R}^*\right) = \frac{1}{16}\sum_{i=1}^{4}\sum_{j=1}^{4} d\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^*\right) = 0.1220.$$

Similarly, $\mathfrak{S}_2^* = 0.0892$ and $\mathfrak{S}_3^* = 0.1249$.

$\mathfrak{S}_1^- = 0.1415, \mathfrak{S}_2^- = 0.0778, \mathfrak{S}_3^- = 0.1107.$

$\mathfrak{S}_1^+ = 0.1242, \mathfrak{S}_2^+ = 0.1679, \mathfrak{S}_3^+ = 0.1625.$

Next, the relative closeness index CI_k $(k = 1, 2, 3)$ of the DMs are calculated as:

$$\mathrm{CI}_1 = \frac{\mathfrak{S}_1^- + \mathfrak{S}_1^+}{\mathfrak{S}_1^* + \mathfrak{S}_1^- + \mathfrak{S}_1^+} = 0.6852.$$

Similarly, $CI_2 = 0.7336$, $CI_3 = 0.6862$.

Thus, the weight of the kth $(k = 1, 2, 3)$ DM is found as

$$\Omega_1 = \frac{\mathrm{CI}_1}{\sum_{k=1}^{3} \mathrm{CI}_k} = 0.3255.$$

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Table 9 Influence of Archimedean parameters on decision results

t-CN&*t*-Ns parameter		Score of the alternatives				Ranking order
		$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	
L*q*-ROFHPWHM	$\rho = 1$	6.4865	6.3328	6.5261	6.9199	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\rho = 2$	6.4682	6.3247	6.5047	6.9235	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\rho = 5$	6.4385	6.3131	6.4630	6.9337	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\rho = 10$	6.4184	6.3064	6.4315	6.9526	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\rho = 20$	6.4048	6.3044	6.4085	6.9857	$A_4 \succ A_3 \succ A_1 \succ A_2$
L*q*-ROFDPWHM	$\tau = 0.5$	6.4424	6.3589	6.4155	6.8174	$A_4 \succ A_1 \succ A_3 \succ A_2$
	$\tau = 1$	6.5369	6.4091	6.5395	6.9119	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\tau = 5$	6.9610	6.6422	6.9692	7.2863	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\tau = 10$	7.0484	6.7266	7.0519	7.4115	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\tau = 15$	7.0779	6.7580	7.0801	7.4525	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\tau = 20$	7.0926	6.7743	7.0942	7.4725	$A_4 \succ A_3 \succ A_1 \succ A_2$
L*q*-ROFFPWHM	$\zeta = 1.01$	3.6787	3.4262	3.9838	4.7692	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\zeta = 5$	3.7608	3.6088	4.1892	4.9594	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\zeta = 10$	3.8213	3.7156	4.3013	5.0616	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\zeta = 15$	3.8641	3.7844	4.3722	5.1241	$A_4 \succ A_3 \succ A_1 \succ A_2$
	$\zeta = 20$	3.8978	3.8355	4.4198	5.1690	$A_4 \succ A_3 \succ A_1 \succ A_2$

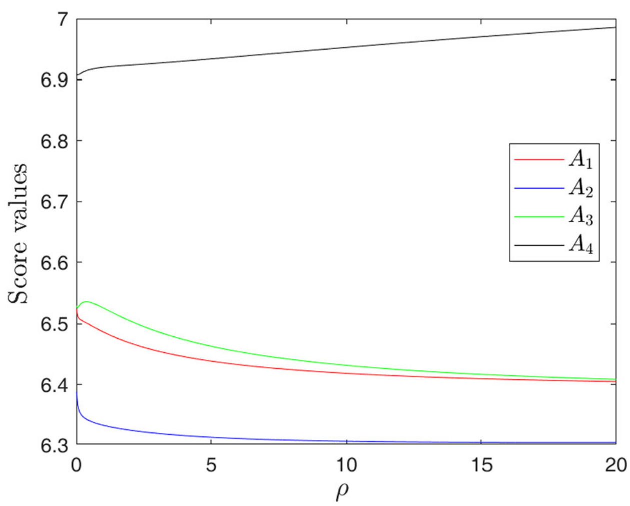

Fig. 10 Score values of alternatives using L*q*-ROFHPWHM operator varying ρ

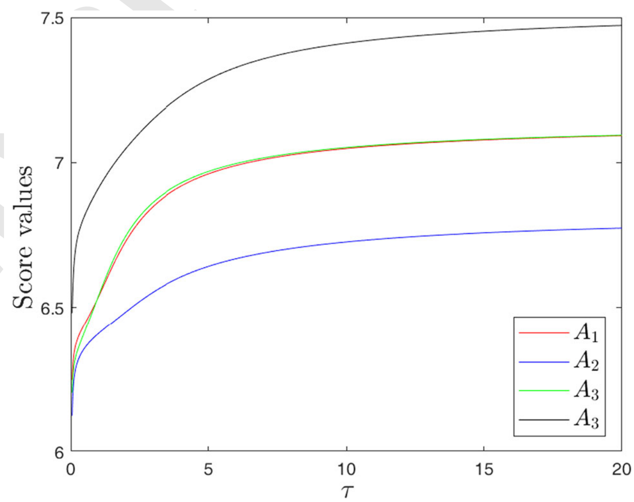

Fig. 11 Score values of alternatives using L*q*-ROFDPWHM operator varying τ

Similarly, $\Omega_2 = 0.3485$, $\Omega_3 = 0.3260$ are found as the weight of 2nd and 3rd DMs.

Step 5 The power weight $\varpi_{ij}^{(k)}$ corresponding to the L*q*-ROFN $\tilde{r}_{ij}^{(k)}$ $(i,j = 1,2,3,4; k = 1,2,3)$ is calculated as:

$$\varpi_{11}^{(1)} = \frac{\Omega_1\left(1 + T\left(\tilde{r}_{11}^{(1)}\right)\right)}{\sum_{k=1}^{3}\Omega_k\left(1 + T\left(\tilde{r}_{11}^{(k)}\right)\right)} = 0.2978.$$

Similarly, other power weights can be found, and subsequently, the following power weight matrices are obtained.

$$\varpi^{(1)} = \begin{bmatrix} 0.29780.33000.33510.3420 \\ 0.33010.32820.32300.3277 \\ 0.34070.33020.33020.2970 \\ 0.31210.32950.33160.2914 \end{bmatrix},$$

$$\varpi^{(2)} = \begin{bmatrix} 0.36170.35700.36190.3165 \\ 0.34270.34060.35850.3425 \\ 0.36670.35560.34960.3666 \\ 0.35580.34640.35580.3568 \end{bmatrix},$$

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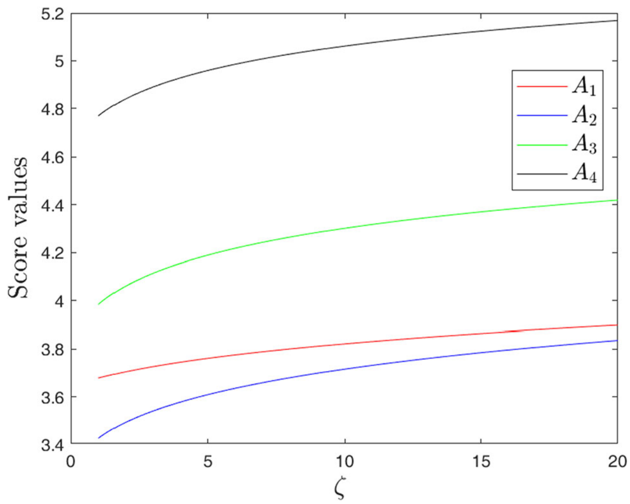

Fig. 12 Score values of alternatives using Lq-ROFFPWHM operator varying ζ

and $\varpi^{(3)} = \begin{bmatrix} 0.34050.31310.30300.3416 \\ 0.32730.33120.31860.3298 \\ 0.29260.31420.32020.3363 \\ 0.33200.32410.31260.3518 \end{bmatrix}$.

Step 6 Aggregating individual Lq-ROFDMs $\widetilde{R}^{(k)} = \left[\widetilde{r}_{ij}^{(k)}\right]_{4\times4}$ $(k = 1,2,3)$ into a collective Lq-ROFDM $\widetilde{R} = [\widetilde{r}_{ij}]_{4\times4}$ using Eq. (34), the aggregated matrix is presented in Table 7.

Step 7 Calculate the support degree column matrices $\widehat{\mathrm{Sup}}_{ju}$ $(j,u = 1,2,3,4)$ based on Eq. (35) as follows:

$$\widehat{\mathrm{Sup}}_{12} = \widehat{\mathrm{Sup}}_{21} = \begin{bmatrix} 0.9157 \\ 0.9110 \\ 0.9148 \\ 0.8234 \end{bmatrix}, \widehat{\mathrm{Sup}}_{13} = \widehat{\mathrm{Sup}}_{31} = \begin{bmatrix} 0.9846 \\ 0.9178 \\ 0.9140 \\ 0.9999 \end{bmatrix}, \widehat{\mathrm{Sup}}_{14} = \widehat{\mathrm{Sup}}_{41} = \begin{bmatrix} 0.8064 \\ 0.8882 \\ 0.9838 \\ 0.9616 \end{bmatrix}, \widehat{\mathrm{Sup}}_{23} = \widehat{\mathrm{Sup}}_{31} = \begin{bmatrix} 0.9056 \\ 0.9871 \\ 0.9466 \\ 0.8233 \end{bmatrix}, \widehat{\mathrm{Sup}}_{24} = \widehat{\mathrm{Sup}}_{42} = \begin{bmatrix} 0.8021 \\ 0.8732 \\ 0.9025 \\ 0.7857 \end{bmatrix}, \widehat{\mathrm{Sup}}_{34} = \widehat{\mathrm{Sup}}_{43} = \begin{bmatrix} 0.8160 \\ 0.8861 \\ 0.8978 \\ 0.9616 \end{bmatrix}$$

Step 8 Based on Eq. (36), calculate $\left[T\left(\widetilde{r}_{ij}\right)\right]_{4\times1} = \sum_{u=1,j\neq u}^{4} \widehat{\mathrm{Sup}}_{ju}$ $(i = 1,2,3,4; j = 1,2,3,4)$ and form the matrix $T = \left[T\left(\widetilde{r}_{ij}\right)\right]_{4\times4}$ as

$$T = \begin{bmatrix} 2.7067 & 2.6234 & 2.7061 & 2.4245 \\ 2.7170 & 2.7713 & 2.7909 & 2.6474 \\ 2.8126 & 2.7639 & 2.7584 & 2.7841 \\ 2.7849 & 2.4324 & 2.7849 & 2.7089 \end{bmatrix}.$$

Step 9 Here, the weight of attributes is completely unknown. Then, following Algorithm 2, the weight of attributes Θ_j are computed. To do this, first, calculate the entropy measure matrix as

Table 10 Comparison with existing methods

Problem	Score values	Rankings
LIFWPBM (Liu and Liu 2017)	$S(A_1) = 3.6485$, $S(A_2) = 3.2927$, $S(A_3) = 3.6449$, $S(A_4) = 5.0931$	$A_4 \succ A_1 \succ A_3 \succ A_2$
LIFWGPBM (Liu and Liu 2017)	$S(A_1) = 4.5457$, $S(A_2) = 4.1293$, $S(A_3) = 4.4301$, $S(A_4) = 5.8934$	$A_4 \succ A_1 \succ A_3 \succ A_2$
LIFWPA (Liu and Qin 2017)	$S(A_1) = 4.2533$, $S(A_2) = 3.8960$, $S(A_3) = 4.3071$, $S(A_4) = 5.8075$	$A_4 \succ A_3 \succ A_1 \succ A_2$
LIFWPG (Liu and Qin 2017)	$S(A_1) = 3.5748$, $S(A_2) = 3.4820$, $S(A_3) = 3.7837$, $S(A_4) = 5.5703$	$A_4 \succ A_3 \succ A_1 \succ A_2$
Lq-ROFWPBM (Liu and Liu 2018b) ($q = 3$)	$S(A_1) = 6.1936$, $S(A_2) = 6.0956$, $S(A_3) = 6.2382$, $S(A_4) = 6.7231$	$A_4 \succ A_3 \succ A_1 \succ A_2$
Lq-ROFWPGBM (Liu and Liu 2018b) ($q = 3$)	$S(A_1) = 6.5734$, $S(A_2) = 6.3653$, $S(A_3) = 6.4801$, $S(A_4) = 6.9889$	$A_4 \succ A_1 \succ A_3 \succ A_2$
LIFWA (Zhang 2014)	$S(A_1) = 4.2318$, $S(A_2) = 3.9057$, $S(A_3) = 4.3652$, $S(A_4) = 5.8012$	$A_4 \succ A_3 \succ A_1 \succ A_2$
LIFWG (Zhang 2014)	$S(A_1) = 3.5359$, $S(A_2) = 3.4938$, $S(A_3) = 3.8106$, $S(A_4) = 5.5586$	$A_4 \succ A_3 \succ A_1 \succ A_2$
LPFEWA (Rong et al. 2020)	$S(A_1) = 5.7941$, $S(A_2) = 5.6121$, $S(A_3) = 5.8842$, $S(A_4) = 6.6415$	$A_4 \succ A_3 \succ A_1 \succ A_2$
LPFEWG (Rong et al. 2020)	$S(A_1) = 5.3549$, $S(A_2) = 5.3370$, $S(A_3) = 5.5631$, $S(A_4) = 6.5073$	$A_4 \succ A_3 \succ A_1 \succ A_2$
Lq-ROFWA (Lin et al. 2019b) ($q = 3$)	$S(A_1) = 6.4696$, $S(A_2) = 6.3650$, $S(A_3) = 6.5197$, $S(A_4) = 6.9009$	$A_4 \succ A_3 \succ A_1 \succ A_2$
Lq-ROFWG (Lin et al. 2019b) ($q = 3$)	$S(A_1) = 6.1269$, $S(A_2) = 6.1639$, $S(A_3) = 6.2989$, $S(A_4) = 6.7640$	$A_4 \succ A_3 \succ A_2 \succ A_1$
LPFWA (Garg 2018)	$S(A_1) = 5.8448$, $S(A_2) = 5.6445$, $S(A_3) = 5.9344$, $S(A_4) = 6.6681$	$A_4 \succ A_3 \succ A_1 \succ A_2$
LPFWG (Garg 2018)	$S(A_1) = 5.2932$, $S(A_2) = 5.3096$, $S(A_3) = 5.5449$, $S(A_4) = 6.4868$	$A_4 \succ A_3 \succ A_2 \succ A_1$
The proposed Lq-ROFAPWHM ($q = 3$)	$S(A_1) = 6.4553$, $S(A_2) = 6.3195$, $S(A_3) = 6.4874$, $S(A_4) = 6.9267$	$A_4 \succ A_3 \succ A_1 \succ A_2$

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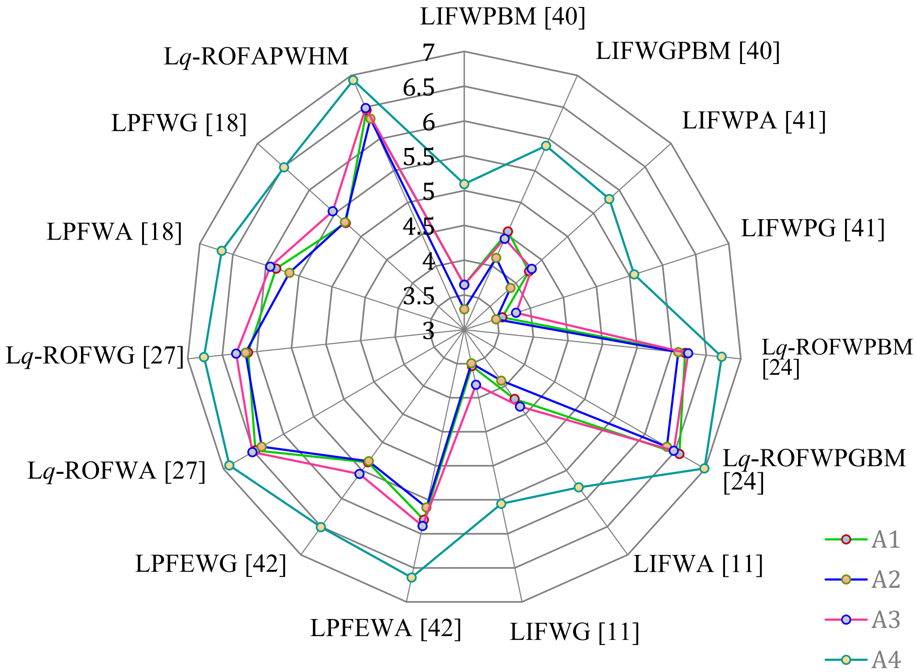

Fig. 13 The radar chart of results on solving with various existing methods. (The scale of the grid is the scores)

Table 11 Lq-ROFDM $\widetilde{\mathcal{X}}^{(2)}$ provided by the DM $\mathfrak{D}^{(2)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$(\mathcal{S}_3, \mathcal{S}_2)$	$(\mathcal{S}_4, \mathcal{S}_1)$	$(\mathcal{S}_3, \mathcal{S}_4)$	$(\mathcal{S}_2, \mathcal{S}_3)$
A_2	$(\mathcal{S}_5, \mathcal{S}_2)$	$(\mathcal{S}_2, \mathcal{S}_1)$	$(\mathcal{S}_3, \mathcal{S}_4)$	$(\mathcal{S}_2, \mathcal{S}_5)$
A_3	$(\mathcal{S}_{7.5}, \mathcal{S}_{0.3})$	$(\mathcal{S}_3, \mathcal{S}_3)$	$(\mathcal{S}_1, \mathcal{S}_2)$	$(\mathcal{S}_3, \mathcal{S}_3)$
A_4	$(\mathcal{S}_{0.1}, \mathcal{S}_{7.9})$	$(\mathcal{S}_3, \mathcal{S}_3)$	$(\mathcal{S}_5, \mathcal{S}_2)$	$(\mathcal{S}_4, \mathcal{S}_1)$

$$E = \begin{bmatrix} 0.77820.89040.78250.6821 \\ 0.84710.96680.94930.7886 \\ 0.82940.88700.96350.8069 \\ 0.59830.88810.59820.5490 \end{bmatrix},$$

where $E_{11}(\widetilde{r}_{11}) = \frac{1}{2}\left(1 + \left(\frac{\pi_{\widetilde{r}_{11}}}{\ell}\right)^q\right)\left(1 - d_{Lq-ROFS}\left(\widetilde{r}_{11}, \widetilde{r}_{11}^C\right)\right) = 0.7782.$

Similarly, other entropy measures have been calculated. Now, the weights of attributes are computed as

Table 12 Ranking results by different methods for Example 3

Operators	Score values	Rankings
LIFWPBM (Liu and Liu 2017)	$S(A_1) = 3.6485$, $S(A_2) = 3.2927$, $S(A_3) = 3.9993$, $S(A_4) = 4.5066$	$A_4 \succ A_3 \succ A_1 \succ A_2$
LIFWGPBM (Liu and Liu 2017)	$S(A_1) = 4.5457$, $S(A_2) = 4.1293$, $S(A_3) = 4.9814$, $S(A_4) = 5.5780$	$A_4 \succ A_3 \succ A_1 \succ A_2$
LIFWPA (Liu and Qin 2017)	$S(A_1) = 4.2533$, $S(A_2) = 3.8960$, $S(A_3) = 5.3925$, $S(A_4) = 5.4699$	$A_4 \succ A_3 \succ A_1 \succ A_2$
Lq-ROFWPBM (Liu and Liu 2018b) ($q = 3$)	$S(A_1) = 6.1936$, $S(A_2) = 6.0956$, $S(A_3) = 6.3866$, $S(A_4) = 6.5685$	$A_4 \succ A_3 \succ A_1 \succ A_2$
Lq-ROFWPGBM (Liu and Liu 2018b) ($q = 3$)	$S(A_1) = 6.5734$, $S(A_2) = 6.3653$, $S(A_3) = 6.6595$, $S(A_4) = 6.9138$	$A_4 \succ A_3 \succ A_1 \succ A_2$
LIFWA (Zhang 2014)	$S(A_1) = 4.2318$, $S(A_2) = 3.9057$, $S(A_3) = 5.4988$, $S(A_4) = 5.3885$	$A_3 \succ A_4 \succ A_1 \succ A_2$
LIFWG (Zhang 2014)	$S(A_1) = 3.5359$, $S(A_2) = 3.4938$, $S(A_3) = 4.2540$, $S(A_4) = 2.9933$	$A_3 \succ A_1 \succ A_2 \succ A_4$
LPFEWA (Rong et al. 2020)	$S(A_1) = 5.7941$, $S(A_2) = 5.6121$, $S(A_3) = 6.4859$, $S(A_4) = 6.4474$	$A_3 \succ A_4 \succ A_1 \succ A_2$
LPFEWG (Rong et al. 2020)	$S(A_1) = 5.3549$, $S(A_2) = 5.3370$, $S(A_3) = 5.7684$, $S(A_4) = 4.9386$	$A_3 \succ A_1 \succ A_2 \succ A_4$
Lq-ROFWA (Lin et al. 2019b) ($q = 3$)	$S(A_1) = 6.4696$, $S(A_2) = 6.3650$, $S(A_3) = 6.9439$, $S(A_4) = 6.8324$	$A_3 \succ A_4 \succ A_1 \succ A_2$
Lq-ROFWG (Lin et al. 2019b) ($q = 3$)	$S(A_1) = 6.1269$, $S(A_2) = 6.1639$, $S(A_3) = 6.3588$, $S(A_4) = 5.4444$	$A_3 \succ A_2 \succ A_1 \succ A_4$
LPFWA (Garg 2018)	$S(A_1) = 5.8448$, $S(A_2) = 5.6445$, $S(A_3) = 6.6101$, $S(A_4) = 6.5089$	$A_3 \succ A_4 \succ A_1 \succ A_2$
LPFWG (Garg 2018)	$S(A_1) = 5.2932$, $S(A_2) = 5.3096$, $S(A_3) = 5.7218$, $S(A_4) = 4.5634$	$A_3 \succ A_2 \succ A_1 \succ A_4$
The proposed Lq-ROFAPWHM ($\rho = 3, q = 3$)	$S(A_1) = 6.4623$, $S(A_2) = 6.3195$, $S(A_3) = 6.8009$, $S(A_4) = 6.8844$	$A_4 \succ A_3 \succ A_1 \succ A_2$

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$$\Theta_1 = \frac{1 - \frac{1}{4}\sum_{i=1}^{4} E_{i1}(\tilde{r}_{i1})}{\sum_{j=1}^{4}\left(1 - \frac{1}{4}\sum_{i=1}^{4} E_{ij}(\tilde{r}_{ij})\right)} = 0.2964.$$

Similarly, $\Theta_2 = 0.1151$, $\Theta_3 = 0.2212$, and $\Theta_4 = 0.3673$.

Step 10 The power weight Ξ_{ij} is calculated as:

$$\Xi_{11} = \frac{\Theta_1(1 + T(\tilde{r}_{11}))}{\sum_{j=1}^{n}\Theta_j(1 + T(\tilde{r}_{1j}))} = \frac{0.2964 \times (1 + 2.7067)}{(0.2964 \times (1 + 2.7067) + 0.1151 \times (1 + 2.6234) + 0.2212 \times (1 + 2.7061) + 0.3673 \times (1 + 2.4245))} = 0.3058.$$

Similarly, other power weights can be found.

Then, the power weight matrix

$$\Xi = \begin{pmatrix} 0.30580.11600.22810.3501 \\ 0.29670.11690.22570.3607 \\ 0.29860.11450.21960.3673 \\ 0.30190.10630.22520.3666 \end{pmatrix}.$$

Step 11 The overall performance value of alternative A_i $(i = 1, 2, 3, 4)$ over all attributes is calculated as:

$$\tilde{r}_1 = \mathrm{L}q - \mathrm{ROFAPWHM}(\tilde{r}_{11}, \tilde{r}_{12}, \tilde{r}_{13}, \tilde{r}_{14}) = (\mathcal{S}_{3.8207}, \mathcal{S}_{3.0996}).$$

Similarly, $\tilde{r}_2 = (\mathcal{S}_{3.1487}, \mathcal{S}_{3.3757})$, $\tilde{r}_3 = (\mathcal{S}_{3.8674}, \mathcal{S}_{2.8762})$ and $\tilde{r}_4 = (\mathcal{S}_{5.4590}, \mathcal{S}_{2.1553})$.

Step 12 The score values of alternatives are obtained as

$$S(\tilde{r}_1) = 6.4553, S(\tilde{r}_2) = 6.3195, S(\tilde{r}_3) = 6.4874 \text{and} S(\tilde{r}_4) = 6.9267.$$

Therefore, the rank of the alternatives is found as $A_4 \succ A_3 \succ A_1 \succ A_2$.

6.1 Influence of rung parameter, *q* on achieved results

It is to be noted here that different variants of fuzzy sets are generated by varying the rung parameter q. In this subsection, the influence of that parameter q on the decision results is investigated. For convenience, the value of HM parameters is considered as $\phi = \psi = 1$, and the value of Hamacher, Dombi, Frank parameters are individually fixed at $\rho = \tau = \zeta = 3$, respectively. Using the software Matlab R2020a, by varying q in $[1, 10]$ and considering the developed Lq-ROFHPWHM, Lq-ROFDPWHM, Lq-ROFFPWHM operators, individually, the achieved results are presented via Figs. 3, 4, 5, respectively.

Figures 3, 4, 5 convey that the increase of the rung parameter q results in increase of score values for each of the proposed operators. Thus, the optimistic or pessimistic view of the DMs may be controlled by the rung parameter.

According to Figs. 3 and 4, two types of orderings are obtained corresponding to Lq-ROFHPWHM and Lq-ROFDPWHM operators as $A_4 \succ A_1 \succ A_3 \succ A_2$ and $A_4 \succ A_3 \succ A_1 \succ A_2$. The cut points of lines representing the score values corresponding to the alternatives A_1 and A_3 are found as $q = 2.1051$ and $q = 1.9868$ using Lq-ROFHPWHM and Lq-ROFDPWHM operators, respectively.

From Fig. 5, it is seen that three cut points are obtained when Lq-ROFFPWHM operator is used. As a consequence, four different rankings, viz. $A_4 \succ A_1 \succ A_2 \succ A_3$, $A_4 \succ A_1 \succ A_3 \succ A_2$, $A_4 \succ A_3 \succ A_1 \succ A_2$ and $A_4 \succ A_3 \succ A_2 \succ A_1$, are obtained for $q \in [1, 1.2632)$, $(1.2632, 1.9168)$, $(1.9168, 4.5963)$, and $(4.5963, 10]$, respectively.

So, it can be ascertained that the best alternative remains the same as A_4 using the proposed operators. Thus, a suitable aggregation operator can be applied for a specific problem according to its characteristics.

6.2 Influence of HM parameters, ϕ, ψ on achieved results

In this subsection, the influence of the HM parameters ϕ, ψ on the score values and the rankings of the alternatives are investigated. For convenience, fixing the rung parameter $q = 3$, the proposed Lq-ROFHWPHM, Lq-ROFDWPHM and Lq-ROFFWPHM operators are individually applied with the consideration of the value of Hamacher parameter $\rho = 3$, Dombi parameter $\tau = 3$, Frank parameter $\zeta = 3$. The score values of the individual alternatives using Lq-ROFHPWHM, Lq-ROFDPWHM and Lq-ROFFPWHM operators are presented through Figs. 6, 7, 8 by varying ϕ, ψ in $[0, 10]$.

It is clear from all these figures that for each of the cases, alternative A_4 achieves the highest score, and consequently, it stands as the best alternative using the

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proposed operators. For ease of presentation, several values of ϕ, ψ are considered, and the score values and ranking orders of the alternatives utilizing Lq-ROFHPWHM, Lq-ROFDPWHM and Lq-ROFFPWHM operators are presented in Table 8. Slight changes in ranking orders between the alternatives A_3 and A_1 are found using Lq-ROFHPWHM and Lq-ROFDPWHM operators. On the other hand, the ranking positions of the alternatives A_2 and A_1 slightly vary when using Lq-ROFFPWHM operator.

It should be noted that some changes in ranking orders of the alternatives are occurred based on specific values of ϕ, ψ. Thus, the parameters ϕ and ψ play an important role in the final ranking of alternatives. So, DMs might choose proper values of ϕ, ψ, to obtain desired orderings based on their needs and state of situations. Therefore, the Lq-ROFHPWHM, L***q***-ROFDPWHM and L***q***-ROFFPWHM operators make the aggregation process considerably flexible with the use of HM parameters.

For further analysis, if $\phi = \psi = c$ is considered where $c \in [1, 10]$, the score values of the alternatives obtained using Lq-ROFHPWHM operator increase with the increasing values of c, which is clearly viewed in Fig. 9. Thus, DMs' accepted risk factor can be reflected through the choice of specific parameter c. To take an optimistic decision, the DMs can utilize Lq-ROFHPWHM operator by considering a larger value of c. This also demonstrates the capability to capture the DMs' cognitive activity in the aggregation processes using the proposed operators.

6.3 Influence of the parameters of A*t*-CN&*t*-Ns on achieved results

Next, the aim is to analyse the impact of assigning different Archimedean parameter values on the resulting outcomes. To do so, the Lq-ROFHPWHM, Lq-ROFDPWHM, and Lq-ROFFPWHM operators, respectively, are used, respectively, to compute the aggregated results (considering $q = 3$, $\phi = \psi = 1$, $\varepsilon = 3$). The consequences of the experiment are summarized in Table 9. It is viewed from the table that a stable ranking result $A_4 \succ A_3 \succ A_1 \succ A_2$ is found using Lq-ROFHPWHM and Lq-ROFFPWHM operators. A variation in ranking result $A_4 \succ A_1 \succ A_3 \succ A_2$ is found for using Lq-ROFDPWHM operator with Dombi parameter $\tau = 0.5$. Excluding that case, the ranking result is similar for both Lq-ROFHPWHM and Lq-ROFFPWHM operators.

To portray the influences of At-CN&t-Ns parameters ρ, τ, ζ more clearly, the score values corresponding to the alternatives based on Lq-ROFHPWHM, Lq-ROFDPWHM, and Lq-ROFFPWHM operators by changing different parameter values are depicted through Figs. 10, 11, 12. It is found that using Lq-ROFHPWHM operator, the ranking is always $A_4 \succ A_3 \succ A_1 \succ A_2$. Based on L$q$-ROFDPWHM operator, a slight change in the ranking of alternatives is found.

For $\tau \in (0, 0.9482)$ the ranking is $A_4 \succ A_1 \succ A_3 \succ A_2$ and for $\tau \in [0.9482, 20]$, the ranking is $A_4 \succ A_3 \succ A_1 \succ A_2$. Again, using L$q$-ROFFPWHM operator, the ranking is always $A_4 \succ A_3 \succ A_1 \succ A_2$. Thus, it can be ascertained that although a minor change in the ranking of alternatives was found, the best alternative remains the same as A_4 for each of the operators, which indicates the sustainability of the proposed method.

7 Comparative analysis

In this section, the validity and effectiveness of the proposed method are illustrated by comparing it with some existing methods based on LIFWPBM (Liu and Liu 2017), LIFWGPBM (Liu and Liu 2017), LIFWPA (Liu and Qin 2017), LIFWPG (Liu and Qin 2017), Lq-ROFWPBM (Liu and Liu 2018b), Lq-ROFWPGBM (Liu and Liu 2018b), LIFWA (Zhang 2014), LIFWG (Zhang 2014), LPFEWA (Rong et al. 2020), LPFEWG (Rong et al. 2020), Lq-ROFWA (Lin et al. 2019b), Lq-ROFWG (Lin et al. 2019b), LPFWA (Garg 2018), LPFWG (Garg 2018) operators. All these methods are considered to solve the same Example 3, and the evaluation results using these methods are compared and analysed in the following.

In Table 10 and Fig. 13, the results of the comparison are provided comprehensively. It is seen from Table 10 and Fig. 13 that although slight changes are found in the ranking orders, the best alternative obtained by different methods is always identical to the method proposed in this paper. So, the feasibility of the proposed method is confirmed by this instance.

It is to be mentioned here that in the process of decision-making, if there exists some biased DMs, the influence of those DMs on the overall decision-making results can be removed using the proposed method. To establish that phenomenon, Example 3 is slightly modified and solved.

7.1 Relieving the impact on decision results caused by biased DMs

Suppose the DM $\mathfrak{D}^{(2)}$ is biased against alternatives A_3 and A_4 in Example 3 due to some obscure cause. That DM provides some extreme values while evaluating those alternatives A_3 (with the optimistic view) and A_4 (with the pessimistic view). The Lq-ROFDM provided by $\mathfrak{D}^{(2)}$ is presented in Table 11.

Different methods are utilized to aggregate the DM's evaluation information presented in Tables 1, 3 and 11, and the obtained results are listed in Table 12. From Table 12,

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it is seen that the selected best alternative of the method developed by Zhang (Zhang 2014), Rong et al. (Rong et al. 2020), Lin et al. (Lin et al. 2019b) and in Garg (Garg 2018) changes from A_4 to A_3. However, the optimal choice remains the same as A_4, utilizing the methods based on some existing operators, viz. LIFWPBM (Liu and Liu 2017), LIFWGPBM (Liu and Liu 2017), LIFWPA (Liu and Qin 2017), Lq-ROFWPBM (Liu and Liu 2018b), Lq-ROFWPGBM (Liu and Liu 2018b) ($q = 3$) and the proposed operator Lq-ROFAPWHM. The reason behind the changes in optimal choice while using some existing methods is that the influence of biased DM $\mathfrak{D}^{(2)}$ on alternative A_3 and A_4. Power aggregation operators can reduce the effect of extreme values given by any biased DMs during the evaluation process. Having the advantage of power aggregation operator, LIFWPBM (Liu and Liu 2017), LIFWGPBM (Liu and Liu 2017), LIFWPA (Liu and Qin 2017), Lq-ROFWPBM (Liu and Liu 2018b), Lq-ROFWPGBM (Liu and Liu 2018b) ($q = 3$) operators and the developed operators Lq-ROFAPWHM can erase the influence of unreasonable extreme values by calculating the supports by producing the rational outcome. But, the case of other operators in Table 12 fails to diminish the effect of unreasonable data caused by the biased DMs, which reflects on the decision results.

7.2 Expressing DMs' evaluation information widely

As discussed above, the developed method has an important characteristic of reducing the influence of extreme decision values caused by biased DMs. Additionally, the proposed method can capture Lq-ROF information, which has the ability to express a greater degree of uncertainty by allowing the sum of q th power of linguistic membership and linguistic non-membership degrees not exceeding the cardinality of the LTS. The existing methods developed by LIFWPBM (Liu and Liu 2017), LIFWGPBM (Liu and Liu 2017), LIFWPA (Liu and Qin 2017), LIFWPG (Liu and Qin 2017), LIFWA (Zhang 2014), LIFWG (Zhang 2014), LPFEWA (Rong et al. 2020), LPFEWG (Rong et al. 2020), LPFWA (Garg 2018), LPFWG (Garg 2018) operators are unable to address the following situations.

For instance, let in Example 3, a DM provides his/her evaluation value as $(\mathcal{S}_7, \mathcal{S}_6)$ on evaluating an alternative A_i with respect to some attribute C_j. In these cases, the existing methods (Zhang 2014; Garg 2018; Liu and Liu 2017; Liu and Qin 2017; Rong et al. 2020) fail to describe evaluating such attribute values because LIFNs must satisfy $\xi + \eta \leq \ell$ and LPFNs must satisfy $\xi^2 + \eta^2 \leq \ell^2$. But here, it is found that $7 + 6 \geq 8$ and $7^2 + 6^2 \geq 8^2$, for which this assessment information cannot be captured under LIF and LPF environments. Whereas our proposed method can deal with such a situation efficiently by considering $q = 3$ in the process of aggregation method. Hence, the proposed method is more general and flexible in resolving uncertainty on a broader range.

7.3 Reflecting the interrelationship among attribute values

Compared with the methods based on LIFWPA (Liu and Qin 2017), LIFWPG (Liu and Qin 2017), LIFWA (Zhang 2014), LIFWG (Zhang 2014), LPFEWA (Rong et al. 2020), LPFEWG (Rong et al. 2020), Lq-ROFWA (Lin et al. 2019b) Lq-ROFWG (Lin et al. 2019b), LPFWA (Garg 2018) and LPFWG (Garg 2018) operators, which only carry simple weighted functions. The proposed method in this paper can take interrelationship among attributes into account by considering the HM parameters ϕ and ψ that can address real decision-making situations more rationally. For instance, an individual wishes to invest in an investment company by considering two factors, viz. "company's management level," and "growth ability," which are interrelated attributes. So ignoring their interrelationship in the selection process may result in a significant loss. Zhang (2014), Garg (2018), Lin et al. (2019b), Liu and Qin (2017), and Rong et al. (Rong et al. 2020) did not consider such interrelationships in their presented approaches. As a consequence, these methods would generate an unreasonable ranking result. The proposed method has the ability to consider the relationship between attributes in a more reasonable manner than the existing methods.

The methods proposed by LIFWPBM (Liu and Liu 2017), LIFWGPBM (Liu and Liu 2017), Lq-ROFWPBM (Liu and Liu 2018b), Lq-ROFWPGBM (Liu and Liu 2018b) are based on BM operator. Though BM operator can capture interrelationship between any pair of attributes, it has the drawback of redundancy. More specifically, the BM operator simultaneously considers the correlation between C_i and C_j $(i \neq j)$ and again C_j and C_i $(i \neq j)$, which is quite unreasonable. Moreover, BM operator also neglects the interrelationship between the attribute C_i and itself. The method proposed in this paper removes all the drawbacks of the existing approaches (Liu and Liu 2018b, 2017) with efficacy by having HM operator in the aggregation process.

7.4 Making the aggregation process flexible using Archimedean parameters

The aggregation operators used in the existing methods (Zhang 2014; Garg 2018; Liu and Liu 2018b, 2017; Lin

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et al. 2019b; Liu and Qin 2017; Rong et al. 2020) are based on some specific At-N&t-CNs, which are not general and flexible in nature. The proposed aggregation method employs a family of At-N&t-CNs including algebraic, Einstein, Hamacher, Dombi, Frank, etc., classes. Thus, the developed operators possess the ability to make the aggregation process more robust and smooth by including various types of At-N&t-CNs in the aggregation functions. Further, the proposed aggregation operators, viz. Lq-ROFHPWA, Lq-ROFDPWA, Lq-ROFFPWA, Lq-ROFHPWHM, Lq-ROFDPWHM and Lq-ROFFPWHM, include various flexible parameters that can reflect the attitudes of DMs allowing their risk preferences. Moreover, the designed aggregation operators in this paper can generate a list of aggregation operators as their special cases by considering specific decreasing generators. Hence, the proposed method is more general than the existing methods (Zhang 2014; Garg 2018; Liu and Liu 2018b, 2017; Lin et al. 2019b; Liu and Qin 2017; Rong et al. 2020).

8 Conclusions and scope for future studies

The proposed MAGDM method can deal with biased evaluation values and also can consider different interrelationships between the attributes. By varying the At-CN&t-Ns parameters connected with the proposed Lq-ROFAPWHM operators, a large number of aggregation operators are defined in this paper, which are subsequently used by the DMs according to their needs. With the use of developed novel distance and entropy measures for Lq-ROFSs, the unknown weights of DMs and attributes can be evaluated efficiently. An approach for solving the Lq-ROF-based MAGDM problems has been proposed. Then, to illustrate the utility of proposed method a case study based on a real-life phenomenon has been investigated. Further, the influence of rung parameter q, HM parameter ϕ, ψ and At-CN&t-Ns parameters ρ, τ, ζ on the decision results have been discussed thoroughly in this paper. The figures and achieved results clearly demonstrate the effectiveness of the developed approach. In addition, the results of the comparisons show that the suggested approach can deal with real-life uncertainty more comprehensively than the existing approaches.

Future studies can extend the proposed approach in the following directions.

- In this paper, the proposed decision-making framework is illustrated under the Lq-ROF context. The proposed study can be extended to several fuzzy variants, such as linguistic interval-valued q-ROFSs (Sajjad Ali Khan et al. 2021), linguistic spherical fuzzy sets (Liu et al. 2020b), and q-ROF uncertain linguistic fuzzy sets (Liu et al. 2019b), weighted dual hesitant q-ROFSs (Sarkar et al. 2023), q-rung orthopair trapezoidal fuzzy set (Gayen et al. 2022) and so on.
- In addition, more innovative aggregation operators can be developed under Lq-ROF environment. It will be interesting to combine power partitioned HM (Zhong et al. 2019), power Maclaurin symmetric mean (Chen and Zhang 2022; Yang and Garg 2021; Garg and Arora 2021), power Muirhead mean (Liu and Liu 2019), prioritized operator (Deb et al. 2022), Volterra and Fredholm integral (Abu 2017; Alshammari et al. 2020; Abu Arqub et al. 2021a, 20212b) aggregation operators with Lq-ROF variants for generating significant aggregation operators.
- The proposed method can be applied to solve various real-life decision-making problems in the fields of fuzzy cluster analysis (Zhang et al. 2022), pattern recognition (Singh and Ganie 2022), business decision-making and budget allocation (Çağlar and Gürel 2019), medical decision-making (Sun et al. 2021) and so on for better resolutions than existing ones.

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A novel Aczel-Alsina triangular norm-based group decision-making approach under dual hesitant q-rung orthopair fuzzy context for parcel lockers' location selection

Souvik Gayen [a], Animesh Biswas [a], Arun Sarkar [b,*], Tapan Senapati [c], Sarbast Moslem [d]

[a] Department of Mathematics, University of Kalyani, Kalyani, 741235, India
[b] Department of Mathematics, Heramba Chandra College, Kolkata, 700029, India
[c] School of Mathematics and Statistics, Southwest University, Beibei, Chongqing, 400715, China
[d] School of Architecture Planning and Environmental Policy, University College Dublin, Belfield, Dublin 15, D04 V1W8, Ireland

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ABSTRACT

For complex last mile problems, the parcel lockers play an important role in urban areas. So, selecting a suitable location is very much crucial to provide optimal service and better logistical performance. From that viewpoint, a multicriteria group decision making method is developed in this article using dual hesitant q-rung orthopair fuzzy (DHq-ROF) set which is more effective than existing variations of fuzzy sets. The Aczel-Alsina (AA) class of t-conorms and t-norms (AAt-CNs&t-Ns) emerged as an important class of union and intersection operations due to its greater flexibility in the information fusion process. To take advantage of the benefits of AAt-CNs&t-Ns and hybrid aggregation operators in DHq-ROF environments, several fundamental operations based on AAt-CNs&t-Ns are first defined. Following the introduction of these stated operations, a series of aggregation operators, viz., DHq-ROF AA weighted averaging, ordered weighted averaging, hybrid averaging and their geometric versions with DHq-ROF information, has been proposed. Based on these operators, a creative approach to handle multi-attribute group decision-making problems has been framed. The parcel lockers' location selection problem is estimated to validate the created strategy and show its applicability and efficacy. The achieved results establish that the post office is the best location for locating parcel lockers.

1. Introduction

The last-mile delivery industry has undergone several abrupt changes and advancements during the past 20 years, largely as a result of the expansion of e-commerce. In Ireland, the B2C market is predicted to experience an annual growth rate of 17.16%, reaching US$15,915.1 million in 2022. The growth story of the B2C e-commerce industry in Ireland looks quite appealing in the medium-to-long term. Over the forecast period, the B2C e-commerce industry is expected to grow steadily and record a compound annual growth rate of 12.80% from 2022–2026. Over the next few years, the B2C Ecommerce Gross Merchandise Value of the country will increase from US$13,583.9 million (2021) to US$25,769.1 million in 2026. Customers are less willing to wait for deliveries because they want the freedom to buy things whenever and wherever they want. Additionally, the need for 24/7 service availability by consumers makes delivery operations for the courier, express and parcel delivery firms more difficult (Grosman, 2018).

Parcel lockers are considered as a sustainable solution for customers to receive, return and ship all their parcels from one location. This new system gives customers greater control over their shopping and reduces the risk of missing or stolen parcels. Retailers can provide a more seamless and simple delivery service if a locker is placed in the proper location. Customers do not need to take any additional procedures to pick up their packages. The new logistics system benefits customers and retailers, providing the convenience of pick up and secure, reliable delivery anytime, anywhere. However, researchers have adopted several mathematical models for selecting the optimal location for parcel lockers (Bengtsson and Vikingson, 2015; Iwan et al., 2016; Lachapelle et al., 2018; Oliveira et al., 2019).

Selecting the appropriate location is a complex issue, and the multicriteria decision-making (MCDM) methodologies have been the most efficient methods to deal with complicated problems (Akram et al., 2023; Alkan and Kahraman, 2022; Ayyildiz, 2022; Choudhury et al., 2022; Karaşan et al., 2020; Mihajlović et al., 2019; Moslem et al.,

* Corresponding author.
E-mail addresses: souvikgayen99@gmail.com (S. Gayen), abiswaskln@rediffmail.com (A. Biswas), asarkarmth@gmail.com (A. Sarkar), math.tapan@gmail.com (T. Senapati), sarbast.moslem@ucd.ie (S. Moslem).

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Table 1
Recent research work based on AAt-N&t-CNs operations.

Reference	Aggregation operators	Application/Problem
Senapati et al. (2022a)	IFAAWA, IFAAOWA, and IFAAHA	MCDM human resource selection
Senapati et al. (2022b)	Pythagorean fuzzy AA- weighted average, order weighted average and hybrid average operators	A group strategy on monetary system of a multinational company in China.
Senapati et al. (2023b)	IFAAWG, IFAAOWG, and IFAAHG	To choice best health-care waste (HCW) disposal method
Ali and Naeem (2023)	p, q-ROFAAWA; p, q-ROFAAOWA; p, q-ROFAAHA; p, q-ROFAAWG; p, q-ROFAAOWG; p, q-ROFAAHG	A tool for determining the impact of this social hazard based on its causes
Hussain et al. (2023)	IFAAHM, IFAAWHM, IFAAGHM, and IFAAWGHM	To choose best solar panels
Jabeen et al. (2023)	q-ROFAAPBM and q-ROFAAWPBM	Disease is diagnosed with the MADM
Senapati et al. (2023c)	IF AA power weighted geometric and arithmetic operators	A case study on sustainable transportation of Novi Sad
Karabacak (2023)	Interval neutrosophic AA- weighted arithmetic, ordered weighted average, weighted geometric, and hybrid weighted average operators	To solve an emerging technology selection problem
Farid and Riaz (2023)	q-ROFAAWA, q-ROFAAOWA, q-ROFAAHA, q-ROFAAWG, q-ROFAAOWG, and q-ROFAAHG operators	For selecting the best green supplier

2023a; Nalan Bilişik and Baraçlı, 2023; Sarkar et al., 2023b). Recently, MCDM tools were widely employed to evaluate and solve transport complex problems (Moslem et al., 2023b), such as, evaluating the quality service in urban transport (Gündoğdu et al., 2021; Moslem and Çelikbilek, 2020), evaluating mode choice (Moslem, 2023c; Moslem et al., 2020a) and traffic safety issues (Moslem et al., 2020b). In current era, it is a big part of decision science and is used a lot in evaluating suppliers, judging their performance, choosing where to build things, and more (Athar Farid et al., 2023; Senapati et al., 2023a; Riaz et al., 2021; Gayen et al., 2023). An MCDM problem is usually called multicriteria group decision-making (MCGDM) problem when more than one person makes a decision. The MCGDM solution is unquestionably more dependable and crucial for individuals and businesses. Real-life decision-making situations make it hard to make decisions based on unclear information. People's minds are fuzzy, and decision-making environments are complicated. In his theory of fuzzy sets, Zadeh (1965) introduced just membership degree (MD). After that Atanassov (1986) invented intuitionistic fuzzy sets (IFSs) that included the idea of non-membership degree (NMD), which satisfy the constraint $\mu+\nu \leq 1$, where μ and ν represent the MD and NMD, respectively.

Furthermore, suppose a DM wants to assign an MD of 0.8 and an NMD of 0.3 for a decision-making problem. In that case, it is evident that they would fail when considering IFS. To solve this kind of issues, Yager (2013, 2014) broadened the concept of Pythagorean fuzzy set (PFS), in which the sum of MD^2 and NMD^2 is not exceeded by 1. So, it is clear that PFS is more important and necessary for solving MCGDM problems. PFS occasionally fails to communicate evaluation data in MCGDM issues adequately. For instance, PFS cannot consider the values (0.9, 0.7) as a pair of MD and NMD since $0.9 + 0.7 \geq 1$. To survive this challenge, Yager (2017) developed the q-rung orthopair fuzzy (q-ROF) sets (q-ROFSs), a generalized variant of IFS and PFS, that satisfies the restriction $\mu^q + \nu^q \leq 1$ where μ and ν denote the MD and NMD, respectively. When $q = 2$ or 1, q-ROFS reduces to PFS or IFS, respectively. Therefore, q-ROFS can describe hazy phenomena in a wider range than IFS and PFS.

Data fusion, also known as information fusion, involves combining data from multiple sources to obtain a more comprehensive and reliable representation of the underlying phenomenon. In this context, aggregation operators play a crucial role in data fusion by effectively combining and integrating information from diverse sources. These operators enable the fusion of data with varying degrees of uncertainty, imprecision, and conflicting information. One such class of aggregation operators widely used in data fusion is t-norms and t-conorms, which provide a framework for aggregating fuzzy information. Menger (2003) first discover the idea of triangular norms in his "Theory of probabilistic metric spaces". It has been revealed that the t-norms (t-Ns) and their associated t-conorms (t-CNs), such as Dombi t-norm and t-conorm, Frank t-norm and t-conorm, Einstein t-norm and t-conorm, Hamacher t-norm and t-conorm and as well as others, are significant operations in fuzzy sets and systems. A recent comprehensive analysis conducted by Klement and colleagues thoroughly examines the characteristics and related aspects of triangular norms. Based on Archimedean t-norms and t-conorms, Liu and Wang (2018) introduced q-ROF weighted geometric and averaging operators to solve MADM problems under q-ROF environments. Further, some weighted averaging neutral aggregation operators are presented by Garg and Chen (2020) to aggregate the q-ROFNs. Based on Yager norm operations, Akram and Shahzadi (2021) developed six aggregation operators, namely q-ROF Yager weighted arithmetic, ordered weighted arithmetic and hybrid operators, along with their geometric operators. Utilizing Einstein norm, Farid and Riaz (2021) introduced a family of aggregation operators, viz., q-ROF Einstein interactive weighted geometric, generalized q-ROF Einstein interactive weighted geometric operator and generalized q-ROF Einstein interactive hybrid geometric operators. Further, Gayen et al. (2022) developed Hamacher aggregation operators by utilizing Hamacher operations, in q-rung orthopair trapezoidal fuzzy environment to solve MCGDM problems.

In 1982, Aczél and Alsina (1982) introduced innovative operations known as Aczel-Alsina t-norm and Aczel-Alsina t-conorm, which emphasize changeability with adjustable parameter activity. According to the needs of the DMs, Aczel-Alsina parameters can be adjusted. Moreover, During the decision-making process, it facilitates the DMs' ability to make optimistic or pessimistic conclusions for the purpose of risk management. Thus, AA operations have higher level of flexibility compare to other existing operations due to the existence of an adjustable parameter. Due to the fact that AAt-N&t-CNs involve a specific parameter, it is more adaptable throughout the process of information fusion and is more suited to mimic real-world decision making challenges. Many researchers have recently been interested on AAt-N&t-CNs in various fuzzy domain and is presented in Table 1.

Moreover, decision-making becomes more complex in real life scenario. The aforementioned methodologies does not capture human hesitancy. In these situations, DMs may hesitate to choose the decision values among some of the possible values. In 2010, Torra (2010) introduced the concept of hesitant fuzzy sets (HFS) as a way to capture and represent uncertainty or hesitancy in decision-making. HFS allows for the consideration of multiple plausible values within the range of [0, 1] for each element, instead of a single precise value. By incorporating this notion of hesitancy, HFS provides a more flexible and expressive framework for handling and modeling uncertain or imprecise information in decision-making processes.

After that, Zhu et al. (2012) identified a limitation of hesitant fuzzy sets (HFS) in that they only consider MDs. To overcome this drawback, they proposed a new concept called dual hesitant fuzzy (DHF) set

Table 2
Comparison among DHq-ROFS and other existing fuzzy sets.

Aspect	Hesitation	Vagueness	Falsity	Whether captures conditions of the sum of MD and NMD is more than one	Whether captures conditions of the square sum of MD and NMD is more than one
FS	○	●	○	○	○
IFS	○	●	●	○	○
PFS	○	●	●	●	○
HFS	●	●	○	○	○
DHFS	●	●	●	○	○
DHPFS	●	●	●	●	○
DHq-ROFS	●	●	●	●	●

(DHFS), which incorporates both MDs and NMDs. DHFS allows for a finite number of MDs and NMDs to represent the hesitancy in decision-making. Building upon the advantages of DHFS and q-ROFS, Xu et al. (2018) further extended the concept and introduced the notion of dual hesitant q-ROF (DHq-ROF) set (DHq-ROFS). In DHq-ROFS, the sum of qth $(q \geq 1)$ power of the greatest MD and NMD to an element is not greater than 1. The DHq-ROFS provides the large spaces that can convey hazy information in a more thorough manner while taking the greater value of q into account. It is clear that DHq-ROFS is more effective than other FS versions, such as q-ROFS, PFS, DHFS, and so on, for resolving hesitation associated with decision making processes. The characteristic comparison of DHq-ROFS with HFS, IFS, PFS, DHFS, DHPFS and q-ROFS is exhibited in Table 2. After the inception of DHq-ROFS, many research works have been developed. DHq-ROF Hamacher hybrid weighted averaging (DHq-ROFHHWA) and DHq-ROF Hamacher hybrid weighted geometric (DHq-ROFHHWG) operators were suggested by Wang et al. (2019b) based on the Hamacher operation. Also, the geometric version of the above operators was proposed. Furthermore, dual Muirhead mean operators were provided by Wang et al. (2019a) and their weighted means were also constructed for aggregating DHq-ROFNs. Again, based on Heronian mean (Wang et al., 2020) introduced some q-rung orthopair hesitant fuzzy interaction aggregation operators. Later, Hussain et al. (2020) presented a MCDM technique through hesitant q-rung orthopair fuzzy (Hq-ROF) information by utilizing two developed Hq-ROF WG and WA aggregation operators. Afterwards, combining Dombi t-conorms and t-norms and Bonferroni mean (BM), Sarkar and Biswas (2021) expanded interrelationships-based aggregation operators under DHq-ROF environment. Recently, Sarkar et al. (2023a) introduced weighted dual hesitant q-rung orthopair fuzzy sets to solve MCGDM problems.

Additionally, ordered weighted averaging or geometric aggregation operators solely consider the weight assigned to the position of each argument, disregarding the significance of the individual arguments themselves, in contrast to weighted average or geometric operators that also take into account the weight of the corresponding arguments.

To address these situations, Xu and Da (2003) introduced the concept of hybrid operators, which consider both the weight assigned to the argument values and their ordered positions. These hybrid operators aim to incorporate the significance of both factors in the aggregation process, allowing for a more comprehensive and accurate representation of the input information. By considering both the weight and the ordered positions of the arguments, these operators provide a balanced approach that combines the advantages of both weighted and position-based aggregation methods. This enables a more nuanced and refined aggregation outcome, enhancing the decision-making process or data analysis in various domains. Thus, it would be pertinent to do research on the benefits of hybrid operators and the adaptability of AAt-CN&t-Ns under DHq-ROF environments. Meanwhile, using the DHq-ROF environment to solve MCGDM problems might be more appropriate because it captures DM hesitance across a wider range. Consequently, all the advantages of Aczel-Alsina operations and hybrid aggregation operators would be a great significant study on DHq-ROFSs for solving MCGDM problems.

From the perspective of selecting the location of parcel lockers, it is difficult to assess traffic impact, security, reliability, accessibility, etc. on commercial areas, private car parkings, high populous areas, post offices, public transport stops and other crowded areas in a precise manners. The available information are also imprecise. Hence the use of DHq-ROFSs adds an extra dimension for information processing than other variants of fuzzy sets. Also the input arguments are mostly interelated and depends on the attitude of the DMs. So, it is most relevant to use Aczel-Alsina aggregation operators for information aggregation.

1.1. Contributions of the study

Even though several MCGDM problems have been developed under the environment of DHq-ROFSs, there is no study on developing aggregation operators based on AAt-CN&t-Ns. In light of the complexity and intricacy inherent in modeling uncertain and imprecise information, it becomes necessary to introduce the operators based on AAt-N&t-CNs in DHq-ROF environment as a powerful framework for capturing and representing the multifaceted nature of uncertainty. The present study's main contributions are pointed out as follows:

1. This research makes a substantial contribution by utilizing the parametric and adaptable Aczel-Alsina operations framework in a DHq-ROF context to process complicated data for decision-making.
2. Based on AAt-CN&t-Ns, certain fundamental operational laws, viz., addition, multiplication, scaler multiplication and exponential are introduced.
3. A series of aggregation operators viz., DHq-ROF AA weighted average (DHq-ROFAAWA), DHq-ROF AA weighted geometric (DHq-ROFAAWG), DHq-ROF AA ordered weighted average (DH q-ROFAAOWA), DHq-ROF AA ordered weighted geometric (DH q-ROFAAOWG) operator have been developed to aggregate the uncertainty in decision making situations based on the Aczel-Alsina operational laws, which also have been designed.
4. To overcome the drawback of using weighted and ordered weighted aggregation operators, DHq-ROFAAHA and DHq-ROFAAHG operators are also presented for considering the weights of the provided arguments and their ordered positions simultaneously.

5. An MCGDM method is developed for selecting the specific priority of options for handling MCGDM problems using the capability and potential of the indicated operators.
6. A case study related to Parcel Lockers' Location Selection has been solved by the developed method to exhibit the applicability and efficacy of the proposed approach.
7. The quality of the provided method is demonstrated through a comparative analysis to confirm the supremacy of the given operators over existing operators.

1.2. Organization of the study

This article is organized as follows: Some basic definitions and properties are discussed in Section 2. In Section 3, Some aggregation operators viz., DHq-ROFAAWA, DHq-ROFAAOWA, DHq-ROFAAWG, DHq-ROFAAOWG, DHq-ROFAAHA, DHq-ROFAAHG for DHq-ROF information are presented along with their desired properties. Section 4 presents an MCGDM method that uses the newly defined operators and is based on the operators. A parcel lockers' location selection problem is considered to show the validity and superiority of the proposed method in Section 5. Also, the sensitivity analysis is investigated in Section 6. A brief comparative analysis is presented in Section 7. Finally, conclusion and future directions are stated in Section 8.

2. Preliminaries

In order to build the suggested technique, several fundamental terminologies related to q-ROFSs (Yager, 2017) and DHq-ROFSs (Xu et al., 2018) are discussed in this section, along with their operational rules.

Definition 2.1 (*Yager, 2017*). A q-ROFS, $\tilde{\mathcal{P}}$ on a universe of discourse, $\mathcal{X}$ is presented as

$$\tilde{\mathcal{P}} = \left\{ \left(x, \mu_{\tilde{\mathcal{P}}}(x), \nu_{\tilde{\mathcal{P}}}(x)\right) | x \in \mathcal{X} \right\},$$

where $\mu_{\tilde{\mathcal{P}}}$ and $\nu_{\tilde{\mathcal{P}}}$ indicate the MD and NMD, lie within the range $[0,1]$, respectively, of the element $x \in \mathcal{X}$ to the set $\tilde{\mathcal{P}}$, and fulfills the condition $0 \le \left(\mu_{\tilde{\mathcal{P}}}(x)\right)^q + \left(\nu_{\tilde{\mathcal{P}}}(x)\right)^q \le 1$ for $q \ge 1$.

The degree of hesitancy is defined by $\pi_{\tilde{\mathcal{P}}}(x) = \sqrt[q]{1 - \left(\mu_{\tilde{\mathcal{P}}}(x)\right)^q - \left(\nu_{\tilde{\mathcal{P}}}(x)\right)^q}$.

For convenience, $\left(\mu_{\tilde{\mathcal{P}}}(x), \nu_{\tilde{\mathcal{P}}}(x)\right)$ is referred as a q-ROFN and is indicated by $\tilde{\mathfrak{p}} = (\mu, \nu)$. Wei et al. (2018) and Liu and Wang (2018) defined accuracy and score functions for comparing q-ROFNs in the following manner.

Definition 2.2 (*Wei et al., 2018; Liu and Wang, 2018*). Let $\tilde{\mathfrak{p}} = (\mu, \nu)$ be a q-ROFN, the score function can be defined as $S(\tilde{\mathfrak{p}}) = \frac{1}{2}(1 + \mu^q - \nu^q)$. Thus, $S(\tilde{\mathfrak{p}}) \in [0,1]$. Further, the accuracy function is defined as $A(\tilde{\mathfrak{p}}) = \mu^q + \nu^q$.

The following is a comparison rule for q-ROFNs.

Definition 2.3 (*Liu and Wang, 2018*). Assuming that $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$ be any two q-ROFNs, the ordering of $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$ determined by the following principles.

(i) $\tilde{\mathfrak{p}}_2 \succ \tilde{\mathfrak{p}}_1$ when $S(\tilde{\mathfrak{p}}_2) > S(\tilde{\mathfrak{p}}_1)$;
(ii) If $S(\tilde{\mathfrak{p}}_2) = S(\tilde{\mathfrak{p}}_1)$, then

- $A(\tilde{\mathfrak{p}}_2) \succ A(\tilde{\mathfrak{p}}_1)$ implies $\tilde{\mathfrak{p}}_2 \succ \tilde{\mathfrak{p}}_1$;
- $\tilde{\mathfrak{p}}_2 \approx \tilde{\mathfrak{p}}_1$ when $A(\tilde{\mathfrak{p}}_2) = A(\tilde{\mathfrak{p}}_1)$.

Combining the notions of DHFS (Zhu et al., 2012) and q-ROFS (Yager, 2017), Xu et al. (2018) introduced the concept of DHq-ROFS and described some basic fundamental laws on DHq-ROFS.

Definition 2.4 (*Xu et al., 2018*). A DHq-ROFS, $\widetilde{\mathcal{K}}$ on a universe of discourse, $\mathcal{X}$ is depicted as:

$$\widetilde{\mathcal{K}} = \left(\left\langle x, \tilde{h}_{\widetilde{\mathcal{K}}}(x), \widetilde{g}_{\widetilde{\mathcal{K}}}(x)\right\rangle \middle| x \in \mathcal{X}\right), \tag{1}$$

where $\tilde{h}_{\widetilde{\mathcal{K}}}(x) = \bigcup_{\gamma \in \tilde{h}_{\widetilde{\mathcal{K}}}(x)} \{\gamma\}$ and $\widetilde{g}_{\widetilde{\mathcal{K}}}(x) = \bigcup_{\eta \in \widetilde{g}_{\widetilde{\mathcal{K}}}(x)} \{\eta\}$ are two sets of finite real numbers in $[0,1]$, denoting the probable MD and NMD to the element $x \in \mathcal{X}$ to the set $\widetilde{\mathcal{K}}$ and satisfy the condition that

$$0 \le \gamma, \eta \le 1 \text{ and } 0 \le \left(\max_{\gamma \in \tilde{h}_{\widetilde{\mathcal{K}}}(x)} \{\gamma\}\right)^q + \left(\max_{\eta \in \widetilde{g}_{\widetilde{\mathcal{K}}}(x)} \{\eta\}\right)^q \le 1.$$

For convenience, pair $\widetilde{\mathcal{K}} = \left\langle \tilde{h}_{\widetilde{\mathcal{K}}}(x), \widetilde{g}_{\widetilde{\mathcal{K}}}(x)\right\rangle$ is recognized as a DHq-ROF number (DHq-ROFN) (Xu et al., 2018) and it is indicated by $\tilde{\kappa} = \left\langle \tilde{h}, \widetilde{g}\right\rangle$.

Definition 2.5 (*Xu et al., 2018*). Let $\tilde{\kappa} = \left\langle \tilde{h}, \widetilde{g}\right\rangle$ be a DHq-ROFN. The score function, $S(\tilde{\kappa})$, and accuracy function, $A(\tilde{\kappa})$ is established by

$$S(\tilde{\kappa}) = \frac{1}{2}\left(1 + \frac{1}{\ell_{\tilde{h}}} \sum_{\gamma \in \tilde{h}} \gamma^q - \frac{1}{\ell_{\widetilde{g}}} \sum_{\eta \in \widetilde{g}} \eta^q\right). \tag{2}$$

and

$$A(\tilde{\kappa}) = \frac{1}{\ell_{\tilde{h}}} \sum_{\gamma \in \tilde{h}} \gamma^q + \frac{1}{\ell_{\widetilde{g}}} \sum_{\eta \in \widetilde{g}} \eta^q, \tag{3}$$

where $\ell_{\tilde{h}}$ and $\ell_{\widetilde{g}}$ indicating the number of elements in $\tilde{h}$ and $\widetilde{g}$, respectively.

The technique for ranking DHq-ROFNs are presented as follows: Let $\tilde{\kappa}_1 = \left(\tilde{h}_1, \widetilde{g}_1\right)$ and $\tilde{\kappa}_2 = \left(\tilde{h}_2, \widetilde{g}_2\right)$ be any two DHq-ROFNs,

(i) If $S(\tilde{\kappa}_1) > S(\tilde{\kappa}_2)$, then $\tilde{\kappa}_1$ is superior to $\tilde{\kappa}_2$, indicated by $\tilde{\kappa}_1 \succ \tilde{\kappa}_2$;
(ii) If $S(\tilde{\kappa}_1) = S(\tilde{\kappa}_2)$, then

- If $A(\tilde{\kappa}_1) > A(\tilde{\kappa}_2)$, then $\tilde{\kappa}_1 \succ \tilde{\kappa}_2$;
- If $A(\tilde{\kappa}_1) = A(\tilde{\kappa}_2)$, then $\tilde{\kappa}_1$ is equivalent to $\tilde{\kappa}_2$, denoted by $\tilde{\kappa}_1 \approx \tilde{\kappa}_2$.

Definition 2.6 (*Xu et al., 2018*). Suppose $\tilde{\kappa} = \left(\tilde{h}, \widetilde{g}\right)$, $\tilde{\kappa}_1 = \left(\tilde{h}_1, \widetilde{g}_1\right)$, and $\tilde{\kappa}_2 = \left(\tilde{h}_2, \widetilde{g}_2\right)$ represent any three DHq-ROFNs and $\lambda > 0$. Then,

(1) $\tilde{\kappa}_1 \oplus \tilde{\kappa}_2 = \left\langle \bigcup_{\substack{\gamma_i \in \tilde{h}_i \\ i=1,2}} \left\{ \left(\gamma_1^q + \gamma_2^q - \gamma_1^q \gamma_2^q\right)^{\frac{1}{q}} \right\}, \bigcup_{\substack{\eta_i \in \tilde{k}_i \\ i=1,2}} \{\eta_1 \eta_2\} \right\rangle$,
(2) $\tilde{\kappa}_1 \otimes \tilde{\kappa}_2 = \left\langle \bigcup_{\substack{\gamma_i \in \tilde{h}_i \\ i=1,2}} \{\gamma_1 \gamma_2\}, \bigcup_{\substack{\eta_i \in \tilde{k}_i \\ i=1,2}} \left\{ \left(\eta_1^q + \eta_2^q - \eta_1^q \eta_2^q\right)^{\frac{1}{q}} \right\} \right\rangle$,
(3) $\lambda \tilde{\kappa} = \left\langle \bigcup_{\gamma \in \tilde{h}} \left\{ \left(1 - (1 - \gamma^q)^\lambda\right)^{\frac{1}{q}} \right\}, \bigcup_{\eta \in \tilde{k}} \{\eta^\lambda\} \right\rangle$,
(4) $\tilde{\kappa}^\lambda = \left\langle \bigcup_{\gamma \in \tilde{h}} \{\gamma^\lambda\}, \bigcup_{\eta \in \tilde{k}} \left\{ \left(1 - (1 - \eta^q)^\lambda\right)^{\frac{1}{q}} \right\} \right\rangle$.

2.1. AAt-N&t-CNs

Definition 2.7 (*Aczél and Alsina, 1982*). A mapping $\mathcal{T}_{AA}^{\rho} : [0,1] \times [0,1] \to [0,1]$ is called an AAt-N if it satisfies

$$\mathcal{T}_{AA}^{\rho}(x, y) = \begin{cases} \mathcal{T}_D(x, y) & if\ \rho = 0 \\ \min(x, y) & if\ \rho = \infty \\ e^{-\left((-\log x)^\rho + (-\log y)^\rho\right)^{\frac{1}{\rho}}} & otherwise \end{cases}, \tag{4}$$

where $x, y \in [0,1]$ and ρ is a positive constant and $\mathcal{T}_D$ is drastic t-N defined as

$$\mathcal{T}_D(x, y) = \begin{cases} x & if\ y = 1 \\ y & if\ x = 1 \\ 0 & otherwise \end{cases}.$$

Definition 2.8 (*Aczél and Alsina, 1982*). A mapping $\mathcal{S}^{r}_{A}: [0,1]\times[0,1] \to [0,1]$ is called an AAt-CN if it satisfies

$$\mathcal{S}^{\rho}_{AA}(x,y)=\begin{cases}\mathcal{S}_D(x,y) & if\ \rho=0\\ \max(x,y) & if\ \rho=\infty,\\ 1-e^{-((-\log(1-x))^{\rho}+(-\log(1-y))^{\rho})^{\frac{1}{\rho}}} & otherwise\end{cases} \tag{5}$$

where $x,y\in[0,1]$ and ρ is the positive constant and $\mathcal{S}_D$ is drastic t-CN defined as

$$\mathcal{S}_D(x,y)=\begin{cases}x & if\ y=0\\ y & if\ x=0\ .\\ 1 & otherwise\end{cases}$$

3. Development of DHq-ROF AOs based on Aczel-Alsina operations

In this section, DHq-ROF weighted averaging and geometric aggregation operators are established based on AAt-CN&t-Ns.

3.1. Fundamental operational rules of DHq-ROFNs based on Aczel-Alsina operations

In accordance with AAt-CNs&t-Ns, some basic mathematical operational laws of DHq-ROFNs are determined in the following manner.

Definition 3.1. Assume that $\tilde{\kappa}=(\tilde{h},\tilde{g})$, $\tilde{\kappa}_1=(\tilde{h}_1,\tilde{g}_1)$ and $\tilde{\kappa}_2=(\tilde{h}_2,\tilde{g}_2)$ represent any three DHq-ROFNs and $\tau>0$ be an Aczel-Alsina parameter, then AAt-CNs&t-Ns-based mathematical laws, viz., addition "$\oplus_{AA}$", multiplication "$\otimes_{AA}$", exponent and scalar multiplication are presented in this way:

(1) $$\tilde{\kappa}_1\oplus_{AA}\tilde{\kappa}_2=\left\langle \bigcup_{\gamma_i\in\tilde{h}_i|i=1,2}\left\{\left(1-e^{-\left(\left(-\log\left(1-\gamma_1^q\right)\right)^{\tau}+\left(-\log\left(1-\gamma_2^q\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\ \bigcup_{\eta_i\in\tilde{g}_i|i=1,2}\left\{\left(e^{-\left(\left(-\log\eta_1^q\right)^{\tau}+\left(-\log\eta_2^q\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle;$$

(2) $$\tilde{\kappa}_1\otimes_{AA}\tilde{\kappa}_2=\left\langle \bigcup_{\gamma_i\in\tilde{h}_i|i=1,2}\left\{\left(e^{-\left(\left(-\log\gamma_1^q\right)^{\tau}+\left(-\log\gamma_2^q\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\ \bigcup_{\eta_i\in\tilde{g}_i|i=1,2}\left\{\left(1-e^{-\left(\left(-\log\left(1-\eta_1^q\right)\right)^{\tau}+\left(-\log\left(1-\eta_2^q\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle;$$

(3) $$\lambda\odot_{AA}\tilde{\kappa}=\left\langle \bigcup_{\gamma\in\tilde{h}}\left\{\left(1-e^{-\left(\lambda\left(-\log\left(1-\gamma^q\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\ \bigcup_{\eta\in\tilde{g}}\left\{\left(e^{-\left(\lambda\left(-\log\eta^q\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle,\ \lambda>0;$$

(4) $$\tilde{\kappa}^{\lambda}=\left\langle \bigcup_{\gamma\in\tilde{h}}\left\{\left(e^{-\left(\lambda\left(-\log\gamma^q\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\ \bigcup_{\vartheta\in\tilde{g}}\left\{\left(1-e^{-\left(\lambda\left(-\log\left(1-\eta^q\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle,\ \lambda>0.$$

The following example is presented for better understanding the aforementioned algebraic operations on DHq-ROFNs.

Example 3.1. Let $\tilde{\kappa}_1=\langle\{0.5,0.8\},\{0.3,0.6\}\rangle$, and $\tilde{\kappa}_2=\langle\{0.5,0.65,0.85\},\{0.4,0.55\}\rangle$ be two DHq-ROFNs. Taking $\tau=2$, $q=3$ and $\lambda=4$, the aforesaid operations are accomplished on $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$, and the results are as follows:

$$\tilde{\kappa}_1\oplus_{AA}\tilde{\kappa}_2=\left\langle \bigcup_{\substack{\gamma_1\in\{0.5,0.8\}\\ \gamma_2\in\{0.5,0.65,0.85\}}}\left\{\left(1-e^{-\left(\left(-\log\left(1-\gamma_1^q\right)\right)^{\tau}+\left(-\log\left(1-\gamma_2^q\right)\right)^{2}\right)^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right\},\ \bigcup_{\substack{\eta_1\in\{0.3,0.6\}\\ \eta_2\in\{0.4,0.55\}}}\left\{\left(e^{-\left(\left(-\log\eta_1^q\right)^{\tau}+\left(-\log\eta_2^q\right)^{2}\right)^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right\}\right\rangle$$

$$=\langle\{0.5562,0.6647,0.8516,0.8031,0.8165,0.8864\},\ \{0.2203,0.2607,0.3503,0.4555\}\rangle.$$

In the similar way other operations, viz., product, scalar product and exponential can be found as $\tilde{\kappa}_1\otimes_{AA}\tilde{\kappa}_2=\langle\{0.3752,0.4422,0.4907,0.4828,0.6156,0.7588\},\{0.4103,0.5519,0.6063,0.6399\}\rangle$, $4\tilde{\kappa}_1=\langle\{0.6166,0.9133\},\{0.0900,0.3600\}\rangle$ and $\tilde{\kappa}_1^4=\langle\{0.3763,0.7277\},\{0.25,0.6400\}\rangle$, respectively.

3.2. Development of AAt-CN&t-N based aggregation operators under DHq-ROF context

Now, a series of aggregation operators, including DHq-ROFAAWA, DHq-ROFAAOWA, DHq-ROFAAWG, DHq-ROFAAOWG, DHq- ROFA AHA, and DHq-ROFAAHG are defined in this subsection utilizing afore mentioned fundamental laws of DHq-ROFNs.

Definition 3.2. Let $\tilde{\kappa}_i=\langle\tilde{h}_i,\tilde{g}_i\rangle$ $(i=1,2,\ldots,n)$ be a set of n DHq-ROFNs. If a function DHq-ROFAAWA is defined using $\oplus_{AA}$ operation as

$$DHq\text{-}ROFAAWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\oplus_{AA}{}_{i=1}^{n}(\omega_i\tilde{\kappa}_i),$$

then $DHq\text{-}ROFAAWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$ is called DHq-ROFAAWA operator, where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ be the weighted vector of $\tilde{\kappa}_i$ $(i=1,2,\ldots,n)$, $\omega_i>0$, and $\sum_{i=1}^{n}\omega_i=1$.

Theorem 3.1. *Suppose $\{\tilde{\kappa}_i=\langle\tilde{h}_i,\tilde{g}_i\rangle|i=1,2,\ldots,n\}$ be any set of n DHq-ROFNs and Aczel-Alsina parameter $\tau>0$, then the aggregating element utilizing DHq-ROFAAWA is also a DHq-ROFN and is given as follows:*

$$DHq\text{-}ROFAAWA(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\oplus_{AA}{}_{i=1}^{n}(\omega_i\tilde{\kappa}_i)$$
$$=\left\langle \bigcup_{\gamma_i\in\tilde{h}_i|i=1,2,\ldots,n}\left\{\left(1-e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\left(1-\gamma_i^q\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\ \bigcup_{\substack{\eta_i\in\tilde{g}_i\\ i=1,\ldots,n}}\left\{\left(e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\eta_i^q\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle, \tag{6}$$

where $\{\omega_i|i=1,2,\ldots,n\}$ be the weighted vector of $\tilde{\kappa}_i$ $(i=1,2,\ldots,n)$ and $\sum_{i=1}^{n}\omega_i=1$, $\omega_i>0$.

Proof. Principle of mathematical induction is used here to prove this Theorem.

$$\omega_1\tilde{\kappa}_1=\omega_1\odot_{AA}\tilde{\kappa}_1=\left\langle \bigcup_{\gamma_1\in\tilde{h}_1}\left\{\left(1-e^{-\left(\omega_1\left(-\log\left(1-\gamma_1^q\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\ \bigcup_{\eta_1\in\tilde{g}_1}\left\{\left(e^{-\left(\omega_1\left(-\log\eta_1^q\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle,$$

$$\omega_2\tilde{\kappa}_2=\omega_2\odot_{AA}\tilde{\kappa}_2=\left\langle\bigcup_{\gamma_2\in\tilde{h}_2}\left\{\left(1-e^{-\left(\omega_2\left(-\log\left(1-\gamma_2^q\right)\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\bigcup_{\eta_2\in\tilde{g}_2}\left\{\left(e^{-\left(\omega_2\left(-\log\eta_2^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle;$$

Let $\omega_i\tilde{\kappa}_i=\left\langle\tilde{h}_i',\tilde{g}_i'\right\rangle$ where $\tilde{h}_i'=\bigcup_{\alpha_i\in\tilde{h}_i'}\{\alpha_i\}$ and $\tilde{g}_i'=\bigcup_{\beta_i\in\tilde{g}_i'}\{\beta_i\}$ for $i=1,2$

where $\alpha_i=\left(1-e^{-\left(\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}$, and $\beta_i=\left(e^{-\left(\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}$.

$$\text{Now, } \oplus_{AA}{}_{i=1}^{2}\left(\omega_i\tilde{\kappa}_i\right)=\left\langle\bigcup_{\substack{\alpha_i\in\tilde{h}_i'\\ i=1,2}}\left\{\left(1-e^{-\left(\left(-\log\left(1-\alpha_1^q\right)\right)^\tau+\left(-\log\left(1-\alpha_2^q\right)\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\bigcup_{\substack{\beta_i\in\tilde{g}_i'\\ i=1,2}}\left\{\left(e^{-\left(\left(-\log\beta_1^q\right)^\tau+\left(-\log\beta_2^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle \tag{7}$$

Now, $\left(1-\alpha_i^q\right)=1-\left(1-e^{-\left(\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}\right)=e^{-\left(\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}$

$-\log\left(1-\alpha_1^q\right)=-\log e^{-\left(\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}=\left(\omega_i\left(-\log\left(1-\gamma_i^q\right)\right)^\tau\right)^{\frac{1}{\tau}}$

i.e., $\left(-\log\left(\left(1-\alpha_i^q\right)\right)\right)^\tau=\omega_i\left(-\log\left(1-\gamma_i^q\right)\right)^\tau$;

so, $\left(-\log\left(1-\alpha_1^q\right)\right)^\tau+\left(-\log\left(1-\alpha_2^q\right)\right)^\tau=\omega_1\left(-\log\left(1-\gamma_1^q\right)\right)^\tau$
$+\omega_2\left(-\log\left(1-\gamma_2^q\right)\right)^\tau$
$=\sum_{i=1}^2\omega_i\left(-\log\left(1-\gamma_i^q\right)\right)^\tau$

Again, $-\log\beta_i^q=-\log\left(e^{-\left(\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)=\left(\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}$,

$\left(-\log\beta_i^q\right)^\tau=\omega_i\left(-\log\eta_i^q\right)^\tau$
$\left(-\log\beta_1^q\right)^\tau+\left(-\log\beta_2^q\right)^\tau=\omega_1\left(-\log\eta_1^q\right)^\tau+\omega_2\left(-\log\eta_2^q\right)^\tau$
$=\sum_{i=1}^2\omega_i\left(-\log\eta_i^q\right)^\tau$

From (7)

$$=\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\(i=1,2)}}\left\{\left(1-e^{-\left(\sum_{i=1}^2\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\bigcup_{\substack{\eta_i\in\tilde{g}_i\\(i=1,2)}}\left\{\left(e^{-\left(\sum_{i=1}^2\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle.$$

Therefore, Theorem is true for $n=2$.
Now, assume that it is true for $n=\upsilon$,

i.e., $DHq\text{-}ROFAAWA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_\upsilon\right)=\oplus_{AA}{}_{i=1}^{\upsilon}\left(\omega_i\tilde{\kappa}_i\right)$

$$=\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\(i=1,2,\ldots,\upsilon)}}\left\{\left(1-e^{-\left(\sum_{i=1}^\upsilon\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\bigcup_{\substack{\eta_i\in\tilde{g}_i\\(i=1,2,\ldots,\upsilon)}}\left\{\left(e^{-\left(\sum_{i=1}^n\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle.$$

Assume $DHq\text{-}ROFAAWA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_\upsilon\right)=\left\langle\tilde{h}_i'',\tilde{g}_i''\right\rangle$
where $\tilde{h}_i''=\bigcup_{\phi_i\in\tilde{h}_i''}\{\phi_i\}$ and $\tilde{g}_i''=\bigcup_{\psi_i\in\tilde{g}_i''}\{\psi_i\}$ for $i=1,2,\ldots,\upsilon$.

Then, $DHq\text{-}ROFAAWA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_\upsilon,\tilde{\kappa}_{\upsilon+1}\right)=\oplus_{AA}{}_{i=1}^{\upsilon}\left(\omega_i\tilde{\kappa}_i\right)\oplus_{AA}\left(\omega_{\upsilon+1}\tilde{\kappa}_{\upsilon+1}\right)$

$=\left\langle\tilde{h}_i'',\tilde{g}_i''\right\rangle\oplus_{AA}\left\langle\tilde{h}_i',\tilde{g}_i'\right\rangle$

$$=\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\(i=1,2,\ldots,\upsilon+1)}}\left\{\left(1-e^{-\left(\sum_{i=1}^\upsilon\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\bigcup_{\substack{\eta_i\in\tilde{g}_i\\(i=1,2,\ldots,\upsilon+1)}}\left\{\left(e^{-\left(\sum_{i=1}^n\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle;$$

Thus, it is holds for $n=\upsilon+1$, and consequently, it is valid for all $n\in\mathbb{N}$.

It must now be demonstrated that the aggregate value corresponds to a DHq-ROFN as follows:
i.e., it must be demonstrated that

$0\le\gamma_i^q+\eta_i^q\le1$

$\Leftrightarrow\log\eta_i^q\le\log\left(1-\gamma_i^q\right)$

$\Leftrightarrow\left(-\log\eta_i^q\right)^\tau\ge\left(-\log\left(1-\gamma_i^q\right)\right)^\tau$

$$\Leftrightarrow\sum_{i=1}^n\omega_i\left(-\log\eta_i^q\right)^\tau\ge\sum_{i=1}^n\omega_i\left(-\log\left(1-\gamma_i^q\right)\right)^\tau$$

$$\Leftrightarrow-\left(\sum_{i=1}^n\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}\le-\left(\sum_{i=1}^n\omega_i\left(-\log\left(1-\gamma_i^q\right)\right)^\tau\right)^{\frac{1}{\tau}}$$

$$\Leftrightarrow e^{-\left(\sum_{i=1}^n\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\le e^{-\left(\sum_{i=1}^n\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}$$

$$\Leftrightarrow e^{-\left(\sum_{i=1}^n\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}-e^{-\left(\sum_{i=1}^n\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\ge0$$

$$\text{Now }\left(\left(1-e^{-\left(\sum_{i=1}^\upsilon\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right)^q+\left(\left(e^{-\left(\sum_{i=1}^n\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right)^q$$

$$=1-\left(e^{-\left(\sum_{i=1}^\upsilon\omega_i\left(-\log(1-\gamma_i^q)\right)^\tau\right)^{\frac{1}{\tau}}}-e^{-\left(\sum_{i=1}^n\omega_i\left(-\log\eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)\le1.$$

Thus, this is a DHq-ROFN.
Hence the proof is completed.

The following example is taken into consideration to demonstrate the significance of the aforementioned theorem.

Example 3.2. Let $P=\left\{\tilde{\kappa}_1=\langle\{0.5,0.7,0.8\},\{0.3,0.4\}\rangle,\tilde{\kappa}_2=\langle\{0.7,0.9\},\{0.3,0.5\}\rangle,\tilde{\kappa}_3=\langle\{0.65,0.75\},\{0.5,0.6\}\rangle\right\}$ be a set of DHq-ROFNs, which contains three elements. The weights of three corresponding elements related with P is taken as 0.2, 0.35 and 0.45. Now utilize the proposed DHq-ROFAAWA operator for aggregating those three elements $\tilde{\kappa}_1$, $\tilde{\kappa}_2$, and $\tilde{\kappa}_3$ of P as follows (suppose $\tau=2$, $q=3$):

$DHq\text{-}ROFAAWA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\tilde{\kappa}_3\right)$

$$=\left\langle\bigcup_{\substack{\gamma_i\in\tilde{h}_i\\(i=1,2,3)}}\left\{\left(1-e^{-\left(\sum_{i=1}^3\omega_i\left(-\log\left(1-\gamma_i^3\right)\right)^2\right)^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right\},\bigcup_{\substack{\eta_i\in\tilde{g}_i\\(i=1,2,3)}}\left\{\left(e^{-\left(\sum_{i=1}^3\omega_i\left(-\log\eta_i^3\right)^2\right)^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right\}\right\rangle$$

$$=\left\langle\begin{array}{c}\{0.6574,0.7120,0.8206,0.8319,0.6806,0.7258,0.8248,0.8356,\\0.7168,0.7499,0.8332,0.8429\},\\\{0.3654,0.3843,0.4399,0.4683,0.3890,0.4105,0.4756,0.5100\}\end{array}\right\rangle.$$

Now, the characteristics of the proposed DHq-ROFAAWA operator are designed in the following manner.

Theorem 3.2 (*Idempotency*). *Let* $\{\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i) \,|\, i = 1, 2, \ldots, n\}$ *be a collection of* n *DHq-ROFNs. If* $\tilde{\kappa}_i = \tilde{\kappa} = (\tilde{h}, \tilde{g})$ $\forall i$*, then* $DHq\text{-}ROFAAWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \tilde{\kappa}$.

Proof. $DHq\text{-}ROFAAWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \oplus_{AA}{}_{i=1}^{n} \omega_i \tilde{\kappa}_i$

$$= \left\langle \bigcup_{\substack{\gamma_i \in \tilde{h}_i \\ (i=1,2,\ldots,n)}} \left\{ \left(1 - e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log\left(1-\gamma_i^q\right)\right)^\tau\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\}, \bigcup_{\substack{\eta_i \in \tilde{g}_i \\ (i=1,\ldots,n)}} \left\{ \left(e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log \eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\} \right\rangle$$

Since $\tilde{\kappa}_i = \tilde{\kappa} = (\tilde{h}, \tilde{g})$ $\forall\ i = 1, 2, \ldots, n$,

$$DHq\text{-}ROFAAWA(\tilde{\kappa}, \tilde{\kappa}, \ldots, \tilde{\kappa}) = \oplus_{AA}{}_{i=1}^{n} (\omega_i \tilde{\kappa})$$

$$= \left\langle \bigcup_{\gamma \in \tilde{h}} \left\{ \left(1 - e^{-\left(\left(-\log(1-\gamma^q)\right)^\tau \sum_{i=1}^{n} \omega_i\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\}, \bigcup_{\eta \in \tilde{g}} \left\{ \left(e^{-\left(\left(-\log \eta^q\right)^\tau \sum_{i=1}^{n} \omega_i\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\} \right\rangle$$

$$= \left\langle \bigcup_{\gamma \in \tilde{h}} \left\{ \left(1 - e^{-\left(\left(-\log(1-\gamma^q)\right)^\tau\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\}, \bigcup_{\eta \in \tilde{g}} \left\{ \left(e^{-\left(\left(-\log \eta^q\right)^\tau\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\} \right\rangle$$

$$= \left\langle \bigcup_{\gamma \in \tilde{h}} \left\{ \left(1 - e^{-\left(\left(\log(1-\gamma^q)\right)^{-\tau}\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\}, \bigcup_{\eta \in \tilde{g}} \left\{ \left(e^{\log \eta^q} \right)^{\frac{1}{q}} \right\} \right\rangle$$

$$= \left\langle \bigcup_{\gamma \in \tilde{h}} \left\{ \left(1 - e^{-\log(1-\gamma^q)^{-1}} \right)^{\frac{1}{q}} \right\}, \bigcup_{\eta \in \tilde{g}} \left\{ \left(e^{\log \eta^q} \right)^{\frac{1}{q}} \right\} \right\rangle$$

$$= \left\langle \bigcup_{\gamma \in \tilde{h}} \left\{ \left(1 - (1-\gamma^q)\right)^{\frac{1}{q}} \right\}, \bigcup_{\eta \in \tilde{g}} \left\{ (\eta^q)^{\frac{1}{q}} \right\} \right\rangle$$

$$= \langle \tilde{h}, \tilde{g} \rangle = \tilde{\kappa}.$$

Therefore, the Theorem is proved.

Theorem 3.3 (*Monotonicity*). *Let* $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)$ *and* $\tilde{\kappa}'_i = (\tilde{h}'_i, \tilde{g}'_i)$ $(i = 1, 2, \ldots, n)$ *be two collections of DHq-ROFNs and if* $\gamma_i \le \gamma'_i$ *and* $\vartheta_i \ge \vartheta'_i$ $\forall\ i$*, then*

$$DHq\text{-}ROFAAWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) \le DHq\text{-}ROFAAWA(\tilde{\kappa}'_1, \tilde{\kappa}'_2, \ldots, \tilde{\kappa}'_n)$$

Proof. Here, $DHq\text{-}ROFAAWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$

$$= \left\langle \bigcup_{\gamma_i \in \tilde{h}_i} \left\{ \left(1 - e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log\left(1-\gamma_i^q\right)\right)^\tau\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\}, \bigcup_{\eta_i \in \tilde{g}_i} \left\{ \left(e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log \eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\} \right\rangle,$$

and $DHq\text{-}ROFAAWA(\tilde{\kappa}'_1, \tilde{\kappa}'_2, \ldots, \tilde{\kappa}'_n) =$

$$\left\langle \bigcup_{\gamma'_i \in \tilde{h}'_i} \left\{ \left(1 - e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log\left(1-\gamma_i'^q\right)\right)^\tau\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\}, \bigcup_{\eta'_i \in \tilde{g}'_i} \left\{ \left(e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log \eta_i'^q\right)^\tau\right)^{\frac{1}{\tau}}} \right)^{\frac{1}{q}} \right\} \right\rangle.$$

Now, for $i = 1, 2, \ldots, n$, $\gamma_i \le \gamma'_i$

$\Leftrightarrow 1 - \gamma_i^q \ge 1 - \gamma_i'^q$,

$\Leftrightarrow \left(-\log\left(1-\gamma_i^q\right)\right)^\tau \le \left(-\log\left(1-\gamma_i'^q\right)\right)^\tau$

$$\Leftrightarrow -\left(\sum_{i=1}^{n} \omega_i \left(-\log\left(1-\gamma_i^q\right)\right)^\tau\right)^{\frac{1}{\tau}} \ge -\left(\sum_{i=1}^{n} \omega_i \left(-\log\left(1-\gamma_i'^q\right)\right)^\tau\right)^{\frac{1}{\tau}}.$$

Hence, $\left(1 - e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log\left(1-\gamma_i^q\right)\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}} \le \left(1 - e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log\left(1-\gamma_i'^q\right)\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}$

Similarly, it can be shown that if $\eta_i \ge \eta'_i$ then

$$\left(e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log \eta_i^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}} \ge \left(e^{-\left(\sum_{i=1}^{n} \omega_i \left(-\log \eta_i'^q\right)^\tau\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}.$$

Then by Definition 2.5.

$$DHq\text{-}ROFAAWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) \le DHq\text{-}ROFAAWA(\tilde{\kappa}'_1, \tilde{\kappa}'_2, \ldots, \tilde{\kappa}'_n).$$

Theorem 3.4 (*Boundedness*). *Let* $\{\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n\}$ *be a collection of* n *DHq-ROFNs. If* $\tilde{\kappa}^+ = \cup_{\gamma_i \in \tilde{h}_i, \eta_i \in \tilde{g}_i} \{\{max_j \gamma_j\}, \{min_j \eta_j\}\}$ *and* $\tilde{\kappa}^- = \cup_{\gamma_i \in \tilde{h}_i, \eta_i \in \tilde{g}_i} \{\{min_j \gamma_j\}, \{max_j \eta_j\}\}$ *then*

$\tilde{\kappa}^- \le DHq\text{-}ROFAAWA(\kappa_1, \kappa_2, \ldots, \kappa_n) \le \tilde{\kappa}^+$.

Proof. Since $\min_j \gamma_j \le \gamma_j \le \max_j \gamma_j$ and $\min_j \eta_j \le \eta_j \le \max_j \eta_j$ then $\tilde{\kappa}^- \le \tilde{\kappa}_i$ for $i = 1, 2, \ldots, n$.

Then by monotonicity, $DHq\text{-}ROFAAWA(\tilde{\kappa}^-, \tilde{\kappa}^-, \ldots, \tilde{\kappa}^-) \le DHq\text{-}ROFAAWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$.

By idempotency, $\tilde{\kappa}^- \le DHq\text{-}ROFAAWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$.

Similarly, $DHq\text{-}ROFAAWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) \le \tilde{\kappa}^+$.

So, $\tilde{\kappa}^- \le DHq\text{-}ROFAAWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) \le \tilde{\kappa}^+$.

Note 1. For $\tau = 1$, DHq-ROFAAWA aggregation operator converted to DHq-ROFWA operator (Wang et al., 2019b) as

$$DHq\text{-}ROFWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \left\langle \bigcup_{\substack{\gamma_i \in \tilde{h}_i \\ (i=1,2,\ldots,n)}} \left\{ \left(1 - \prod_{i=1}^{n} \left(1-\gamma_i^q\right)^{\omega_i} \right)^{\frac{1}{q}} \right\}, \bigcup_{\substack{\eta_i \in \tilde{g}_i \\ (i=1,2,\ldots,n)}} \left\{ \prod_{i=1}^{n} \eta_i^{\omega_i} \right\} \right\rangle.$$

Now, develop the DHq-ROFAAOWA operator.

Definition 3.3. Suppose $\{\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i) \,|\, i = 1, 2, \ldots, n\}$ be a set of DHq-ROFNs. If a function DHq-ROFAAOWA is defined using $\oplus_{AA}$ operation as

$$DHq\text{-}ROFAAOWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \oplus_{AA}{}_{i=1}^{n} \left(\omega_i \tilde{\kappa}_{\sigma(i)}\right),$$

then $DHq\text{-}ROFAAOWA(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$ is called DHq-ROFAAOWA operator, where σ is a permutation defined on $\{1, 2, \ldots, n\}$ in such a way that $\tilde{\kappa}_{\sigma(i-1)} \ge \tilde{\kappa}_{\sigma(i)}$ $\forall\ i = 2, 3, \ldots, n$. Here, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the weighted vector of DHq-ROFNs with $\omega_i > 0$, and $\sum_{i=1}^{n} \omega_i = 1$.

Theorem 3.5. *Let* $\{\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i) \,|\, i = 1, 2, \ldots, n\}$ *represents a set of* n *DHq-ROFNs and Aczel-Alsina parameter* $\tau > 0$*, then the aggregating element* *utilizing DHq-ROFAAOWA operator is also a DHq-ROFN and is represented in the following manner:*

$$DHq\text{-}ROFAAOWA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)=\oplus_{AA}{}_{i=1}^{n}\left(\omega_i\tilde{\kappa}_{\sigma(i)}\right)$$
$$=\left\langle \bigcup_{\substack{\gamma_{\sigma(i)}\in\tilde{h}_{\sigma(i)}\\(i=1,2,\ldots,n)}}\left\{\left(1-e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\left(1-\gamma_{\sigma(i)}^{q}\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\right.$$
$$\left.\bigcup_{\substack{\eta_{\sigma(i)}\in\tilde{g}_{\sigma(i)}\\(i=1,2,\ldots,n)}}\left\{\left(e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\eta_{\sigma(i)}^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle. \quad (8)$$

where σ is a permutation on $\{1,2,\ldots,n\}$ in such a way that $\tilde{\kappa}_{\sigma(i-1)}\geq\tilde{\kappa}_{\sigma(i)}$ $\forall\, i=2,3,\ldots,n$, and $\left\{\omega_i|i=1,2,\ldots,n\right\}$ be the associated weight vector such that $\omega_i>0$ and $\sum_{i=1}^{n}\omega_i=1$.

Proof. The proof of Theorem 3.5 is identical to the proof of Theorem 3.1.

If Example 3.2 is solved utilizing DHq-ROFAAOWA operator, then the aggregating value is obtained as

$$DHq\text{-}ROFAAOWA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\tilde{\kappa}_3\right)$$
$$=\left\langle \bigcup_{\substack{\gamma_{\sigma(i)}\in\tilde{h}_{\sigma(i)}\\(i=1,2,3)}}\left\{\left(1-e^{-\left(\sum_{i=1}^{3}\omega_i\left(-\log\left(1-\gamma_{\sigma(i)}^{3}\right)\right)^{2}\right)^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right\},\right.$$
$$\left.\bigcup_{\substack{\eta_{\sigma(i)}\in\tilde{g}_{\sigma(i)}\\(i=1,2,3)}}\left\{\left(e^{-\left(\sum_{i=1}^{3}\omega_i\left(-\log\eta_{\sigma(i)}^{3}\right)^{2}\right)^{\frac{1}{2}}}\right)^{\frac{1}{3}}\right\}\right\rangle$$

(From Definition 2.5, $\tilde{\kappa}_2>\tilde{\kappa}_1>\tilde{\kappa}_3$. So, $\tilde{\kappa}_{\sigma(1)}=\tilde{\kappa}_2$, $\tilde{\kappa}_{\sigma(2)}=\tilde{\kappa}_1$, $\tilde{\kappa}_{\sigma(3)}=\tilde{\kappa}_3$)

$$=\left\langle \begin{array}{c}\{0.6355,0.7001,0.6806,0.7258,0.7371,0.7645,\\ 0.7753,0.7941,0.7875,0.8040,0.8089,0.8219\},\\ \{0.3654,0.3843,0.4088,0.4328,0.4044,0.4278,0.4591,0.4905\}\end{array}\right\rangle.$$

Note 2. The DHq-ROFAAOWA operator also satisfies the monotonicity, idempotency and boundedness requirements, and this can be demonstrated similarly.

Note 3. DHq-ROFOWA (Wang et al., 2019b) operator can be generated, for considering $\tau=1$.

Here, DHq-ROFAAWG operator is presented.

Definition 3.4. Suppose $\left\{\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right\}$ be a collection of DHq-ROFNs. If DHq-ROFAAWG is defined using $\otimes_{AA}$ operation as

$$DHq\text{-}ROFAAWG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)=\otimes_{AA}{}_{i=1}^{n}\left(\tilde{\kappa}_i\right)^{\omega_i},$$

then $DHq\text{-}ROFAAWG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)$ is called DHq-ROFAAWG operator, where $\left\{\omega_i|i=1,2,\ldots,n\right\}$ be the weighted vector of $\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)$ and $\omega_i>0$, and $\sum_{i=1}^{n}\omega_i=1$.

Theorem 3.6. *Let $\left\{\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right\}$ be a collection of DHq-ROFNs. and $\tau>0$, then the aggregating value of $\tilde{\kappa}_i$ utilizing DHq-ROFAAWG operator is also a DHq-ROFN and can be defined as*

$$DHq\text{-}ROFAAWG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)=\otimes_{AA}{}_{i=1}^{n}\left(\tilde{\kappa}_i\right)^{\omega_i}$$
$$=\left\langle \bigcup_{\substack{\gamma_i\in\tilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left(e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\gamma_i^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\right.$$
$$\left.\bigcup_{\substack{\eta_i\in\tilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left(1-e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\left(1-\eta_i^{q}\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle, \quad (9)$$

where $\left\{\omega_i|i=1,2,\ldots,n\right\}$ be weighted vector of $\tilde{\kappa}_i$ $(i=1,2,\ldots,n)$ where $\omega_i>0$, $\sum_{i=1}^{n}\omega_i=1$.

Proof. The proof of Theorem 3.6 is alike to the proof of Theorem 3.1.

If Example 3.2 is solved utilizing DHq-ROFAAWG operator, then the aggregating value is obtained as

$$DHq\text{-}ROFAAWG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\tilde{\kappa}_3\right)=$$
$$\left\langle \begin{array}{c}\{0.6229,0.6559,0.6516,0.6904,0.6759,0.7208,0.7147,\\ 0.7726,0.6897,0.7387,0.7320,0.7977\},\\ \{0.4444,0.5333,0.4836,0.5498,0.4502,0.5355,0.4874,5516\}\end{array}\right\rangle.$$

Note 4. It is also possible to demonstrate that the DHq-ROFAAWG operator satisfies the idempotency, monotonicity, and boundedness characteristics.

Note 5. DHq-ROFWG (Wang et al., 2019b) operator can be generated for taking $\tau=1$.

DHq-ROFAAOWG aggregation operator is presented as

Definition 3.5. Suppose $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)|i=1,2,\ldots,n\right\}$ be a set of n DHq-ROFNs. If a function DHq-ROFAAOWG is defined using $\otimes_{AA}$ operation as

$$DHq\text{-}ROFAAOWG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)=\otimes_{AA}{}_{i=1}^{n}\left(\tilde{\kappa}_{\sigma(i)}\right)^{\omega_i},$$

then $DHq\text{-}ROFAAOWG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)$ is called DHq-ROFAAOWG operator, where σ is a permutation defined on $\{1,2,\ldots,n\}$ in such a way that $\tilde{\kappa}_{\sigma(i-1)}\geq\tilde{\kappa}_{\sigma(i)}$ for all $i=2,3,\ldots,n$, and $\left\{\omega_i|i=1,2,\ldots,n\right\}$ be the weight vector such that $\omega_i>0$, and $\sum_{i=1}^{n}\omega_i=1$.

Theorem 3.7. *Suppose $\tilde{\kappa}_i=\left\langle\tilde{h}_i,\tilde{g}_i\right\rangle$ $(i=1,2,\ldots,n)$ be a collection of DHq-ROFNs and $\tau>0$, using DHq-ROFAAOWG operator the aggregated value is also a DHq-ROFN and can be given as follows:*

$$DHq\text{-}ROFAAOWG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)=\otimes_{AA}{}_{i=1}^{n}\left(\tilde{\kappa}_{\sigma(i)}\right)^{\omega_i}$$
$$=\left\langle \bigcup_{\substack{\gamma_{\sigma(i)}\in\tilde{h}_{\sigma(i)}\\(i=1,2,\ldots,n)}}\left\{\left(e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\gamma_{\sigma(i)}^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\right.$$
$$\left.\bigcup_{\substack{\eta_{\sigma(i)}\in\tilde{g}_{\sigma(i)}\\(i=1,2,\ldots,n)}}\left\{\left(1-e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\left(1-\eta_{\sigma(i)}^{q}\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle, \quad (10)$$

where σ and ω convey the same meaning as in Definition 3.5.

Proof. The proof is similar to Theorem 3.1.

If Example 3.2 is solved utilizing DHq-ROFAAOWG operator, then the aggregating value is obtained as

$$DHq\text{-}ROFAAOWG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\tilde{\kappa}_3\right)=$$
$$\left\langle \begin{array}{c}\{0.5907,0.6185,0.6759,0.7208,0.7008,0.7535,\\ 0.6042,0.6340,0.6970,0.7484,0.7253,0.7878\},\\ \{0.4444,0.5353,0.4544,0.5371,0.4690,0.5431,0.4766,0.5465\}\end{array}\right\rangle.$$

Note 6. It is also possible to demonstrate that the DHq-ROFAAOWG operator satisfies the idempotency, monotonicity, and boundedness characteristics.

Note 7. DHq-ROFOWG (Wang et al., 2019b) operator can be found for taking $\tau=1$, in DHq-ROFAAOWG operator.

Now, DHq-ROFAAHA operator is defined below.

Definition 3.6. Let $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)|i=1,2,\ldots,n\right\}$ be a collection of DHq-ROFNs. If a function DHq-ROFAAHA is defined using $\oplus_{AA}$ operation as

$$DHq\text{-}ROFAAHA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)=\oplus_{AA}{}_{i=1}^{n}\left(\omega_i\tilde{\kappa}_{\sigma(i)}\right),$$

then $DHq\text{-}ROFAAHA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)$ is called DHq-ROFAAHA operator, where n is the balancing co-efficient and $\omega=\left(\omega_1,\omega_2,\ldots,\omega_n\right)$ be associated weight vector such that $\omega_i>0$, and $\sum_{i=1}^{n}\omega_i=1$. $\tilde{\kappa}_{\sigma(i)}$ is the i^{th} largest element of the DHq-ROF arguments $\tilde{\kappa}_i$ $\left(\tilde{\kappa}_i=n\Omega_i\tilde{\kappa}_i=\left(n\Omega_i\tilde{h}_i,n\Omega_i\tilde{g}_i\right)\right)$, $\Omega=\left(\Omega_i|i=1,2,\ldots,n\right)$ is the weight vector of DHq-ROF arguments $\tilde{\kappa}_i$ with $\sum_{i=1}^{n}\Omega_i=1$ and $\Omega_i>0$.

Moreover, if $\omega=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$ (or $\Omega=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$) then DH$q$-ROFAAHA operator is reduced to DHq-ROFAAWA (or DHq- ROFAA OWA) operator, respectively.

Theorem 3.8. *Let $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)|i=1,2,\ldots,n\right\}$ represents a set of DHq-ROFNs and Aczel-Alsina parameter $\tau>0$, then the aggregated number utilizing DHq-ROFAAHA operator is also a DHq-ROFN and can be defined as*

$$DHq\text{-}ROFAAHA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)=\oplus_{AA}{}_{i=1}^{n}\left(\omega_i\tilde{\kappa}_{\sigma(i)}\right)$$
$$=\left\langle\bigcup_{\substack{\dot{\gamma}_i\in\tilde{h}_i\\(i=1,2,\ldots,n)}}\left\{\left(1-e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\left(1-\dot{\gamma}^q_{\sigma(i)}\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\right.$$
$$\left.\bigcup_{\substack{\dot{\eta}_i\in\tilde{g}_i\\(i=1,2,\ldots,n)}}\left\{\left(e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\dot{\eta}^q_{\sigma(i)}\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle,\tag{11}$$

where $\tilde{\kappa}_i=n\Omega_i\tilde{\kappa}_i=\left(n\Omega_i\tilde{h}_i,n\Omega_i\tilde{g}_i\right)=\left(\tilde{h},\tilde{g}\right)$.

Proof. The proof of Theorem 3.8 is similar to Theorem 3.1.

If Example 3.2 is solved utilizing DHq-ROFAAHA operator, then the aggregating value is obtained as (taking $\Omega=(0.4,0.3,0.3)$)

$$DHq\text{-}ROFAAHA\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\tilde{\kappa}_3\right)=$$
$$\left\langle\begin{array}{c}\{0.6280,0.6913,0.6833,0.7239,0.7466,0.7694,\\0.7660,0.7848,0.7819,0.7978,0.8084,0.8203\},\\\{0.3564,0.3725,0.4070,0.4282,0.3893,0.4086,0.4511,0.4780\}\end{array}\right\rangle.$$

Note 8. It is also possible to demonstrate that the DHq-ROFAAHA operator satisfies the idempotency, monotonicity, and boundedness characteristics.

Note 9. For $\tau=1$, the DHq-ROFAAHA operator is changed to DHq-ROFHA.

Now, the aggregation operator DHq-ROFAAHG is defined in the following manner.

Definition 3.7. Assume $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)|i=1,2,\ldots,n\right\}$ be a collection of DHq-ROFNs. If $DHq\text{-}ROFAAHG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)=\otimes_{AA}{}_{i=1}^{n}\left(\tilde{\kappa}_{\sigma(i)}\right)^{\omega_i}$, then $DHq\text{-}ROFAAHG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)$ is called DHq-ROFAAHG operator, where n is the balancing co-efficient and $\tilde{\kappa}_{\sigma(i)}$ is the i^{th} largest element of the DHq-ROF arguments $\tilde{\kappa}_i$ $\left(\tilde{\kappa}_i=\tilde{\kappa}_i^{n\Omega_i}=\left(\tilde{h}_i^{n\Omega_i},\tilde{g}_i^{n\Omega_i}\right)\right)$. $\omega=\left(\omega_1,\omega_2,\ldots,\omega_n\right)$ represents the associate weighted vector such that $\sum_{i=1}^{n}\omega_i=1$ and $\omega_i>0$, $\Omega=\left(\Omega_1,\Omega_2,\ldots,\Omega_n\right)$ is the weight vector of DHq-ROF arguments $\tilde{\kappa}_i$ with $\Omega_i>0$, and $\sum_{i=1}^{n}\Omega_i=1$.

Moreover, if $\omega=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$ then DHq-ROFAAHG operator is converted to DHq-ROFAAWG operator. Moreover, aggregation operator DHq-ROFAAOWG can be obtained from the developed operator DHq-ROFAAHG if $\Omega=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$.

Theorem 3.9. *Let $\left\{\tilde{\kappa}_i=\left(\tilde{h}_i,\tilde{g}_i\right)|i=1,2,\ldots,n\right\}$ represents a set of DHq-ROFNs, $\tau>0$, using DHq-ROFAAHG operator, the aggregated value is also a DHq-ROFN and can be represented as*

$$DHq\text{-}ROFAAHG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right)=\otimes_{AA}{}_{i=1}^{n}\left(\omega_i\tilde{\kappa}_{\sigma(i)}\right)$$
$$=\left\langle\bigcup_{\dot{\gamma}_i\in\tilde{h}_i,i=1,2,\ldots,n}\left\{\left(e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\dot{\gamma}^q_{\sigma(i)}\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\},\right.$$
$$\left.\bigcup_{\substack{\dot{\eta}_i\in\tilde{g}\\(i=1,2,\ldots,n)}}\left\{\left(1-e^{-\left(\sum_{i=1}^{n}\omega_i\left(-\log\left(1-\dot{\eta}^q_{\sigma(i)}\right)\right)^{\tau}\right)^{\frac{1}{\tau}}}\right)^{\frac{1}{q}}\right\}\right\rangle,\tag{12}$$

where $\tilde{\kappa}_i=\tilde{\kappa}_i^{n\Omega_i}=\left(\tilde{h}_i^{n\Omega_i},\tilde{g}_i^{n\Omega_i}\right)=\left(\tilde{h},\tilde{g}\right)$.

Proof. The proof is similar to proof of Theorem 3.1.

If Example 3.2 is solved utilizing DHq-ROFAAHG operator, then the aggregating value is obtained as (taking $\Omega=(0.4,0.3,0.3)$)

$$DHq\text{-}ROFAAHG\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\tilde{\kappa}_3\right)=$$
$$\left\langle\begin{array}{c}\{0.5783,0.6016,0.6776,0.7179,0.7083,0.7572,\\0.5897,0.6144,0.6967,0.7421,0.7311,0.7886\},\\\{0.4377,0.5250,0.4505,0.5299,0.4618,0.5346,0.4715,0.5391\}\end{array}\right\rangle.$$

4. Methodological developments of MCGDM in DHq-ROF environment using the proposed AAt-CN&t-Ns-based operators

In this section, an MCGDM method is developed using the developed operators. Suppose, $A=\left\{A_1,A_2,\ldots,A_m\right\}$ be a finite collection of m alternatives, $C=\left\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\right\}$ be a set of n criteria and $D=\left\{D^{(1)},D^{(1)},\ldots,D^{(p)}\right\}$ be a set consisting of p number of DMs. Let $\omega=\left(\omega_1,\omega_2,\ldots,\omega_n\right)^T$ be the weight vector of criteria such that $\sum_{i=1}^{n}\omega_i=1$ where $\omega_i\in[0,1]$ and the weight vector of DMs is given as $\Omega=\left(\Omega_1,\Omega_1,\ldots,\Omega_p\right)^T$ such that $\sum_{j=1}^{p}\Omega_j=1$ and $\Omega_j>0$. Suppose $\mathcal{K}^{(l)}=\left[\tilde{\kappa}_{ij}^{(l)}\right]_{m\times n}=\left(\tilde{h}_{ij}^{(l)},\tilde{g}_{ij}^{(l)}\right)_{m\times n}$ represents the DHq-ROF decision matrix (DHq-ROFDM). In which $\tilde{h}_{ij}^{(l)}$ and $\tilde{g}_{ij}^{(l)}$ denote, set of possible MD and NMD, respectively, of the ith alternative for the jth criterion evaluated by the lth DM.

Step 1: Cost type and benefit type are the two types of criteria in a decision-making problem. In terms of cost type, a lower value is preferable, while a higher number is preferable in terms of benefit type. In this way, the evaluation matrix is normalized as

$$R^{(l)}=\tilde{r}_{ij}^{(l)}=\begin{cases}\tilde{\kappa}_{ij}^{(l)} & \text{for benefit type } \mathcal{C}_j,\\ \left(\tilde{\kappa}_{ij}^{(l)}\right)^c & \text{for cost types } \mathcal{C}_j\end{cases}$$

$i=1,2,\ldots,n;\ j=1,2,\ldots,m;\ l=1,2,\ldots,p.$

$\left(\tilde{\kappa}_{ij}^{(l)}\right)^c$ represents the complement of $\tilde{\kappa}_{ij}^{(l)}$, i.e., $\left(\tilde{\kappa}_{ij}^{(l)}\right)^c=\left(\tilde{g}_{ij}^{(l)},\tilde{h}_{ij}^{(l)}\right)$.

Step 2: Utilize DHq-ROFAAWA (or DHq-ROFAAWG) operator for aggregating all the individual DHq-ROFDM $R^{(l)}=\left(\tilde{r}_{ij}^{(l)}\right)_{m\times n}$ into a single DHq-ROFDM, $R=\left[\tilde{r}_{ij}\right]_{m\times n}$. $\tilde{r}_{ij}=DHq\text{-}ROFAAWA\left(\tilde{r}_{ij}^{(1)},\tilde{r}_{ij}^{(2)},\ldots,\tilde{r}_{ij}^{(p)}\right)$ (or $\tilde{r}_{ij}=DHq\text{-}ROFAAWG\left(\tilde{r}_{ij}^{(1)},\tilde{r}_{ij}^{(2)},\ldots,\tilde{r}_{ij}^{(p)}\right)$).

Step 3: To get overall $\tilde{d}_i$, utilize the DHq-ROFAAWA (or DHq-ROFAAWG) operator for aggregating all the values r_{ij} $(j=1,2,\ldots,n)$ of the alternative A_i.

$$\tilde{d}_i=DHq\text{-}ROFAAWA\left(\tilde{r}_{i1},\tilde{r}_{i2},\ldots,\tilde{r}_{in}\right)$$
$$\left(\text{or } \tilde{d}_i=DHq\text{-}ROFAAWG\left(\tilde{r}_{i1},\tilde{r}_{i2},\ldots,\tilde{r}_{in}\right)\right).$$

Step 4: Compute the scores $S(\tilde{d}_i)$ $(i = 1, 2, \ldots, m)$ of the overall DHq-ROFN $\tilde{d}_i$ $(i = 1, 2, \ldots, n)$.

Step 5: Rank the all alternatives A_i $(i = 1, 2, \ldots, m)$ and select the best ones(s) in based with the scores $\left(\tilde{d}_i\right)$ $(i = 1, 2, \ldots, m)$.

It is worthy to mention here that the weights of the criteria in MCGDM may be known in advance based on their importances. However, some situations may be raised when the criteria weights are unknown or partially known by the DMs. This method is developed based on known criteria weights. However, if the weights are unknown, it can be evaluated from the decision matrices based on some known techniques (Maghrabie et al., 2019; Seikh and Mandal, 2022; Liu et al., 2021b.) which have not been considered in this article.

5. Case study

5.1. Problem description

The use of the novel approach has been applied to evaluate the parcel lockers location selection problem within Dublin city. This model is based on a hierarchical structure composed of criteria and alternatives. By conducting a comprehensive literature review and consulting with field experts, the possible locations for parcel lockers have been identified along with the relevant criteria. After careful consideration, five locations were selected as the most suitable for the logistics requirements ("On commercial area, A_1", "Private car parking A_2", "High population density area A_3", "Post office A_4" and "Public transport stops A_5"). The interviews with the experts and the analysis have been conducted in November 2022 in Dublin city, where the evaluators were experts in the related field. Taking into consideration the five mentioned alternatives and the following four main criteria are defined:

- **Traffic Impact** $(\mathcal{C}_1)$ — The use of parcel lockers can contribute to the decrease of traffic impact in cities, which can lead to lower CO_2 emissions (Iwan et al., 2016; Lagorio and Roberto, 2020).
- **Security** $(\mathcal{C}_2)$ — Maintaining users' privacy and security is crucial when using a delivery system. The system is able to verify delivery and 24/7 cameras are connected to police stations to record any incidents (Keeling et al., 2021).
- **Reliability** $(\mathcal{C}_3)$ — The ability of a system to meet consumers' daily demands based on the quantity, size, and quick service provided can be assessed through the use of parcel lockers. These lockers offer an efficient solution for non-delivery cases and are available 24 h a day, every day of the week (Kilibarda et al., 2020).
- **Accessibility** $(\mathcal{C}_4)$ — The positioning of a service point is important in terms of accessibility. Positioning it in a location that is easily accessible by walk, cycle, public transport, or private vehicle will encourage users to utilize it (Lagorio and Roberto, 2020).

The hierarchy structure of the parcel locker location selection had been created considering four main criteria and five alternatives, as it is depicted in Fig. 1.

5.2. Solution technique

We use the developed MCGDM methodology to parcel locker location selection with DHq-ROF data. Here $\mathcal{C}_2$, $\mathcal{C}_4$ are benefit type and $\mathcal{C}_1$ and $\mathcal{C}_3$ are cost-type criteria. The investor discussed with the four experts $E^{(l)}$ $(l = 1, 2, 3, 4)$. Based on their knowledge and expertise, the weight vector of experts is $\Omega = (0.1, 0.2, 0.3, 0.4)^T$. The judgement value of the experts is shown in Tables 3–6.

Step 1: Since, $\mathcal{C}_1$ and $\mathcal{C}_3$ are two cost-type criteria, Normalized DHq-ROFDMs are given in the following Tables 7–10.

Step 2: Utilize DHq-ROFAAWA operator, by considering $q = 3$, $\tau = 2$, for aggregating individual DHq-ROFDMs $R^{(l)} = \left(\tilde{r}_{ij}^{(l)}\right)_{m\times n}$ into a single DHq-ROFDM, $R = \left[\tilde{r}_{ij}\right]_{m\times n}$, as shown in Table 11.

Step 3: Utilize the DHq-ROFAAWA operator for aggregating $\tilde{r}_{ij}$ $(j = 1, 2,$
$\ldots, 4)$ and the values of $\tilde{d}_i$ $(i = 1, 2, \ldots, 5)$ are found as follows:

$$\tilde{d}_1 = \left\langle \begin{Bmatrix} 0.6076, 0.6098, 0.6085, 0.6107, 0.6146, 0.6166, 0.6154, 0.6175, 0.6123, 0.6144, \\ 0.6132, 0.6153, 0.6190, 0.6209, 0.6198, 6218 \end{Bmatrix}, \begin{Bmatrix} 0.6085, 0.6153, 0.6338, 0.6414, 0.6366, 0.6443, 0.6660, 0.6751, 0.6095, 0.6162, \\ 0.6349, 0.6426, 0.6377, 0.6455, 0.6673, 0.6764, 0.6095, 0.6162, 0.6349, 0.6426, \\ 0.6377, 0.6455, 0.6673, 0.6764, 0.6105, 0.6172, 0.6360, 0.6437, 0.6388, 0.6466, \\ 0.6686, 0.6777, 0.6272, 0.6346, 0.6551, 0.6637, 0.6582, 0.6669, 0.6915, 0.7018, \\ 0.6282, 0.6357, 0.6563, 0.6649, 0.6595, 0.6682, 0.6929, 0.7034, 0.6282, 0.6357, \\ 0.6563, 0.6649, 0.6595, 0.6682, 0.6929, 0.7034, 0.6293, 0.6367, 0.6575, 0.6662, \\ 0.6607, 0.6695, 0.6944, 0.7049 \end{Bmatrix} \right\rangle$$

$$\tilde{d}_2 = \langle \{0.6945, 0.6960, 0.6951, \ldots (64\ values)\}, \{0.4030, 0.4304, 0.5004, \ldots (32\ values)\} \rangle$$

$$\tilde{d}_3 = \langle \{0.6031, 0.6370, \ldots (1024\ values)\}, \{0.4816, 0.4823, 0.4874, \ldots (192\ values)\} \rangle;$$

$$\tilde{d}_4 = \langle \{0.7361, 0.7432, 0.8025, \ldots (384\ values)\}, \{0.4257, 0.4290, 0.4303, 0.4337\} \rangle;$$

$$\tilde{d}_5 = \langle \{0.6432, 0.6433, 0.6439, \ldots (8\ values)\}, \{0.5049, 0.5051, \ldots (128\ values)\} \rangle;$$

Step 4: The score values of each $\tilde{d}_i$ $(i = 1, 2, 3, 4, 5)$ are calculated as follows:

$S\left(\tilde{d}_1\right) = 0.4767, S\left(\tilde{d}_2\right) = 0.6389, S\left(\tilde{d}_3\right) = 0.6218, S\left(\tilde{d}_4\right) = 0.7043,$
$S\left(\tilde{d}_5\right) = 0.5599.$

Step 5: So, ordering of the location alternatives is appeared as $A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$ and A_4 is identified as the best location alternative.

It is to be mentioned here that the given problem is solved by utilizing the other developed operators and the proposed MCGDM method. After solving the problem, the results are given in Table 12.

6. Sensitivity analysis (impact of different parameters on decision making results)

Now, the impact of Aczel-Alsina parameters, τ and rung q, on group decision making outcomes are investigated using the proposed method. It is observed that the ranking of alternatives is greatly influenced by those parameters. By setting different values to the parameters, various score values are obtained. Varying the parameter $q \in [3, 10]$ and $\tau \in [1, 10]$, the variation of the score values are found by using the developed operators and is shown in Tables 13–16 and Figs. 2–9.

When the DHq-ROFAAWA operator is used, the corresponding score values for the several location alternatives are displayed in Fig. 2. It shows that the score values of all alternatives decrease when the parameter q's values rise from 3 to 10 with the exception of location alternative A_1. Although A_1 is increasing, but it gives a lower score value among all the location alternatives for all the $q \in [3, 10]$.

Furthermore, the following observations are found:

1. when $q \in [3, 6.02]$ the ordering of the five location alternatives is $A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$ and the best choice is found as A_4.
2. when $q \in [6.02, 12]$ the ordering of five location alternatives is $A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$ and the best choice is identified as A_4.

So, in this situation A_4 is the best location alternative.

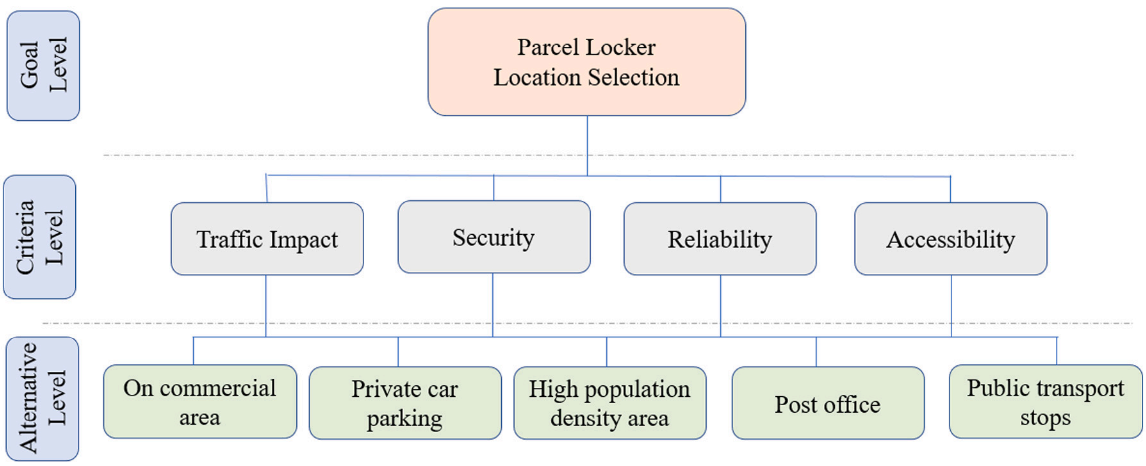

Fig. 1. The framework for selecting the parcel locker location.

Table 3
DHq-ROFDM of $E^{(1)}$.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.5\},\{0.8\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.4\},\{0.7\}\rangle$	$\langle\{0.3,0.5\},\{0.7\}\rangle$
A_2	$\langle\{0.4\},\{0.8,0.9\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.5,0.6\},\{0.8\}\rangle$	$\langle\{0.3\},\{0.8\}\rangle$
A_3	$\langle\{0.6\},\{0.5\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.2,0.5\},\{0.7,0.9\}\rangle$	$\langle\{0.3,0.8\},\{0.8\}\rangle$
A_4	$\langle\{0.5\},\{0.6,0.7\}\rangle$	$\langle\{0.4\},\{0.6\}\rangle$	$\langle\{0.3,0.4\},\{0.6,0.9\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$
A_5	$\langle\{0.8\},\{0.6\}\rangle$	$\langle\{0.6\},\{0.6,0.8\}\rangle$	$\langle\{0.3\},\{0.7\}\rangle$	$\langle\{0.2,0.5\},\{0.6,0.8\}\rangle$

Table 4
DHq-ROFDM of $E^{(2)}$.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.4,0.7\},\{0.6\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_2	$\langle\{0.2,0.3\},\{0.7\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_3	$\langle\{0.6\},\{0.6,0.8\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.3,0.6\},\{0.6,0.9\}\rangle$	$\langle\{0.4,0.8\},\{0.5,0.7\}\rangle$
A_4	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.6,0.8\},\{0.6\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.5\},\{0.8\}\rangle$
A_5	$\langle\{0.4\},\{0.6\}\rangle$	$\langle\{0.2\},\{0.7\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$

Table 5
DHq-ROFDM of $E^{(3)}$.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.3,0.5\},\{0.6,0.7\}\rangle$	$\langle\{0.4\},\{0.7\}\rangle$	$\langle\{0.5,0.8\},\{0.3,0.6\}\rangle$	$\langle\{0.4,0.5\},\{0.6\}\rangle$
A_2	$\langle\{0.3\},\{0.8\}\rangle$	$\langle\{0.3,0.6\},\{0.7\}\rangle$	$\langle\{0.1,0.3\},\{0.8\}\rangle$	$\langle\{0.2\},\{0.9\}\rangle$
A_3	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.6\},\{0.6\}\rangle$	$\langle\{0.2,0.4\},\{0.7,0.9\}\rangle$	$\langle\{0.3,0.7\},\{0.4,0.6,0.8\}\rangle$
A_4	$\langle\{0.5\},\{0.7,0.8\}\rangle$	$\langle\{0.5,0.6,0.7\},\{0.8\}\rangle$	$\langle\{0.1\},\{0.9\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_5	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.2,0.4\},\{0.8\}\rangle$	$\langle\{0.3\},\{0.7,0.9\}\rangle$

Table 6
DHq-ROFDM of $E^{(4)}$.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.3\},\{0.8\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.4,0.5\},\{0.7\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$
A_2	$\langle\{0.2,0.3\},\{0.7,0.8\}\rangle$	$\langle\{0.3\},\{0.8\}\rangle$	$\langle\{0.1,0.3\},\{0.8,0.9\}\rangle$	$\langle\{0.2,0.5\},\{0.8\}\rangle$
A_3	$\langle\{0.6\},\{0.6\}\rangle$	$\langle\{0.6,0.8\},\{0.8\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.3,0.7\},\{0.6,0.8\}\rangle$
A_4	$\langle\{0.4\},\{0.7\}\rangle$	$\langle\{0.6,0.8\},\{0.8\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.5,0.9\},\{0.3,0.7\}\rangle$
A_5	$\langle\{0.4,0.6\},\{0.6\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.2,0.5\},\{0.5,0.8\}\rangle$	$\langle\{0.2,0.3\},\{0.8,0.9\}\rangle$

Table 7
Normalized DHq-ROFDM $R^{(1)}$.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.8\},\{0.5\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.7\},\{0.4\}\rangle$	$\langle\{0.3,0.5\},\{0.7\}\rangle$
A_2	$\langle\{0.8,0.9\},\{0.4\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$	$\langle\{0.8\},\{0.5,0.6\}\rangle$	$\langle\{0.3\},\{0.8\}\rangle$
A_3	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.7,0.9\},\{0.2,0.5\}\rangle$	$\langle\{0.3,0.8\},\{0.8\}\rangle$
A_4	$\langle\{0.6,0.7\},\{0.5\}\rangle$	$\langle\{0.4\},\{0.6\}\rangle$	$\langle\{0.6,0.9\},\{0.3,0.4\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$
A_5	$\langle\{0.6\},\{0.8\}\rangle$	$\langle\{0.6\},\{0.6,0.8\}\rangle$	$\langle\{0.7\},\{0.3\}\rangle$	$\langle\{0.2,0.5\},\{0.6,0.8\}\rangle$

Table 8
Normalized DHq-ROFDM $\boldsymbol{R}^{(2)}$.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.6\},\{0.5\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.6\},\{0.4,0.7\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_2	$\langle\{0.7\},\{0.2,0.3\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$	$\langle\{0.7\},\{0.5\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_3	$\langle\{0.6,0.8\},\{0.6\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.6,0.9\},\{0.3,0.6\}\rangle$	$\langle\{0.4,0.8\},\{0.5,0.7\}\rangle$
A_4	$\langle\{0.7\},\{0.5\}\rangle$	$\langle\{0.6,0.8\},\{0.6\}\rangle$	$\langle\{0.7,0.8\},\{0.4\}\rangle$	$\langle\{0.5\},\{0.8\}\rangle$
A_5	$\langle\{0.6\},\{0.4\}\rangle$	$\langle\{0.2\},\{0.7\}\rangle$	$\langle\{0.8\},\{0.4\}\rangle$	$\langle\{0.4\},\{0.8\}\rangle$

Table 9
Normalized DHq-ROFDM $\boldsymbol{R}^{(3)}$.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.6,0.7\},\{0.3,0.5\}\rangle$	$\langle\{0.4\},\{0.7\}\rangle$	$\langle\{0.3,0.6\},\{0.5,0.8\}\rangle$	$\langle\{0.4,0.5\},\{0.6\}\rangle$
A_2	$\langle\{0.8\},\{0.3\}\rangle$	$\langle\{0.3,0.6\},\{0.7\}\rangle$	$\langle\{0.8\},\{0.1,0.3\}\rangle$	$\langle\{0.2\},\{0.9\}\rangle$
A_3	$\langle\{0.6\},\{0.5\}\rangle$	$\langle\{0.6\},\{0.6\}\rangle$	$\langle\{0.7,0.9\},\{0.2,0.4\}\rangle$	$\langle\{0.3,0.7\},\{0.4,0.6,0.8\}\rangle$
A_4	$\langle\{0.7,0.8\},\{0.5\}\rangle$	$\langle\{0.5,0.6,0.7\},\{0.8\}\rangle$	$\langle\{0.9\},\{0.1\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$
A_5	$\langle\{0.7\},\{0.5\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.8\},\{0.2,0.4\}\rangle$	$\langle\{0.3\},\{0.7,0.9\}\rangle$

Table 10
Normalized DHq-ROFDM $\boldsymbol{R}^{(4)}$.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.8\},\{0.3\}\rangle$	$\langle\{0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.7\},\{0.4,0.5\}\rangle$	$\langle\{0.5\},\{0.7\}\rangle$
A_2	$\langle\{0.7,0.8\},\{0.2,0.3\}\rangle$	$\langle\{0.3\},\{0.8\}\rangle$	$\langle\{0.8,0.9\},\{0.1,0.3\}\rangle$	$\langle\{0.2,0.5\},\{0.8\}\rangle$
A_3	$\langle\{0.6\},\{0.6\}\rangle$	$\langle\{0.6,0.8\},\{0.8\}\rangle$	$\langle\{0.7,0.8\},\{0.4\}\rangle$	$\langle\{0.3,0.7\},\{0.6,0.8\}\rangle$
A_4	$\langle\{0.7\},\{0.4\}\rangle$	$\langle\{0.6,0.8\},\{0.8\}\rangle$	$\langle\{0.8\},\{0.2\}\rangle$	$\langle\{0.5,0.9\},\{0.3,0.7\}\rangle$
A_5	$\langle\{0.6\},\{0.4,0.6\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.5,0.8\},\{0.2,0.5\}\rangle$	$\langle\{0.2,0.3\},\{0.8,0.9\}\rangle$

Now, utilizing the DHq-ROFAAWA operator and changing the Aczel-Alsina parameter $\tau \in [1,10]$, the obtained score values, $S(A_i)$ of several location alternatives A_i are portrayed in Fig. 3. Fig. 3 shows that all of the location alternatives' score values are rising.

Additionally, the following cases are found:

1. when $\tau \in [1,1.22]$ the ranking of the five options is $A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$ and A_4 is identified as the best location.
2. when $\tau \in [1.22,7.61]$ the ordering of the five options is $A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$ and A_4 is identified as the best location.
3. when $\tau \in [7.61,10]$ the ranking of the five options is $A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$ and A_4 is identified as the best location.

Thus, taking into account all cases A_4 is the best location alternative.

Now, when the DHq ROFAAWG operator is utilized and varying $q \in [3,10]$, the corresponding score values of various location alternatives are shown in Fig. 4. From this figure, it is notable here that the score value of A_1, A_2, A_5 are increasing and A_3, A_4 are decreasing. But the ordering of the location alternative remains same $(\text{i.e}, A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5)$ for all $q \in [3,10]$. The DHq ROFAAWG operator produces A_4 as the best alternative. Moreover, as ranking of all the alternatives remains same, this shows that DHq ROFAAWG operator gives a stable solution under the condition $\tau = 2$.

Now, using the DHq-ROFAAWG operator and changing the parameter $\tau \in [1,10]$, the obtained score values for the different alternatives are displayed in Fig. 5. From this figure, it is observed that all location alternatives' scores are decreasing.

Furthermore, the following observations are found:

1. when $\tau \in [1,2.97]$ the ranking of the five alternatives is $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$ and A_4 is identified as the best location.
2. when $\tau \in [2.97,3.68]$ the ranking of the five alternatives is $A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$ and A_3 is identified as the best location.
3. when $\tau \in [3.68,6.54]$ the ranking of the five alternatives is $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$ and A_4 is identified as the best location.
4. when $\tau \in [6.54,10]$ the ordering of the five alternatives is $A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$ and A_4 is identified as the best location.

So, here A_3 is the best alternative location for $\tau \in [2.47,5.15]$ and A_4 is the best alternative location for $\tau \notin [2.47,5.15]$. Thus, considering all cases, it can be said that A_4 is the best alternative location.

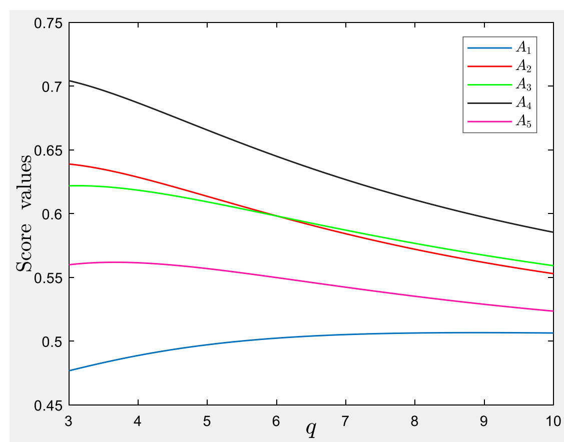

Fig. 2. Score value using variation of rung parameter q for fixed Aczel-Alsina parameter $\tau = 2$ (DHq-ROFAAWA).

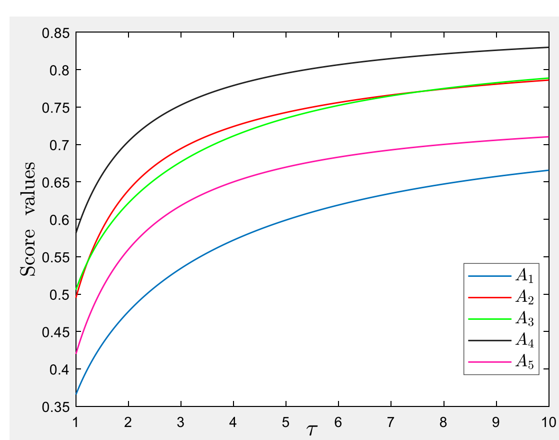

Fig. 3. Score value using variation of Aczel-Alsina parameter τ for fixed $q = 3$ (DHq-ROFAAWA).

Table 11
Aggregated DHq-ROFDM $\boldsymbol{R}$.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.7457, 0.7566\}, \{0.4270, 0.5178\}\rangle$	$\langle\{0.4000\}, \{0.7624, 0.7737, 0.7737, 0.7858\}\rangle$	$\langle\{0.6475, 0.6623\}, \{0.5002, 0.5136, 0.5531, 0.5706, 0.5594, 0.5774, 0.6336, 0.6602\}\rangle$	$\langle\{0.4704, 0.4922, 0.4803, 0.5000\}, \{0.6738\}\rangle$
A_2	$\langle\{0.7535, 0.7871, 0.7836, 0.8086\}, \{0.3203, 0.3372, 0.3558, 0.3766\}\rangle$	$\langle\{0.3800, 0.5118\}, \{0.7737\}\rangle$	$\langle\{0.7871, 0.8515\}, \{0.2220, 0.2539, 0.3488, 0.4270, 0.2237, 0.2559, 0.3525, 0.4326\}\rangle$	$\langle\{0.3143, 0.4415, 0.3891, 0.4621\}, \{0.8214\}\rangle$
A_3	$\langle\{0.5936, 0.6824\}, \{0.6074\}\rangle$	$\langle\{0.5794, 0.7245\}, \{0.6124, 0.6627\}\rangle$	$\langle\{0.6866, 0.7462, 0.8156, 0.8333, 0.7940, 0.8164, 0.8531, 0.8643, 0.7517, 0.7858, 0.8353, 0.8493, 0.8189, 0.8360, 0.8656, 0.8750\}, \{0.3050, 0.3904, 0.3387, 0.4474, 0.3345, 0.4401, 0.3751, 0.5167\}\rangle$	$\langle\{0.3352, 0.6171, 0.5922, 0.6674, 0.6504, 0.6986, 0.6886, 0.7235, 0.5900, 0.6664, 0.6522, 0.6997, 0.6878, 0.7229, 0.7153, 0.7427\}, \{0.5403, 0.5498, 0.6356, 0.6509, 0.6859, 0.7062, 0.5739, 0.5851, 0.6927, 0.7138, 0.7654, 0.7990\}\rangle$
A_4	$\langle\{0.6936, 0.7387, 0.7000, 0.7427\}, \{0.5433\}\rangle$	$\langle\{0.5687, 0.7222, 0.5915, 0.7273, 0.6365, 0.7404, 0.6732, 0.7544, 0.6815, 0.7580, 0.7019, 0.7676\}, \{0.7310\}\rangle$	$\langle\{0.8346, 0.8420, 0.8515, 0.8574\}, \{0.2298, 0.2347\}\rangle$	$\langle\{0.4939, 0.8265\}, \{0.6138, 0.7222\}\rangle$
A_5	$\langle\{0.6419\}, \{0.5309, 0.5567\}\rangle$	$\langle\{0.4206\}, \{0.7654, 0.7990\}\rangle$	$\langle\{0.7442, 0.7938\}, \{0.3130, 0.3441, 0.4035, 0.4571\}\rangle$	$\langle\{0.3210, 0.3326, 0.3744, 0.3800\}, \{0.7547, 0.7600, 0.8101, 0.8177, 0.7858, 0.7923, 0.8586, 0.8700\}\rangle$

Table 12
Results utilizing proposed operators.

Operators	Score values					Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	
DHq-ROFAAOWA	0.4531	0.5680	0.5271	0.5675	0.4636	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
DHq-ROFAAWG	0.4488	0.3738	0.5267	0.5307	0.3293	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
DHq-ROFAAOWG	0.4273	0.3411	0.4775	0.4538	0.2945	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
DHq-ROFAAHA	0.4485	0.5448	0.5494	0.6263	0.4608	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
DHq-ROFAAHG	0.4129	0.3139	0.4885	0.4884	0.2808	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$

Table 13
The impact of rung parameter q utilizing DHq-ROFAAWA operator ($\tau = 2$).

Parameter	$S(\tilde{d}_1)$	$S(\tilde{d}_2)$	$S(\tilde{d}_3)$	$S(\tilde{d}_4)$	$S(\tilde{d}_5)$	Ordering
$q = 3$	0.4767	0.6389	0.6218	0.7043	0.5599	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$q = 4$	0.4887	0.6285	0.6183	0.6868	0.5614	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$q = 5$	0.4971	0.6135	0.6092	0.6655	0.5569	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$q = 6$	0.5023	0.5982	0.5981	0.6450	0.5498	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$q = 7$	0.5051	0.5842	0.5870	0.6266	0.5423	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$q = 8$	0.5063	0.5720	0.5766	0.6107	0.5351	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$q = 9$	0.5066	0.5616	0.5673	0.5971	0.5289	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$q = 10$	0.5063	0.5529	0.5591	0.5853	0.5235	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$

Further, when the DHq-ROFAAHA operator is used, the acquired score values for the alternatives are graphically displayed in Figs. 6 and 7. In Fig. 6, the rung parameter is varied from 3 to 10 and fixed the Aczel-Alsina parameter $\tau = 2$.

Table 14
The impact of Aczel-Alsina parameter τ utilizing DHq-ROFAAWA operator.

Parameter ($q = 3$)	$S(\tilde{d}_1)$	$S(\tilde{d}_2)$	$S(\tilde{d}_3)$	$S(\tilde{d}_4)$	$S(\tilde{d}_5)$	Ordering
$\tau = 1$	0.3657	0.4951	0.5058	0.5815	0.4199	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$\tau = 2$	0.4767	0.6389	0.6218	0.7043	0.5599	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau = 3$	0.5345	0.6946	0.6769	0.7528	0.6184	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau = 4$	0.5722	0.7241	0.7114	0.7789	0.6501	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau = 5$	0.5990	0.7429	0.7351	0.7953	0.6697	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau = 6$	0.6191	0.7561	0.7523	0.8066	0.6831	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau = 7$	0.6347	0.7662	0.7650	0.8148	0.6928	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
$\tau = 8$	0.6472	0.7742	0.7748	0.8211	0.7001	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$\tau = 9$	0.6572	0.7807	0.7826	0.8260	0.7058	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
$\tau = 10$	0.6656	0.7861	0.7888	0.8299	0.7103	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$

Table 15
The significance of the rung parameter q (fixed Aczel-Alsina parameter $\tau = 2$) utilizing DHq-ROFAAWG operator.

Parameter	$S(\tilde{d}_1)$	$S(\tilde{d}_2)$	$S(\tilde{d}_3)$	$S(\tilde{d}_4)$	$S(\tilde{d}_5)$	Ordering
$q = 3$	0.4488	0.3738	0.5267	0.5307	0.3293	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q = 4$	0.4499	0.3719	0.5149	0.5186	0.3382	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q = 5$	0.4546	0.3779	0.5059	0.5092	0.3538	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q = 6$	0.4605	0.3875	0.4997	0.5026	0.3709	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q = 7$	0.4667	0.3983	0.4959	0.4982	0.3872	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q = 8$	0.4723	0.4090	0.4938	0.4956	0.4018	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q = 9$	0.4773	0.4191	0.4929	0.4942	0.4147	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$q = 10$	0.4815	0.4283	0.4927	0.4936	0.4257	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$

Table 16
The significance of the Aczel-Alsina parameter τ (fixed $q = 3$) utilizing DHq-ROFAAWG operator.

Parameter	$S(\tilde{d}_1)$	$S(\tilde{d}_2)$	$S(\tilde{d}_3)$	$S(\tilde{d}_4)$	$S(\tilde{d}_5)$	Ordering
$\tau = 1$	0.5454	0.5245	0.6460	0.6633	0.4664	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau = 2$	0.4488	0.3738	0.5267	0.5307	0.3293	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau = 3$	0.4067	0.3061	0.4641	0.4640	0.2773	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
$\tau = 4$	0.3808	0.2676	0.4231	0.4237	0.2505	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau = 5$	0.3626	0.2427	0.3940	0.3969	0.2340	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau = 6$	0.3490	0.2252	0.3725	0.3778	0.2227	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
$\tau = 7$	0.3384	0.2124	0.3564	0.3637	0.2145	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$
$\tau = 8$	0.3299	0.2027	0.3439	0.3528	0.2083	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$
$\tau = 9$	0.3230	0.1952	0.3342	0.3442	0.2035	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$
$\tau = 10$	0.3172	0.1891	0.3263	0.3372	0.1996	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$

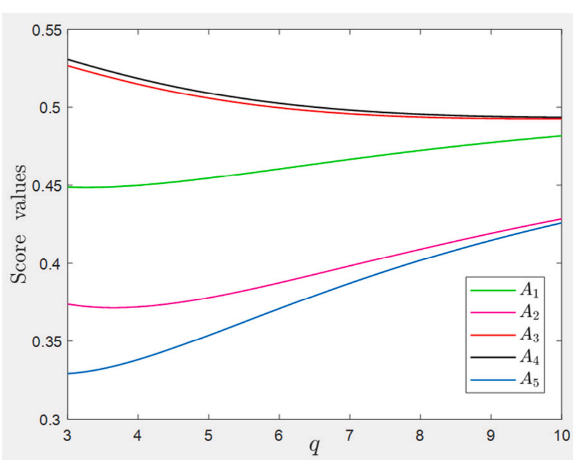

Fig. 4. Score value varying rung parameter q for fixed Aczel-Alsina parameter $\tau = 2$ (DHq-ROFAAWG).

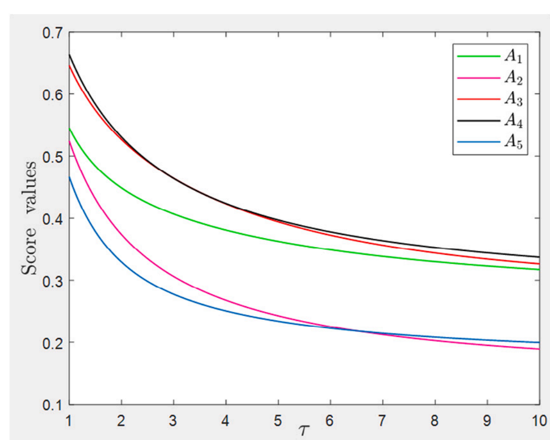

Fig. 5. Score value varying Aczel-Alsina parameter τ for fixed $q = 3$ (DHq-ROFAAWG).

Again, Fig. 7 represents the score values for Aczel-Alsina parameter varied from 1 to 10 with fixed rung parameter q. Graphically it is shown that in all the cases the location alternative A_4 is the best alternative.

Again, when the DHq-ROFAAHG operator is used, the acquired score values for the alternatives are graphically displayed in Figs. 8 and 9. In Fig. 8, the rung parameter is varied from 3 to 10 and fixed the Aczel-Alsina parameter $\tau = 2$.

Again, Fig. 9 represents the score values for Aczel-Alsina parameter varied from 1 to 10 with fixed rung parameter q. Graphically it is shown that in all the cases the alternative A_4 and A_3 show similar nature and in some certain conditions A_4 appeared as the best, whereas, in other conditions A_3 stood as the best.

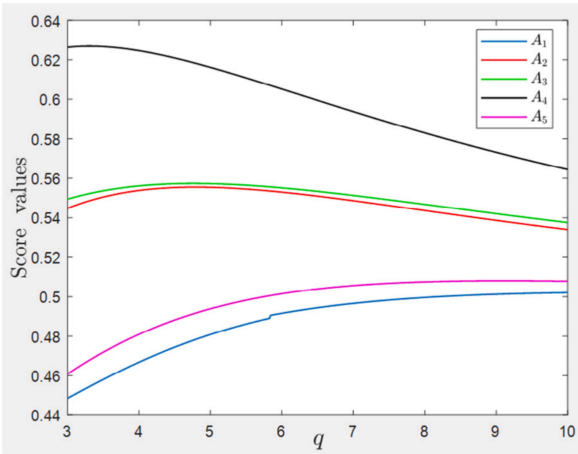

Fig. 6. Score value varying rung parameter q for fixed Aczel-Alsina parameter $\tau = 2$ (DHq-ROFAAHA).

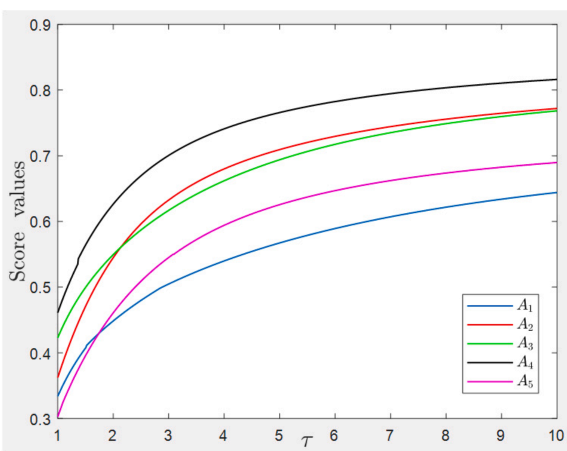

Fig. 7. Score value varying Aczel-Alsina parameter τ for fixed $q = 3$ (DHq-ROFAAHA).

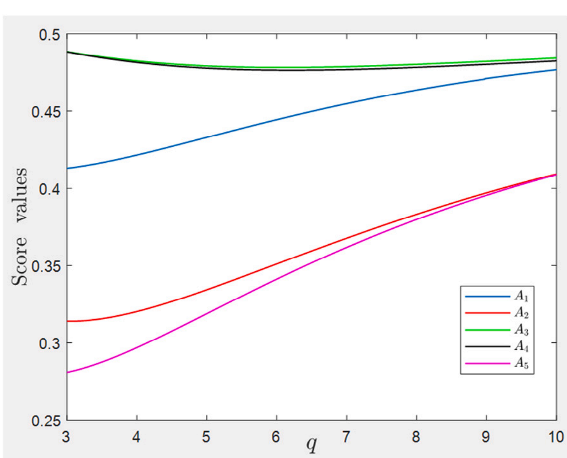

Fig. 8. Score value varying rung parameter q for fixed Aczel-Alsina parameter $\tau = 2$ (DHq-ROFAAHG).

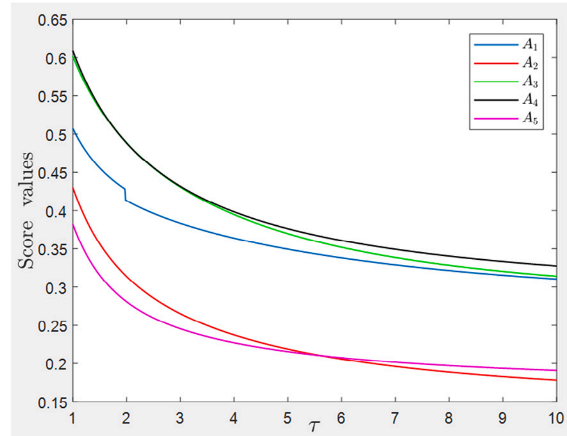

Fig. 9. Score value varying Aczel-Alsina parameter τ for fixed $q = 3$ (DHq-ROFAAHG).

7. Comparative analysis

To determine the efficacy of the proposed method, the case study as considered above, has been solved by several existing MCGDM technique using different aggregation operators, viz., PFEHGA (Rahman and Ali, 2020), DHq-ROFHWA (Wang et al., 2019b), DHq-ROFHWG (Wang et al., 2019b), DHq-ROFHHWA (Wang et al., 2019b), DHq-ROFHHWG (Wang et al., 2019b), DHPFHWA (Wei and Lu, 2017), DHPFHWG (Wei and Lu, 2017), DHPFHOWA (Wei and Lu, 2017), DHPFHOWG (Wei and Lu, 2017), DHPFHHA (Wei and Lu, 2017), DHPFHHG (Wei and Lu, 2017), DHq-ROFWDBM (Sarkar and Biswas, 2021), DHq-ROFWDGBM (Sarkar and Biswas, 2021) and by conforming the data in the respective environments. The comparisons are conducted using two distinct methods. Initially, the comparisons are based on the characteristics of the operators, evaluating their specific attributes and features. Subsequently, the comparisons are based on the achieved results, assessing the outcomes or performance of the operators in practical applications or experiments. This two-step approach allows for a comprehensive evaluation of the operators from both theoretical and empirical perspectives.

When comparing the method based on the characteristics of the operators, it is important to note that all the mentioned existing operators, as well as the developed operators, have the ability to capture hesitant fuzzy information except PFEHGA operator. The PFEHGA operator (Rahman and Ali, 2020) is developed for PF environments which can be viewed as a special case of DHq-ROF contexts by considering $q = 2$. With the exception of the developed operators, none of the aforementioned operators take into account Aczel-Alsina operations, which provide more flexibility in the decision aggregation process. The inclusion of Aczel-Alsina operations allows decision-makers to adjust parameters and tailor the aggregation process according to their specific needs and preferences. By incorporating Aczel-Alsina operations, the decision aggregation process becomes more adaptable and customizable, offering a valuable advantage over the other mentioned operators. Combining hybrid operator with Aczel-Alsina operations in DHq-ROF context, the proposed operators become extra flexible and powerful than the current existing operators. Table 17 presents the characteristics of the current mentioned operators. The presented table demonstrates the broader scope and coverage of the proposed operators compared to the existing operators. It indicates that the proposed operators offer a wider range of capabilities and functionalities, allowing for a more comprehensive approach to the problem. This expanded coverage suggests that the proposed operators have the potential to handle a greater variety of scenarios and provide more versatile solutions compared to the existing operators.

Table 17
Some characteristics of existing operators.

Operator	Consideration of weighs both the given fuzzy value and its ordered position	Consideration of hesitancy	Flexibility due to Aczel-Alsina operation	Capturing information by q-rung fuzzy number
PFEHGA (Rahman and Ali, 2020)	Yes	No	No	No
DHq-ROFHWA (Wang et al., 2019b)	No	Yes	No	Yes
DHq-ROFHWG (Wang et al., 2019b)	No	Yes	No	Yes
DHPFHWA (Wei and Lu, 2017)	No	Yes	No	No
DHPFHOWA (Wei and Lu, 2017)	No	Yes	No	Yes
DHPFHHA (Wei and Lu, 2017)	Yes	Yes	No	Yes
DHPFHWG (Wei and Lu, 2017)	No	Yes	No	No
DHPFHOWG (Wei and Lu, 2017)	No	Yes	No	Yes
DHPFHHG (Wei and Lu, 2017)	Yes	Yes	No	Yes
DHq-ROFWDBM (Sarkar and Biswas, 2021)	No	Yes	No	Yes
DHq-ROFWDGBM (Sarkar and Biswas, 2021)	No	Yes	No	Yes
Proposed method	Yes	Yes	Yes	Yes

Table 18
Compared to Rahman and Ali's method.

Method	Score values					Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	
Rahman and Ali (Rahman and Ali, 2020)	−0.221	−0.177	−0.1540	−0.099	−0.225	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
DHq-ROFAAWA ($q = 2, \tau = 2$)	0.4854	0.5710	0.4650	0.6611	0.5909	$A_4 \succ A_5 \succ A_2 \succ A_1 \succ A_3$
DHq-ROFAAWA ($q = 3, \tau = 2$)	0.4962	0.5835	0.4766	0.6586	0.5977	$A_4 \succ A_5 \succ A_2 \succ A_1 \succ A_3$
DHq-ROFAAHA ($q = 2, \tau = 2$)	0.4401	0.4682	0.3866	0.5502	0.4691	$A_4 \succ A_5 \succ A_2 \succ A_1 \succ A_3$
DHq-ROFAAHA ($q = 3, \tau = 2$)	0.4562	0.4935	0.4046	0.5623	0.4940	$A_4 \succ A_5 \succ A_2 \succ A_1 \succ A_3$

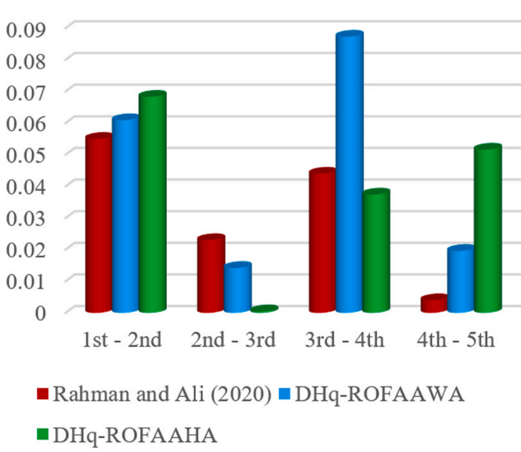

Fig. 10. Bar diagram of differences of the alternatives' score values $q = 2, \tau = 2$ DHqROFAAWA.

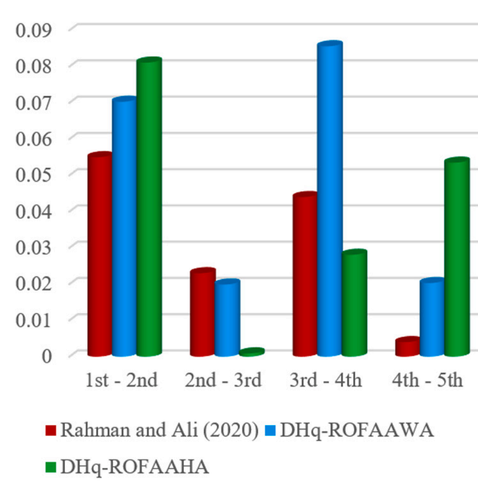

Fig. 11. Bar diagram of differences of the alternatives' score values $q = 3, \tau = 2$ DHqROFAAWA.

Now, the achieved results through the proposed method is compared with the results obtained by Rahman and Ali (2020).

Table 18 depicts the ranking and score values of the alternative using Rahman and Ali (2020) method and the suggested method. Although the rankings in both techniques fluctuate significantly, the best location alternative A_4 remains the same. The proposed method has a larger score value difference between two successive alternatives (rank-wise) than Rahman and Ali's method. Hence, the suggested approach is more effective in terms of selecting the best alternative than Rahman and Ali's approach. The difference in score values is graphically represented in Figs. 10 and 11.

As a results, The suggested approach superior to Rahman and Ali's method.

Afterwards, the problem under consideration is solved using the proposed method by considering the existing operators (Rahman and Ali, 2020; Wang et al., 2019b; Wei and Lu, 2017; Sarkar and Biswas, 2021). It is to be noted that operators provided by Wei and Lu (2017) in DHPF environment. To solve the problem DHq-ROF data are converted to DHPF data by adjustments. Similarly, for PFEHGA operator data are converted to PFN. Also, all the problems are solved by considering the rung values, $q = 3$.

The achieved results are compared with the solution achieved through the proposed method and is shown in Table 19 and Fig. 12. Graphically, it is observed that the location alternative A_4 is the best choice over other location alternatives in most of the cases. It differs for the proposed DHq-ROFAAHG operator. It is worth mentioning that few geometric operators (DHPFHWG, DHPFHOWG, DHPFHHG, DHq-ROFWDBM) also produce A_3 as the best location alternative like the proposed DHq-ROFAAHG operator. Although, A_3 is best solution but the difference between score values of A_3 and A_4 are very small. Again, the existing PFEHGA, DHq-ROFWDGBM operators produced A_1, A_2, respectively, as best solution.

Considering the above cases it is easily realized that the developed operators possess greater capability of capturing uncertainties in wide domains also are capable of solving real life problems in significant manners than the existing methods.

Table 19
Comparison with existing operators.

Operators	Score values					Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	
PFEHGA (Rahman and Ali, 2020)	0.4615	0.3106	0.3821	0.4576	0.3666	$A_1 \succ A_4 \succ A_3 \succ A_5 \succ A_2$
DHq-ROFHWA (Wang et al., 2019b)	0.3595	0.4444	0.4484	0.5327	0.3787	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
DHq-ROFHWG (Wang et al., 2019b)	0.3381	0.3889	0.3979	0.4511	0.3226	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
DHq-ROFHHWA (Wang et al., 2019b)	0.3250	0.3601	0.3789	0.4320	0.3161	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
DHq-ROFHHWG (Wang et al., 2019b)	0.5231	0.3851	0.5526	0.5591	0.3628	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
DHPFHWA (Wei and Lu, 2017)	0.4094	0.4571	0.4785	0.5401	0.4015	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
DHPFHOWA (Wei and Lu, 2017)	0.3865	0.4096	0.4364	0.4806	0.3515	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
DHPFHHA (Wei and Lu, 2017)	0.3725	0.3821	0.4096	0.4541	0.3400	$A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$
DHPFHWG (Wei and Lu, 2017)	0.5344	0.4883	0.5946	0.5911	0.4680	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
DHPFHOWG (Wei and Lu, 2017)	0.5211	0.4784	0.5825	0.5533	0.4471	$A_3 \succ A_4 \succ A_1 \succ A_1 \succ A_5$
DHPFHHG (Wei and Lu, 2017)	0.4734	0.3808	0.5459	0.5308	0.3678	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
DHq-ROFWDBM (Sarkar and Biswas, 2021)	0.2713	0.3361	0.2968	0.3340	0.1991	$A_3 \succ A_4 \succ A_2 \succ A_1 \succ A_5$
DHq-ROFWDGBM (Sarkar and Biswas, 2021)	0.2438	0.2845	0.2635	0.2819	0.1708	$A_2 \succ A_4 \succ A_3 \succ A_1 \succ A_5$
DHq-ROFAAWA	0.4767	0.6389	0.6218	0.7043	0.5599	$A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$
DHq-ROFAAOWA	0.4531	0.5680	0.5271	0.5675	0.4636	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
DHq-ROFAAWG	0.4488	0.3738	0.5267	0.5307	0.3293	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
DHq-ROFAAOWG	0.4273	0.3411	0.4775	0.4538	0.2945	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
DHq-ROFAAHA	0.4485	0.5448	0.5494	0.6263	0.4608	$A_4 \succ A_3 \succ A_2 \succ A_5 \succ A_1$
DHq-ROFAAHG	0.4129	0.3139	0.4885	0.4884	0.2808	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$

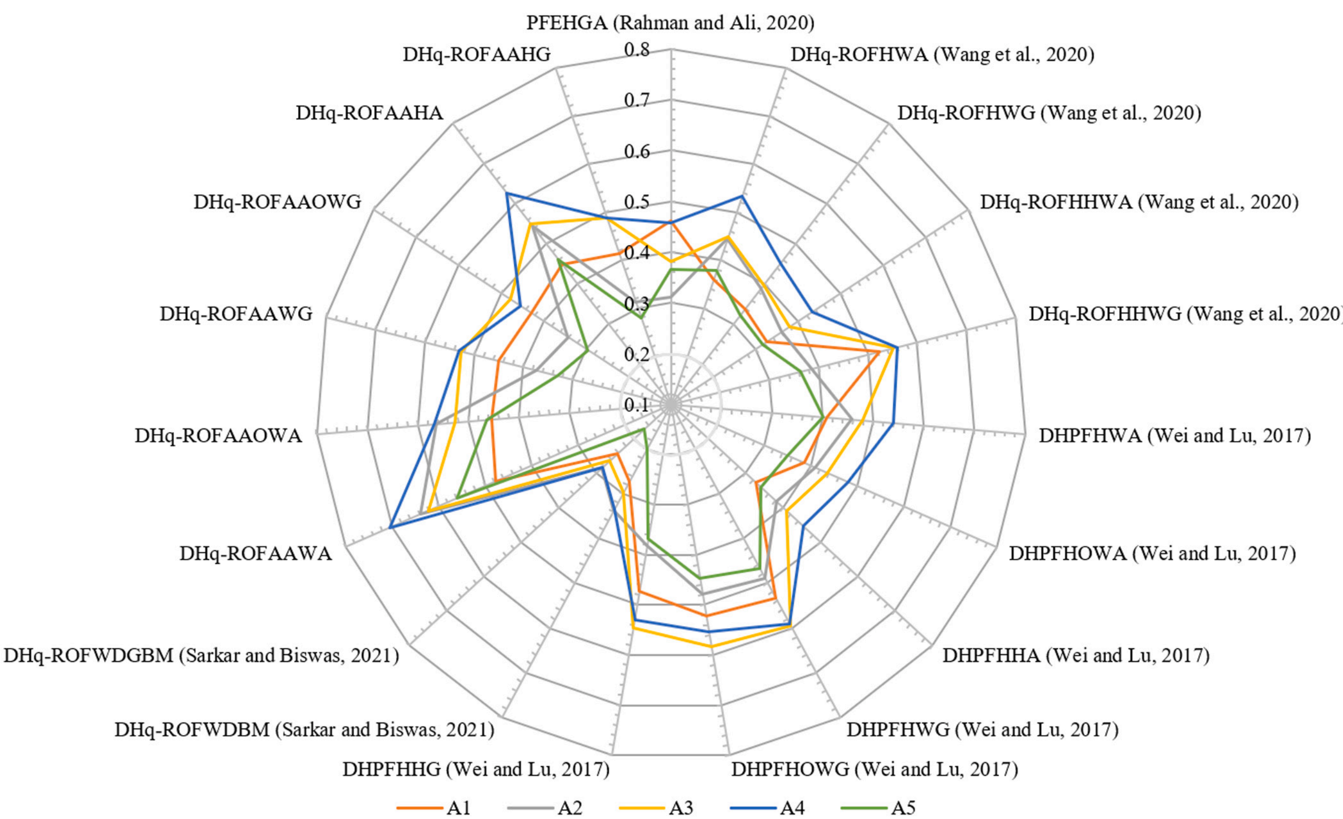

Fig. 12. The radar plot of the results on solving with various existing methods. (The scale of the grid is the scores).

8. Conclusions and future research directions

In this paper, an MCGDM technique is developed based on AAt-CN&t-Ns, DHq-ROFAAWA, DHq-ROFAAWG, DHq-ROFAAOWA, DHq-ROFAAOWG, DHq-ROFAAHA and DHq-ROFAAHG aggregation operators. As mentioned above, the DHq-ROFS is a generalization of the DHFSs, IFSs, and PFSs, as well as the q-ROFSs. So, DHq-ROFS contains more information (both membership and non-membership degrees) than the DHFSs, IFSs, PFSs and q-ROFSs.

The utilization of the developed hybrid operators offers a significant advantage in that they simultaneously consider both the weight of all DHq-ROF arguments and their ordered positions. This combined consideration allows for a more comprehensive and accurate aggregation of the information.

Additionally, by altering the Aczel-Alsina parameter, τ, all generalized cases employing AAt-CN&t-Ns are taken into account. These operators can therefore be utilized to solve location selection problems more effectively because they are far more reliable compared to other current aggregation operators on those sets. The suggested operators can capture human hesitation and the relationship among integrated arguments; in addition, the proposed methods may spontaneously adjust the parameter's value relying on the decision-maker's risk tolerance levels.

One of the best ways to solve the problem of last-mile delivery is to use parcel lockers, which can be accessed 24/7. However, they should be placed in easy-to-reach places. In Dublin, most post offices are located in densely populated neighborhoods, and the obtained outcomes illustrated the post offices as the best location for locating parcel lockers.

The example of choosing the location of the selected parcel lockers shows that our model is correct and has a scientific basis. At the end, it has also been established by comparison with existing operators and methods that the proposed operators are more efficient for solving MCGDM problems.

The fact that the unknown weights of the DMs or criteria are not taken into account is a limitation of the proposed study. In the future, there is a possibility of developing a novel model that can address the limitation by incorporating an unknown weight approach within the DHq-ROF environment. Also, consensus of the group decision making has not been considered in this study. It would be considered in future. As potential expansion of the developed method, future studies could explore the following aspects: the proposed operators can be developed for group decision-making with complete or incomplete probabilistic linguistic preference relations scenarios (Liu et al., 2021a, 2023a; Wang et al., 2021), multi-criteria large-scale group decision making (Liu et al., 2022, 2023b,c). Also, hybrid aggregation operators would be established in various fuzzy sets viz., hesitant picture fuzzy sets, hesitant T-spherical fuzzy sets, linguistic DHq-ROFSs, Linear Diophantine fuzzy sets, Spherical linear Diophantine fuzzy sets, etc.

CRediT authorship contribution statement

Souvik Gayen: Methodology, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Animesh Biswas:** Investigation, Writing – original draft, Writing – review & editing. **Arun Sarkar:** Conceptualization, Visualization, Investigation, Writing – original draft, Writing – review & editing. **Tapan Senapati:** Methodology, Software, Validation, Writing – original draft, Writing – review & editing. **Sarbast Moslem:** Case study selection, Investigation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data is available in the manuscript.

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Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Original paper

A hybridized correlation coefficient technique and its application in classification process under intuitionistic fuzzy setting

P. A. Ejegwa[1], C. F. Ajogwu[2] and A. Sarkar[3]

[1,2] *Department of Mathematics, University of Agriculture, P.M.B. 2373, Makurdi, Nigeria*
[3] *Department of Mathematics, Heramba Chandra College, Kolkata - 700029, India*

ejegwa.augustine@uam.edu.ng, itzfranciscachinelo9@gmail.com, as5mathku17@klyuniv.ac.in

Abstract

Intuitionistic fuzzy set (IFS) is a reliable device for resolving uncertainty and haziness encountered in decision-making process. In most cases, the significance of IFSs are explored based on correlation measures in myriad of areas like in engineering, image segmentation, pattern recognition, diagnostic analysis, etc. Some methods for computing intuitionistic fuzzy correlation coefficient (IFCC) have been investigated, however with some inadequacies. In this present work, a new method of IFCC is developed to correct the drawbacks in some existing techniques in terms of mathematical presentation and the exclusion of the hesitation parameter to enhance reasonable output. A comparative analysis is presented to ascertain the edge of the new technique over some similar approaches. In addition, the new correlation coefficient technique is applied to discuss some pattern recognition problems. This new IFCC method could be investigated based on spherical fuzzy data, q-rung orthopair fuzzy data, and picture fuzzy data.

Keywords: Correlation measure, intuitionistic fuzzy sets, decision-making, pattern recognition.

1 Introduction

Pattern recognition has to do with the grouping of data based on an already gained knowledge for the purpose of inference. Pattern recognition is the art of categorizing patterns based on machine learning algorithm. Most often, pattern recognition process is enmeshed with uncertainties, which justifies the use of soft computing approach of IFSs [1]. IFS expands the sphere of fuzzy set [55] by including non-membership degree with the likelihood of hesitation margin to the membership degree of fuzzy set, and thereby enlarges the scope of fuzzy set to enhance its participation as a reliable soft computing tool in decision-making, pattern recognition, etc. Because of the practicality of IFS, the construct has been applied in medical diagnosis based on composite relation [9], distance measures [8], and similarity measures [33, 41, 45]. IFSs have been applied in numerous areas namely; career determination [15], decision-making [13, 37], etc. Some decision making approaches have been discussed based on intuitionistic fuzzy information [7, 26, 43, 47], and the concept of time series forecasting was discussed under intuitionistic fuzzy domain [39].

Many researchers have discussed the application of IFSs in pattern recognition using various soft computing tools. Some novel approaches for the calculation of similarity between IFSs were discussed and applied to pattern recognition [10, 36, 54]. In [32], a pattern recognition problem was discussed based on some new construction for similarity measures between IFSs, and Boran and Akay [3] presented a two-parametric similarity measure on IFSs and discussed its applications in the problems of pattern recognition. In [4], a new approach of calculation similarity between IFSs were discussed based on transformation techniques with pattern recognition application. Similarly, the idea of pattern recognition has been discussed based on distance measure using intuitionistic fuzzy information [27, 50]. Some measuring association tools between two fuzzy random variables with applications have discussed [42, 44].

In recent time, the idea of IFSs has been discussed in the education sphere [6, 34] and medical domain [16, 19, 35], respectively. Duan and Li [11] constructed intuitionistic similarities using implication operator and the corresponding

Corresponding Author: P. A. Ejegwa

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logical metric spaces with application to solving a pattern recognition problem. In [21], some intuitionistic fuzzy distances were constructed and applied in decision making, and an application of IFS-TOPSIS on the level assessment of the surrounding socks was discussed [25]. In [14], an improved intuitionistic fuzzy similarity operator was constructed and used to discuss pattern recognition and management of emergency. Some applications of IFSs were discussed in [5, 20, 58, 56].

Correlation analysis is a statistical technique used to assess the strength of association between two, numerically continuous variables. This kind of analysis is deployed whenever a researcher wants to investigate whether there are possible relations between two variables. Similarly, the statistical measure that computes the strength of the association of two variables is called correlation coefficient. The construct of correlation analysis has been encapsulated with intuitionistic fuzzy information to enhance the applicability of IFSs in real life problems [24]. Intuitionistic fuzzy correlation analysis has been studied in probability spaces [28]. Hung [30] studied IFCC from statistical perspective, and the idea of IFCC based on centroid method has been studied [31].

Xu [51] introduced a new IFCC approach and applied the concept to disease diagnosis. The approach in [51] was modified by including the complete convention parameters of IFSs to boost accuracy [52]. Because of the drawbacks in the approaches in [51, 52], Huang and Guo [29] introduced a robust approach, however by considering only two parameters of IFSs. In [40, 46], the approach in [30] was independently modified by the inclusion of hesitation margin to avoid error of omission. Similarly, Zeng and Li [57] developed an intuitionistic fuzzy correlation coefficient approach which modified [24] by the inclusion of hesitation margin. Similar approaches of computing IFCC were studied in [53]. Some statistical approaches of computing IFCC based on variance and covariance have been studied and applied in cases of decision-making [12, 38, 48, 49]. In [17, 18], some new approaches of IFCC were computed based on JAVA computer programming. Certain correlation coefficient approaches based on connection number of set pair analysis and TOPSIS method with applications to decision-making problems have been discussed [22, 23]. The motivation of this paper is informed by the following:

- The IFCC methods in [24, 53, 57] lack the ability to compute the correlation coefficient between some IFSs like $\mathcal{A}_1 = \{\langle x_1, 1, 0\rangle, \langle x_2, 0, 0.3\rangle\}$ and $\mathcal{A}_2 = \{\langle x_1, 0, 0.3\rangle, \langle x_2, 1, 0\rangle\}$ in $X = \{x_1, x_2\}$.
- The IFCC methods in [51, 52] yield $0/0$, which is mathematically undefined whenever the IFSs are equal. Normally, the correlation coefficient of equal IFSs should be 1. Also, the approaches yield a perfect correlation coefficient even when the IFSs are not equal.
- The IFCC method in [29] does not include the definitive parameters of IFSs and so its result cannot be trusted.

In this work, we develop an efficient method to compute IFCC. This is obtained by the inclusion of the hesitation margin and the number of the parameters of IFSs to enhance reliability unlike the approach in [29]. This study seeks to hybridize the IFCC approaches in [29, 51, 52] to birth a new approach with reliable accuracy, reasonable interpretation, sound mathematical correctness, and in order to avoid error of omission, the approach includes the complete parameters of IFSs. The new approach is a hybridized method of the approaches in [29, 51, 52] because it crossbred the existing approaches with an enhanced performance and interpretation by

- extending the approach in [29] through the inclusion of hesitation margin and parametric number of IFSs, and
- employing parametric absolute difference, minimum parametric absolute difference, and maximum parametric absolute difference, respectively as seen in [29, 51, 52].

In this present study, we;

(i) reiterate and appraise the IFCC approaches in [29, 51, 52] to pinpoint their drawbacks.

(ii) develop a hybridized IFCC approach with reliable output, reasonable interpretation, mathematical correctness, and inclusive of the complete parameters of IFSs.

(iii) apply the hybridized IFCC approach in pattern recognition analysis of mineral fields and building materials.

(iv) present a comparative analysis between the hybridized IFCC approach and the obtainable techniques.

The organization of the paper is as follows: Section 2 discusses the preliminaries of IFSs and some existing IFCC approaches with the highlights of drawbacks of the existing IFCC approaches; Section 3 introduces the hybridized IFCC approach, characterizes some of its properties, presents its computational processes; Section 4 discusses pattern recognition in terms of the classifications of mineral fields and building materials; and Section 5 summarises the findings of the paper and gives recommendations for further research.

2 Intuitionistic fuzzy sets and their correlation measures

In this section, the fundamentals of IFSs are recalled for reference. Afterward, some existing IFCC approaches are enlisted and their limitations itemized to justify the development of a new IFCC method.

2.1 Preliminaries on IFSs

We take X as the universe of discourse in this work. Firstly, we reiterate the definition of fuzzy set as follows.

Definition 2.1. [55] *A fuzzy set represented by F in X is defined by*

$$F = \{\langle x, \alpha_F(x)\rangle \mid x \in X\},$$

where $\alpha_F(x)$ is a function $\alpha_F : X \to [0,1]$, which explains the degree of membership of $x \in X$.

Definition 2.2. [1] *An intuitionistic fuzzy set represented by L in X is of the form*

$$L = \{\langle x, \alpha_L(x), \beta_L(x)\rangle \mid x \in X\},$$

where $\alpha_L(x)$ and $\beta_L(x)$ are defined by the functions $\alpha_L : X \to [0,1]$ and $\beta_L : X \to [0,1]$, to describe the degrees of membership and non-membership of $x \in X$ with the property, $0 \le \alpha_L(x) + \beta_L(x) \le 1$.

The hesitation margin of an IFS L in X is defined by $\gamma_L(x) = 1 - \alpha_L(x) - \beta_L(x)$. The hesitation margin expresses the knowledge of the degree to whether $x \in X$ or $x \notin X$.

Definition 2.3. [1] *Given that L and M are IFSs in X, we define the following properties of IFSs:*

(i) *Complement;* $\overline{L} = \{\langle x, \beta_L(x), \alpha_L(x)\rangle | x \in X\}$, $\overline{M} = \{\langle x, \beta_M(x), \alpha_M(x)\rangle | x \in X\}$.

(ii) *Union;* $L \cup M = \{\langle x, \max\{\alpha_L(x), \alpha_M(x)\}, \min\{\beta_L(x), \beta_M(x)\}\rangle | x \in X\}$.

(iii) *Intersection;* $L \cap M = \{\langle x, \min\{\alpha_L(x), \alpha_M(x)\}, \max\{\beta_L(x), \beta_M(x)\}\rangle | x \in X\}$.

(iv) *Equality;* $L = M$ *iff* $\alpha_L(x) = \alpha_M(x)$ *and* $\beta_L(x) = \beta_M(x)$ *for all* $x \in X$.

(v) *Inclusion;* $L \subseteq M$ *iff* $\alpha_L(x) \le \alpha_M(x)$ *and* $\beta_L(x) \ge \beta_M(x)$ *for all* $x \in X$.

Definition 2.4. [2] *Intuitionistic fuzzy values (IFVs) are ordered pairs of the form $\langle l, m\rangle$ with the property $l + m \le 1$, where $l, m \in [0,1]$. In the IFVs $\langle l, m\rangle$, l represents the degree of membership, and m represents the degree of nonmembership, respectively.*

2.2 Some correlation coefficients of IFSs

Some existing approaches of finding correlation coefficient for IFSs are reiterated before the introduction of the new approach.

Definition 2.5. [24] *If L and M are IFSs in X, then the coefficient of correlation between L and M denoted by $\rho(L, M)$ is a function ρ: $IFS \times IFS \to [0,1]$ such that the following conditions hold:*

(i) $0 \le \rho(L, M) \le 1$,

(ii) $\rho(L, M) = 1$ *iff* $L = M$,

(iii) $\rho(L, M) = \rho(L, M)$.

When $\rho(L, M)$ reaches 1, it shows that L and M have strong correlation. Again, if $\rho(L, M)$ reaches 0 then L and M have weak correlation. But if $\rho(L, M) = 0$ then L and M have no correlation. Hence, greater correlation coefficient shows better performance rating.

Now, for IFSs L and M in $X = \{x_1, \cdots, x_n\}$ where $n < \infty$, the following approaches are recalled.

2.2.1 Gerstenkon and Manko [24]

The notion of correlation coefficient was first discussed by Gerstenkon and Manko [24], and it is given as follows:

$$\rho_1(L,M) = \frac{\mathcal{C}(L,M)}{\sqrt{\mathcal{T}(L)}\sqrt{\mathcal{T}(M)}}, \tag{1}$$

where

$$\left.\begin{aligned} \mathcal{C}(L,M) &= \sum_{i=1}^{N}\Big(\alpha_L(x_i)\alpha_M(x_i)+\beta_L(x_i)\beta_M(x_i)\Big) \\ \mathcal{T}(L) &= \sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)\Big) \\ \mathcal{T}(M) &= \sum_{i=1}^{N}\Big(\alpha_M^2(x_i)+\beta_M^2(x_i)\Big) \end{aligned}\right\}. \tag{2}$$

The obvious limitation of (1) is the omission of hesitation margin from the computation and the inability to measure the correlation of some IFSs, and thus the output from this approach cannot be trusted.

Example 2.6. *Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are IFSs given by $\mathcal{A}_1 = \{\langle x_1, 1, 0\rangle, \langle x_2, 0, 0.3\rangle\}$ and $\mathcal{A}_2 = \{\langle x_1, 0, 0.3\rangle, \langle x_2, 1, 0\rangle\}$ in $X = \{x_1, x_2\}$.*

Applying (1) we get $\mathcal{C}(\mathcal{A}_1,\mathcal{A}_2) = 0$, $\mathcal{T}(\mathcal{A}_1) = \mathcal{T}(\mathcal{A}_2) = 1.09$, and so $\rho_1(\mathcal{A}_1,\mathcal{A}_2) = \frac{0}{\sqrt{1.09 \times 1.09}} = 0$. Clearly, this output is a misinformation of the correlation between $\mathcal{A}_1$ and $\mathcal{A}_2$.

2.2.2 Zeng and Li [57]

By considering the limitation in the approach of Gerstenkon and Manko [24], a new approach was introduced by Zeng and Li [57] taking into account hesitation margin as seen in (3).

$$\rho_2(L,M) = \frac{\mathcal{C}(L,M)}{\sqrt{\mathcal{T}(L)}\sqrt{\mathcal{T}(M)}}, \tag{3}$$

where

$$\left.\begin{aligned} \mathcal{C}(L,M) &= \frac{\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)\alpha_M(x_i)+\beta_L(x_i)\beta_M(x_i)+\gamma_L(x_i)\gamma_M(x_i)\Big)}{N} \\ \mathcal{T}(L) &= \frac{\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)+\gamma_L^2(x_i)\Big)}{N} \\ \mathcal{T}(M) &= \frac{\sum_{i=1}^{N}\Big(\alpha_M^2(x_i)+\beta_M^2(x_i)+\gamma_M^2(x_i)\Big)}{N} \end{aligned}\right\}. \tag{4}$$

The limitation of (3) is the inability to measure the correlation of some IFSs. Applying (3) to Example 2.6, we get $\mathcal{C}(\mathcal{A}_1,\mathcal{A}_2) = \frac{0}{2} = 0$, $\mathcal{T}(\mathcal{A}_1) = \mathcal{T}(\mathcal{A}_2) = \frac{1.58}{2} = 0.79$, and so $\rho_2(\mathcal{A}_1,\mathcal{A}_2) = \frac{0}{\sqrt{0.79 \times 0.79}} = 0$. Similarly, this output is a misinformation of the correlation between $\mathcal{A}_1$ and $\mathcal{A}_2$.

2.2.3 Xu et al. [53]

Three methods of computing correlation coefficient between IFSs were discussed in [53]. The first approach modified the approach in [24], and it is given by

$$\rho_3(L,M) = \frac{\mathcal{C}(L,M)}{\max\Big\{\sqrt{\mathcal{T}(L)}, \sqrt{\mathcal{T}(M)}\Big\}}, \tag{5}$$

where

$$\left.\begin{aligned}\mathcal{C}(L,M)=\sum_{i=1}^{N}\Big(\alpha_L(x_i)\alpha_M(x_i)+\beta_L(x_i)\beta_M(x_i)\Big)\\ \mathcal{T}(L)=\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)\Big)\\ \mathcal{T}(M)=\sum_{i=1}^{N}\Big(\alpha_M^2(x_i)+\beta_M^2(x_i)\Big)\end{aligned}\right\}. \tag{6}$$

Similarly, (5) discards hesitation margin from the computation. Applying (5) to Example 2.6, we get $\mathcal{C}(\mathcal{A}_1,\mathcal{A}_2)=0$, $\mathcal{T}(\mathcal{A}_1)=\mathcal{T}(\mathcal{A}_2)=1.09$, and so $\rho_3(\mathcal{A}_1,\mathcal{A}_2)=\dfrac{0}{\max\{\sqrt{1.09},\sqrt{1.09}\}}=0$. Similarly, this output is a misinformation of the correlation between $\mathcal{A}_1$ and $\mathcal{A}_2$.

The other two approaches in [53] were based on the approach in [24], namely:

$$\rho_4(L,M)=\frac{\mathcal{C}(L,M)}{\max\Big(\sqrt{\mathcal{T}(L)},\sqrt{\mathcal{T}(M)}\Big)}, \tag{7}$$

$$\rho_5(L,M)=\frac{\mathcal{C}(L,M)}{\sqrt{\mathcal{T}(L)}\sqrt{\mathcal{T}(M)}}, \tag{8}$$

where

$$\left.\begin{aligned}\mathcal{C}(L,M)=\sum_{i=1}^{N}\Big(\alpha_L(x_i)\alpha_M(x_i)+\beta_L(x_i)\beta_M(x_i)+\gamma_L(x_i)\gamma_M(x_i)\Big)\\ \mathcal{T}(L)=\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)+\gamma_L^2(x_i)\Big)\\ \mathcal{T}(M)=\sum_{i=1}^{N}\Big(\alpha_M^2(x_i)+\beta_M^2(x_i)+\gamma_M^2(x_i)\Big)\end{aligned}\right\}. \tag{9}$$

By simplification, it is observed that (3) and (8) are equivalent. It is worthy to note that (5) and (7) are not reliable correlation measures because they do not yield perfect correlation coefficient whenever the IFSs are equal. To see this, let us recall the following:

$$\rho_3(L,M)=\frac{\sum_{i=1}^{N}\Big(\alpha_L(x_i)\alpha_M(x_i)+\beta_L(x_i)\beta_M(x_i)\Big)}{\max\Big(\sqrt{\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)\Big)},\sqrt{\sum_{i=1}^{N}\Big(\alpha_M^2(x_i)+\beta_M^2(x_i)\Big)}\Big)},$$

If $L=M$, then

$$\begin{aligned}\rho_4(L,M)&=\frac{\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)\Big)}{\max\Big(\sqrt{\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)\Big)},\sqrt{\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)\Big)}\Big)}\\ &=\frac{\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)\Big)}{\sqrt{\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)\Big)}}\\ &=\sqrt{\sum_{i=1}^{N}\Big(\alpha_L^2(x_i)+\beta_L^2(x_i)\Big)}\\ &\neq 1.\end{aligned}$$

Similarly, $\rho_4(L,M)\neq 1$ if $L=M$.

Applying (7) and (8) to Example 2.6, we get $\mathcal{C}(\mathcal{A}_1,\mathcal{A}_2)=0$, $\mathcal{T}(\mathcal{A}_1)=\mathcal{T}(\mathcal{A}_2)=1.58$, and so $\rho_4(\mathcal{A}_1,\mathcal{A}_2)=\dfrac{0}{\max\{\sqrt{1.58},\sqrt{1.58}\}}=0$ and $\rho_5(\mathcal{A}_1,\mathcal{A}_2)=\dfrac{0}{\sqrt{1.58\times 1.58}}=0$. Again, these outputs are misinformation of the correlation between $\mathcal{A}_1$ and $\mathcal{A}_2$.

2.2.4 Xu [51]

In [51], an approach for estimating correlation coefficient between IFSs were developed from different perspective. The approach is as follows:

$$\rho_6(L,M)=\frac{1}{2N}\sum_{i=1}^{N}\Big(\frac{\Delta\alpha_{\min}+\Delta\alpha_{\max}}{\Delta\alpha_i+\Delta\alpha_{\max}}+\frac{\Delta\beta_{\min}+\Delta\beta_{\max}}{\Delta\beta_i+\Delta\beta_{\max}}\Big), \tag{10}$$

where

$$\left.\begin{aligned}\Delta\alpha_i=|\alpha_L(x_i)-\alpha_M(x_i)|,\ \Delta\beta_i=|\beta_L(x_i)-\beta_M(x_i)|\\ \Delta\alpha_{\min}=\min_i|\alpha_L(x_i)-\alpha_M(x_i)|,\ \Delta\beta_{\min}=\min_i|\beta_L(x_i)-\beta_M(x_i)|\\ \Delta\alpha_{\max}=\max_i|\alpha_L(x_i)-\alpha_M(x_i)|,\ \Delta\beta_{\max}=\max_i|\beta_L(x_i)-\beta_M(x_i)|\end{aligned}\right\}. \tag{11}$$

The approach lacks reliability due to information loss occasioned by the omission of hesitation margin and its inability to measure the correlation of some IFSs.

Example 2.7. *Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are IFSs given by $\mathcal{B}_1=\{\langle x_1,0.4,0.3\rangle,\langle x_2,0.3,0.2\rangle\}$ and $\mathcal{B}_2=\{\langle x_1,0.3,0.2\rangle,\langle x_2,0.2,0.1\rangle\}$ in $X=\{x_1,x_2\}$.*

Applying (10), we get the correlation coefficient using the information in Table 1.

Table 1: Computational Process

X	$\Delta\alpha_i$	$\Delta\beta_i$
x_1	0.1	0.1
x_2	0.1	0.1

We see that

$$\Delta\alpha_{\min}=\Delta\alpha_{\max}=0.1,\Delta\beta_{\min}=\Delta\beta_{\max}=0.1.$$

Hence

$$\begin{aligned}\rho_6(\mathcal{B}_1,\mathcal{B}_2)&=\frac{1}{4}\left[\frac{(0.1+0.1)}{(0.1+0.1)}+\frac{(0.1+0.1)}{(0.1+0.1)}+\frac{(0.1+0.1)}{(0.1+0.1)}+\frac{(0.1+0.1)}{(0.1+0.1)}\right]\\&=1.\end{aligned}$$

This result does not corroborate with Definition 2.5, and so it is not reliable.

2.2.5 Xu and Cai [52]

Due to the limitation of the approach in [51], an enhanced correlation measure was envisaged to mitigate the setback and improve reliability. The approach for measuring correlation coefficient in [52] is:

$$\rho_7(L,M)=\frac{1}{3N}\sum_{i=1}^{N}\Big(\frac{\Delta\alpha_{\min}+\Delta\alpha_{\max}}{\Delta\alpha_i+\Delta\alpha_{\max}}+\frac{\Delta\beta_{\min}+\Delta\beta_{\max}}{\Delta\beta_i+\Delta\beta_{\max}}+\frac{\Delta\gamma_{\min}+\Delta\gamma_{\max}}{\Delta\gamma_i+\Delta\gamma_{\max}}\Big), \tag{12}$$

where

$$\left.\begin{aligned}\Delta\alpha_i=|\alpha_L(x_i)-\alpha_M(x_i)|,\ \Delta\beta_i=|\beta_L(x_i)-\beta_M(x_i)|\\ \Delta\gamma_i=|\gamma_L(x_i)-\gamma_M(x_i)|\\ \Delta\alpha_{\min}=\min_i|\alpha_L(x_i)-\alpha_M(x_i)|,\ \Delta\beta_{\min}=\min_i|\beta_L(x_i)-\beta_M(x_i)|\\ \Delta\gamma_{\min}=\min_i|\gamma_L(x_i)-\gamma_M(x_i)|\\ \Delta\alpha_{\max}=\max_i|\alpha_L(x_i)-\alpha_M(x_i)|,\ \Delta\beta_{\max}=\max_i|\beta_L(x_i)-\beta_M(x_i)|\\ \Delta\gamma_{\max}=\max_i|\gamma_L(x_i)-\gamma_M(x_i)|\end{aligned}\right\}. \tag{13}$$

The approach lacks reliability due to its inability to measure the correlation of some IFSs. Applying (12) to Example 2.7, we get the correlation coefficient using the information in Table 2.

Table 2: Computational Process

X	$\Delta\alpha_i$	$\Delta\beta_i$	$\Delta\gamma_i$
x_1	0.1	0.1	0.2
x_2	0.1	0.1	0.2

It follows that

$$\Delta\alpha_{\min} = 0.1,\ \Delta\beta_{\min} = 0.1,\ \Delta\gamma_{\min} = 0.2$$

$$\Delta\alpha_{\max} = 0.1,\ \Delta\beta_{\max} = 0.1,\ \Delta\gamma_{\max} = 0.2.$$

Hence

$$\begin{aligned}\rho_7(\mathcal{B}_1, \mathcal{B}_2) &= \frac{1}{6}\left[\frac{(0.1+0.1)}{(0.1+0.1)} + \frac{(0.1+0.1)}{(0.1+0.1)} + \frac{(0.1+0.1)}{(0.1+0.1)} + \frac{(0.1+0.1)}{(0.1+0.1)} + \frac{(0.2+0.2)}{(0.2+0.2)} + \frac{(0.2+0.2)}{(0.2+0.2)}\right] \\ &= 1.\end{aligned}$$

This result does not corroborate with Definition 2.5 since $\mathcal{B}_1 \neq \mathcal{B}_2$, and so it is not reliable.

2.2.6 Huang and Guo [29]

The approaches in [51, 52] were observed to have some limitations. First, the approaches yield 0/0, which is mathematically undefined whenever the IFSs are equal. Normally, the correlation coefficient of equal IFSs should be 1. Secondly, the approaches yield a perfect correlation coefficient 1 even when the IFSs are not equal. Due to these setbacks, Huang and Guo [29] introduced a novel approach as follows:

$$\rho_8(L, M) = \frac{1}{2N}\sum_{i=1}^{N}\Big(\mu_i(1-\Delta\alpha_i) + \nu_i(1-\Delta\beta_i)\Big), \tag{14}$$

where

$$\left.\begin{aligned}\mu_i &= \frac{c - \Delta\alpha_i - \Delta\alpha_{\max}}{c - \Delta\alpha_{\min} - \Delta\alpha_{\max}} \\ \nu_i &= \frac{c - \Delta\beta_i - \Delta\beta_{\max}}{c - \Delta\beta_{\min} - \Delta\beta_{\max}}\end{aligned}\right\}, \tag{15}$$

for $c > 2$, and

$$\left.\begin{aligned}\Delta\alpha_i = |\alpha_L(x_i) - \alpha_M(x_i)|,\ \Delta\beta_i = |\beta_L(x_i) - \beta_M(x_i)| \\ \Delta\alpha_{\min} = \min_i |\alpha_L(x_i) - \alpha_M(x_i)|,\ \Delta\beta_{\min} = \min_i |\beta_L(x_i) - \beta_M(x_i)| \\ \Delta\alpha_{\max} = \max_i |\alpha_L(x_i) - \alpha_M(x_i)|,\ \Delta\beta_{\max} = \max_i |\beta_L(x_i) - \beta_M(x_i)|\end{aligned}\right\}. \tag{16}$$

One cannot rely on the results from this method because it omits the hesitation margin component in the computation.

Example 2.8. *Suppose* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are IFSs given by* $\mathcal{C}_1 = \{\langle x_1, 0.9, 0.1\rangle, \langle x_2, 0.7, 0.2\rangle\}$ *and* $\mathcal{C}_2 = \{\langle x_1, 0.1, 0.8\rangle, \langle x_2, 0.6, 0.4\rangle\}$ *in* $X = \{x_1, x_2\}$.

Applying (14), we get the correlation coefficient using the information in Table 3.

Table 3: Computational Process

X	$\Delta\alpha_i$	$\Delta\beta_i$
x_1	0.8	0.7
x_2	0.1	0.2

We see that

$$\Delta\alpha_{\min} = 0.1, \Delta\alpha_{\max} = 0.8,$$

$$\Delta\beta_{\min} = 0.2, \Delta\beta_{\max} = 0.7.$$

Thus,

$$\mu_1 = \frac{3-0.8-0.8}{3-0.1-0.8} = 0.6667,\ \nu_1 = \frac{3-0.7-0.7}{3-0.2-0.7} = 0.7619,$$

$$\mu_2 = \frac{3-0.1-0.8}{3-0.1-0.8} = 1,\ \nu_2 = \frac{3-0.2-0.7}{3-0.2-0.7} = 1.$$

Hence

$$\begin{aligned}\rho_8(\mathcal{C}_1, \mathcal{C}_2) &= \frac{1}{4}\Big[0.6667(1-0.8) + (1-0.1) + 0.7619(1-0.7) + (1-0.2)\Big] \\ &= 0.5155.\end{aligned}$$

This result shows that a minimum correlation exists between the IFSs. That is, the correlation coefficient has a low performance index.

3 Hybridized intuitionistic fuzzy correlation coefficient

In this work, we introduce and discuss an efficient method to compute the correlation coefficient for IFSs. This is obtained by the inclusion of the hesitation margin and the number of the parameters of IFSs to enhance reliability unlike the approach in [29]. In fact, this approach hybridizes the intuitionistic fuzzy correlation coefficient approaches in [29, 51, 52]. The new approach is a hybridized method of the approaches in [29, 51, 52] because it crossbred the existing approaches with an enhanced performance and interpretation by

- extending the approach in [29] through the inclusion of hesitation margin and parametric number of IFSs, and
- employing parametric absolute difference, minimum parametric absolute difference, and maximum parametric absolute difference, respectively as seen in [29, 51, 52].

By combining the attributes of the approaches in [29, 51, 52], a new method is developed which resolves the limitations of the approaches in [29, 51, 52].

Assume there are two arbitrary IFSs L and M in $X = \{x_1, \cdots, x_n\}$ where $n < \infty$, then the correlation coefficient for the IFSs can be measured by:

$$\tilde{\rho}(L, M) = \frac{1}{3N}\sum_{i=1}^{N}\Big(\mu_i(1-\Delta\alpha_i) + \nu_i(1-\Delta\beta_i) + \pi_i(1-\Delta\gamma_i)\Big), \tag{17}$$

where

$$\left.\begin{aligned}\mu_i &= \frac{c-\Delta\alpha_i-\Delta\alpha_{\max}}{c-\Delta\alpha_{\min}-\Delta\alpha_{\max}} \\ \nu_i &= \frac{c-\Delta\beta_i-\Delta\beta_{\max}}{c-\Delta\beta_{\min}-\Delta\beta_{\max}} \\ \pi_i &= \frac{c-\Delta\gamma_i-\Delta\gamma_{\max}}{c-\Delta\gamma_{\min}-\Delta\gamma_{\max}}\end{aligned}\right\}, \tag{18}$$

for $c > 2$, and

$$\left.\begin{aligned}\Delta\alpha_i = |\alpha_L(x_i) - \alpha_M(x_i)|,\ \Delta\beta_i = |\beta_L(x_i) - \beta_M(x_i)| \\ \Delta\gamma_i = |\gamma_L(x_i) - \gamma_M(x_i)| \\ \Delta\alpha_{\min} = \min_i |\alpha_L(x_i) - \alpha_M(x_i)|,\ \Delta\beta_{\min} = \min_i |\beta_L(x_i) - \beta_M(x_i)| \\ \Delta\gamma_{\min} = \min_i |\gamma_L(x_i) - \gamma_M(x_i)| \\ \Delta\alpha_{\max} = \max_i |\alpha_L(x_i) - \alpha_M(x_i)|,\ \Delta\beta_{\max} = \max_i |\beta_L(x_i) - \beta_M(x_i)| \\ \Delta\gamma_{\max} = \max_i |\gamma_L(x_i) - \gamma_M(x_i)|\end{aligned}\right\}. \tag{19}$$

In some cases, it is of necessity to consider the weights of elements of X while computing the correlation coefficient. For instance, in multi-attribute decision-making cases, every feature does has dissimilar significant and so needs to be apportioned a dissimilar weight. By considering the weights of the elements of X, (17) becomes

$$\tilde{\rho}_\omega(L,M) = \frac{1}{3}\sum_{i=1}^{N}\omega_i\Big(\mu_i(1-\Delta\alpha_i)+\nu_i(1-\Delta\beta_i)+\pi_i(1-\Delta\gamma_i)\Big), \tag{20}$$

where the parameters are the same as in (18) and (19), and $\omega_i \geq 0$ for $\sum_{i=1}^{N}\omega_i = 1$. If $\omega = \left(\frac{1}{N}, \frac{1}{N}, \cdots, \frac{1}{N}\right)^T$, then (17) and (20) are the same.

3.1 Numerical illustrations of the new IFCC approach

Some examples of IFSs are considered to illustrate the steps involve in the computation of correlation coefficient based on the new approach.

Example 3.1. *Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are IFSs in $X = \{x_1, x_2, x_3\}$ defined by*

$$\mathcal{L}_1 = \{\langle x_1, 0.1, 0.2, 0.7\rangle, \langle x_2, 0.2, 0.1, 0.7\rangle, \langle x_3, 0.1, 0.6, 0.3\rangle\},$$

$$\mathcal{L}_2 = \{\langle x_1, 0.3, 0.0, 0.7\rangle, \langle x_2, 0.2, 0.2, 0.6\rangle, \langle x_3, 0.3, 0.0, 0.7\rangle\}.$$

By mere observation, $\mathcal{L}_1$ and $\mathcal{L}_2$ are related since $\mathcal{L}_1 \subseteq \mathcal{L}_2$. We calculate the correlation coefficient concerning the IFSs via the new approach using the information in Table 4.

Table 4: Computational Process

X	$\Delta\alpha_i$	$\Delta\beta_i$	$\Delta\gamma_i$
x_1	0.2	0.2	0.0
x_2	0.0	0.1	0.1
x_3	0.2	0.6	0.4

where

$$\Delta\alpha_{\min} = 0.0,\ \Delta\beta_{\min} = 0.1,\ \Delta\gamma_{\min} = 0.0,$$

$$\Delta\alpha_{\max} = 0.2,\ \Delta\beta_{\max} = 0.6,\ \Delta\gamma_{\max} = 0.4.$$

Thus

$$\mu_1 = \frac{3-0.2-0.2}{3-0.0-0.2} = 0.9286,\ \nu_1 = \frac{3-0.2-0.6}{3-0.1-0.6} = 0.9565,\ \pi_1 = \frac{3-0.0-0.4}{3-0.0-0.4} = 1,$$

$$\mu_2 = \frac{3-0.0-0.2}{3-0.0-0.2} = 1,\ \nu_2 = \frac{3-0.1-0.6}{3-0.1-0.6} = 1,\ \pi_2 = \frac{3-0.1-0.4}{3-0.0-0.4} = 0.9615,$$

$$\mu_3 = \frac{3-0.2-0.2}{3-0.0-0.2} = 0.9286,\ \nu_3 = \frac{3-0.6-0.6}{3-0.1-0.6} = 0.7826,\ \pi_3 = \frac{3-0.4-0.4}{3-0.0-0.4} = 0.8462.$$

Hence

$$\begin{aligned}\tilde{\rho}(\mathcal{L}_1,\mathcal{L}_2) &= \frac{1}{9}\Big((0.9286\times 0.8)+(0.9565\times 0.8)+(1\times 1)+(1\times 1)+(1\times 0.9)+(0.9615\times 0.9)\\ &+ (0.9286\times 0.8)+(0.7826\times 0.4)+(0.8462\times 0.6)\Big)\\ &= 0.7597.\end{aligned}$$

This result corroborates the relationship between $\mathcal{L}_1$ and $\mathcal{L}_2$.

Example 3.2. *Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are IFSs in $X = \{x_1, x_2, x_3, x_4\}$ defined by*

$$\mathcal{M}_1 = \{\langle x_1, 0.7, 0.2, 0.1\rangle, \langle x_2, 0.6, 0.1, 0.3\rangle, \langle x_4, 0.5, 0.4, 0.1\rangle\},$$

$$\mathcal{M}_2 = \{\langle x_1, 0.8, 0.1, 0.1\rangle, \langle x_2, 0.7, 0.1, 0.2\rangle, \langle x_3, 0.3, 0.4, 0.3\rangle\}.$$

Similarly, we compute the correlation coefficient between the IFSs through the new approach using the information in Table 5.

Table 5: Computational Process

X	$\Delta\alpha_i$	$\Delta\beta_i$	$\Delta\gamma_i$
x_1	0.1	0.1	0.0
x_2	0.1	0.0	0.1
x_3	0.2	0.6	0.3
x_4	0.5	0.6	0.1

where

$$\Delta\alpha_{\min} = 0.1,\ \Delta\beta_{\min} = 0.0,\ \Delta\gamma_{\min} = 0.0,$$

$$\Delta\alpha_{\max} = 0.5,\ \Delta\beta_{\max} = 0.6,\ \Delta\gamma_{\max} = 0.3.$$

So,

$$\mu_1 = \frac{3-0.1-0.5}{3-0.1-0.5} = 1,\ \nu_1 = \frac{3-0.1-0.6}{3-0.0-0.6} = 0.9583,\ \pi_1 = \frac{3-0.0-0.3}{3-0.0-0.3} = 1,$$

$$\mu_2 = \frac{3-0.1-0.5}{3-0.1-0.5} = 1,\ \nu_2 = \frac{3-0.0-0.6}{3-0.0-0.6} = 1,\ \pi_2 = \frac{3-0.1-0.3}{3-0.0-0.3} = 0.963,$$

$$\mu_3 = \frac{3-0.2-0.5}{3-0.1-0.5} = 0.9583,\ \nu_3 = \frac{3-0.6-0.6}{3-0.0-0.6} = 0.75,\ \pi_3 = \frac{3-0.3-0.3}{3-0.0-0.3} = 0.8889,$$

$$\mu_4 = \frac{3-0.5-0.5}{3-0.1-0.5} = 0.8333,\ \nu_4 = \frac{3-0.6-0.6}{3-0.0-0.6} = 0.75,\ \pi_4 = \frac{3-0.1-0.3}{3-0.0-0.3} = 0.963.$$

Hence

$$\begin{aligned}\tilde{\rho}(\mathcal{M}_1, \mathcal{M}_2) &= \frac{1}{12}\Big((1\times 0.9) + (0.9583\times 0.9) + (1\times 1) + (1\times 0.9) + (1\times 1) + (0.963\times 0.9) \\ &+ (0.9583\times 0.8) + (0.75\times 0.4) + (0.8889\times 0.7) + (0.8333\times 0.5) + (0.75\times 0.4) + (0.963\times 0.9)\Big) \\ &= 0.7334,\end{aligned}$$

which interprets the correlation between $\mathcal{M}_1$ and $\mathcal{M}_2$.

3.2 Comparative analysis

The superiority of the new IFCC method over the existing IFCC methods is unveiled by presenting a comparative analysis as follows. By applying the new IFCC method to Example 2.6, we have $\tilde{\rho}(\mathcal{A}_1, \mathcal{A}_2) = 0.3333$, while the IFCC methods in [24, 53, 57] give $\rho_1(\mathcal{A}_1, \mathcal{A}_2) = 0.0$, $\rho_2(\mathcal{A}_1, \mathcal{A}_2) = 0.0$, $\rho_3(\mathcal{A}_1, \mathcal{A}_2) = 0.0$, $\rho_4(\mathcal{A}_1, \mathcal{A}_2) = 0.0$, and $\rho_5(\mathcal{A}_1, \mathcal{A}_2) = 0.0$.

Though the correlation between $\mathcal{A}_1$ and $\mathcal{A}_2$ is weak by mere observation, the IFCC methods in [24, 53, 57] give a misleading interpretation that the correlation does not exist at all. On the contrary, the new IFCC method gives a correlation value that tallies with the mere observation. This proves the advantage of the new IFCC methods over the methods in [24, 53, 57].

By applying the new IFCC method to Example 2.7, we have a correlation coefficient $\tilde{\rho}(\mathcal{B}_1, \mathcal{B}_2) = 0.8667$, while the IFCC methods in [51, 52] give correlation coefficients $\rho_6(\mathcal{B}_1, \mathcal{B}_2) = 1$ and $\rho_7(\mathcal{B}_1, \mathcal{B}_2) = 1$. Of course, a strong correlation exists between $\mathcal{B}_1$ and $\mathcal{B}_2$ but certainly not perfect. Correlation coefficient can only be perfect if $\mathcal{B}_1 = \mathcal{B}_2$. This speaks to the limitation of the IFCC methods in [51, 52]. Again, this proves the advantage of the new IFCC methods over the methods in [51, 52].

Finally, we apply the new IFCC method to Example 2.8, and get a correlation coefficient $\tilde{\rho}(\mathcal{C}_1, \mathcal{C}_2) = 0.6437$. By applying the IFCC method in [29], we have $\rho_8(\mathcal{C}_1, \mathcal{C}_2) = 0.5155$. The new IFCC method is more reliable compare to the IFCC method [29] because it take account of all the parametric definition of the concerned IFSs. It is observed that as the hesitation margin becomes smaller, the new IFCC method yields a result with high performance index compare to the IFCC method in [29], which underscores the limitation of the IFCC method [29] and proves the advantage of the new IFCC methods.

The new approach is an improved version of the method in [29] with high performance rating and reliability because it does not provide any leeway for information loss as seen in [24, 29, 51].

3.3 Some properties of the new IFCC approach

To substantiate the validity of the new approach, we present some of its properties.

Theorem 3.3. *Suppose L and M are two IFSs in X, then the new IFCC $\tilde{\rho}(L, M)$ satisfies the commutative property:*

(i) $\tilde{\rho}(L, M) = \tilde{\rho}(M, L)$,

(ii) $\tilde{\rho}_\omega(L, M) = \tilde{\rho}_\omega(M, L)$.

Proof. To prove (i), we recall that

$$\tilde{\rho}(L, M) = \frac{1}{3N} \sum_{i=1}^{N} \Big(\mu_i(1 - \Delta\alpha_i) + \nu_i(1 - \Delta\beta_i) + \pi_i(1 - \Delta\gamma_i) \Big),$$

and then

$$\begin{aligned}
\tilde{\rho}(L, M) &= \frac{1}{3N} \sum_{i=1}^{N} \bigg(\mu_i\Big(1 - |\alpha_L(x_i) - \alpha_M(x_i)|\Big) + \nu_i\Big(1 - |\beta_L(x_i) - \beta_M(x_i)|\Big) + \pi_i\Big(1 - |\gamma_L(x_i) - \gamma_M(x_i)|\Big) \bigg) \\
&= \frac{1}{3N} \sum_{i=1}^{N} \bigg(\mu_i\Big(1 - |\alpha_M(x_i) - \alpha_L(x_i)|\Big) + \nu_i\Big(1 - |\beta_M(x_i) - \beta_L(x_i)|\Big) + \pi_i\Big(1 - |\gamma_M(x_i) - \gamma_L(x_i)|\Big) \bigg) \\
&= \tilde{\rho}(M, L),
\end{aligned}$$

which proofe (i). The prove of (ii) is similar to (i). □

Theorem 3.4. *If L and M are two IFSs in X, then the new correlation coefficient $\tilde{\rho}(L, M)$ satisfies*

(i) $\tilde{\rho}(L, M) = 1$ *iff* $L = M$,

(ii) $\tilde{\rho}_\omega(L, M) = 1$ *iff* $L = M$.

Proof. First, we establish (i). Suppose $L = M$. Then $|\alpha_L(x_i) - \alpha_M(x_i)| = 0$, $|\beta_L(x_i) - \beta_M(x_i)| = 0$, and $|\gamma_L(x_i) - \gamma_M(x_i)| = 0$. Deductively,

$$\Delta\alpha_i = \Delta\beta_i = \Delta\gamma_i = 0,$$

$$\Delta\alpha_{\min} = \Delta\beta_{\min} = \Delta\gamma_{\min} = 0, \text{ and}$$

$$\Delta\alpha_{\max} = \Delta\beta_{\max} = \Delta\gamma_{\max} = 0.$$

Thus $\mu_i = \nu_i = \pi_i = 1$, and thus $\tilde{\rho}(L, M) = \frac{1}{3N} \sum_{i=1}^{N} 3N = 1$.

Conversely, if $\tilde{\rho}(L, M) = 1$ then L and M have perfect relation, and so $L = M$. Hence, (i) holds. The prove of (ii) is similar to (i). □

Theorem 3.5. *Suppose $\tilde{\rho}(L, M)$ and $\tilde{\rho}_\omega(L, M)$ are correlation coefficients between IFSs L and M in X, then $\tilde{\rho}(L, M) \in [0, 1]$ and $\tilde{\rho}_\omega(L, M) \in [0, 1]$.*

Proof. We need to prove that $0 \le \tilde{\rho}(L, M) \le 1$, i.e. $\tilde{\rho}(L, M) \ge 0$ and $\tilde{\rho}(L, M) \le 1$. Certainly, $\tilde{\rho}(L, M) \ge 0$. Now, we show that $\tilde{\rho}(L, M) \le 1$.

To see this, let us assume that

$$\sum_{i=1}^{N} \Big(\mu_i(1 - \Delta\alpha_i) \Big) = \xi, \ \sum_{i=1}^{N} \Big(\nu_i(1 - \Delta\beta_i) \Big) = \eta,$$

$$\sum_{i=1}^{N} \Big(\pi_i(1 - \Delta\gamma_i) \Big) = \kappa.$$

By Cauchy-Schwarz inequality's principle, we get

$$\begin{aligned}\tilde{\rho}(L,M) &= \frac{1}{3N}\sum_{i=1}^{N}\Big(\mu_i(1-\Delta\alpha_i)+\nu_i(1-\Delta\beta_i)+\pi_i(1-\Delta\gamma_i)\Big)\\ &\leq \frac{\sum_{i=1}^{N}\Big(\mu_i(1-\Delta\alpha_i)\Big)+\sum_{i=1}^{N}\Big(\nu_i(1-\Delta\beta_i)\Big)+\sum_{i=1}^{N}\Big(\pi_i(1-\Delta\gamma_i)\Big)}{3N}\\ &= \frac{\xi+\eta+\kappa}{3N}.\end{aligned}$$

Thus,

$$\begin{aligned}\tilde{\rho}(L,M)-1 &= \frac{\xi+\eta+\kappa}{3N}-1=\frac{\xi+\eta+\kappa-3N}{3N}=-\frac{(3N-\xi-\eta-\kappa)}{3N}\\ &\leq 0,\end{aligned}$$

which implies that $\tilde{\rho}(L,M)\leq 1$. Hence, $\tilde{\rho}(L,M)\in[0,1]$. Similarly, the proof of $\tilde{\rho}_\omega(L,M)\in[0,1]$ follows. □

4 Application examples

We demonstrate the hand-on relevance of the new correlation coefficient approach and the similar approaches [29, 51, 52] in cases of pattern recognition to project the viability of the new approach. To start with, pattern recognition is the art of categorizing patterns based on machine learning algorithm. Pattern recognition has to do with the grouping of data based on an already gained knowledge to aid inference. In most cases, the art of pattern recognition is enmeshed with uncertainties, which justifies the use of soft computing approach of IFCC technique.

To achieve this, we suppose there are known patterns within a sample space and an unknown pattern within the same space that needed to be grouped into any of the similar known pattern using IFCC technique. The correlation concerning the known pattern and the unknown pattern which yields the greatest correlation coefficient value determine the grouping or classification. The intuitionistic fuzzy data presented in [50] is employed for the application discussions.

4.1 Pattern recognition of mineral fields

Table 6: Mineral Fields as IFVs

	Feature Space					
Patterns	s_1	s_2	s_3	s_4	s_5	s_6
$\alpha_{\hat{C}_1}$	0.739	0.033	0.188	0.492	0.020	0.739
$\beta_{\hat{C}_1}$	0.125	0.818	0.626	0.358	0.628	0.125
$\gamma_{\hat{C}_1}$	0.136	0.149	0.186	0.150	0.352	0.136
$\alpha_{\hat{C}_2}$	0.124	0.030	0.048	0.136	0.019	0.300
$\beta_{\hat{C}_2}$	0.665	0.825	0.800	0.648	0.823	0.653
$\gamma_{\hat{C}_2}$	0.211	0.145	0.152	0.216	0.158	0.047
$\alpha_{\hat{C}_3}$	0.449	0.662	1.000	1.000	1.000	1.000
$\beta_{\hat{C}_3}$	0.387	0.298	0.000	0.000	0.000	0.000
$\gamma_{\hat{C}_3}$	0.164	0.040	0.000	0.000	0.000	0.000
$\alpha_{\hat{C}_4}$	0.280	0.521	0.470	0.295	0.188	0.735
$\beta_{\hat{C}_4}$	0.715	0.368	0.423	0.658	0.806	0.118
$\gamma_{\hat{C}_4}$	0.005	0.111	0.107	0.047	0.006	0.147
$\alpha_{\hat{C}_5}$	0.326	1.000	0.182	0.156	0.049	0.675
$\beta_{\hat{C}_5}$	0.452	0.000	0.725	0.765	0.896	0.263
$\gamma_{\hat{C}_5}$	0.222	0.000	0.093	0.079	0.055	0.062
$\alpha_{\hat{M}}$	0.629	0.524	0.210	0.218	0.069	0.658
$\beta_{\hat{M}}$	0.303	0.356	0.689	0.753	0.876	0.256
$\gamma_{\hat{M}}$	0.068	0.120	0.101	0.029	0.055	0.086

Firstly, we think through a case of pattern recognition of certain mineral fields. Given there are five categories of mineral fields which are featured in the content of six minerals, and there is a category of typical hybrid mineral.

We represent the five categories of the typical hybrid mineral by IFSs $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$, $\hat{C}_4$, and $\hat{C}_5$ in the feature space $S = \{s_1, \cdots, s_6\}$. Assuming there is another unclassified category of hybrid mineral $\hat{M}$, then we determine which field this unclassified category of hybrid mineral $\hat{M}$ can be classified with. The IFVs of the mineral fields are given in Table 6.

In order to classify the unknown hybrid mineral $\hat{M}$, we compute its correlation coefficients with each $\hat{C}_i$, for $i = 1, 2, 3, 4, 5$ using the new correlation coefficient approach and similar approaches [29, 51, 52] to obtain the following results:

Our new approach yields;

$$\tilde{\rho}(\hat{C}_1, \hat{M}) = 0.7880,\ \tilde{\rho}(\hat{C}_2, \hat{M}) = 0.7541,\ \tilde{\rho}(\hat{C}_3, \hat{M}) = 0.6121,$$

$$\tilde{\rho}(\hat{C}_4, \hat{M}) = 0.8532,\ \tilde{\rho}(\hat{C}_5, \hat{M}) = 0.8723,$$

which shows that the unknown hybrid mineral $\hat{M}$ can be classified with $\hat{C}_5$ since $\tilde{\rho}(\hat{C}_5, \hat{M})$ is the greatest.
The approach of Xu [51] yields;

$$\rho_6(\hat{C}_1, \hat{M}) = 0.7934,\ \rho_6(\hat{C}_2, \hat{M}) = 0.7602,\ \rho_6(\hat{C}_3, \hat{M}) = 0.7602,$$

$$\rho_6(\hat{C}_4, \hat{M}) = 0.7595,\ \rho_6(\hat{C}_5, \hat{M}) = 0.8455.$$

The approach of Xu and Cai [52] yields;

$$\rho_7(\hat{C}_1, \hat{M}) = 0.8074,\ \rho_7(\hat{C}_2, \hat{M}) = 0.7718,\ \rho_7(\hat{C}_3, \hat{M}) = 0.7596,$$

$$\rho_7(\hat{C}_4, \hat{M}) = 0.7580,\ \rho_7(\hat{C}_5, \hat{M}) = 0.8210.$$

The approach of Huang and Guo [29] yields;

$$\rho_8(\hat{C}_1, \hat{M}) = 0.7480,\ \rho_8(\hat{C}_2, \hat{M}) = 0.6871,\ \rho_8(\hat{C}_3, \hat{M}) = 0.4626,$$

$$\rho_8(\hat{C}_4, \hat{M}) = 0.8016,\ \rho_8(\hat{C}_5, \hat{M}) = 0.8473.$$

Table 7 presents the results of the IFCC values.

Table 7: Results for Mineral Fields Classification

IFCC Methods	$(\hat{C}_1, \hat{M})$	$(\hat{C}_2, \hat{M})$	$(\hat{C}_3, \hat{M})$	$(\hat{C}_4, \hat{M})$	$(\hat{C}_5, \hat{M})$	Classifications
New IFCC	0.7880	0.7541	0.6121	0.8532	0.8723	$\hat{M}$ belongs to $\hat{C}_5$
Xu [51]	0.7934	0.7602	0.7602	0.7595	0.8455	$\hat{M}$ belongs to $\hat{C}_5$
Xu and Cai [52]	0.8074	0.7718	0.7596	0.7580	0.8210	$\hat{M}$ belongs to $\hat{C}_5$
Huang and Guo [29]	0.7480	0.6871	0.4626	0.8016	0.8473	$\hat{M}$ belongs to $\hat{C}_5$

Similarly, the existing IFCC approaches yield the same pattern recognition, but the new approach shows that a better correlation exists between the unknown hybrid mineral $\hat{M}$ and the mineral field $\hat{C}_5$.

4.2 Pattern recognition of building materials

In this second case, a pattern recognition problem regarding the classification/grouping of some building materials is considered. Assuming there are four given classes of building material, which are represented by IFSs $\hat{M}_1$, $\hat{M}_2$, $\hat{M}_3$, and $\hat{M}_4$ in the feature space $S = \{s_1, s_2, \cdots, s_{12}\}$.
Given another kind of unknown building material $\hat{N}$, we seek to associate the unknown pattern $\hat{N}$ with any of the appropriate known patterns based on the intuitionistic fuzzy correlation measures. The intuitionistic fuzzy information of the patterns are presented in Table 8.

Table 8: Building Materials as IFVs

	Feature Space											
Patterns	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}
$\alpha_{\hat{M}_1}$	0.173	0.102	0.530	0.965	0.420	0.008	0.331	1.000	0.215	0.432	0.750	0.432
$\beta_{\hat{M}_1}$	0.524	0.818	0.326	0.008	0.351	0.956	0.512	0.000	0.625	0.534	0.126	0.432
$\gamma_{\hat{M}_1}$	0.303	0.080	0.144	0.027	0.229	0.036	0.157	0.000	0.160	0.034	0.124	0.136
$\alpha_{\hat{M}_2}$	0.510	0.627	1.000	0.125	0.026	0.732	0.556	0.650	1.000	0.145	0.047	0.760
$\beta_{\hat{M}_2}$	0.365	0.125	0.000	0.648	0.823	0.153	0.303	0.267	0.000	0.762	0.923	0.231
$\gamma_{\hat{M}_2}$	0.125	0.248	0.000	0.227	0.151	0.115	0.141	0.083	0.000	0.093	0.030	0.009
$\alpha_{\hat{M}_3}$	0.495	0.603	0.987	0.073	0.037	0.690	0.147	0.213	0.501	1.000	0.324	0.045
$\beta_{\hat{M}_3}$	0.387	0.298	0.006	0.849	0.923	0.268	0.812	0.653	0.284	0.000	0.483	0.912
$\gamma_{\hat{M}_3}$	0.118	0.099	0.007	0.078	0.040	0.042	0.041	0.134	0.215	0.000	0.193	0.043
$\alpha_{\hat{M}_4}$	1.000	1.000	0.857	0.734	0.021	0.076	0.152	0.113	0.489	1.000	0.386	0.028
$\beta_{\hat{M}_4}$	0.000	0.000	0.123	0.158	0.896	0.912	0.712	0.756	0.389	0.000	0.485	0.912
$\gamma_{\hat{M}_4}$	0.000	0.000	0.020	0.108	0.083	0.012	0.136	0.131	0.122	0.000	0.129	0.060
$\alpha_{\hat{N}}$	0.978	0.980	0.798	0.693	0.051	0.123	0.152	0.113	0.494	0.987	0.376	0.012
$\beta_{\hat{N}}$	0.003	0.012	0.132	0.213	0.876	0.756	0.721	0.732	0.368	0.000	0.423	0.897
$\gamma_{\hat{N}}$	0.019	0.008	0.070	0.094	0.073	0.121	0.127	0.155	0.138	0.013	0.201	0.091

To obtain the grouping of the unknown building material $\hat{N}$ with $\hat{M}_i$, for $i = 1, 2, 3, 4$, we calculate its correlation coefficients with each $\hat{M}_i$ using the new approach and similar approaches [29, 51, 52] to get the following results:
New approach yields;

$$\tilde{\rho}(\hat{M}_1, \hat{N}) = 0.6414,\ \tilde{\rho}(\hat{M}_2, \hat{N}) = 0.6118,$$

$$\tilde{\rho}(\hat{M}_3, \hat{N}) = 0.8143,\ \tilde{\rho}(\hat{M}_4, \hat{N}) = 0.9632,$$

which shows that the unknown building material $\hat{N}$ can be associated with building material $\hat{M}_4$ because the correlation coefficient between $(\hat{M}_4, \hat{N})$ is the greatest.
The approach of Xu [51] yields;

$$\rho_6(\hat{M}_1, \hat{N}) = 0.8098,\ \rho_6(\hat{M}_2, \hat{N}) = 0.7030,$$

$$\rho_6(\hat{M}_3, \hat{N}) = 0.8086,\ \rho_6(\hat{M}_4, \hat{N}) = 0.8113.$$

The approach of Xu and Cai [52] yields;

$$\rho_7(\hat{M}_1, \hat{N}) = 0.8195,\ \rho_7(\hat{M}_2, \hat{N}) = 0.7184,$$

$$\rho_7(\hat{M}_3, \hat{N}) = 0.7854,\ \rho_7(\hat{M}_4, \hat{N}) = 0.8289.$$

The approach of Huang and Guo [29] yields;

$$\rho_8(\hat{M}_1, \hat{N}) = 0.5182,\ \rho_8(\hat{M}_2, \hat{N}) = 0.4818,$$

$$\rho_8(\hat{M}_3, \hat{N}) = 0.7550,\ \rho_8(\hat{M}_4, \hat{N}) = 0.9642.$$

Table 9 presents the results of the IFCC values.

Table 9: Results for Classification of Building Materials

IFCC Methods	$(\hat{M}_1, \hat{N})$	$(\hat{M}_2, \hat{N})$	$(\hat{M}_3, \hat{N})$	$(\hat{M}_4, \hat{N})$	Classifications
New IFCC	0.6414	0.6118	0.8143	0.9632	$\hat{N}$ belongs to $\hat{M}_4$
Xu [51]	0.8098	0.7030	0.8086	0.8113	$\hat{N}$ belongs to $\hat{M}_4$
Xu and Cai [52]	0.8195	0.7184	0.7854	0.8289	$\hat{N}$ belongs to $\hat{M}_4$
Huang and Guo [29]	0.5182	0.4818	0.7550	0.9642	$\hat{N}$ belongs to $\hat{M}_4$

From the existing IFCC approaches, it follows that the unknown building material $\hat{N}$ can be associated with building material $\hat{M}_4$, akin to the interpretation from the new approach. In the whole, our approach is better than the approach in [29, 51, 52].

5 Conclusions

In this work, we have developed a new IFCC approach which measures correlation reliably better than the existing approaches [29, 51, 52]. The new approach is an improved version of the method in [29] with a better reliability rating because it does not provide any leeway for information loss unlike the approaches in [24, 29, 51]. The properties of the new approach were discussed, and easy to follow illustrative examples of the approach were provided. By comparative analysis, it has been shown that where the existing approaches fail, the new IFCC approach gives a better measure of correlation. Finally, the new approach was applied to tackle problems of pattern recognition because of its flexibility in decision-making. The following are some of advantages of the new approach; (i) it incorporates the complete parameters of IFSs to avoid error of omission, (ii) it hybridizes the IFCC approaches in [29, 51, 52] with reliable output, reasonable interpretation, and mathematical correctness, (iii) it can correctly measure the correlation between two similar IFSs, and also two equal IFSs unlike [51, 52], (iv) it possesses better performance rating which enhances reliable interpretation than the other tri-parametric approaches in [52, 53, 57]. In future studies, the new approach could be investigated in TOPSIS method, multiple criteria decision-making, and multiple group attributes decision-making based on spherical fuzzy data, q-rung orthopair fuzzy data, and picture fuzzy data.

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A hybrid approach based on dual hesitant q-rung orthopair fuzzy Frank power partitioned Heronian mean aggregation operators for estimating sustainable urban transport solutions

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6 authors, including:

Arun Sarkar

Heramba Chandra College

Sarbast Moslem

University College Dublin

Domokos Esztergár-Kiss

Budapest University of Technology and Economics

Muhammad Akram

University of the Punjab

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A hybrid approach based on dual hesitant q-rung orthopair fuzzy Frank power partitioned Heronian mean aggregation operators for estimating sustainable urban transport solutions

Arun Sarkar[a], Sarbast Moslem[b,c], Domokos Esztergár-Kiss[b], Muhammad Akram[d], LeSheng Jin[e], Tapan Senapati[f,*]

[a] Department of Mathematics, Heramba Chandra College, Kolkata, 700029, India
[b] Department of Transport Technology and Economics, Budapest University of Technology and Economics, 1111, Budapest, Hungary
[c] School of Architecture Planning and Environmental Policy, University College Dublin, D04 V1W8, Belfield, Dublin, Ireland
[d] Department of Mathematics, University of the Punjab, New Campus, Lahore 4590, Pakistan
[e] Business School, Nanjing Normal University, Nanjing, China
[f] School of Mathematics and Statistics, Southwest University, Beibei, Chongqing, 400715, China

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ABSTRACT

Transportation systems are a key part of sustainable development, and they need to be carefully evaluated to show that they have a strong impact on the target area's social, environmental, and economic sustainability. For this reason, involving the developed decision support systems helps to shed light on the users' demand and provide unblemished policy decisions considering the existing situation. The "q-rung orthopair fuzzy set (q-ROFS)" is a generalization of the "intuitionistic fuzzy sets (IFSs)" and "Pythagorean fuzzy sets (PFSs)" that expresses vague and uncertain data more efficiently. In the interim, the notion of "dual hesitant q-rung orthopair fuzzy set (DHq-ROFS)" is presented to account for human hesitancy, which may be more applicable to genuine "multicriteria group decision-making (MCGDM)" situations. The main goal of this study is to address MCGDM problems using Heronian mean (HM) and DHq-ROF data. The first step is to introduce the Frank t-norm and t-conorm-based DHq-ROF HM (DHq-ROFFHM) operator. DHq-ROFFHM's features are next described in depth. In addition, the DHq-ROF Frank weighted HM (DHq-ROFFWHM) operator is presented, which takes into account different degrees of liking for input arguments. The DHq-ROF Frank weighted power partitioned HM model is then used to come up with a way to solve models in MCGDM problems where individual arguments are grouped together and have relationships with each other. A final example shows how the established model can be implemented and how well it works.

1. Introduction

Estimating the quality of service in the urban transportation system is important for making users happier, increasing productivity, and using more profitable methods. There are many ways to measure the service quality of urban transportation, such as how reliable it is, how easy it is to get to, etc. But the goal of the evaluation process in all organizations is to ensure that the system is sustainable and that users are happy with it. This is done by ensuring that the system is well organized and that all users get good, efficient service.

In the works that came before, a variety of models were used to analyze the resource quality of urban transportation networks. The most adopted models for this target are: factor analysis (Jomnonkwao and Ratanavaraha, 2016); structural equation modeling (Eboli and Mazzulla, 2007); SERVQUAL framework (Too and Earl, 2010), "multiple linear regression and logit and cluster analysis (Pina and Torres, 2001)". Đalić et al. (2021) introduced a "novel integrated MCDM-SWOT-TOWS model for the strategic decision analysis of a transportation company". Respectively, decision support models were adopted for estimating and ameliorating the service quality of the public transport system, Gündoğdu et al. (2021) integrated the Picture Fuzzy AHP and linear assignment models to evaluate public transportation service quality in the city of Budapest. Moslem

* Corresponding author at: School of Mathematics and Statistics, Southwest University, Beibei, Chongqing, 400715, China.
E-mail addresses: asarkarmth@gmail.com (A. Sarkar), sarbast.moslem@ucd.ie (S. Moslem), esztergar@mail.bme.hu (D. Esztergár-Kiss), m.akram@pucit.edu.pk (M. Akram), jls1980@163.com (L. Jin), math.tapan@gmail.com (T. Senapati).

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List of Abbreviations

DM	Decision maker
HFS	Hesitant fuzzy set HFS
HM	Heronian mean
IFS	Intuitionistic fuzzy set
MCDM	Multicriteria decision making
MCGDM	Multicriteria group decision-making
PFS	Pythagorean fuzzy set
q-ROF	q-rung orthopair fuzzy
q-ROFN	q-ROF number
q-ROFS	q-rung orthopair fuzzy set
DHFS	Dual hesitant fuzzy set
DHq-ROFN	Dual hesitant q-ROF number
DHq-ROFS	Dual hesitant q-ROF set
PA	Power average
PHM	Power Heronian mean
DHq-ROFDM	DHq-ROF decision matrix
DHq-ROFFPA	DHq-ROF Frank power average
DHq-ROFFPG	DHq-ROF Frank power geometric
DHq-ROFFPWA	DHq-ROF Frank power weighted average
DHq-ROFFPWG	DHq-ROF Frank power weighted geometric
DHq-ROFFPHM	DHq-ROF Frank power Heronian mean
DHq-ROFFPGHM	DHq-ROF Frank power geometric Heronian mean
DHq-ROFFPWHM	DHq-ROF Frank power weighted Heronian mean
DHq-ROFFPWGHM	DHq-ROF Frank power weighted geometric Heronian mean
DHq-ROFFWPPHM	DHq-ROF Frank weighted power partitioned Heronian mean
HPFWA/HPFWG	Hesitant Pythagorean fuzzy weighted average/geometric
DHPFHWA/ DHPFHWG	Dual hesitant Pythagorean fuzzy Hamacher weighted average/geometric
DHPFWBM/ DHPFGWHM	Dual hesitant Pythagorean fuzzy weighted Bonferroni mean/Heronian mean
DHq-ROFWA/ DHq-ROFWG	DHq-ROF weighted average/geometric
DHq-ROFWDBM/ DHq-ROFWDGBM	DHq-ROF weighted Dombi Bonferroni mean/geometric
q-RDHFPWDMSM	q-rung dual hesitant fuzzy power weighted dual Maclaurin symmetric mean
q-RDHFWHM/ q-RDHFWGHM	q-rung dual hesitant fuzzy weighted Heronian mean/geometric
WHPFMSM	Hesitant Pythagorean fuzzy weighted Maclaurin symmetric mean

and Çelikbilek (2020) conducted a combined model of AHP with "Multi Objective Optimization Method by Ratio Analysis (MOORA)" in a grey environment to estimate the service quality of the public transport system in Budapest, Hungary. Tumsekcali et al. (2021) integrated AHP with "Weighted Aggregated Sum Product Assessment (WASPAS)" in an "interval-valued intuitionistic fuzzy (IVIF)" environment to evaluate public transportation service quality in Istanbul, Turkey. Duleba et al. (2021) employed AHP with the interval-valued spherical fuzzy sets to spot the light on the users, non-user citizens' preferences regarding developing the public transport in Mersin, Turkey. Çelikbilek et al. (2022) applied Best Worst Method with AHP and MOORA in a fuzzy environment to detect the most important criteria in the public transportation system of Budapest. Alkharabsheh et al. (2022) spotted the demand for developing the public transport system in Amman, Jordan, by testing AHP in a fuzzy environment.

Because the scenario research we did for putting the new model into place focused on qualities and options for improving urban transportation, it was important to make a model with criteria and possible solutions. Nassereddine and Eskandari (2017), who calculated waiting time, trip time, safety, suitability, and accessibility, conducted one of the most comprehensive analyses of the relevant parameters of urban transportation. Eboli and Mazzulla (2015) shed light on the most important aspects of a public transportation system's dependability and connectivity. Information supply before and during travel has also occurred in several notable works (Felleson and Friman, 2012; Mouwen, 2015). Niknejad et al. (2020) gave a broad overview of smart wearables, talking about things like the state-of-the-art literature, new developments, and problems that are still to come.

On the basis of this research and the publicly available model by Duleba and Moslem (2019), we have combined these criteria and developed a hierarchical model that is depicted in the case study section. In addition to the criteria, upgrade options for public transportation have been identified. The four action plan simulations (Nassereddine and Eskandari, 2017) were based on the work of experts in the field and on what was written about them. Mardani et al. (2016) talked about a systematic evaluation of strategies for making decisions in transportation systems that take many factors into account.

1.1. The motivation for developing DHq-ROF Frank weighted power partitioned HM model

How to ensure efficient service quality and how to inspire individuals and businesses to engage in the decision-making process are crucial questions for policymakers. For this aim, numerous models have been adopted, however, the most tremendous methodologies are MCDM methodologies, such as, the "analytic hierarchy process (AHP)" and "best-worst method (BWM)". These methodologies have been applied in order to specify the weights of the criteria in the decision process, however, the MCDM methodologies were combined with other methods to determine the best alternative for the examined problem. Because of the multifaceted nature of modern challenges in decision-making, a single "decision-maker (DM)" cannot analyze all relevant information for all decision objectives. Thus, numerous decision-making challenges in the real world necessitate "multicriteria group decision-making (MCGDM)".

DMs have difficulty describing the obscure and ambiguous attributes of the decision-making process, which is often complex and uncertain. Consequently, the evaluation values of options offered by DMs are frequently marred by substantial ambiguity and uncertainties. Yager (2013, 2014) suggested the PFS as an efficient technique for representing imprecision and uncertainty. As an expansion of IFSs, PFSs are much more effective and ideal for handling complex fuzzy data. PFSs cannot manage circumstances in which the square sum of membership and non-membership degrees exceeds one, which is their most significant deficiency. Yager (2016) presented a new idea, q-rung orthopair fuzzy sets, in order to successfully manage such situations (q-ROFSs). Please see Fig. 1 for further details. Saha et al. (2022) developed q-ROF improved power weighted aggregation operators (AOs) and associated implementations in MCGDM issues. Mahmood et al. (2022) introduced Choquet-Frank AOs relying on q-ROF values and related implementation in MCDM issues. After q-ROFSs were introduced, researchers began to look into their applications in several directions, such as evaluating and emphasizing sustainable urban transport services (Deveci et al., 2022a), personal mobility in the metaverse with autonomous vehicles (Deveci et al., 2022b), "a comprehensive model for socially responsible rehabilitation of mining sites" (Deveci et al., 2022c), "floating offshore wind farm site selection in Norway" (Deveci et al., 2022d), "safe E-scooter operation alternative prioritization" (Deveci et al., 2022e), for solving renewable energy source selection problems (Krishankumar et al., 2021), to solving green supplier selection problem (Krishankumar et al., 2020), evaluation of renewable energy sources (Krishankumar et al., 2019).

In other terms, DMs frequently describe their decision data with reluctance, and they prefer to employ a collection of single values to represent the "membership degree" and "non-membership degree". Consequently, Zhu et al. (2012) developed "dual hesitant fuzzy sets (DHFSs)", which permit DMs to communicate their assessment data via a collection of possible membership values. Using the concepts of DHFS and q-ROFS, Xu et al. (2018) came up with the concept of "dual hesitant q-ROF (DHq-ROF) sets (DHq-ROFS)". In their study, Naz et al. (2022) showed a numerical simulation they made to help find the right medicine to stop COVID-19 epidemics. Akram et al. (2021) applied these sets to fuzzy graph theory. Nevertheless, in many real-life decision-making difficulties, some scenarios in which the criteria interact with one another are frequently observed. Beliakov et al. (2007) characterized the "Heronian mean (HM)" operator, which has the advantageous characteristic of reflecting the correlation of the aggregated variables, to deal with such cases. In numerous fuzzy contexts, the HM operator was effectively implemented.

The primary objective of this study is to develop a series of HM-based DHq-ROF AOs and investigate some of their intriguing features. It is to be mentioned here that a generalized distance measure has been presented for generating DMs' power weights and attributes' power weights. The created operators are then employed to address MCDM problems in DHq-ROF settings using interactive criteria. In order to show the efficacy of the suggested method, a real-world problem involving the evaluation of the quality of urban transport services in Budapest is studied and resolved.

1.2. Contributions of this study

Due to the ever-increasing complexity of real-world decision-making scenarios, the following aspects must be addressed when developing an effective DHq-ROF information aggregation tool to handle MCGDM problems:

(1) An adverse effect is found on the aggregation result caused by some extreme attribute values provided by a biassed DM. To resolve that issue, Yager (2001) introduced the "power average (PA)" AO, which reduced the effect of unduly low and high arguments. PA reinforces the unreasonable evaluation values by calculating the support measures, and assigning them to produce different power weights. So, PA operators can be used as a good way to get rid of this kind of bias in the assessment process when things are not clear.

(2) In practical MCGDM problems, the attributes are not always independent, i.e., interrelationships between attributes are often needed to be considered. The AOs, having assumed that the arguments to be aggregated would have to be independent, cannot produce exact decision results. Meanwhile, there exist some novel aggregation operations, viz., "Bonferroni mean (BM)" (Bonferroni, 1950), HM (Beliakov et al., 2007), etc., which can consider the correlation between input arguments. Yet, HM possesses more advantages than BM, as it ignores the calculation redundancy and takes the correlation between an attribute and itself into account.

(3) The preferences of DMs usually change dynamically according to their pessimistic or optimistic views towards an evaluated object. The required AOs for MCGDM must be sufficiently comprehensive and adaptable to capture all DMs' preferences while aggregating evaluation values.

Frank's t-norm and t-conorm (Frank, 1979), which are intriguing expansions of Lukasiewicz and probabilistic t-norm and t-conorm, are a class of continuous triangular norms that are both comprehensive and adaptable. Due to the fact that the Frank t-norm and t-conorm incorporate a parameter, they are more adaptable in the procedure of data integration and better suited to modeling logical decision-making situations.

Therefore, with the above discussions in mind, this paper is aimed at defining some Frank t-norm & t-conorm based DHq-ROF PA AOs, viz., DHq-ROF Frank PA (DHq-ROFFPA), DHq-ROF Frank power weighted average (DHq-ROFFPWA), DHq-ROF Frank power geometric (DHq-ROFFPG) and DHq-ROF Frank power weighted geometric (DHq-ROFFPWG) operators. Also, combining PA with HM operators, a series of AOs, viz., DHq-ROF Frank power HM (DHq-ROFFPHM) and DHq-ROF Frank power weighted HM (DHq-ROFFPWHM) operators, are introduced in order to develop an MCGDM approach.

1.3. Organization of the study

In order to achieve the above objectives, this paper is organized as follows: Section 2 introduces some fundamental concepts related to DHq-ROFSs. In Section 3, firstly, Frank t-norm & t-conorm based operational laws of the DHq-ROFNs are defined, and then a generalized distance measure for DH-ROFSs is established. Based on the new operational rules of DHq-ROFNs and PA and HM operators, several AOs, viz., DHq-ROFFPA, DHq-ROFFPWA, DHq-ROFFPHM and DHq-ROFFPWHM; (geometric form) DHq-ROFFPG, DHq-ROFFPWG, DHq-ROFFPGHM and DHq-ROFFPWGHM operators, are introduced, followed by a discussion of their characteristics and unique instances in Section 4. Based on these AOs, Section 5 develops the DHq-ROF Frank weighted power partitioned HM model. Section 6 defines and answers the research-based real-world decision-making challenge. The ranking explanation and sensitivity analysis are covered in Section 7. In Section 8, comparisons are made between the proposed method and other pertinent methods to establish its quality. In Section 9, the conclusion and consequences of the planned study are presented.

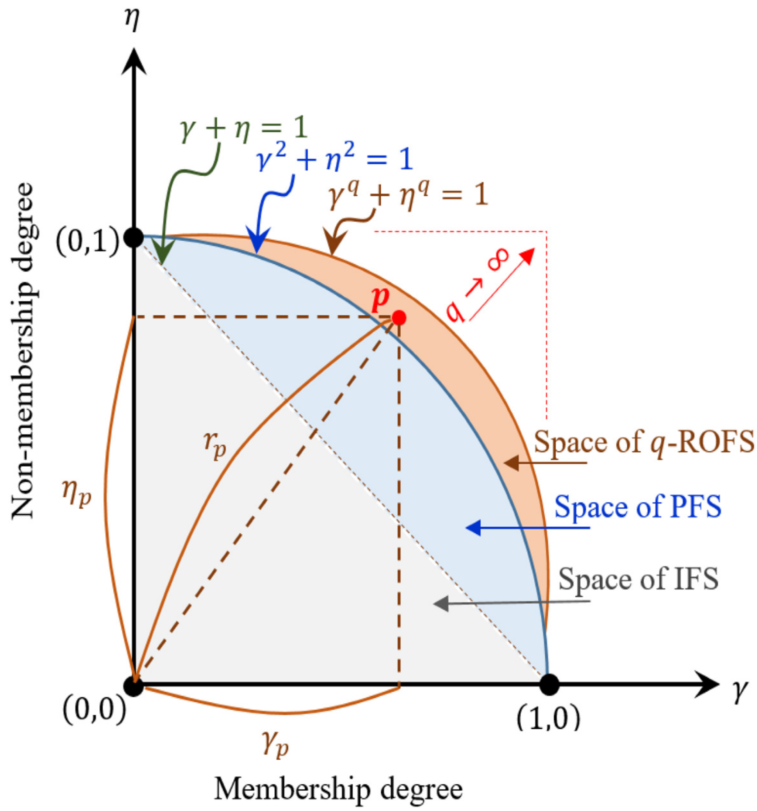

Fig. 1. Comparison of space ranges of IFSs, PFSs, and q-ROFSs.

2. Preliminaries

In this section, the fundamental ideas of q-ROFS and DHq-ROFS (Yager, 2013, 2014) are described briefly. Following this, a novel distance measure for DHq-ROFNs is created.

2.1. q-ROFS

Definition 1 (*Yager, 2016*). Let X be a universe of discourse. A q-ROFS, $\mathfrak{R}$ on X is represented by $\mathfrak{R} = \left\{\left(x, \mu_{\mathfrak{R}}(x), \nu_{\mathfrak{R}}(x)\right) | x \in X\right\}$, where $\mu_{\mathfrak{R}} : X \to [0,1]$ and $\nu_{\mathfrak{R}} : X \to [0,1]$ represent the membership and non-membership functions, respectively, to describe the degree of belongingness of the element $x \in X$ to the set $\mathfrak{R}$, meeting the requirement that

$$0 \leq \left(\mu_{\mathfrak{R}}(x)\right)^q + \left(\nu_{\mathfrak{R}}(x)\right)^q \leq 1, q \geq 1.,$$

The formula for the degree of indeterminacy is

$$\pi_{\mathfrak{R}}(x) = \left[\left(\mu_{\mathfrak{R}}(x)\right)^q + \left(\nu_{\mathfrak{R}}(x)\right)^q - \left(\mu_{\mathfrak{R}}(x)\right)^q \left(\nu_{\mathfrak{R}}(x)\right)^q\right]^{\frac{1}{q}}.$$

For convenience, Yager (2016) designated $\left(\mu_{\mathfrak{R}}(x), \nu_{\mathfrak{R}}(x)\right)$ as a q-ROFN and marked it as $\tilde{r} = (\mu, \nu)$.

2.2. DHq-ROFS

Based on the q-ROFSs (Yager, 2016) and DHFSs (Zhu et al., 2012), Xu et al. (2018) suggested the idea and fundamental operations of the DHq-ROFSs.

Definition 2 (*Xu et al., 2018*). Let X be a fixed set. A DHq-ROFS $\tilde{\mathfrak{D}}$ on X is described as:

$$\tilde{\mathfrak{D}} = \left(\left\langle x, \tilde{h}_{\tilde{\mathfrak{D}}}(x), \tilde{g}_{\tilde{\mathfrak{D}}}(x)\right\rangle | x \in \mathfrak{U}\right), \quad (1)$$

where $\tilde{h}_{\tilde{\mathfrak{D}}}(x)$ and $\tilde{g}_{\tilde{\mathfrak{D}}}(x)$ are two sets of real numbers in $[0,1]$, reflecting the possible membership degrees, $\gamma \in [0,1]$, and non-membership degrees, $\eta \in [0,1]$, respectively, of the element $x \in X$ to the set $\tilde{\mathfrak{D}}$ fulfilling the conditions:

$$0 \leq \left(max_{\gamma \in \tilde{h}_{\tilde{\mathfrak{D}}}(x)}\{\gamma\}\right)^q + \left(max_{\eta \in \tilde{g}_{\tilde{\mathfrak{D}}}(x)}\{\eta\}\right)^q \leq 1. \quad (2)$$

The degree of indeterminacy is given as $\pi_{\tilde{\mathfrak{d}}} = \left(1 - \frac{1}{|\tilde{h}|}\sum_{\gamma \in \tilde{h}} \gamma^q - \frac{1}{|\tilde{g}|}\sum_{\eta \in \tilde{g}} \eta^q\right)^{\frac{1}{q}}$.

For convenience, Xu et al. (2018) called the pair $\tilde{\mathfrak{D}} = \left(\tilde{h}_{\tilde{\mathfrak{D}}}(x), \tilde{g}_{\tilde{\mathfrak{D}}}(x)\right)$ as a DHq-ROF number (DHq-ROFN) denoted by $\tilde{\mathfrak{d}} = \left(\tilde{h}, \tilde{g}\right)$.

Definition 3 (Xu et al., 2018)**.** Let $\tilde{\eth} = \left(\tilde{h}, \tilde{g}\right)$ be a DHq-ROFN. Then the score function $S\left(\tilde{\eth}\right)$ of $\tilde{\eth}$, given by

$$S\left(\tilde{\eth}\right) = \frac{1}{2}\left(1 + \frac{1}{\left|\tilde{h}\right|}\sum_{\gamma\in\tilde{h}}\gamma^q - \frac{1}{|\tilde{g}|}\sum_{\eta\in\tilde{g}}\eta^q\right), \tag{3}$$

and, the accuracy function $A\left(\tilde{\eth}\right)$ of $\tilde{\eth}$, is given by

$$A\left(\tilde{\eth}\right) = \left(\frac{1}{\left|\tilde{h}\right|}\sum_{\gamma\in\tilde{h}}\gamma^q + \frac{1}{|\tilde{g}|}\sum_{\eta\in\tilde{g}}\eta^q\right), \tag{4}$$

where $\left|\tilde{h}\right|$ and $|\tilde{g}|$ are the number of elements in $\tilde{h}$ and $\tilde{g}$, respectively.

The following is how DHq-ROFNs are ordered:

Definition 4 (Xu et al., 2018)**.** Let $\tilde{\eth}_i = \left(\tilde{h}_i, \tilde{g}_i\right)$ $(i = 1, 2)$ be any two DHq-ROFNs,

(i) If $S\left(\tilde{\eth}_1\right) > S\left(\tilde{\eth}_2\right)$, then $\tilde{\eth}_1$ is superior to $\tilde{\eth}_2$, denoted by $\tilde{\eth}_1 \succ \tilde{\eth}_2$;
(ii) If $S\left(\tilde{\eth}_1\right) = S\left(\tilde{\eth}_2\right)$, then

- If $A\left(\tilde{\eth}_1\right) > A\left(\tilde{\eth}_2\right)$, then $\tilde{\eth}_1 \succ \tilde{\eth}_2$;
- If $A\left(\tilde{\eth}_1\right) = A\left(\tilde{\eth}_2\right)$, then $\tilde{\eth}_1$ is equivalent to $\tilde{\eth}_2$, denoted by $\tilde{\eth}_1 \approx \tilde{\eth}_2$.

2.3. Frank operation

Frank operations (Frank, 1979) include the "Frank product and Frank sum", which are examples of "triangular norms and triangular conforms", respectively.

"Frank product $\otimes_F$ is a t-norm and Frank sum $\oplus_F$ is a t-conorm", both of which are classified as follows ($\xi > 1$):

$$x \oplus_F y = 1 - \log_\xi\left(1 + \frac{\left(\xi^{1-x} - 1\right)\left(\xi^{1-y} - 1\right)}{\xi - 1}\right) \quad \forall (x, y) \in [0, 1] \times [0, 1] \tag{5}$$

$$x \otimes_F y = \log_\xi\left(1 + \frac{(\xi^x - 1)(\xi^y - 1)}{\xi - 1}\right) \quad \forall (x, y) \in [0, 1] \times [0, 1] \tag{6}$$

We can reasonably claim some intriguing results using limit theory:

(i) "If $\xi \to 1$, then $x \oplus_F y \to x + y - xy$, $x \otimes_F y \to xy$, the Frank product and Frank sum are simplified to the probabilistic product (product) and probabilistic sum";
(ii) "If $\xi \to \infty$, then $x \oplus_F y \to \min(x + y, 1)$, $x \otimes_F y \to \max(0, x + y - 1)$, the Frank product and Frank sum are simplified to the Lukasiewicz product and Lukasiewicz sum, respectively".

2.4. HM operator

Definition 5 (Beliakov et al., 2007)**.** Let a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$HM\left(a_1, a_2, \ldots, a_n\right) = \frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n}\sqrt{a_i a_j},$$

then $HM\left(a_1, a_2, \ldots, a_n\right)$ is called the Heronian mean (HM).

By implementing two parameters ψ and ϕ, Sýkora (2009) developed the basic HM to a more generalized version:

$$HM_\omega^{\psi,\phi}\left(a_1, a_2, \ldots, a_n\right) = \left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n}\left(\omega_i a_i\right)^\psi\left(\omega_j a_j\right)^\phi\right)^{\frac{1}{\psi+\phi}}.$$

HM can take into account the interdependencies between any two input variables. In many circumstances, input statements can be partitioned into multiple unique kinds, where the members of the same type are interdependent, and the members of the other kinds are independent. The PHM operator, given this context, is suggested by Liu et al. (2018).

2.5. PHM operator

Definition 6 (Liu et al., 2018)**.** Let $\left(a_1, a_2, \ldots, a_n\right)$ be a collection of input arguments, which is partitioned into δ distinct sorts $P_1, P_2, \ldots, P_\delta$, where $P_t = \left\{a_{t_1}, a_{t_2}, \ldots, a_{t_{|P_t|}}\right\}$ $(t = 1, 2, \ldots, \delta)$, $\sum_{t=1}^{\delta}\left|P_t\right| = n$ and $\left|P_t\right|$ denotes the cardinality of P_t. For any, $\psi, \phi \geq 0$, the PHM operator is presented as follows:

$$PHM^{\psi,\varphi}\left(a_1, a_2, \ldots, a_n\right) = \frac{1}{\delta}\left(\sum_{t=1}^{\delta}\left(\frac{2}{\left|P_t\right|\left(\left|P_t\right| + 1\right)}\sum_{\substack{i,j=1\\ i\le j}}^{\left|P_t\right|} a_{t_i}{}^{\psi} a_{t_j}{}^{\phi}\right)\right)^{\frac{1}{\psi+\varphi}} \tag{7}$$

2.6. The power average operator

The power average (PA), which was first made by Yager (2001), is an important AO that helps reduce the negative effects of decision-makers presenting too many or too few arguments. The conventional PA can accumulate a set of discrete values whose weighting vectors rely solely on the input data, and its definition is described in the following:

Definition 7 (*Yager, 2001*)**.** Let $(a_1, a_2, \ldots, a_n)$ be the set of evaluated values, the PA operator is the mapping defined by

$$PA(a_1, a_2, \ldots, a_n) = \sum \frac{(1 + T(a_i))}{\sum_{k=1}^{n} (1 + T(a_k))} a_i,$$

where $T(a_i) = \sum_{\substack{j=1 \\ i \neq j}}^{n} Sup(a_i, a_j)$, $Sup(a_i, a_j) = 1 - d(a_i, a_j)$ and $Sup(a_i, a_j)$ is the support degree for a from b, which satisfies the following three properties:
(1) $Sup(a_i, a_j) \in [0, 1]$;
(2) $Sup(a_i, a_j) = Sup(a_j, a_i)$;
(3) If $(a_i, a_j) \leq d(a_l, a_r)$, then $Sup(a_i, a_j) \geq Sup(a_l, a_r)$, where $d(a_i, a_j)$ represents the distance between a_i and a_j.

3. Frank *t*-norm & *t*-conorm operations on DH*q*-ROFNs

Frank *t*-norms and *t*-conorms, which play an important role in the aggregation of fuzzy numbers, are utilized to resolve a range of decision-making challenges. Many operations that are founded on Frank t-conorms and t-norms in the DHq-ROF context are presented in this section. These operations are specified by the following:

Definition 8. Let $\tilde{\eth}_i = \langle \tilde{h}_i, \tilde{g}_i \rangle$ $(i = 1, 2)$, $\tilde{\eth} = \langle \tilde{h}, \tilde{g} \rangle$ be any three DH*q*-ROFNs, and $\lambda > 0$; then, based on Frank's *t*-conorms and *t*-norms, we derive the following operational rules for the DH*q*-ROFNs ($\zeta > 1$ and $\lambda > 0$):

$$(1)\ \tilde{\eth}_1 \oplus_F \tilde{\eth}_2 = \left\langle \bigcup_{\gamma_i \in \tilde{h}_i, i=1,2} \left\{ \left(1 - \log_\xi \left(1 + \frac{\left(\xi^{1-\gamma_1{}^q} - 1\right)\left(\xi^{1-\gamma_2{}^q} - 1\right)}{\xi - 1}\right)\right)^{\frac{1}{q}} \right\}, \bigcup_{\eta_i \in \tilde{g}_i, i=1,2} \left\{ \left(\log_\xi \left(1 + \frac{\left(\xi^{\eta_1{}^q} - 1\right)\left(\xi^{\eta_2{}^q} - 1\right)}{\xi - 1}\right)\right)^{\frac{1}{q}} \right\} \right\rangle;$$

$$(2)\ \tilde{\eth}_1 \otimes_F \tilde{\eth}_2 = \left\langle \bigcup_{\gamma_i \in \tilde{h}_i, i=1,2} \left\{ \left(\log_\xi \left(1 + \frac{\left(\xi^{\gamma_1{}^q} - 1\right)\left(\xi^{\gamma_2{}^q} - 1\right)}{\xi - 1}\right)\right)^{\frac{1}{q}} \right\}, \bigcup_{\eta_i \in \tilde{g}_i, i=1,2} \left\{ \left(1 - \log_\xi \left(1 + \frac{\left(\xi^{1-\eta_1{}^q} - 1\right)\left(\xi^{1-\eta_2{}^q} - 1\right)}{\xi - 1}\right)\right)^{\frac{1}{q}} \right\} \right\rangle;$$

$$(3)\ \lambda \odot_F \tilde{\eth} = \left\langle \bigcup_{\gamma \in \tilde{h}} \left\{ \left(1 - \log_\xi \left(1 + \frac{\left(\xi^{1-\gamma^q} - 1\right)^\lambda}{(\xi - 1)^{\lambda - 1}}\right)\right)^{\frac{1}{q}} \right\}, \bigcup_{\eta \in \tilde{g}} \left\{ \left(\log_\xi \left(1 + \frac{\left(\xi^{\eta^q} - 1\right)^\lambda}{(\xi - 1)^{\lambda - 1}}\right)\right)^{\frac{1}{q}} \right\} \right\rangle;$$

$$(4)\ \tilde{\eth}^\lambda = \left\langle \bigcup_{\gamma \in \tilde{h}} \left\{ \left(\log_\xi \left(1 + \frac{\left(\xi^{\gamma^q} - 1\right)^\lambda}{(\xi - 1)^{\lambda - 1}}\right)\right)^{\frac{1}{q}} \right\}, \bigcup_{\eta \in \tilde{g}} \left\{ \left(1 - \log_\xi \left(1 + \frac{\left(\xi^{1-\eta^q} - 1\right)^\lambda}{(\xi - 1)^{\lambda - 1}}\right)\right)^{\frac{1}{q}} \right\} \right\rangle.$$

4. Development of DH*q*ROF AOs

In this section, the power AOs are modified to accept DH*q*-ROF data as input. After presenting the idea of distance between DH*q*-ROFNs, we suggest several novel distance measures for DH*q*-ROFNs.

4.1. Distance measure of DHq-ROFNs

Let $\tilde{\eth}_1 = \langle \tilde{h}_1, \tilde{g}_1 \rangle$ and $\tilde{\eth}_2 = \langle \tilde{h}_2, \tilde{g}_2 \rangle$ be any two DH*q*-ROFNs, then the distance measure between $\tilde{\eth}_1$ and $\tilde{\eth}_2$ is defined as $d(\tilde{\eth}_1, \tilde{\eth}_2)$, which satisfies the following properties:

(P1) $0 \leq d(\tilde{\eth}_1, \tilde{\eth}_2) \leq 1$;
(P2) $d(\tilde{\eth}_1, \tilde{\eth}_2) = 0$ if and only if $\tilde{\eth}_1 = \tilde{\eth}_2$;
(P3) $d(\tilde{\eth}_1, \tilde{\eth}_2) = d(\tilde{\eth}_2, \tilde{\eth}_1)$
(P4) Let $\tilde{\eth}_3$ be any DH*q*-ROFN, if $\tilde{\eth}_1 \leq \tilde{\eth}_2 \leq \tilde{\eth}_3$, then $d(\tilde{\eth}_1, \tilde{\eth}_2) \leq d(\tilde{\eth}_1, \tilde{\eth}_3)$ and $d(\tilde{\eth}_2, \tilde{\eth}_3) \leq d(\tilde{\eth}_1, \tilde{\eth}_3)$.

It should be noted that the number of entries in various DH*q*-ROFNs may vary. If we define the number of entries in $h(x)$ as $l_h(d(x)) = \#h$ and the number of entries in $g(x)$ as $l_g(d(x)) = \#g$, then we may write $l(d(x)) = (l_h(d(x)), l_g(d(x))) = (\#h, \#g)$. Let two DH*q*-ROFNs $\tilde{\eth}_1$ and $\tilde{\eth}_2$, in most cases, $l(\tilde{\eth}_1) \neq l(\tilde{\eth}_2)$, i.e., $l_h(\tilde{\eth}_1) \neq l_h(\tilde{\eth}_2)$ and $l_g(\tilde{\eth}_1) \neq l_g(\tilde{\eth}_2)$. For convenience, let $l = l_h + l_g$, where $l_h = \max\{l_h(\tilde{\eth}_1), l_h(\tilde{\eth}_2)\}$; $l_g = \max\{l_g(\tilde{\eth}_1), l_g(\tilde{\eth}_2)\}$. When comparing the two, the smaller one needs to be prolonged until they are both the same length. Repeating the same value several times is the greatest way to lengthen the shorter one. The decision-makers' preferences for risk play a major role in determining this value. Pessimists anticipate negative results and may contribute the least, whilst optimists anticipate beneficial results and may contribute the most. For efficient functioning, we assume that two DH*q*-ROFNs $\tilde{\eth}_1$ and $\tilde{\eth}_2$ have the same length (l_h, l_g). There is also a possibility that the values in a DH*q*-ROFN are not in the correct sequence; nonetheless, we are free to rearrange them in any order we see fit. For a DH*q*-ROFN $\tilde{\eth}$, let $\sigma : (1, 2, \ldots, n) \to (1, 2, \ldots, n)$ be a permutation that satisfies: $\gamma_{\sigma(i)}^{[\tilde{\eth}]} \leq \gamma_{\sigma(i+1)}^{[\tilde{\eth}]}$, $\gamma^{[\tilde{\eth}]} \in h^{[\tilde{\eth}]}$, $i = 1, 2, \ldots, l_h(\tilde{\eth})$; $\eta_{\sigma(i)}^{[\tilde{\eth}]} \leq \eta_{\sigma(i+1)}^{[\tilde{\eth}]}$, $\eta^{[\tilde{\eth}]} \in g^{[\tilde{\eth}]}$, $i = 1, 2, \ldots, l_g(\tilde{\eth})$.

Definition 9. Let two DHq-ROFNs $\tilde{\eth}_1 = \left\langle \tilde{h}_1, \tilde{g}_1 \right\rangle$ and $\tilde{\eth}_2 = \left\langle \tilde{h}_2, \tilde{g}_2 \right\rangle$. Then the distance between $\tilde{\eth}_1$ and $\tilde{\eth}_2$, denoted as $d\left(\tilde{\eth}_1, \tilde{\eth}_2\right)$, is defined as follows:

$$d\left(\tilde{\eth}_1, \tilde{\eth}_2\right) = \left(\frac{1}{l}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon + \sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon\right)\right)^{\frac{1}{\varepsilon}}, \text{ with } \varepsilon > 0. \tag{8}$$

is referred to as the generalized DHq-ROF distance between $\tilde{\eth}_1$ and $\tilde{\eth}_2$.

The following are two particular cases of the generalized DHq-ROF distance $d\left(\tilde{\eth}_1, \tilde{\eth}_2\right)$:

1. If $\varepsilon = 1$, then $d\left(\tilde{\eth}_1, \tilde{\eth}_2\right)$ is transformed to a DHq-ROF Hamming distance

$$d_H\left(\tilde{\alpha}_1, \tilde{\alpha}_2\right) = \frac{1}{l}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right| + \sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|\right)$$

2. if $\varepsilon = 2$, then $d\left(\tilde{\eth}_1, \tilde{\eth}_2\right)$ is transformed to a DHq-ROF Euclidean distance

$$d_E\left(\tilde{\eth}_1, \tilde{\eth}_2\right) = \left(\frac{1}{l}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^2 + \sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^2\right)\right)^{\frac{1}{2}}$$

Next, we shall show that the developed distance measures satisfy axiom definition of distance measure.

Proposition 1. *Let $\tilde{\eth}_1$, $\tilde{\eth}_2$ and $\tilde{\eth}_3$ be any three DHq-ROFNs, then $d\left(\tilde{\eth}_1, \tilde{\eth}_2\right)$ is the distance measure.*

Proof. We prove $d\left(\tilde{\eth}_1, \tilde{\eth}_2\right)$ satisfy axioms (P1)–(P4).

(P1) Let $\tilde{\eth}_1$ and $\tilde{\eth}_2$ be two DHq-ROFNs, then

$$\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon \geq 0 \text{ and } \left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon \geq 0$$

$$\Rightarrow \sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon \geq 0 \text{ and } \sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon \geq 0$$

$$\Rightarrow \left(\frac{1}{l}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon + \sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon\right)\right)^{\frac{1}{\varepsilon}} \geq 0$$

$$\Rightarrow d\left(\tilde{\eth}_1, \tilde{\eth}_2\right) \geq 0.$$

Again from the definition of DHq-ROFS, we have

$$\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon \leq 1 \text{ and } \left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon \leq 1$$

$$\Rightarrow \left(\frac{1}{l}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon + \sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon\right)\right)^{\frac{1}{\varepsilon}} \leq 1$$

$$\Rightarrow d\left(\tilde{\eth}_1, \tilde{\eth}_2\right) \leq 1$$

Hence, $0 \leq d\left(\tilde{\eth}_1, \tilde{\eth}_2\right) \leq 1$.

(P2)

$$d\left(\tilde{\eth}_1, \tilde{\eth}_2\right) = \left(\frac{1}{l}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon + \sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon\right)\right)^{\frac{1}{\varepsilon}}$$

$$= \left(\frac{1}{l}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q\right|^\varepsilon + \sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q\right|^\varepsilon\right)\right)^{\frac{1}{\varepsilon}}$$

$$= d\left(\tilde{\eth}_2, \tilde{\eth}_1\right).$$

(P3) Let $\tilde{\eth}_1 = \tilde{\eth}_2 \Leftrightarrow \gamma_{\sigma(i)}^{[\tilde{\eth}_1]} \leq \gamma_{\sigma(i)}^{[\tilde{\eth}_2]}$ and $\eta_{\sigma(i)}^{[\tilde{\eth}_1]} \leq \eta_{\sigma(i)}^{[\tilde{\eth}_2]}$

$$\Leftrightarrow \left(\frac{1}{l}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon + \sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q - \left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^\varepsilon\right)\right)^{\frac{1}{\varepsilon}} = 0$$

$$d\left(\tilde{\eth}_2, \tilde{\eth}_1\right) = 0$$

(P4) Let $\tilde{\eth}_1 \leq \tilde{\eth}_2 \leq \tilde{\eth}_3 \Rightarrow S\left(\tilde{\eth}_1\right) \leq S\left(\tilde{\eth}_2\right) \leq S\left(\tilde{\eth}_3\right)$

$$\Rightarrow \frac{1}{2}\left(1 + \frac{1}{l_{\tilde{h}_1}}\sum_{\gamma\in\tilde{h}}\left(\gamma^{[\tilde{\eth}_1]}\right)^q - \frac{1}{l_{\tilde{g}_1}}\sum_{\eta\in\tilde{g}}\left(\eta^{[\tilde{\eth}_1]}\right)^q\right) \leq \frac{1}{2}\left(1 + \frac{1}{l_{\tilde{h}_2}}\sum_{\gamma\in\tilde{h}}\left(\gamma^{[\tilde{\eth}_2]}\right)^q - \frac{1}{l_{\tilde{g}_2}}\sum_{\eta\in\tilde{g}}\left(\eta^{[\tilde{\eth}_2]}\right)^q\right) \leq$$

$$\frac{1}{2}\left(1+\frac{1}{l_{\tilde{h}_3}}\sum_{\gamma\in\tilde{h}}\left(\gamma \leq^{[\tilde{\eth}_3]}\right)^q-\frac{1}{l_{\tilde{g}_3}}\sum_{\eta\in\tilde{g}}\left(\eta^{[\tilde{\eth}_3]}\right)^q\right)$$

$$\Rightarrow \frac{1}{l_{\tilde{h}_1}}\sum_{\gamma\in\tilde{h}}\left(\gamma^{[\tilde{\eth}_1]}\right)^q-\frac{1}{l_{\tilde{g}_1}}\sum_{\eta\in\tilde{g}}\left(\eta^{[\tilde{\eth}_1]}\right)^q \leq \frac{1}{l_{\tilde{h}_2}}\sum_{\gamma\in\tilde{h}}\left(\gamma^{[\tilde{\eth}_2]}\right)^q-\frac{1}{l_{\tilde{g}_2}}\sum_{\eta\in\tilde{g}}\left(\eta^{[\tilde{\eth}_2]}\right)^q \leq$$
$$\frac{1}{l_{\tilde{h}_3}}\sum_{\gamma\in\tilde{h}}\left(\gamma^{[\tilde{\eth}_3]}\right)^q-\frac{1}{l_{\tilde{g}_3}}\sum_{\eta\in\tilde{g}}\left(\eta^{[\tilde{\eth}_3]}\right)^q$$
$$-\frac{1}{l_{\tilde{h}_1}}\sum_{\gamma^{[\tilde{\eth}_1]}\in\tilde{h}_1}\left(\gamma^{[\tilde{\eth}_1]}\right)^q+\frac{1}{l_{\tilde{h}_2}}\sum_{\gamma\in\tilde{h}}\left(\gamma^{[\tilde{\eth}_2]}\right)^q+\frac{1}{l_{\tilde{g}_1}}\sum_{\eta^{[\tilde{\eth}_1]}\in\tilde{g}_1}\left(\eta^{[\tilde{\eth}_1]}\right)^q-\frac{1}{l_{\tilde{g}_2}}\sum_{\eta\in\tilde{g}}\left(\eta^{[\tilde{\eth}_2]}\right)^q \leq$$
$$-\frac{1}{l_{\tilde{h}_1}}\sum_{\gamma^{[\tilde{\eth}_1]}\in\tilde{h}_1}\left(\gamma^{[\tilde{\eth}_1]}\right)^q+\frac{1}{l_{\tilde{h}_3}}\sum_{\gamma\in\tilde{h}}\left(\gamma^{[\tilde{\eth}_3]}\right)^q+\frac{1}{l_{\tilde{g}_1}}\sum_{\eta^{[\tilde{\eth}_1]}\in\tilde{g}_1}\left(\eta^{[\tilde{\eth}_1]}\right)^q-\frac{1}{l_{\tilde{g}_3}}\sum_{\eta\in\tilde{g}}\left(\eta^{[\tilde{\eth}_3]}\right)^q$$

$$\Rightarrow\left(\frac{1}{l_{\tilde{\eth}_1\tilde{\eth}_2}}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q-\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^{\varepsilon}+\sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q-\left(\eta_{\sigma(i)}^{[\tilde{\eth}_2]}\right)^q\right|^{\varepsilon}\right)\right)^{\frac{1}{\varepsilon}} \leq$$
$$\left(\frac{1}{l_{\tilde{\eth}_1\tilde{\eth}_3}}\left(\sum_{i=1}^{l_h}\left|\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q-\left(\gamma_{\sigma(i)}^{[\tilde{\eth}_3]}\right)^q\right|^{\varepsilon}+\sum_{i=1}^{l_g}\left|\left(\eta_{\sigma(i)}^{[\tilde{\eth}_1]}\right)^q-\left(\eta_{\sigma(i)}^{[\tilde{\eth}_3]}\right)^q\right|^{\varepsilon}\right)\right)^{\frac{1}{\varepsilon}} \Rightarrow d\left(\tilde{\eth}_1,\tilde{\eth}_2\right)\leq d\left(\tilde{\eth}_1,\tilde{\eth}_3\right).$$

Similarly, we can prove $d\left(\tilde{\eth}_2,\tilde{\eth}_3\right)\leq d\left(\tilde{\eth}_1,\tilde{\eth}_3\right)$.

Hence, $d\left(\tilde{\eth}_1,\tilde{\eth}_2\right)$ is a distance measure.

4.2. DHq-ROFFPWA operator

In this subsection, we apply the PA into DHq-ROFSs and present the DHq-ROFFPWA operator.

Definition 10. Let $\tilde{\eth}_i=\left\langle\tilde{h}_i,\tilde{g}_i\right\rangle$ $(i=1,2,\ldots,n)$ be a collection of DHq-ROFNs. Then the DHq-ROFFPWA operator is given as follows:

$$DHq-ROFFPWA\left(\tilde{\eth}_1,\tilde{\eth}_2,\ldots,\tilde{\eth}_n\right)=\oplus_{F\,i=1}^{\;n}\left(\frac{w_i\left(1+T\left(\tilde{\eth}_i\right)\right)}{\sum_{k=1}^n w_k\left(1+T\left(\tilde{\eth}_k\right)\right)}\tilde{\eth}_i\right)$$
$$=\left\langle\bigcup_{\gamma_i\in\tilde{h}_i}\left\{\left(1-\frac{\log\left(1+\prod_{i=1}^n\left(\xi^{1-\gamma_i{}^q}-1\right)^{\Omega_i}\right)}{\log\xi}\right)^{1/q}\right\},\bigcup_{\eta_i\in\tilde{g}_i}\left\{\left(\frac{\log\left(1+\prod_{i=1}^n\left(\xi^{\eta_i{}^q}-1\right)^{\Omega_i}\right)}{\log\xi}\right)^{1/q}\right\}\right\rangle \tag{9}$$

where $\Omega_i=\frac{w_i(1+T(\tilde{\eth}_i))}{\sum_{k=1}^n w_k(1+T(\tilde{\eth}_k))}$, and $T\left(\tilde{\eth}_i\right)=\sum_{\substack{j=1\\j\neq i}}^n Sup\left(\tilde{\eth}_i,\tilde{\eth}_j\right)$.

To illustrate the applicability of the preceding theorem, the accompanying illustration is examined.

Example 1. Let $P=\{\langle\{0.5,0.6,0.7\},\{0.2,0.3\}\rangle,\langle\{0.7,0.9\},\{0.3,0.4\}\rangle,\langle\{0.5,0.75\},\{0.4,0.5\}\rangle\}$ be a set of DHq-ROFNs, which contains three elements with their corresponding weights 0.25, 0.35 and 0.4, respectively. Now utilize the proposed DHq-ROFFPWA operator to aggregate those three input arguments of P as follows (suppose $q=3$, $\xi=2$):

Let the aggregated value of the elements contained in P is denoted by $\tilde{p}$. Thus

$$DHq-ROFFPWA\left(\tilde{\eth}_1,\tilde{\eth}_2,\tilde{\eth}_3\right)=\oplus_{F\,i=1}^{\;3}\left(\frac{w_i\left(1+T\left(\tilde{\eth}_i\right)\right)}{\sum_{k=1}^3 w_k\left(1+T\left(\tilde{\eth}_k\right)\right)}\tilde{\eth}_i\right)$$
$$=\left\langle\bigcup_{\gamma_i\in\tilde{h}_i}\left\{\left(1-\frac{\log\left(1+\prod_{i=1}^3\left(2^{1-\gamma_i{}^3}-1\right)^{\Omega_i}\right)}{\log 2}\right)^{1/3}\right\},\bigcup_{\eta_i\in\tilde{g}_i}\left\{\left(\frac{\log\left(1+\prod_{i=1}^3\left(2^{\eta_i{}^q}-1\right)^{\Omega_i}\right)}{\log 2}\right)^{1/3}\right\}\right\rangle$$
$$=\left\langle\begin{array}{c}\{0.5752,0.6796,0.6816,0.7682,0.5986,0.6992,0.7011,0.7841,0.6251,0.7214,0.7232,0.8020\},\\\{0.3065,0.3379,0.3379,0.3722,0.3384,0.3727,0.3728,0.4101\}\end{array}\right\rangle.$$

4.3. DHq-ROF Frank weighted power partitioned HM AOs

Definition 11. Let $\tilde{\eth}_i=\left\langle\tilde{h}_i,\tilde{g}_i\right\rangle$ $(i=1,2,\ldots,n)$ be a collection of DHq-ROFNs that is partitioned into δ distinct sorts $P_1,P_2,\ldots,P_\delta$, where $P_t=\left\{\tilde{\eth}_{t_1},\tilde{\eth}_{t_2},\ldots,\tilde{\eth}_{t_{|P_t|}}\right\}$ $(t=1,2,\ldots,\delta)$, $\sum_{t=1}^{\delta}\left|P_t\right|=n$ and $\left|P_t\right|$ denotes the cardinality of P_t. $Sup\left(\tilde{\eth}_i,\tilde{\eth}_j\right)=1-d\left(\tilde{\eth}_i,\tilde{\eth}_j\right)$ is the degree of support for $\tilde{\eth}_i$ from $\tilde{\eth}_j$, where $d\left(\tilde{\eth}_i,\tilde{\eth}_j\right)$ is the distance between $\tilde{\eth}_i$ and $\tilde{\eth}_j$. $Sup\left(\tilde{\eth}_i,\tilde{\eth}_j\right)$ is satisfy (1) $Sup\left(\tilde{\eth}_i,\tilde{\eth}_j\right)\in[0,1]$; (2) $Sup\left(\tilde{\eth}_i,\tilde{\eth}_j\right)=Sup\left(\tilde{\eth}_j,\tilde{\eth}_i\right)$; (3) $Sup\left(\tilde{\eth}_i,\tilde{\eth}_j\right)\geq Sup\left(\tilde{\eth}'_i,\tilde{\eth}'_j\right)$ if $\left|\tilde{\eth}'_i-\tilde{\eth}'_j\right|<\left|\tilde{\eth}_i-\tilde{\eth}_j\right|$. $w=\left(w_1,w_2,\ldots,w_n\right)^T$ be the weights of $\tilde{\eth}_i$ $(i=1,2,\ldots,n)$, $w_i\in[0,1]$, $\sum_{i=1}^n w_i=1$, and $T\left(\tilde{\eth}_i\right)=\sum_{j=1,i\neq j}^n Sup\left(\tilde{\eth}_i,\tilde{\eth}_j\right)$. Clearly, the support (Sup) measure is fundamentally an index of similarity. The greater the similarity and proximity between two values, the greater their mutual support.

The DHq-ROF Frank weighted power partitioned HM (DHq-ROFFWPPHM) operator is defined as follows:

$$DHq-ROFFWPPHM_{w}^{\psi,\varphi}\left(\tilde{\eth}_1,\tilde{\eth}_2,\ldots,\tilde{\eth}_n\right)=$$

$$\frac{1}{\delta}\left(\oplus_{F}{}_{t=1}^{\delta}\left(\frac{2}{|P_t|\left(|P_t|+1\right)}\oplus_{F}{}_{\substack{i,j=1\\i\le j}}^{|P_t|}\left(\left(n\frac{w_{t_i}\left(1+T\left(\tilde{\eth}_{t_i}\right)\right)}{\sum_{v=1}^{n}\left(1+T\left(\tilde{\eth}_v\right)\right)}\tilde{\eth}_{t_i}\right)^{\psi}\otimes_F\left(n\frac{w_{t_j}\left(1+T\left(\tilde{\eth}_{t_j}\right)\right)}{\sum_{v=1}^{n}\left(1+T\left(\tilde{\eth}_v\right)\right)}\tilde{\eth}_{t_j}\right)^{\varphi}\right)\right)\right)^{\frac{1}{\psi+\varphi}} \tag{10}$$

then $DHq-ROFFWPPHM_{w}^{\psi,\varphi}\left(\tilde{\eth}_1,\tilde{\eth}_2,\ldots,\tilde{\eth}_n\right)$ is called the DHq-ROFFWPPHM operator.

To simplify the above expression, let $\varpi_{t_i}=\frac{w_{t_i}\left(1+T\left(\tilde{\eth}_{t_i}\right)\right)}{\sum_{v=1}^{n}w_v\left(1+T\left(\tilde{\eth}_v\right)\right)}$, $\varpi_{t_j}=\frac{w_{t_j}\left(1+T\left(\tilde{\eth}_{t_j}\right)\right)}{\sum_{v=1}^{n}w_v\left(1+T\left(\tilde{\eth}_v\right)\right)}$ then the equation can be written as

$$DHq-ROFFWPPHM_{w}^{p,q}\left(\tilde{\eth}_1,\tilde{\eth}_2,\ldots,\tilde{\eth}_n\right)=$$

$$\frac{1}{\delta}\left(\oplus_{F}{}_{t=1}^{\delta}\left(\frac{2}{|P_t|\left(|P_t|+1\right)}\oplus_{F}{}_{\substack{i,j=1\\i\le j}}^{|P_t|}\left(\left(n\varpi_{t_i}\tilde{\eth}_{t_i}\right)^{\psi}\otimes_F\left(n\varpi_{t_j}\tilde{\eth}_{t_j}\right)^{\varphi}\right)\right)\right)^{\frac{1}{\psi+\varphi}}, \tag{11}$$

where ϖ_i means the power weights of $\tilde{\eth}_i$.

Theorem 1. *Let $\tilde{\eth}_i$ $(i=1,2,\ldots,n)$ be a set of DHq-ROFNs that is partitioned into δ distinct sorts $P_1,P_2,\ldots,P_\delta$, where $P_t=\left\{\tilde{\eth}_{t_1},\tilde{\eth}_{t_2},\ldots,\tilde{\eth}_{t_{|P_t|}}\right\}$ $(t=1,2,\ldots,\delta)$, $\sum_{t=1}^{\delta}|P_t|=n$ and $|P_t|$ denotes the cardinality of P_t. Let $\psi,\varphi>0$ represent any numbers. Consequently, the aggregated value produced by DHq-ROFFWPPHM is also a DHq-ROFN and*

$$DHq-ROFFWPPHM^{\psi,\varphi}\left(\tilde{\eth}_1,\tilde{\eth}_2,\ldots,\tilde{\eth}_n\right)=$$

$$\left\langle\bigcup_{\gamma_{t_i}\in\tilde{h}_{t_i},\gamma_{t_j}\in\tilde{h}_{t_j}}\left\{\left(1-\left(\log\left(\left((\zeta-1)\Big/\left(\frac{\zeta-1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{B}+1\right)}{\log\zeta}}-1}\right)^{\frac{1}{\delta}}\right)+1\right)\Big/\log\zeta\right)\right)^{\frac{1}{q}}\right\},\right.$$

$$\left.\bigcup_{\eta_{t_i}\in\tilde{g}_{t_i},\eta_{t_j}\in\tilde{g}_{t_j}}\left\{\left(\log\left((\zeta-1)\Big/\left(\frac{\zeta-1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{\left(\frac{\zeta-1}{\zeta^{1-D}-1}\right)^{\frac{1}{\psi+\varphi}}}+1\right)}{\log\zeta}}-1}\right)^{\frac{1}{\delta}}+1\right)\Big/\log\zeta\right)^{\frac{1}{q}}\right\}\right\rangle \tag{12}$$

where

$$A=\left((\zeta-1)\Big/\left(\zeta^{\left[1-\frac{\log\left(\frac{\zeta-1}{\left(\frac{\zeta-1}{\zeta^{1-\gamma_{t_i}{}^{q}}-1}\right)^{n\varpi_{t_i}}}+1\right)}{\log\zeta}\right]}-1\right)\right)^{\psi}\left((\zeta-1)\Big/\left(\zeta^{\left[1-\frac{\log\left(\frac{\zeta-1}{\left(\frac{\zeta-1}{\zeta^{1-\gamma_{t_j}{}^{q}}-1}\right)^{n\varpi_{t_j}}}+1\right)}{\log\zeta}\right]}-1\right)\right)^{\phi},$$

$$B=\left((\zeta-1)\Big/\left(\zeta^{\left(1-\left(\log\left(\left((\zeta-1)\Big/\prod_{t=1}^{\delta}\left(\prod_{\substack{i,j=1\\i\le j}}^{|P_t|}\left(\frac{\zeta-1}{\zeta^{1-\frac{\log\left(\frac{\zeta-1}{A}+1\right)}{\log\zeta}}-1}\right)\right)^{\frac{2}{|P_t|\left(|P_t|+1\right)}}\right)+1\right)\Big/\log\zeta\right)\right)}-1\right)\right)^{\frac{1}{\psi+\varphi}}$$

$$C = \left((\zeta - 1) \Big/ \left(\zeta^{1 - \frac{\log\left(\frac{\zeta - 1}{\left(\frac{\zeta - 1}{\zeta^{\eta_i{}^q} - 1} \right)^{n \varpi_{t_i}}} + 1 \right)}{\log \zeta}} - 1 \right) \right)^{\psi} \left((\zeta - 1) \Big/ \left(\zeta^{1 - \frac{\log\left(\frac{\zeta - 1}{\left(\frac{\zeta - 1}{\zeta^{\eta_{t_j}{}^q} - 1} \right)^{n \varpi_{t_j}}} + 1 \right)}{\log \zeta}} - 1 \right) \right)^{\varphi},$$

$$D = \frac{\log\left(\left((\zeta - 1) \Big/ \prod_{t=1}^{\delta} \left(\prod_{\substack{i,j=1 \\ i \le j}}^{|P_t|} \left(\frac{\zeta - 1}{\zeta^{\left(1 - \frac{\log\left(\frac{\zeta - 1}{C} + 1 \right)}{\log \zeta} \right)} - 1} \right) \right)^{\frac{2}{|P_t|(|P_t|+1)}} \right) + 1 \right)}{\log \zeta},$$

where $\varpi_{t_i} = \frac{w_{t_i}\left(1 + T\left(\tilde{\eth}_{t_i}\right)\right)}{\sum_{v=1}^{n} w_v (1 + T(\tilde{\eth}_v))}$, $\varpi_{t_j} = \frac{w_{t_j}\left(1 + T\left(\tilde{\eth}_{t_j}\right)\right)}{\sum_{v=1}^{n} w_v (1 + T(\tilde{\eth}_v))}$.

Proof.

$$\left(n\varpi_{t_i} \tilde{\eth}_i \right)^{\psi} = \left\langle \bigcup_{\gamma_i \in \tilde{h}_i} \left\{ \left(f^{-1}\left(\psi f\left(g^{-1}\left(n\varpi_i g\left(\gamma_i{}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\}, \bigcup_{\eta_i \in \tilde{g}_i} \left\{ \left(g^{-1}\left(\psi g\left(f^{-1}\left(n\varpi_i f\left(\eta_i{}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\} \right\rangle,$$

$$\left(n\varpi_j \tilde{\eth}_j \right)^{\varphi} = \left\langle \bigcup_{\gamma_j \in \tilde{h}_j} \left\{ \left(f^{-1}\left(\varphi f\left(g^{-1}\left(n\varpi_j g\left(\gamma_j{}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\}, \bigcup_{\eta_j \in \tilde{g}_j} \left\{ \left(g^{-1}\left(\varphi g\left(f^{-1}\left(n\varpi_j f\left(\eta_j{}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\} \right\rangle,$$

and then $\left(n\varpi_i \tilde{\eth}_i \right)^{\psi} \otimes_F \left(n\varpi_j \tilde{\eth}_j \right)^{\varphi} =$

$$\left\langle \bigcup_{\gamma_i \in \tilde{h}_i, i=1,2} \left\{ \left(f^{-1}\left(\psi f\left(g^{-1}\left(n\varpi_i g\left(\gamma_i{}^q \right) \right) \right) + \varphi f\left(g^{-1}\left(n\varpi_j g\left(\gamma_j{}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\}, \right.$$

$$\left. \bigcup_{\eta_j \in \tilde{g}_j, i=1,2} \left\{ \left(g^{-1}\left(\psi g\left(f^{-1}\left(n\varpi_i f\left(\eta_i{}^q \right) \right) \right) + \varphi g\left(f^{-1}\left(n\varpi_j f\left(\eta_j{}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\} \right\rangle.$$

By mathematical induction, it can be shown that

$$\oplus_{F}{}_{\substack{i,j=1 \\ i \le j}}^{n} \left(\left(n\varpi_i \tilde{\eth}_i \right)^{\psi} \otimes_F \left(n\varpi_j \tilde{\eth}_j \right)^{\varphi} \right) =$$

$$\left\langle \bigcup_{\gamma_i \in \tilde{h}_i, \gamma_j \in \tilde{h}_j} \left\{ \left(g^{-1}\left(\sum_{\substack{i,j=1 \\ i \le j}}^{n} g\left(f^{-1}\left(\psi f\left(g^{-1}\left(n\varpi_i g\left(\gamma_i{}^q \right) \right) \right) + \varphi f\left(g^{-1}\left(n\varpi_j g\left(\gamma_j{}^q \right) \right) \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\}, \right.$$

$$\left. \bigcup_{\eta_i \in \tilde{g}_i, \eta_j \in \tilde{g}_j} \left\{ \left(f^{-1}\left(\sum_{\substack{i,j=1 \\ i \le j}}^{n} f\left(g^{-1}\left(\psi g\left(f^{-1}\left(n\varpi_i f\left(\eta_i{}^q \right) \right) \right) + \varphi g\left(f^{-1}\left(n\varpi_j f\left(\eta_j{}^q \right) \right) \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\} \right\rangle.$$

Now, $\frac{2}{n(n+1)} \oplus_{F}{}_{\substack{i,j=1 \\ i \le j}}^{n} \left(\left(n\varpi_i \tilde{\eth}_i \right)^{\psi} \otimes_F \left(n\varpi_j \tilde{\eth}_j \right)^{\varphi} \right) =$

$$\left\langle \bigcup_{\gamma_i \in \tilde{h}_i, \gamma_j \in \tilde{h}_j} \left\{ \left(g^{-1}\left(\frac{2}{n(n+1)} \sum_{\substack{i,j=1 \\ i \le j}}^{n} g\left(f^{-1}\left(\psi f\left(g^{-1}\left(n\varpi_i g\left(\gamma_i{}^q \right) \right) \right) + \varphi f\left(g^{-1}\left(n\varpi_j g\left(\gamma_j{}^q \right) \right) \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\}, \right.$$

$$\left. \bigcup_{\eta_i \in \tilde{g}_i, \eta_j \in \tilde{g}_j} \left\{ \left(f^{-1}\left(\frac{2}{n(n+1)} \sum_{\substack{i,j=1 \\ i \le j}}^{n} f\left(g^{-1}\left(\psi g\left(f^{-1}\left(n\varpi_i f\left(\eta_i{}^q \right) \right) \right) + \varphi g\left(f^{-1}\left(n\varpi_j f\left(\eta_j{}^q \right) \right) \right) \right) \right) \right) \right)^{\frac{1}{q}} \right\} \right\rangle.$$

Now,

$$\left(\frac{2}{n(n+1)} \oplus_{F}{}_{\substack{i,j=1 \\ i \le j}}^{n} \left(\left(n\varpi_i \tilde{\eth}_i \right)^{\psi} \otimes_F \left(n\varpi_j \tilde{\eth}_j \right)^{\varphi} \right) \right)^{\frac{1}{p+q}} =$$

$$\left\langle \bigcup_{\gamma_i\in\tilde{h}_i,\gamma_j\in\tilde{h}_j} \left\{ \left(f^{-1}\left(\frac{1}{p+q} f\left(g^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} g\left(f^{-1}\left(\psi f\left(g^{-1}\left(n\varpi_i g\left(\gamma_i{}^q\right)\right)\right)+\varphi f\left(g^{-1}\left(n\varpi_j g\left(\gamma_j{}^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}} \right\},\right.$$

$$\left.\bigcup_{\eta_i\in\tilde{g}_i,\eta_j\in\tilde{g}_j} \left\{ \left(g^{-1}\left(\frac{1}{p+q} g\left(f^{-1}\left(\frac{2}{n(n+1)}\sum_{\substack{i,j=1\\ i\le j}}^{n} f\left(g^{-1}\left(\psi g\left(f^{-1}\left(n\varpi_i f\left(\eta_i{}^q\right)\right)\right)+\varphi g\left(f^{-1}\left(n\varpi_j f\left(\eta_j{}^q\right)\right)\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}} \right\}\right\rangle.$$

Hence the theorem.

5. A novel approach to solving the MCGDM problem using the proposed operators

5.1. Illustration of a major MCGDM issue

Assuming that $A=\{A_1,A_2,\ldots,A_m\}$ is a set of alternatives, MCGDM is the method by which a team of experts selects its most desired alternative consists of a set of attributes $\mathcal{C}=\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_n\}$ by a group of experts $D=\{D_1,D_2,\ldots,D_t\}$. The weight vector that corresponds to attributes is written as $w=\{w_1,w_2,\ldots,w_n\}$ with the condition that $w_i\ge 0$ and $\sum_{i=1}^{n} w_i=1$, and the weight vector that corresponds to experts is written as $\omega=\{\omega_1,\omega_2,\ldots,\omega_t\}$ with the condition that $\omega_k\ge 0$ and $\sum_{k=1}^{t}\omega_k=1$. Because of the ambiguity of data, the attribute value $\mathcal{C}_j$ with regard to alternative A_i as determined by expert D_k is represented as DHq-ROFN $\tilde{\mathfrak{d}}_{ij}=\langle\tilde{h}_{ij},\tilde{g}_{ij}\rangle$, and the DH$q$-ROF decision matrix (DH$q$-ROFDM) $\tilde{R}_k=\left[\tilde{r}_{ij}^k\right]_{m\times n}$ is generated.

Furthermore, depending on the features of attributes, the attributes are separated into d distinct parts $A_1,A_2,\ldots,A_d$, and members of the same part are interrelated, whilst members of different parts are unrelated. Listed below are the steps of the MCGDM approach we have devised for selecting the optimal option.

5.2. The steps of MCGDM method

Step 1. Determine the support in between the DHq-ROFN $\tilde{r}_{ij}^k$ with other DHq-ROFNs $\tilde{r}_{ij}^l$ DHq-ROFN $(k,l=1,2,\ldots,t)$

$$Sup\left(\tilde{r}_{ij}^k,\tilde{r}_{ij}^l\right)=1-d\left(\tilde{r}_{ij}^k,\tilde{r}_{ij}^l\right),(i=1,2,\ldots,m;j=1,2,\ldots,n), \tag{13}$$

where $d\left(\tilde{r}_{ij}^k,\tilde{r}_{ij}^l\right)$ is the distance between DHq-ROFNs $\tilde{r}_{ij}^k$ and $\tilde{r}_{ij}^l$ based on Definition 8.

Step 2. Determine the $T\left(\tilde{r}_{ij}^k\right)$ of the DHq-ROFN $\tilde{r}_{ij}^k$ $(k=1,2,\ldots,t)$.

$$T\left(\tilde{r}_{ij}^k\right)=\sum_{l=1,l\ne k}^{t} Sup\left(\tilde{r}_{ij}^k,\tilde{r}_{ij}^l\right),(i=1,2,\ldots,m;j=1,2,\ldots,n). \tag{14}$$

Step 3. Determine the power weights ϖ_{ij}^k corresponding to the DHq-ROFNs $\tilde{r}_{ij}^k$ $(k=1,2,\ldots,t)$.

$$\varpi_{ij}^k=\frac{\left(1+T\left(\tilde{r}_{ij}^k\right)\right)}{\sum_{k=1}^{t}\left(1+T\left(\tilde{r}_{ij}^k\right)\right)},(i=1,2,\ldots,m;j=1,2,\ldots,n). \tag{15}$$

Step 4. According to the DHq-ROFFPWA operator, aggregate the assessment of attributes C_j reported by decision makers D_k $(k=1,2,\ldots,t)$ for the alternative X_i,

$$\tilde{r}_{ij}=DHq-ROFFPWA\left(\tilde{r}_{ij}^1,\tilde{r}_{ij}^2,\ldots,\tilde{r}_{ij}^t\right), \tag{16}$$

and derive the entire decision matrix $\tilde{R}_k=\left[\tilde{r}_{ij}\right]_{m\times n}$.

Step 5. Determine the support degree $\left(\tilde{r}_{ij},\tilde{r}_{il}\right)$ $(j,l=1,2,\ldots,n)$.

$$Sup\left(\tilde{r}_{ij},\tilde{r}_{il}\right)=1-d\left(\tilde{r}_{ij},\tilde{r}_{il}\right),(i=1,2,\ldots,m). \tag{17}$$

Step 6. Determine the $\left(\tilde{r}_{ij}\right)$ $(i=1,2,\ldots,m)$;

$$T\left(\tilde{r}_{ij}\right)=\sum_{i=1,j\ne l}^{n} Sup\left(\tilde{r}_{ij},\tilde{r}_{il}\right),(j,l=1,2,\ldots,n). \tag{18}$$

Step 7. Determine the power weights ϖ_{ij} corresponding to attribute C_j $(j=1,2,\ldots,n)$,

$$\varpi_{ij}=\frac{\left(1+T\left(\tilde{r}_{ij}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{r}_{ij}\right)\right)},(i=1,2,\ldots,m). \tag{19}$$

Step 8. Determine the overall performance value of alternative X_i $(i=1,2,\ldots,m)$ over all attributes.

$$\tilde{r}_i=DHq-ROFFWPPHM\left(\tilde{r}_{i1},\tilde{r}_{i2},\ldots,\tilde{r}_{in}\right). \tag{20}$$

Step 9. Determine the score function for each alternative, and then rank each alternative using the comparison method described in Section 2.

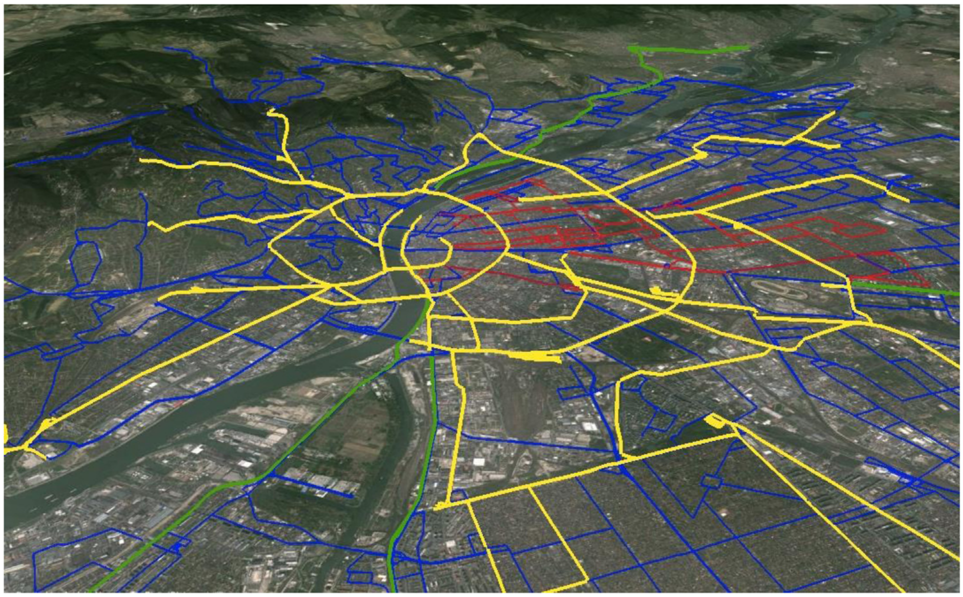

Fig. 2. Aerial view of public transport routes of Budapest, Hungary (Elevation is illustrated with a vertical distortion factor of 2.0.).

Table 1
The main lines of public transport system in Budapest.

No	Route type	Number of routes
1	Daytime bus	240
2	Night-time bus	43
3	Tram	35
4	Trolleybus	16
5	Inner city train	6
6	Subway	4
7	Boat	3

6. Case study

This section sheds light on a real-world case study that was carried out in the capital city of Hungary, Budapest, which has for a very long time been the main focus of the nation as well as an active cultural center. Budapest will have approximately 1,775,000 inhabitants in 2021, with a 0.23% increase from 2020, and this will make Budapest one of the biggest cities in Central Europe. Budapest city has a fairly extensive and efficient urban transport system. It is considered a cheaper transport service than in most Western European cities. The urban transport system consists of five main alternatives: buses, trams, metros, suburban and trolleybuses; all of them use an effective routes (Fig. 2). Where, bus lines are illustrated in blue, trams are illustrated in yellow, trolleys are illustrated in red, and suburban railways are shown in green (the Danube flows approximately from the north (upper right corner) to the south (lower left corner). These five large divisions are operated by BKV "Budapest Transit Company", which is the main urban transport operator in Budapest, Hungary. The company operates the cogwheel railway, the funicular, and the boat service, which are oriented towards tourists. The BKV Company provides transport services to around 1.3 billion commuters a year (BKV Zrt, 2022).

There are more than 347 routes of different urban transport alternatives which generate an easy accessibility for commuters to reach their destinations, Table 1. Illustrate the main routes and their numbers for each alternative (BKV Zrt, 2022).

In our study, we shed light on evaluating the daytime bus system's service quality in order to provide effective and reliable strategies for a sustainable public bus transport system. The daytime bus service itself consists of 240 bus routes and 2611 km of network.

For this aim, we have constructed a hierarchy structure for the service quality (SQ) of the urban bus transport (UBT) system. The hierarchy structure is formed of five main criteria and ten sub-criteria (Fig. 2 show the explanation of each of the criteria). The structure was created based on the points of view of three transportation experts in the Department of Transport Technology and Economics at Budapest University of Technology and Economics in Budapest, Hungary. After several meetings, the experts provided five alternatives (providing new vehicles, providing new lines, modifying timetables, and changing stop locations) which assumed the most suitable solutions for ameliorating the service quality of the public bus transport system in Budapest.

The expert's choice promotes maximizing the objectivity of the proposed solutions. The shown model helps local government officials come up with plans for improving the quality of public bus networks in the future.

The development of public bus transport service quality is related to several criteria. These criteria are defined to cover the real demand in Budapest.

6.1. Definition of criteria of the urban bus transport service quality

The following main criteria are adopted as follows:

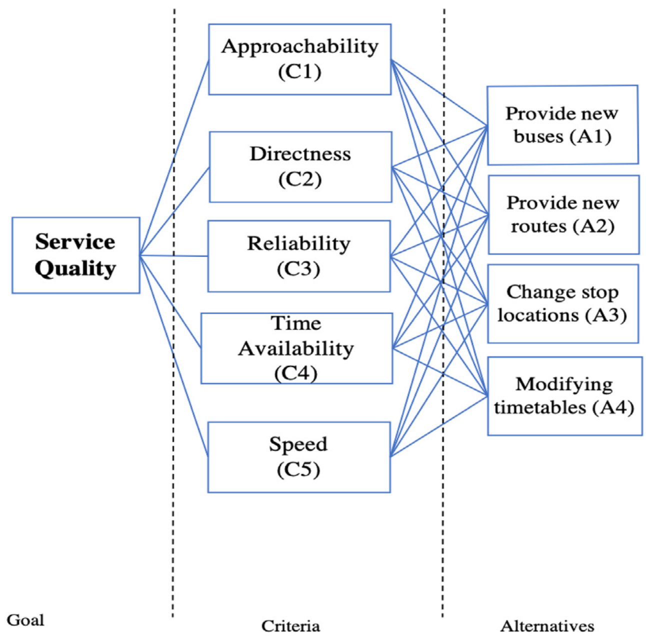

Fig. 3. The service quality structure of the urban bus transportation.

Table 2
DHq-ROFDM $\tilde{\boldsymbol{R}}_k$ provided by the DM $\boldsymbol{D}_k$.

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
1st DM's weight	0.0786	0.1258	0.1590	0.2449	0.3917
2nd DM's weight	0.0993	0.1488	0.3756	0.0626	0.3137
3rd DM's weight	0.2657	0.0728	0.0728	0.1532	0.4355

(1) ***Approachability (*** $\mathcal{C}_1$ ***):*** It is the first major criterion of the UBT system's service quality, and it refers to the service provided before to the start of a trip, including directness, safety, and convenience at bus stops (Saif et al., 2019; Cheranchery et al., 2019; Miao et al., 2019).

(2) ***Directness (*** $\mathcal{C}_2$ ***)***: It is the second most important thing that determines the quality of the service provided by the UBT system. It has to do with whether or not commuters want to switch service types and whether or not there are connections between different urban bus routes or between urban buses and other kinds of urban transportation systems (Duleba and Moslem, 2019; Jin et al., 2019).

(3) ***Reliability (*** $\mathcal{C}_3$ ***):*** It is the third most important service quality criterion for the UBT system. It shows how trustworthy the system is by providing the expected service perfectly and on time (Soza-Parra et al., 2019).

(4) ***Time availability (*** $\mathcal{C}_4$ ***):*** It is the fourth fundamental criterion of the UBT system's service quality, and it corresponds to the amount of time UBT is delivered along a route, as well as the start and end periods of UBT service throughout the day. (Scott et al., 2016; Deng and Yan, 2019).

(5) ***Speed (*** $\mathcal{C}_5$ ***):*** It is the fifth primary criterion of the UBT system's service quality, and it relates to the speed of the entire travel process, which includes the time passengers spend on-board between the departure and arrival points, as well as the pre-journey waiting time at bus stops (Kujala et al., 2018; Ingvardson et al., 2018).

6.2. Definition of alternatives for improving urban bus transport service quality

(1) ***Change stop locations (*** A_1 ***)***: Modifying the locations of the bus stops is an important alternative, and it increases user satisfaction before starting the journey. However, its role in attracting new users is considered an important action.

(2) ***Provide new routes (*** A_2 ***)***: Creating new lines considering a fixed number of buses is considered a logical alternative in the event of the limited budget of the operating company or organization. The effect of providing new routes has a positive impact on users' satisfaction, and it has a significant impact on attracting new users.

(3) ***Provide new buses (*** A_3 ***)***: New bus purchasing is an efficient alternative to increase the reliability of the urban bus transport service quality system, which elevates users' satisfaction and captivates non-users.

(4) ***Modifying timetables (*** A_4 ***)***: Changing the timetables to meet user demand is a critical alternative, and it has a great impact on improving the service quality if the planners consider the real demand not only for the users' side but also, taking into consideration the real demand for the non-user side (see Fig. 3).

For aggregating the Alternative's evaluation values, use the following weights:
Weight of criteria $\mathcal{C}_1 = 0.0786$, $\mathcal{C}_2 = 0.1258$, $\mathcal{C}_3 = 0.1590$, $\mathcal{C}_4 = 0.2449$ and $\mathcal{C}_5 = 0.3917$.

The DHq-ROFDM $\tilde{\boldsymbol{R}}_k$ provided by the DM $\boldsymbol{D}_k$ is represented in Table 2. The three DMs evaluate the four alternatives based on the five criteria and their judgmental values are listed in Tables 3–5.

Without loss of generality this case is analyzed, considering the value of, rung parameter $q = 3$, Frank t-conorm & t-norm parameter $\zeta = 2$, Heronian mean parameters $\psi = \varphi = 1$, and generalized DHq-ROF distance parameter $\epsilon = 2$.

Table 3
DHq-ROF DM D_1.

	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$
A_1	$\langle\{0.8\},\{0.6,0.7\}\rangle$	$\langle\{0.7\},\{0.6\}\rangle$	$\langle\{0.7,0.8\},\{0.5\}\rangle$	$\langle\{0.5\},\{0.3\}\rangle$	$\langle\{0.7\},\{0.2\}\rangle$
A_2	$\langle\{0.7\},\{0.2\}\rangle$	$\langle\{0.8\},\{0.2\}\rangle$	$\langle\{0.3,0.4\},\{0.5\}\rangle$	$\langle\{0.7,0.8\},\{0.1\}\rangle$	$\langle\{0.9\},\{0.2\}\rangle$
A_3	$\langle\{0.6\},\{0.5\}\rangle$	$\langle\{0.5,0.6\},\{0.4,0.5\}\rangle$	$\langle\{0.5\},\{0.6\}\rangle$	$\langle\{0.8\},\{0.5,0.7\}\rangle$	$\langle\{0.8\},\{0.3\}\rangle$
A_4	$\langle\{0.7\},\{0.2\}\rangle$	$\langle\{0.6\},\{0.5\}\rangle$	$\langle\{0.6\},\{0.5\}\rangle$	$\langle\{0.6\},\{0.2\}\rangle$	$\langle\{0.6\},\{0.5,0.6\}\rangle$

Table 4
DHq-ROF DM D_2.

	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$
A_1	$\langle\{0.8\},\{0.3\}\rangle$	$\langle\{0.7\},\{0.4,0.5\}\rangle$	$\langle\{0.7,0.8\},\{0.4\}\rangle$	$\langle\{0.9\},\{0.2\}\rangle$	$\langle\{0.6\},\{0.2\}\rangle$
A_2	$\langle\{0.7,0.8\},\{0.1\}\rangle$	$\langle\{0.7\},\{0.3\}\rangle$	$\langle\{0.3\},\{0.5\}\rangle$	$\langle\{0.6\},\{0.1\}\rangle$	$\langle\{0.8\},\{0.2\}\rangle$
A_3	$\langle\{0.5\},\{0.2\}\rangle$	$\langle\{0.5\},\{0.3\}\rangle$	$\langle\{0.5,0.7\},\{0.3\}\rangle$	$\langle\{0.8\},\{0.6,0.7\}\rangle$	$\langle\{0.9\},\{0.2\}\rangle$
A_4	$\langle\{0.6\},\{0.3\}\rangle$	$\langle\{0.7\},\{0.5\}\rangle$	$\langle\{0.4\},\{0.6\}\rangle$	$\langle\{0.8,0.9\},\{0.3,0.4\}\rangle$	$\langle\{0.7\},\{0.6\}\rangle$

Table 5
DHq-ROF DM D_3.

	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$
A_1	$\langle\{0.8\},\{0.6\}\rangle$	$\langle\{0.7\},\{0.4\}\rangle$	$\langle\{0.6,0.7\},\{0.3,0.4\}\rangle$	$\langle\{0.7,0.9\},\{0.4\}\rangle$	$\langle\{0.6\},\{0.4\}\rangle$
A_2	$\langle\{0.8\},\{0.2\}\rangle$	$\langle\{0.6,0.8\},\{0.1,0.2\}\rangle$	$\langle\{0.3\},\{0.7\}\rangle$	$\langle\{0.8\},\{0.1\}\rangle$	$\langle\{0.7,0.9\},\{0.1\}\rangle$
A_3	$\langle\{0.5,0.6\},\{0.1,0.2\}\rangle$	$\langle\{0.8\},\{0.3,0.5\}\rangle$	$\langle\{0.2\},\{0.6\}\rangle$	$\langle\{0.5\},\{0.2\}\rangle$	$\langle\{0.8\},\{0.2\}\rangle$
A_4	$\langle\{0.4,0.5,0.6\},\{0.7\}\rangle$	$\langle\{0.90\},\{0.2\}\rangle$	$\langle\{0.2\},\{0.5\}\rangle$	$\langle\{0.5\},\{0.5\}\rangle$	$\langle\{0.9\},\{0.6\}\rangle$

Step 1. Using Eq. (13), we obtain the support $\left(\tilde{\eth}_{ij}^{k},\tilde{\eth}_{ij}^{l}\right)$ $(i=1,2,3,4; j=1,2,3,4,5; k=1,2,3)$. For simplify, $\left(Sup\left(\tilde{\eth}_{ij}^{k},\tilde{\eth}_{ij}^{l}\right)\right)_{4\times 5}$ is represented by S^{kl} and indicated in the following way: (Assume that $q=3$)

$$S^{12}=S^{21}=\begin{bmatrix}0.7874 & 0.8977 & 0.9648 & 0.5727 & 0.9102\\ 0.9023 & 0.8797 & 0.9786 & 0.8140 & 0.8466\\ 0.8952 & 0.7508 & 0.8334 & 0.9475 & 0.8460\\ 0.9092 & 0.9102 & 0.8747 & 0.7024 & 0.9098\end{bmatrix}$$

$$S^{13}=S^{31}=\begin{bmatrix}0.9267 & 0.8925 & 0.8796 & 0.6286 & 0.9019\\ 0.8805 & 0.8520 & 0.8723 & 0.9024 & 0.7771\\ 0.9034 & 0.9509 & 0.9173 & 0.6969 & 0.9866\\ 0.7481 & 0.6279 & 0.8529 & 0.8952 & 0.6992\end{bmatrix}$$

$$S^{23}=S^{32}=\begin{bmatrix}0.8664 & 0.9648 & 0.8927 & 0.7748 & 0.9604\\ 0.9023 & 0.8931 & 0.8459 & 0.7907 & 0.8412\\ 0.9544 & 0.7695 & 0.7679 & 0.6810 & 0.8466\\ 0.8189 & 0.7148 & 0.9244 & 0.6367 & 0.7271\end{bmatrix}$$

Step 2. We compute the $T\left(\tilde{r}_{ij}^{k}\right)$. For simplify, $\left[T\left(\tilde{r}_{ij}^{k}\right)\right]_{4\times 5}$ is denoted as T_k and presented as follows:

$$T_1=\begin{bmatrix}1.7141 & 1.7902 & 1.8444 & 1.2013 & 1.8121\\ 1.7828 & 1.7317 & 1.8510 & 1.7165 & 1.6237\\ 1.7986 & 1.7017 & 1.7507 & 1.6443 & 1.8325\\ 1.6573 & 1.5381 & 1.7277 & 1.5976 & 1.6090\end{bmatrix}$$

$$T_2=\begin{bmatrix}1.6538 & 1.8625 & 1.8575 & 1.3475 & 1.8706\\ 1.8047 & 1.7728 & 1.8245 & 1.6047 & 1.6877\\ 1.8496 & 1.5203 & 1.6013 & 1.6285 & 1.6925\\ 1.7281 & 1.6250 & 1.7992 & 1.3391 & 1.6369\end{bmatrix}$$

$$T_3=\begin{bmatrix}1.7930 & 1.8573 & 1.7723 & 1.4035 & 1.8623\\ 1.7828 & 1.7450 & 1.7182 & 1.6931 & 1.6183\\ 1.8577 & 1.7204 & 1.6852 & 1.3779 & 1.8331\\ 1.5670 & 1.3427 & 1.7774 & 1.5319 & 1.4263\end{bmatrix}$$

Table 6
Aggregating DM with DHq-ROFFPWA.

	C_1	C_2	C_3	C_4	C_5
A_1	$\langle\{0.8000\},\{0.5187, 0.5335\}\rangle$	$\langle\{0.7000\},\{0.4632, 0.5097\}\rangle$	$\langle\{0.6904, 0.7000, 0.7610, 0.7679, 0.7231, 0.7314, 0.7846, 0.7907\},\{0.4104, 0.4242\}\rangle$	$\langle\{0.6830, 0.7896\},\{0.3137\}\rangle$	$\langle\{0.6391\},\{0.2613\}\rangle$
A_2	$\langle\{0.7657, 0.7857\},\{0.1711\}\rangle$	$\langle\{0.7277, 0.7629\},\{0.2063, 0.2383\}\rangle$	$\langle\{0.3000, 0.3329\},\{0.5204\}\rangle$	$\langle\{0.7301, 0.7826\},\{0.1000\}\rangle$	$\langle\{0.8186, 0.8790\},\{0.1539\}\rangle$
A_3	$\langle\{0.5210, 0.5811\},\{0.1550, 0.2350\}\rangle$	$\langle\{0.6735, 0.6943\},\{0.3340, 0.3735, 0.3635, 0.4062\}\rangle$	$\langle\{0.4811, 0.6307\},\{0.3958\}\rangle$	$\langle\{0.7434\},\{0.3882, 0.3973, 0.4713, 0.4822\}\rangle$	$\langle\{0.8343\},\{0.2304\}\rangle$
A_4	$\langle\{0.5374, 0.5737, 0.6215\},\{0.4646\}\rangle$	$\langle\{0.7385\},\{0.4201\}\rangle$	$\langle\{0.4619\},\{0.5605\}\rangle$	$\langle\{0.6137, 0.6539\},\{0.2859, 0.2965\}\rangle$	$\langle\{0.7853\},\{0.5631, 0.6000\}\rangle$

Step 3. We compute the power weights $\varpi_{ij}^{k} = \frac{w_i(1+T(\tilde{\alpha}_i))}{\sum_{k=1}^{n} w_k(1+T(\tilde{\alpha}_k))}$. For simplify, $\left(\varpi_{ij}^{k}\right)_{4\times5}$ is denoted as W_k and presented as follows:

$$W_1 = \begin{bmatrix} 0.1750 & 0.3564 & 0.2618 & 0.5114 & 0.3391 \\ 0.1769 & 0.3594 & 0.2648 & 0.5361 & 0.3413 \\ 0.1743 & 0.3723 & 0.2717 & 0.5505 & 0.3480 \\ 0.1798 & 0.3627 & 0.2570 & 0.5435 & 0.3517 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 0.2162 & 0.4324 & 0.6213 & 0.1394 & 0.2772 \\ 0.2252 & 0.4315 & 0.6196 & 0.1314 & 0.2800 \\ 0.2242 & 0.4108 & 0.6069 & 0.1399 & 0.2649 \\ 0.2332 & 0.4436 & 0.6231 & 0.1251 & 0.2847 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 0.6088 & 0.2112 & 0.1168 & 0.3493 & 0.3837 \\ 0.5979 & 0.2090 & 0.1156 & 0.3325 & 0.3787 \\ 0.6016 & 0.2169 & 0.1214 & 0.3097 & 0.3870 \\ 0.5871 & 0.1937 & 0.1198 & 0.3314 & 0.3636 \end{bmatrix}$$

Step 4. For alternative X_j, aggregate the assessments of attributes C_j provided by decision-makers D_k $(k = 1, 2, 3)$ in accordance with Eq. (16) and produce the comprehensive decision matrix shown in Table 6.

Step 5. We compute the $Sup\left(\tilde{r}_{ij}, \tilde{r}_{il}\right)$ $(i = 1, 2, 3, 4; j, l = 1, 2, 3, 4, 5)$ based on Eq. (17). To make things simpler, $\left(Sup\left(\tilde{r}_{ij}, \tilde{r}_{il}\right)\right)_{4\times1}$ is represented as Sup_{jl} and the result is $S_{12} = S_{21} = \begin{bmatrix} 0.8991 \\ 0.9619 \\ 0.9061 \\ 0.8169 \end{bmatrix}$, $S_{13} = S_{31} = \begin{bmatrix} 0.8928 \\ 0.6361 \\ 0.9508 \\ 0.9037 \end{bmatrix}$, $S_{14} = S_{41} = \begin{bmatrix} 0.8733 \\ 0.9653 \\ 0.8460 \\ 0.9265 \end{bmatrix}$, $S_{15} = S_{51} = \begin{bmatrix} 0.8213 \\ 0.8740 \\ 0.7081 \\ 0.7656 \end{bmatrix}$ $S_{23} = S_{32} = \begin{bmatrix} 0.9152 \\ 0.7137 \\ 0.9128 \\ 0.7732 \end{bmatrix}$, $S_{24} = S_{42} = \begin{bmatrix} 0.9027 \\ 0.9808 \\ 0.9390 \\ 0.8888 \end{bmatrix}$, $S_{25} = S_{52} = \begin{bmatrix} 0.9060 \\ 0.8568 \\ 0.8458 \\ 0.8879 \end{bmatrix}$ $S_{34} = S_{43} = \begin{bmatrix} 0.9269 \\ 0.6601 \\ 0.8588 \\ 0.8449 \end{bmatrix}$, $S_{35} = S_{53} = \begin{bmatrix} 0.8514 \\ 0.5158 \\ 0.6676 \\ 0.7761 \end{bmatrix}$, $S_{45} = S_{54} = \begin{bmatrix} 0.8622 \\ 0.8524 \\ 0.8980 \\ 0.7962 \end{bmatrix}$.

Step 6. We derive the $\left(\tilde{r}_{ij}\right)$ $(i = 1, 2, \ldots, m)$;
$T\left(\tilde{r}_{ij}\right) = \sum_{l=1, j\neq i}^{n} Sup\left(\tilde{r}_{ij}, \tilde{r}_{il}\right)$, $(j, l = 1, 2, \ldots, n)$

$$T = \begin{bmatrix} 3.4865 & 3.6229 & 3.5862 & 3.5650 & 3.4409 \\ 3.4373 & 3.5132 & 2.5256 & 3.4586 & 3.0990 \\ 3.4110 & 3.6036 & 3.3900 & 3.5417 & 3.1195 \\ 3.4127 & 3.3668 & 3.2979 & 3.4564 & 3.2259 \end{bmatrix}$$

Step 7. Considering Eq. (19), we determine the power weight vector ϖ_i of alternative X_i $(i = 1, 2, 3, 4)$ with regard to the attributes C_j $(j = 1, 2, \ldots, 5)$ and acquire $\omega = \{0.0786, 0.1258, 0.1590, 0.2449, 0.3917\}$

$$\varpi_{i,j} = \frac{w_j\left(1 + T\left(\tilde{\kappa}_{ij}\right)\right)}{\sum_{v=1}^{n} w_v\left(1 + T\left(\tilde{\kappa}_{iv}\right)\right)}$$

$$W = \begin{bmatrix} 0.0778 & 0.1294 & 0.1620 & 0.2480 & 0.3828 \\ 0.0851 & 0.1392 & 0.1265 & 0.2668 & 0.3824 \\ 0.0800 & 0.1353 & 0.1609 & 0.2589 & 0.3648 \\ 0.0806 & 0.1273 & 0.1576 & 0.2545 & 0.3799 \end{bmatrix}$$

Step 8. Applying the DHq-ROFFPPWHM operator, we derive the overall performance of alternative A_i $(i = 1, 2, 3, 4)$ over all attributes.

$$A_1 = \left\langle \left\{ \begin{matrix} 0.4960, 0.5176, 0.4972, 0.5186, 0.5062, 0.5259, \\ 0.5074, 0.5269, 0.5003, 0.5210, 0.5015, 0.5220, \\ 0.5106, 0.5295, 0.5118, 0.5305 \end{matrix} \right\}, \left\{ \begin{matrix} 0.7365, 0.7379, 0.7427, 0.7442, \\ 0.7377, 0.7392, 0.7440, 0.7454 \end{matrix} \right\} \right\rangle;$$

$$A_2 = \left\langle \left\{ \begin{matrix} 0.5288, 0.5557, 0.5385, 0.5634, 0.5292, 0.5561, \\ 0.5389, 0.5637, 0.5353, 0.5608, 0.5450, 0.5685, \\ 0.5357, 0.5612, 0.5453, 0.5689, 0.5318, 0.5585, \\ 0.5413, 0.5659, 0.5322, 0.5589, 0.5416, 0.5663, \\ 0.5381, 0.5634, 0.5475, 0.5709, 0.5385, 0.5638, \\ 0.5479, 0.5713 \end{matrix} \right\}, \{0.5695, 0.5787\} \right\rangle;$$

$$A_3 = \left\langle \left\{ \begin{matrix} 0.5140, 0.5216, 0.5168, \\ 0.5242, 0.5168, 0.5244, \\ 0.5196, 0.5270 \end{matrix} \right\}, \left\{ \begin{matrix} 0.6703, 0.6716, 0.6814, 0.6827, 0.6767, 0.6779, \\ 0.6875, 0.6888, 0.6751, 0.6764, 0.6860, 0.6873, \\ 0.6815, 0.6827, 0.6921, 0.6933, 0.6867, 0.6880, \\ 0.6980, 0.6993, 0.6931, 0.6944, 0.7042, 0.7054, \\ 0.6915, 0.6929, 0.7026, 0.7040, 0.6980, 0.6993, \\ 0.7088, 0.7101 \end{matrix} \right\} \right\rangle;$$

$$A_4 = \langle \{0.4870, 0.4917, 0.4889, 0.4935, 0.4921, 0.4966\}, \{0.7724, 0.7761, 0.7745, 0.7783\} \rangle.$$

Step 9. We assess the expected function of alternative A_i $(i = 1, 2, 3, 4)$ as $S(A_1) = 0.3646$, $S(A_2) = 0.4892$, $S(A_3) = 0.4060$, $S(A_4) = 0.3264$.

On the basis of the expected value of alternatives, the ranking result of alternatives can be obtained and shown as $A_2 \succ A_3 \succ A_1 \succ A_4$. Thus the most effective and reliable strategies for sustainable public bus transport system is to provide new routes (A_2).

7. Discussion

7.1. Ranking discussion

Our work led to the creation of a new model that helps operators and policymakers better understand what real users want and how to give it to them. It also makes the transit system look different and improves the quality of service at the same time.

In compliance with the adopted outcomes, providing new routes (A_2) is the most significant alternative for improving the service quality of the urban transportation system in Budapest. In terms of sustainable urban transportation, adding new serving lines is seen as a good idea that the operator companies and local government officials should think about carefully along with the other strategic plans they are working on to help the urban transportation network in Budapest improve along with other changes. With this in mind, putting more effort into making the new lines more direct to create a faster system, which will reduce travel time and make the current users happier, will bring in more potential users.

According to the obtained results, the second most significant alternative is changing stop locations (A_1) followed by providing new buses (A_3). Changing where bus stops are located would usually have a big effect on how well the system works, so it needs to be planned carefully to give people the best access to bus stops.

The results show that modifying timetables (A_4) has the lowest influence on improving the quality of the urban transport service, as it was the worst-ranked alternative. The reason is the efficient schedule, which used to meet the needs of commuters and make them happy.

7.2. Sensitivity analysis

The effect of the rung parameter q, Frank parameter, ζ and HM parameters, ψ and φ on decision making outcomes obtained via the DHq-ROFFPPWHM operator is currently being investigated. These parameters are crucial in the ranking of options. Different score values are produced by providing distinct parameter values.

7.2.1. The impact of the q parameter on the ranking outcomes

The effect of the rung parameter q on the decision result utilizing DHq-ROFFPPWHM operator is explained comprehensively in Table 7. It is seen that keeping HM parameter fixed at $\varphi = \psi = 1$, based on different performance values, orderings of the alternatives obtained for different values of q in $[0, 10]$ remain same as $A_2 \succ A_3 \succ A_1 \succ A_4$ when using the DHq-ROFFPPWHM operator. Although the best alternative remains unaltered as A_2.

On the other hand, regardless of how the value of the q parameter is altered when using the DHq-ROFFPPWHM operator, there is no discernible difference in the order in which the alternatives are ranked. Fig. 4 shows how the q parameter affects the ranking result so that you can understand it better.

Table 7
Ranking results varying rung parameter q in DHq-ROFFPPWHM ($\zeta = 3$).

Parameters	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Rankings
$q = 1$	0.2847	0.3662	0.3022	0.2501	$A_2 \succ A_3 \succ A_1 \succ A_4$
$q = 2$	0.3216	0.4403	0.3547	0.2827	$A_2 \succ A_3 \succ A_1 \succ A_4$
$q = 3$	0.3646	0.4892	0.4060	0.3264	$A_2 \succ A_3 \succ A_1 \succ A_4$
$q = 5$	0.4323	0.5243	0.4714	0.4035	$A_2 \succ A_3 \succ A_1 \succ A_4$
$q = 7$	0.4703	0.5261	0.4977	0.4528	$A_2 \succ A_3 \succ A_1 \succ A_4$
$q = 9$	0.4889	0.5293	0.5058	0.4795	$A_2 \succ A_3 \succ A_1 \succ A_4$
$q = 10$	0.4939	0.5311	0.5070	0.4874	$A_2 \succ A_3 \succ A_1 \succ A_4$

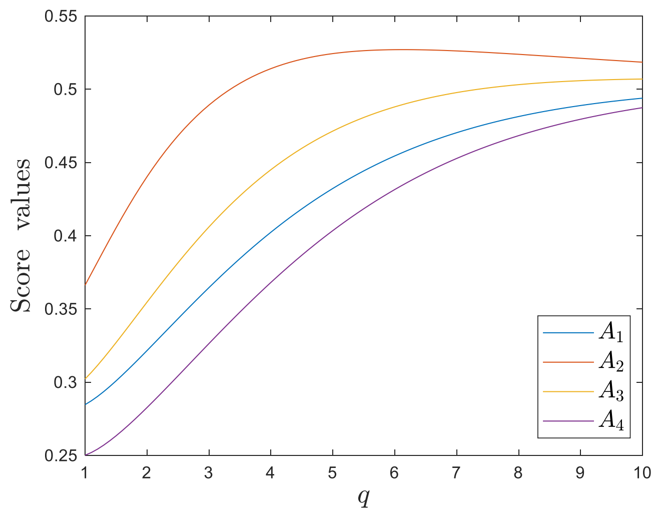

Fig. 4. Score values of A_i by IVDHq-ROFWHM operator based on q parameter ($\varphi, \psi = 1$).

Table 8
Ranking results varying Frank parameter ζ in DHq-ROFFPPWHM ($q = 3$).

Parameters	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Rankings
$\zeta = 1.1$	0.3920	0.5082	0.4310	0.3537	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\zeta = 2$	0.3646	0.4892	0.4060	0.3264	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\zeta = 3$	0.3458	0.4752	0.3881	0.3081	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\zeta = 4$	0.3327	0.4646	0.3751	0.2955	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\zeta = 6$	0.3147	0.4490	0.3567	0.2786	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\zeta = 8$	0.3027	0.4374	0.3439	0.2675	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\zeta = 10$	0.2938	0.4281	0.3343	0.2594	$A_2 \succ A_3 \succ A_1 \succ A_4$

7.2.2. The impact of the Frank parameter ζ

To demonstrate the impact of the Frank parameter ζ in the above example, the developed steps are repeatedly executed with different values of ζ. For convenience, the rung parameter is fixed at $q = 3$ in this case. Table 8 shows both the total score values for the DHq-ROFFPPWHM operator and the ranking results for that operator. Even if different values of the Frank parameter ζ lead to different score values, Table 8 shows that the ranking results are always $A_2 \succ A_3 \succ A_1 \succ A_4$. This can be seen even though the score values are varied.

When the value of the Frank parameter ζ based on the DHq-ROFFPPWHM operator is increased, the score values of the various alternatives experience a considerable and noticeable decrease. In Fig. 5, the change in score values of different alternatives are visualized for $q = 3$ as a fixed value and varying ζ in $[1, 10]$. It appears that the score values are on a downward trend there. As a result, it is possible to establish that a DM's outlook might be either pessimistic or optimistic dependent on the conviction they hold. So, DMs who have a negative outlook on an option based on specific criteria are required to choose a larger value of the Frank parameter ζ.

7.2.3. The influence of the HM parameters φ and ψ on the ranking outcomes

During the aggregation process, the results that are obtained are dependent on the HM parameters, ψ and φ. In the previous step, various values were given to the HM parameters ψ and φ to show how they affect the model. Using the DH-ROFFPPWHM operator, we changed the values of the parameters ψ and φ at the same time in the range [0,10], keeping q and ζ at the same value of 3. Table 9 show the score values of various options A_i $(i = 1, 2, 3, 4)$ and how they were ranked. Using the DHq-ROFFPPWHM operator, distinct score values for each alternatives are obtained. From Table 9, it is observed that using DHq-ROFFPPWHM operator, the ranking result is obtained as $A_2 \succ A_3 \succ A_1 \succ A_4$ varying the HM parameters ψ and φ. The best alternative continues to be A_2.

Through the use of the DHq-ROFFPPWHM operator, the score values of the various alternatives, A_i, are depicted geometrically in Figs. 6–9, which helps provide a more transparent picture.

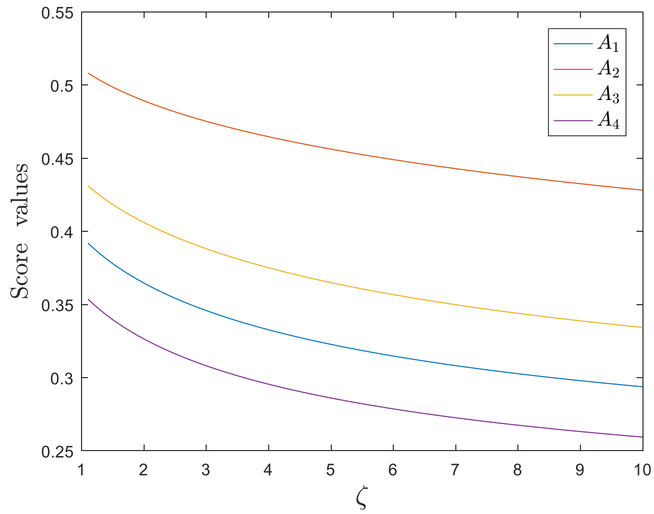

Fig. 5. Effect of Frank parameter $\zeta \in [1, 10]$ on score values of alternatives.

Table 9
Impact of HM parameters ψ and ϕ on decision-making consequences.

	Varying ϕ and ψ	Score values of alternatives				Ranking order
		$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	
DHq-ROFFPPWHM	$\psi = 1, \phi = 1$	0.3646	0.4892	0.4060	0.3264	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 1, \phi = 5$	0.3411	0.4831	0.3993	0.3028	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 1, \phi = 10$	0.3554	0.4874	0.4204	0.3083	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 2, \phi = 2$	0.3259	0.4672	0.3816	0.2968	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 2, \phi = 6$	0.3348	0.4736	0.3987	0.3002	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 2, \phi = 10$	0.3490	0.4790	0.4159	0.3061	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 3, \phi = 3$	0.3202	0.4623	0.3831	0.2939	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 3, \phi = 7$	0.3359	0.4702	0.4028	0.3012	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 3, \phi = 10$	0.3465	0.4747	0.4147	0.3056	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 5, \phi = 5$	0.3301	0.4625	0.3967	0.2983	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 5, \phi = 10$	0.3475	0.4711	0.4155	0.3059	$A_2 \succ A_3 \succ A_1 \succ A_4$
	$\psi = 10, \phi = 10$	0.3356	0.4712	0.4225	0.3084	$A_2 \succ A_3 \succ A_1 \succ A_4$

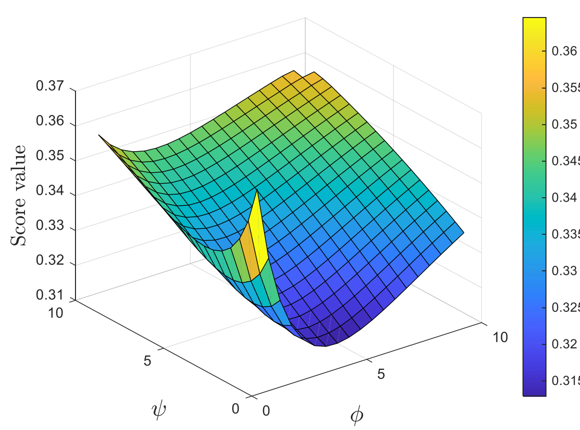

Fig. 6. Score value of change stop locations (A_1).

8. Comparison with existing methods

To evaluate the efficacy of the developed technique, the preceding analytical expression is addressed using many available MCDM techniques using different AOs, based on HPFWA (Garg, 2018), HPFWG (Garg, 2018), WHPFMSM (Garg, 2019), DHPFHWA (Wei and Lu, 2017), DHPFHWG (Wei and Lu, 2017), DHPFWBM (Tang and Wei, 2019), DHPFGWHM (Tang et al., 2019), DHPF weighted Hamy Mean (Wei et al., 2019), DHPF weighted dual Hamy Mean (Wei et al., 2019), DHq-ROFWA (Wang et al., 2019a,b), DHq-ROFWG (Wang et al., 2019a,b), DHq-ROFWMM (Wang et al., 2019a,b), q-RDHFWHM (Xu et al., 2018), DHq-ROFWDBM (Sarkar and Biswas, 2021), DHq-ROFWDGBM (Sarkar and Biswas, 2021) and q-RDHFPWDMSM (Li et al., 2022). The assessments are conducted in two distinct ways.

The assessments are made first based on the operators' characteristics and then on the obtained results.

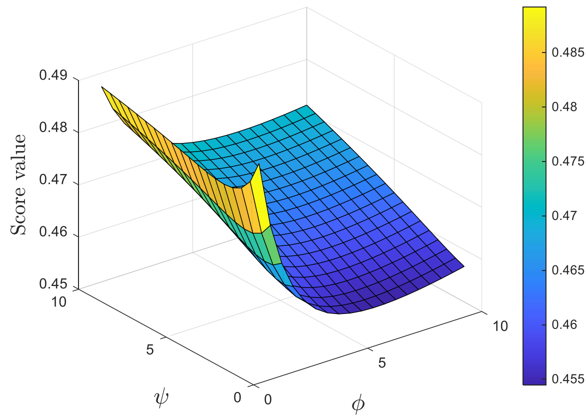

Fig. 7. Score value of provide new routes (A_2).

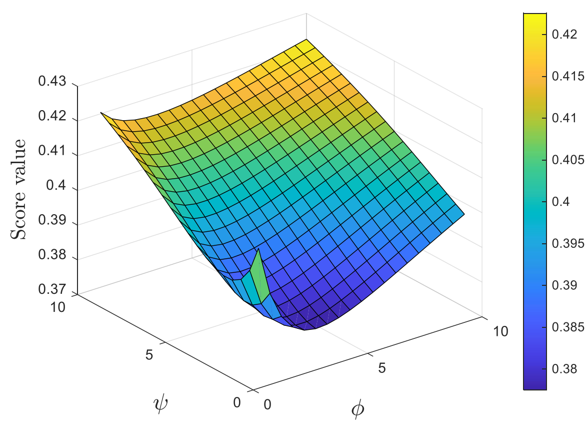

Fig. 8. Score value of provide new buses (A_3).

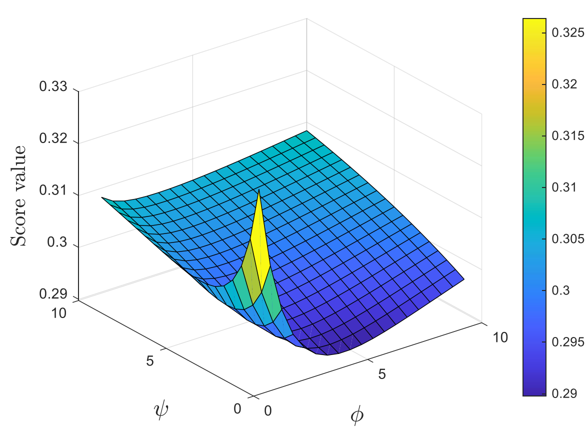

Fig. 9. Score value of modifying timetables (A_4).

When comparing the approach based on the operator's characteristics, it is worth noting that all of the existing AOs, including the established AOs, can acquire hesitant fuzzy data. By taking into account $q = 2$, the existing AOs (Garg, 2018, 2019; Wei and Lu, 2017; Tang and Wei, 2019; Tang et al., 2019; Wei et al., 2019) established in hesitant Pythagorean fuzzy environments can be viewed as a special type of comparable operators in DHq-ROF contexts. Except for the developed operators, none of the aforementioned operators took into account Frank t-norm and t-conorms, which makes the decision aggregation procedure more flexible. When power partition operator, HM and Frank t-norm and t-conorms are combined under DHq-ROF context, the established operators become more versatile and powerful than current operators, considering interdependencies among input arguments. Table 10 summarizes the characteristics of the present operators. This table shows how the proposed approaches cover a wide range of existing methods.

The previous approaches (Garg, 2018, 2019; Wei and Lu, 2017; Tang and Wei, 2019; Tang et al., 2019; Wei et al., 2019; Wang et al., 2019a,b; Xu et al., 2018; Sarkar and Biswas, 2021; Li et al., 2022) will now be compared based on achieved results. Notable is the fact that, for some values of the parameters associated with the established method, the ranking of alternatives from different approaches stays the same as the ranking

Table 10
Considered characteristics of diverse methodologies.

Methods	Consideration of interrelationships	Consideration of hesitancy	Adaptability owing to the Frank operation	Capturing information by q-ROF
HPFWA/HPFWG (Garg, 2018)	No	Yes	No	No
WHPFMSM (Garg, 2019)	Yes	Yes	No	No
DHPFHWA/DHPFHWG (Wei and Lu, 2017)	No	Yes	No	No
DHPFWBM/DHPFWGBM (Tang and Wei, 2019)	Yes	Yes	No	No
DHPFGWHM (Tang et al., 2019)	Yes	Yes	No	No
DHPF weighted Hamy Mean/DHPF weighted dual Hamy Mean (Wei et al., 2019)	Yes	Yes	No	No
DH**q**-ROFWA/DH**q**-ROFWG (Wang et al., 2019a,b)	No	Yes	No	Yes
DH**q**-ROFWMM/DH**q**-ROFWDMM (Wang et al., 2019a,b)	Yes	Yes	No	Yes
q-RDHFWHM (Xu et al., 2018)	Yes	Yes	No	Yes
DH**q**-ROFWDBM/DH**q**-ROFWDGBM (Sarkar and Biswas, 2021)	Yes	Yes	No	Yes
q-RDHFPWDMSM (Li et al., 2022)	Yes	Yes	No	Yes
Proposed method	Yes	Yes	Yes	Yes

Table 11
Comparative analysis with the existing methods with regard to score values and ranking of the alternatives.

AOs	Score values	Ranking
HPFWA (Garg, 2018)	$S(A_1) = 0.7104,\ S(A_2) = 0.7756,\ S(A_3) = 0.7278,\ S(A_4) = 0.6550$	$A_2 \succ A_3 \succ A_1 \succ A_4$
HPFWG (Garg, 2018)	$S(A_1) = 0.6621,\ S(A_2) = 0.6806,\ S(A_3) = 0.6251,\ S(A_4) = 0.5600$	$A_2 \succ A_1 \succ A_3 \succ A_4$
WHPFMSM (Garg, 2019)	$S(A_1) = 0.3740,\ S(A_2) = 0.4289,\ S(A_3) = 0.3977,\ S(A_4) = 0.3749$	$A_2 \succ A_3 \succ A_4 \succ A_1$
DHPFHWA (Wei and Lu, 2017)	$S(A_1) = 0.7019,\ S(A_2) = 0.7631,\ S(A_3) = 0.7130,\ S(A_4) = 0.6364$	$A_2 \succ A_3 \succ A_1 \succ A_4$
DHPFHWG (Wei and Lu, 2017)	$S(A_1) = 0.6723,\ S(A_2) = 0.7043,\ S(A_3) = 0.6496,\ S(A_4) = 0.5784$	$A_2 \succ A_1 \succ A_3 \succ A_4$
DHPFWBM (Tang and Wei, 2019)	$S(A_1) = 0.3955,\ S(A_2) = 0.4498,\ S(A_3) = 0.4011,\ S(A_4) = 0.3789$	$A_2 \succ A_3 \succ A_1 \succ A_4$
DHPFWGBM (Tang and Wei, 2019)	$S(A_1) = 0.7649,\ S(A_2) = 0.7539,\ S(A_3) = 0.7245,\ S(A_4) = 0.7095$	$A_1 \succ A_2 \succ A_3 \succ A_4$
DHPFGWHM (Tang et al., 2019)	$S(A_1) = 0.2449,\ S(A_2) = 0.3173,\ S(A_3) = 0.2519,\ S(A_4) = 0.2030$	$A_2 \succ A_3 \succ A_1 \succ A_4$
DHPF weighted Hamy Mean (Wei et al., 2019)	$S(A_1) = 0.9305,\ S(A_2) = 0.9315,\ S(A_3) = 0.9222,\ S(A_4) = 0.9113$	$A_2 \succ A_1 \succ A_3 \succ A_4$
DHPF weighted dual Hamy Mean (Wei et al., 2019)	$S(A_1) = 0.2093,\ S(A_2) = 0.2752,\ S(A_3) = 0.1936,\ S(A_4) = 0.1631$	$A_2 \succ A_1 \succ A_3 \succ A_4$
DH**q**-ROFWA (Wang et al., 2019a,b)	$S(A_1) = 0.6678,\ S(A_2) = 0.7134,\ S(A_3) = 0.6846,\ S(A_4) = 0.6284$	$A_2 \succ A_3 \succ A_1 \succ A_4$
DH**q**-ROFWG (Wang et al., 2019a,b)	$S(A_1) = 0.6363,\ S(A_2) = 0.6479,\ S(A_3) = 0.6120,\ S(A_4) = 0.5600$	$A_2 \succ A_1 \succ A_3 \succ A_4$
DH**q**-ROFWMM (Wang et al., 2019a,b)	$S(A_1) = 0.4694,\ S(A_2) = 0.4985,\ S(A_3) = 0.4843,\ S(A_4) = 0.4705$	$A_2 \succ A_3 \succ A_4 \succ A_1$
DH**q**-ROFWDMM (Wang et al., 2019a,b)	$S(A_1) = 0.7239,\ S(A_2) = 0.7177,\ S(A_3) = 0.6524,\ S(A_4) = 0.6469$	$A_1 \succ A_2 \succ A_3 \succ A_4$
q-RDHFWHM (Xu et al., 2018)	$S(A_1) = 0.2809,\ S(A_2) = 0.3622,\ S(A_3) = 0.2893,\ S(A_4) = 0.2349$	$A_2 \succ A_3 \succ A_1 \succ A_4$
DH**q**-ROFWDBM (Sarkar and Biswas, 2021)	$S(A_1) = 0.5818,\ S(A_2) = 0.6063,\ S(A_3) = 0.5784,\ S(A_4) = 0.5991$	$A_2 \succ A_4 \succ A_1 \succ A_3$
DH**q**-ROFWDGBM (Sarkar and Biswas, 2021)	$S(A_1) = 0.7377,\ S(A_2) = 0.7703,\ S(A_3) = 0.6601,\ S(A_4) = 0.6460$	$A_2 \succ A_1 \succ A_3 \succ A_4$
q-RDHFPWDMSM (Li et al., 2022)	$S(A_1) = 0.8101,\ S(A_2) = 0.8161,\ S(A_3) = 0.7831,\ S(A_4) = 0.7488$	$A_2 \succ A_1 \succ A_3 \succ A_4$
Proposed operator (DH**q**-ROFPPWHM)	$S(A_1) = 0.3646,\ S(A_2) = 0.4892,\ S(A_3) = 0.4060,\ S(A_4) = 0.3264$	$A_2 \succ A_3 \succ A_1 \succ A_4$

from the established method. In Table 11 and Fig. 10, the results of the comparison are provided comprehensively. Table 11 shows the resulting score values and ranks using all of the existing methods (Garg, 2018, 2019; Wei and Lu, 2017; Tang and Wei, 2019; Tang et al., 2019; Wei et al., 2019; Wang et al., 2019a,b; Xu et al., 2018; Sarkar and Biswas, 2021; Li et al., 2022) under discussion, as well as the suggested approach with the values $\psi = \phi = 1$, $q = 3$, and $\zeta = 3$. Table 11 and Fig. 10 reveal that, despite minor variations in the ranking orders, the best alternative achieved by various methods is nearly equivalent to the strategy provided in this research. Consequently, this case confirms the viability of the suggested strategy.

By adjusting the rung parameter q; Frank parameter ζ; and HM parameters ψ, ϕ, different score values of the alternatives can be produced.

Let us say a DM has a predisposition against some specific alternatives for some mysterious reason. While assessing those alternatives, the DM offers some extreme values (from the pessimistic/optimistic views).

When current all-but-power operators, Li et al. (2022), are used to acquire results, the results would be biased as a result of the DM's impact on the outcomes. Because of the influence of biased DM, D_k on alternative, A_i, changes in optimal choice while applying some existing approaches are the cause. The impact of extreme values provided by any biased DMs during the evaluation process can be lessened by power AOs. By calculating the supports and generating the rational result, the developed operators DHq-ROFFPPWHM, who have the advantage of power AOs, can eliminate the impact of unreasonable extreme values. But the fact that there are other operators does not change the effect that biased DMs' bad data has on how decisions turn out.

There are interrelationships among the attributes in the same subsets, but there are no interrelationships among those in the different subsets. The HPFWA (Garg, 2018), HPFWG (Garg, 2018), DHPFHWA (Wei and Lu, 2017), DHPFHWG (Wei and Lu, 2017), DHq-ROFWA (Wang et al., 2019a,b), and DHq-ROFWG (Wang et al., 2019a,b) operators cannot deal with this kind of complex interrelationships among the attributes, while the proposed operator utilizes the PPHM to capture the interrelationships among the attributes.

The AOs used in the existing methods (Garg, 2018, 2019; Tang and Wei, 2019; Tang et al., 2019; Wei et al., 2019; Wang et al., 2019a,b; Xu et al., 2018; Li et al., 2022) are based on algebraic operations, which are not general and flexible in nature. The proposed aggregation method employs Frank t-norm and t-conorms. Thus, the developed operators possess the ability to make the aggregation process more robust and smooth by varying the parameters of Frank t-norm and t-conorms in the aggregation functions.

As was previously discussed, the developed method has the crucial quality of lessening the impact of excessive evaluation values brought on by biased DMs. The suggested method can also capture DHq-ROF information, which allows the total of the qth powers of membership and non-membership degrees to not exceed one and can represent a higher degree of uncertainty. The existing methods developed by operators (Garg, 2018, 2019; Wei and Lu, 2017; Tang and Wei, 2019; Tang et al., 2019; Wei et al., 2019) are unable to address such situations.

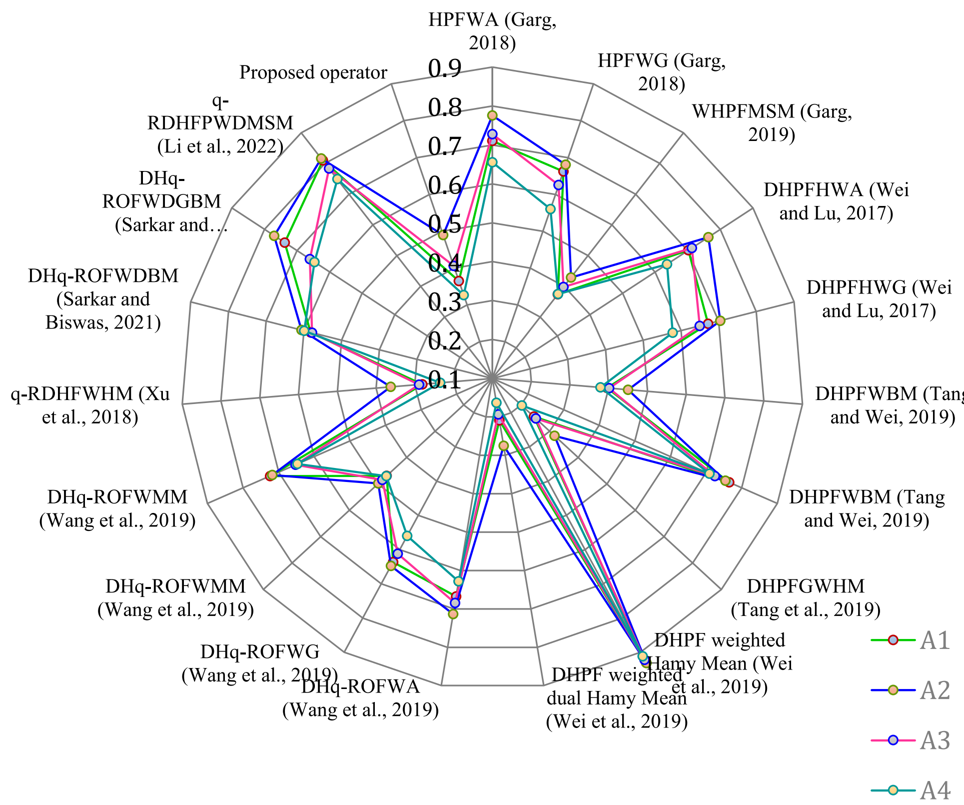

Fig. 10. The radar chart of ranking results on solving with various existing methods.

9. Conclusions

The proposed model is a pioneering model that can make the urban transport system more attractive and sustainable. But local government officials and operator companies should work on improving the quality of service by making sure the public transportation system works well. Our study shed light on MCGDM problems using HM and DHq-ROF data. The generated operator is used to figure out how to solve MCGDM problems with models with relationships between arguments that have been grouped together.

The model was used to figure out how good the public bus system in Budapest is at getting people where they need to go. The research showed that adding new routes (A_2) was the most popular and best-scored way to improve the service quality of Budapest's public transportation system. The obtained results would aid local decision-makers in their future strategic plans for improving the quality of public transportation to improve the city's image and reduce commuter numbers, which is a key factor in reducing private car numbers and pollution. As a suggestion for future research, different stakeholders should be involved so that we can learn more about their needs for improving and making strategic plans for the current system.

Despite this, the study introduced numerous excellent concepts to the subject of decision-making. Researchers will be able to apply this DM approach to a variety of problems, such as renewable energy source selection (Mishra et al., 2022a), biomass crop selection (Mishra et al., 2022b), selecting cold chain logistics distribution centers (Rong et al., 2022), risk investment assessment (Tan et al., 2022), "fuel cell and hydrogen components supplier selection" (Alipour et al., 2021), and medical diagnosis problems, among others. We will try to use the new DM technique of enhanced other fuzzy sets, like "Fermatean fuzzy sets (Senapati and Yager, 2019a,b, 2020), picture fuzzy sets (Jana et al., 2019), complex fuzzy sets, cubic intuitionistic sets (Senapati et al., 2021), bipolar soft sets (Shabir and Naz, 2013; Mahmood, 2020), bipolar complex fuzzy sets (Mahmood and Ur Rehman, 2022), extended Pythagorean fuzzy sets (Saeidi et al., 2022), interval-valued Pythagorean fuzzy sets (Senapati and Chen, 2021), hesitant multi-fuzzy soft set (Dey et al., 2020), 2-tuple linguistic q-rung picture fuzzy sets (Akram et al., 2023), probabilistic linguistic sets (Krishankumar et al., 2022), and dual probabilistic linguistic set (Saha et al., 2021)" to make decisions in the real world in the near future.

CRediT authorship contribution statement

Arun Sarkar: Conceptualization, Methodology, Software, Validation, Visualization, Investigation, Writing – original draft, Writing – review & editing. **Sarbast Moslem:** Conceptualization, Writing – original draft, Writing – review & editing. **Domokos Esztergár-Kiss:** Writing – original draft, Writing – review & editing. **Muhammad Akram:** Conceptualization, Visualization, Writing – original draft, Writing – review & editing. **LeSheng Jin:** Writing – original draft, Visualization, Writing – review & editing. **Tapan Senapati:** Conceptualization, Methodology, Software, Validation, Visualization, Investigation, Writing – original draft, Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Linguistic q-rung orthopair fuzzy prioritized aggregation operators based on Hamacher t-norm and t-conorm and their applications to multicriteria group decision making

Nayana DEB, Arun SARKAR and Animesh BISWAS

The linguistic q-rung orthopair fuzzy (Lq-ROF) set is an important implement in the research area in modelling vague decision information by incorporating the advantages of q-rung orthopair fuzzy sets and linguistic variables. This paper aims to investigate the multicriteria decision group decision making (MCGDM) with Lq-ROF information. To do this, utilizing Hamacher t-norm and t-conorm, some Lq-ROF prioritized aggregation operators viz., Lq-ROF Hamacher prioritized weighted averaging, and Lq-ROF Hamacher prioritized weighted geometric operators are developed in this paper. The defined operators can effectively deal with different priority levels of attributes involved in the decision making processes. In addition, Hamacher parameters incorporated with the proposed operators make the information fusion process more flexible. Some prominent characteristics of the developed operators are also well-proven. Then based on the proposed aggregation operators, an MCGDM model with Lq-ROF context is framed. A numerical example is illustrated in accordance with the developed model to verify its rationality and applicability. The impacts of Hamacher and rung parameters on the achieved decision results are also analyzed in detail. Afterwards, a comparative study with other representative methods is presented in order to reflect the validity and superiority of the proposed approach.

Key words: linguistic q-rung orthopair fuzzy set, multicriteria group decision making, Hamacher operations, prioritized aggregation operator

1. Introduction

Multicriteria decision making (MCDM) has emerged as an important branch in modern decision science. It refers to find a suitable choice based on the

N. Deb (corresponding author, e-mail: nayanadeb1@gmail.com) and A. Biswas are with Department of Mathematics, University of Kalyani, Kalyani – 741235, India.

A. Sarkar is with Department of Mathematics, Heramba Chandra College, Kolkata – 700029, India.

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evaluation information by a decision-maker (DM) from a collection of alternatives under a set of criteria. If the evaluation of alternatives against a certain criterion is performed under multiple DMs instead of a single DM, then the process is termed as multicriteria group decision making (MCGDM). With the increase in vagueness of environment day by day and the inherent fuzziness connected with human perception, decision information cannot always be provided using crisp numbers. In response to this issue Zadeh [1] first introduced the notion of fuzzy set. After that, several extensions of fuzzy set were developed, including intuitionistic fuzzy sets (IFSs) [2], interval-valued IFSs [3], Pythagorean fuzzy sets (PFSs) [4,5], interval-valued PFS [6,7], Fermatin fuzzy sets (FFSs) [8] etc. Since these extensions of fuzzy sets appear, they have received more and more attention in solving decision-making problems [9–15]. By enlarging the scope of IFS, PFS and FFS, recently, another variant of fuzzy set, q-rung orthopair fuzzy (q-ROF) set (q-ROFS) [16], has been developed as an efficient tool in terms of capturing uncertainty during the process of MCGDM. For q-ROFSs membership degree μ and non-membership degree ν satisfy the condition that sum of their q-th power is less than or equal to 1, i.e., $\mu^q + \nu^q \leqslant 1$. As a more generalized fuzzy set, q-ROFS include fuzzy sets, IFSs, PFSs, and FFSs as special cases with certain conditions. For instance, q-ROFS reduces to IFS, PFS, FFS by taking the value of rung parameter $q = 1, 2, 3$, respectively. So q-ROFS is the most valuable and focused extension of fuzzy sets in which DMs can modify the range of their judgement values by varying rung parameter q based on different indeterminate degrees.

So far, q-ROFSs have attracted many scholars attention. Liu and Wang [17] investigated multi-attribute decision making (MADM) problems with q-ROF information on developing q-ROF weighted averaging (WA) and weighted geometric (WG) operators. They [18] further extended Archimedean Bonferroni mean operators to q-ROF environment. Heronian mean was utilized to fuse q-ROF data, and thereby a MADM approach was developed by Wei et al. [19]. On the basis of the cosine function, Wang et al. [20] studied novel similarity measures for q-ROFSs. Further, a study on induced logarithmic distance measures for q-ROFSs was conducted by Zeng et al. [21]. In recent days, a variety of applications [22–28] on q-ROFSs have been developed by numerous researchers.

However, q-ROFS theory has successfully been applied in several decision-making processes, but in real-world issues, many attribute values are present that are often difficult to express quantitatively. In such cases, it seems suitable to express them using a qualitative form. To address such situations, Liu and Liu [29] invented linguistic q-ROF (Lq-ROF) set (Lq-ROFS), following the advantage of q-ROFS and linguistic variables [30], which is a generalization of linguistic intuitionistic fuzzy (LIF) set (LIFS) [31] and linguistic Pythagorean fuzzy (LPF) set (LPFS) [32]. In recent years, several significant researches on Lq-ROFS have been carried out, along with numerous decision-making theories.

In short, Lq-ROFS have been studied effectually from different perspectives, including information measures [33, 34], traditional decision techniques [35, 36], aggregation operators [29, 37–41]. Nevertheless, to generate the ranking of alternatives, aggregation operators usually can address decision making situations more effectively than conventional decision techniques because aggregation operators can produce a ranking of alternatives along with their collective evaluation values. In contrast, traditional techniques can be only able to produce ranking results. Liu and Liu [29] introduced some aggregation operators based on power Bonferroni mean and utilized them for MCGDM under Lq-ROF environment. An interactional partitioned Heronian mean based decision method with Lq-ROF information has been developed by Lin et al. [37]. Further, Liu and Liu [38] investigated Lq-ROF power Muirhead mean aggregation operators for MCGDM. Recently, Akram et al. [39] proposed an Einstein model in order to build a Lq-ROF group decision-making framework, and Liu et al. [40] developed some generalized point weighted aggregation operators for Lq-ROF group decision-making context as well.

It is important that in the process of MCDM, the required aggregation operators must be general and flexible enough to capture the relationship between the different criteria when aggregating the values of attributes. Assuming that the criteria are at the same priority level may lead to serious loss of information. Yager [42] introduced the prioritized averaging operator to overcome these issues, which may take into account various priority levels of criteria during the aggregating procedure. However, so far, the aggregated operators to fuse Lq-ROF information have not taken prioritization relation among criteria into account. Thus, introducing the concept prioritized aggregation (PA) operator in Lq-ROF environment for developing some MCGDM techniques would be a useful study in Literature. It is important to point out that among the existing aggregation operators for Lq-ROF numbers (Lq-ROFNs), most of the aggregation functions involve algebraic sum and product in order to carry the aggregation process. However, the operational rules play an important role in aggregating decision information. Hamacher operations [43], a generalized form of algebraic and Einstein operations [44], have significant importance in the aggregation process by means of a flexible parameter. Several achievements [45–47] have been discovered in the past decades employing Hamacher operational rules in the aggregation process. Therefore motivated by the idea of Hamacher t-norms and t-conorms with PA operators, some Lq-ROF aggregation operators, viz., Lq-ROF Hamacher prioritized WA (Lq-ROFHPWA), and Lq-ROF Hamacher prioritized WG (Lq-ROFHPWG) operators have been developed in this paper.

The paper is structured as follows. Section 2 reviews several fundamental concepts such as Lq-ROFSs, Hamacher t-norms and t-conorms and PA operators. Hamacher operational laws for Lq-ROFNs are proposed in Section 3. Section 4 introduces some newly Lq-ROF PA operators based on Hamacher operations, viz., Lq-ROFHPWA and Lq-ROFHPWG operators. Further, some characteristics of these developed operators are also exhibited in this section. Section 5 illustrates an MCGDM approach utilizing the proposed aggregation operators. A numerical example utilizing the developed approach has been provided in Section 6. Comparative and sensitivity analyses are discussed in Section 7. Finally, an overall summarization and scope for future studies have been demonstrated in Section 8.

2. Preliminaries

Some basic ideas of linguistic term set (LTS), Lq-ROFS, PA operator, and Hamacher t-norm and t-conorm are briefly discussed in this section.

2.1. LTS

Definition 1 [48] *Let* $\mathfrak{S} = \{\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_t\}$ *be a finite-ordered discrete set with odd cardinality and the terms* $\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_t$ *can be specified in terms of various real-world scenarios. Then* $\mathfrak{S}$ *is said to be a LTS if it satisfies the following conditions:*

(i) If $i > j$*, then* $\mathfrak{S}_i > \mathfrak{S}_j$*, implies* $\mathfrak{S}_i$ *is superior than* $\mathfrak{S}_j$ *(Ordered);*

(ii) $\neg(\mathfrak{S}_i) = \mathfrak{S}_j$*, where* $j = t - i$ *(Negation);*

(iii) If $i \leqslant j$*, that is,* $\mathfrak{S}_i \leqslant \mathfrak{S}_j$*, then* $\min\left(\mathfrak{S}_i, \mathfrak{S}_j\right) = \mathfrak{S}_i$ *(Min operator);*

(iv) If $i \geqslant j$*, that is,* $\mathfrak{S}_i \geqslant \mathfrak{S}_j$*, then* $\max\left(\mathfrak{S}_i, \mathfrak{S}_j\right) = \mathfrak{S}_i$ *(Max operator).*

For example, when an expert wants to evaluate the quality of comforts of a car, he/she may feel more convenient to assess it using LTS as

$$\begin{aligned}\mathfrak{S} &= \{\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_6\} \\ &= \{\text{extreme low, very low, low, medium, high, very high, extreme high}\}.\end{aligned}$$

Further, Xu [49] prolonged the notion of discrete LTS $\mathfrak{S}$ to continuous LTS (CLTS) $\overline{\mathfrak{S}}$ such that $\overline{\mathfrak{S}} = \{\mathfrak{S}_\hbar | \mathfrak{S}_0 \leqslant \mathfrak{S}_\hbar \leqslant \mathfrak{S}_t, \hbar \in [0, t]\}$ and the components likewise meet all of the preceding requirements.

2.2. Lq-ROFS

Definition 2 [29] *An* Lq-ROFS $\widetilde{\mathcal{B}}$ *defined in a universe of discourse* $X = \{x_1, x_2, \ldots, x_n\}$ *is represented by*

$$\widetilde{\mathcal{B}} = \left\{\left\langle x, \mathfrak{S}_{\gamma_{\widetilde{\mathcal{B}}}}(x), \mathfrak{S}_{\zeta_{\widetilde{\mathcal{B}}}}(x)\right\rangle \middle| \, x \in X\right\}, \tag{1}$$

where $\mathfrak{S}_{\gamma_{\widetilde{\mathcal{B}}}}(x), \mathfrak{S}_{\zeta_{\widetilde{\mathcal{B}}}}(x) \in \mathfrak{S}_{[0,t]}$ *denote the linguistic membership and non-membership degrees, respectively satisfying the condition* $0 \leqslant \left(\gamma_{\widetilde{\mathcal{B}}}\right)^q + \left(\zeta_{\widetilde{\mathcal{B}}}\right)^q \leqslant t^q$ $(q \geqslant 1)$ *for every* $x \in X$. *For convenience, Liu and Liu [29] represents a L*q*-ROFN as* $\widetilde{\beta} = \langle \mathfrak{S}_\gamma, \mathfrak{S}_\zeta \rangle$. *The linguistic indeterminacy degree of* x *to* $\widetilde{\beta}$ *is presented as* $\mathfrak{S}_{\pi_{\widetilde{\beta}}}(x) = \mathfrak{S}_{(t^q - \gamma^q - \zeta^q)^{\frac{1}{q}}}$.

Definition 3 [29] *Let* $\widetilde{\beta} = \langle \mathfrak{S}_\gamma, \mathfrak{S}_\zeta \rangle$ *be an L*q*-ROFN, the score function,* $S\left(\widetilde{\beta}\right)$*, and accuracy function,* $A\left(\widetilde{\beta}\right)$*, of the L*$q$*-ROFN can be defined as*

$$S\left(\widetilde{\beta}\right) = \left(\frac{t^q + \gamma^q - \zeta^q}{2}\right)^{\frac{1}{q}}, \tag{2}$$

and

$$A\left(\widetilde{\beta}\right) = \left(\gamma^q + \zeta^q\right)^{\frac{1}{q}}. \tag{3}$$

The following comparison method based on the score and accuracy functions is presented to compare any two Lq-ROFNs.

Definition 4 [29] *Let* $\widetilde{\beta}_1 = \langle \mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\zeta_1} \rangle$, $\widetilde{\beta}_2 = \langle \mathfrak{S}_{\gamma_2}, \mathfrak{S}_{\zeta_2} \rangle$ *be any two L*q*-ROFNs*

(i) If $S\left(\widetilde{\beta}_1\right) < S\left(\widetilde{\beta}_2\right)$, *then* $\widetilde{\beta}_1 \prec \widetilde{\beta}_2$;

(ii) If $S\left(\widetilde{\beta}_1\right) = S\left(\widetilde{\beta}_2\right)$, *then*

- *if* $A\left(\widetilde{\beta}_1\right) < A\left(\widetilde{\beta}_2\right)$, *then* $\widetilde{\beta}_1 \prec \widetilde{\beta}_2$ *which means* $\widetilde{\beta}_2$ *is better than* $\widetilde{\beta}_1$;
- *if* $A\left(\widetilde{\beta}_1\right) = A\left(\widetilde{\beta}_2\right)$, *then* $\widetilde{\beta}_1 \approx \widetilde{\beta}_2$, *which means* $\widetilde{\beta}_1$ *is equal to* $\widetilde{\beta}_2$.

2.3. PA operator

Yager [42] originally introduced the PA operator, which is presented in the following:

Definition 5 [42] *Consider* $\{C_i | i = 1, 2, \ldots, n\}$ *as a collection of criteria, the linear ordering* $C_1 \succ C_2 \succ \ldots \succ C_n$ *represents their priority. This ordering reveals that if* $j < k$ *then criteria* C_j *has a higher priority than* C_k. $C_j(x) \in [0, 1]$ *denotes the assessment value of any alternative* x *evaluated on the criteria* C_j.

If $PA\left(C_j(x)\right) = \sum_{j=1}^{n} w_j C_j(x)$, *where* $w_j = \frac{T_j}{\sum_{j=1}^{n} T_j}$, $T_j = \prod_{k=1}^{j-1} C_k(x)$

$(j = 2, \ldots, n)$, $T_1 = 1$. *Then* $PA\left(C_j(x)\right)$ *is called the PA operator.*

2.4. Hamacher t-norms and t-conorms

In 1978, Hamacher [43] introduced one of generalized t-norm and t-conorm, which is known as Hamacher t-norms and t-conorms, and expressed as ($\varsigma > 0$):

- Hamacher t-norm: $T_{\varsigma}^{H}(x, y) = \dfrac{xy}{\varsigma + (1-\varsigma)\,(x + y - xy)}$,
- Hamacher t-conorm: $S_{\varsigma}^{H}(x, y) = \dfrac{x + y - xy - (1-\varsigma)\,xy}{1-(1-\varsigma)\,xy}$.

3. Hamacher t-norms and t-conorms based operational laws for Lq-ROFNs

According to the Hamacher t-norms and t-conorms, the following operational rules of Lq-ROFNs are defined as follows.

Definition 6 *Let $\overline{\mathfrak{S}} = \{\mathfrak{S}_{\hbar} : \hbar \in [0, t]\}$ be a CLTS, $\widetilde{\beta}_1 = \langle \mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\zeta_1} \rangle$, $\widetilde{\beta}_2 = \langle \mathfrak{S}_{\gamma_2}, \mathfrak{S}_{\zeta_2} \rangle$ and $\widetilde{\beta} = \langle \mathfrak{S}_{\gamma}, \mathfrak{S}_{\zeta} \rangle$ be three Lq-ROFNs. Then, the Hamacher operational laws of Lq-ROFNs are defined as ($\lambda > 0$)*

(i) $$\widetilde{\beta}_1 \oplus_H \widetilde{\beta}_2 = \left\langle \mathfrak{S}_{t\left(\frac{t^q\gamma_1^q + t^q\gamma_2^q - \gamma_1^q\gamma_2^q - (1-\varsigma)\gamma_1^q\gamma_2^q}{t^{2q} - (1-\varsigma)\gamma_1^q\gamma_2^q}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{\zeta_1^q\zeta_2^q}{\varsigma t^{2q} + (1-\varsigma)\left(t^q\zeta_1^q + t^q\zeta_2^q - \zeta_1^q\zeta_2^q\right)}\right)^{\frac{1}{q}}} \right\rangle;$$

(ii) $$\widetilde{\beta}_1 \otimes_H \widetilde{\beta}_2 = \left\langle \mathfrak{S}_{t\left(\frac{\gamma_1^q\gamma_2^q}{\varsigma t^{2q} + (1-\varsigma)\left(t^q\gamma_1^q + t^q\gamma_2^q - \gamma_1^q\gamma_2^q\right)}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{t^q\zeta_1^q + t^q\zeta_2^q - \zeta_1^q\zeta_2^q - (1-\varsigma)\zeta_1^q\zeta_2^q}{t^{2q} - (1-\varsigma)\zeta_1^q\zeta_2^q}\right)^{\frac{1}{q}}} \right\rangle;$$

(iii) $$\lambda\widetilde{\beta} = \left\langle \mathfrak{S}_{t\left(\frac{(t^q + \gamma^q(\varsigma-1))^\lambda - (t^q - \gamma^q)^\lambda}{(t^q + \gamma^q(\varsigma-1))^\lambda + (\varsigma-1)(t^q - \gamma^q)^\lambda}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{\varsigma\zeta^{q\lambda}}{(t^q + (\varsigma-1)(t^q - \zeta^q))^\lambda + (\varsigma-1)\zeta^{q\lambda}}\right)^{\frac{1}{q}}} \right\rangle;$$

(iv) $$\widetilde{\beta}^{\lambda} = \left\langle \mathfrak{S}_{t\left(\frac{\varsigma\gamma^{q\lambda}}{(t^q + (\varsigma-1)(t^q - \gamma^q))^\lambda + (\varsigma-1)\gamma^{q\lambda}}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{(t^q + \zeta^q(\varsigma-1))^\lambda - (t^q - \zeta^q)^\lambda}{(t^q + \zeta^q(\varsigma-1))^\lambda + (\varsigma-1)(t^q - \zeta^q)^\lambda}\right)^{\frac{1}{q}}} \right\rangle.$$

4. Development of Hamacher operations-based PA operators on Lq-ROF environment

In the following, utilizing Hamacher operations, the PA operator is extended into Lq-ROFNs and Lq-ROFHPWA and Lq-ROFHPWG operators are proposed.

Definition 7 *Let* $\left\{\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_n\right\}$ *represents a collection of Lq-ROFNs, where* $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle$ *(*$i = 1, 2, \ldots, n$*) and* $q \geqslant 1$. *Then Lq-ROFHPWA operator is defined as*

$$Lq\text{-}ROFHPWA\left(\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_n\right) = \bigoplus_{i=1}^{n}{}_H \left(\frac{T_i}{\sum_{i=1}^{n} T_i} \widetilde{\beta}_i \right) \tag{4}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ *is the weight vectors of* $\widetilde{\beta}_i$ *with* $\omega_i \in [0,\ 1]$ *and* $\omega_i = \frac{T_i}{\sum_{i=1}^{n} T_i}$, $T_i = \prod_{k=1}^{i-1} \frac{S\left(\widetilde{\beta}_k\right)}{t}$ *(*$i = 2, \ldots, n$*),* $T_1 = 1$ *and* $S\left(\widetilde{\beta}_i\right)$ *is the score of* $\widetilde{\beta}_i$.

Theorem 1 *Let* $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle$ *(*$i = 1, 2, \ldots, n$*) represents a collection of Lq-ROFNs. Then, the aggregated result is also a Lq-ROFN based on Lq-ROFHPWA operator, and*

$$Lq\text{-}ROFHPWA\left(\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_n\right) = \bigoplus_{i=1}^{n}{}_H \left(\frac{T_i}{\sum_{i=1}^{n} T_i} \widetilde{\beta}_i \right)$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma \prod_{i=1}^{n} \zeta_i^{\frac{T_i}{\sum_{i=1}^{n} T_i} q}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\prod_{i=1}^{n} \zeta_i^{\frac{T_i}{\sum_{i=1}^{n} T_i} q}} \right)^{\frac{1}{q}}} \right\rangle.$$

Proof. Based on Definition 6,

$$\frac{T_i}{\sum\limits_{i=1}^{n} T_i}\widetilde{\beta}_i = \left\langle \mathfrak{S}_t\left(\frac{\left(t^q+\gamma_i^q(\varsigma-1)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}-\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\left(t^q+\gamma_i^q(\varsigma-1)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}+(\varsigma-1)\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right)^{\frac{1}{q}}, \mathfrak{S}_t\left(\frac{\varsigma\zeta_i^{q\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}+(\varsigma-1)\zeta_i^{q\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right)^{\frac{1}{q}}\right\rangle.$$

Then, it can be obtained that

$$\bigoplus_{i=1}^{2}{}_H\left(\frac{T_i}{\sum\limits_{i=1}^{2} T_i}\widetilde{\beta}_i\right) = \frac{T_1}{\sum\limits_{i=1}^{2} T_i}\widetilde{\beta}_1 \oplus_H \frac{T_2}{\sum\limits_{i=1}^{2} T_i}\widetilde{\beta}_2 =$$

$$= \left\langle \mathfrak{S}_t\left(\frac{\left(t^q+\gamma_1^q(\varsigma-1)\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}}-\left(t^q-\gamma_1^q\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}}}{\left(t^q+\gamma_1^q(\varsigma-1)\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}}+(\varsigma-1)\left(t^q-\gamma_1^q\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}}}\right)^{\frac{1}{q}}, \mathfrak{S}_t\left(\frac{\varsigma\zeta_1^{\frac{T_1}{\sum_{i=1}^{2} T_i}q}}{\left(t^q+(\varsigma-1)\left(t^q-\zeta_1^q\right)\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}}+(\varsigma-1)\zeta_1^{\frac{T_1}{\sum_{i=1}^{2} T_i}q}}\right)^{\frac{1}{q}}\right\rangle \oplus_H$$

$$\left\langle \mathfrak{S}_t\left(\frac{\left(t^q+\gamma_2^q(\varsigma-1)\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}}-\left(t^q-\gamma_2^q\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}}}{\left(t^q+\gamma_2^q(\varsigma-1)\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}}+(\varsigma-1)\left(t^q-\gamma_2^q\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}}}\right)^{\frac{1}{q}}, \mathfrak{S}_t\left(\frac{\varsigma\zeta_2^{\frac{T_2}{\sum_{i=1}^{2} T_i}q}}{\left(t^q+(\varsigma-1)\left(t^q-\zeta_2^q\right)\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}}+(\varsigma-1)\zeta_2^{\frac{T_2}{\sum_{i=1}^{2} T_i}q}}\right)^{\frac{1}{q}}\right\rangle$$

$$= \left\langle \mathfrak{S}_t\left(\frac{\prod\limits_{i=1}^{2}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}}-\prod\limits_{i=1}^{2}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}}}{\prod\limits_{i=1}^{2}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}}+(\varsigma-1)\prod\limits_{i=1}^{2}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}}}\right)^{\frac{1}{q}}, \mathfrak{S}_t\left(\frac{\varsigma\prod\limits_{i=1}^{2}\zeta_i^{\frac{T_i}{\sum_{i=1}^{2} T_i}q}}{\prod\limits_{i=1}^{2}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}}+(\varsigma-1)\prod\limits_{i=1}^{2}\zeta_i^{\frac{T_i}{\sum_{i=1}^{2} T_i}q}}\right)^{\frac{1}{q}}\right\rangle.$$

So, the theorem is true for $n = 2$.

Now let theorem is true for $n = m$, i.e.,

$$\text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_m\right) = \bigoplus_{i=1}^{m}{}_H \left(\frac{T_i}{\sum_{i=1}^{m} T_i} \widetilde{\beta}_i \right)$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{m}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}} - \prod_{i=1}^{m}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}}}{\prod_{i=1}^{m}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}} + (\varsigma-1)\prod_{i=1}^{m}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}}} \right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma \prod_{i=1}^{m} \zeta_i^{\frac{T_i}{\sum_{i=1}^{m}T_i} q}}{\prod_{i=1}^{m}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}} + (\varsigma-1)\prod_{i=1}^{m}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m}T_i} q}} \right)^{\frac{1}{q}}} \right\rangle.$$

Now would show that it is true for $n = m + 1$,

$$\text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_m, \widetilde{\beta}_{m+1}\right)$$
$$= \left(\text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_m\right)\right) \oplus_H \frac{T_{m+1}}{\sum_{i=1}^{m+1} T_i} \widetilde{\beta}_{m+1}$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{m}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}} - \prod_{i=1}^{m}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}}}{\prod_{i=1}^{m}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}} + (\varsigma-1)\prod_{i=1}^{m}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}}} \right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma \prod_{i=1}^{m} \zeta_i^{\frac{T_i}{\sum_{i=1}^{m}T_i} q}}{\prod_{i=1}^{m}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{m}T_i}} + (\varsigma-1)\prod_{i=1}^{m}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m}T_i} q}} \right)^{\frac{1}{q}}} \right\rangle \oplus_H$$

$$\left\langle \mathfrak{S}_{t\left(\frac{\left(t^q+\gamma_i^q(\varsigma-1)\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1} T_i}}-\left(t^q-\gamma_i^q\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1} T_i}}}{\left(t^q+\gamma_i^q(\varsigma-1)\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1} T_i}}+(\varsigma-1)\left(t^q-\gamma_i^q\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1} T_i}}}\right)^{\frac{1}{q}}}, \mathfrak{S}_{t\left(\frac{\varsigma\zeta_i^{q\frac{T_{m+1}}{\sum_{i=1}^{m+1} T_i}}}{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1} T_i}}+(\varsigma-1)\zeta_i^{q\frac{T_{m+1}}{\sum_{i=1}^{m+1} T_i}}}\right)^{\frac{1}{q}}} \right\rangle$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{m+1}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m+1} T_i}}-\prod_{i=1}^{m+1}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m+1} T_i}}}{\prod_{i=1}^{m+1}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m+1} T_i}}+(\varsigma-1)\prod_{i=1}^{m+1}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m+1} T_i}}}\right)^{\frac{1}{q}}},\right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma\prod_{i=1}^{m+1}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m+1} T_i}q}}{\prod_{i=1}^{m+1}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{m+1} T_i}}+(\varsigma-1)\prod_{i=1}^{m+1}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m+1} T_i}q}}\right)^{\frac{1}{q}}} \right\rangle.$$

Since it is valid for $n = m + 1$, theorem is proved for all n. □

In the next, some particular cases, concerning parameter ς, for Lq-ROFHPWA operator are discussed.

- When $\varsigma = 1$, Lq-ROFHPWA operator reduces to the Lq-ROF weighted average (Lq-ROFPWA) operator as follows:

$$\text{L}q\text{-ROFPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) = \left\langle \mathfrak{S}_{t\left(\frac{t^q-\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{t^q}\right)^{\frac{1}{q}}}, \mathfrak{S}_{t\left(\prod_{i=1}^{n}\left(\frac{\zeta_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}q}\right)^{\frac{1}{q}}} \right\rangle.$$

- When $\varsigma = 2$, Lq-ROFHPWA operator reduces to the Lq-ROF Einstein weighted average (Lq-ROFEPWA) operator as follows:

$$\mathrm{L}q\text{-ROFEPWA}\left(\widetilde{\beta}_1,\widetilde{\beta}_2,\ldots,\widetilde{\beta}_n\right)$$

$$=\left\langle \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{n}\left(t^q+\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(t^q+\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{2\prod_{i=1}^{n}\zeta_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}{\prod_{i=1}^{n}\left(t^q+\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\zeta_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}\right)^{\frac{1}{q}}} \right\rangle.$$

Example 1 Let $\widetilde{\beta}_1=\langle\mathfrak{S}_4,\mathfrak{S}_4\rangle$, $\widetilde{\beta}_2=\langle\mathfrak{S}_6,\mathfrak{S}_2\rangle$, $\widetilde{\beta}_3=\langle\mathfrak{S}_5,\mathfrak{S}_3\rangle$ and $\widetilde{\beta}_4=\langle\mathfrak{S}_7,\mathfrak{S}_2\rangle$ be four Lq-ROFNs on LTS $\left\{S_i\middle|\, i=0,1,\ldots,8\right\}$. Utilizing the score function of Lq-ROFNs, $S\left(\widetilde{\beta}_1\right)=6.3496$, $S\left(\widetilde{\beta}_2\right)=7.1138$, $S\left(\widetilde{\beta}_3\right)=6.7313$ and $S\left(\widetilde{\beta}_4\right)=7.5096$ are obtained. So, $T_1=1$, $T_2=0.7937$, $T_3=0.7058$ and $T_4=0.5939$. Then using Lq-ROFHPWA operator, the aggregated value of $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, $\widetilde{\beta}_3$ and $\widetilde{\beta}_4$ is calculated as (Considering $\varsigma=3$, $q=3$)

$$\mathrm{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1,\widetilde{\beta}_2,\widetilde{\beta}_3,\widetilde{\beta}_4\right)=\left\langle \mathfrak{S}_{8\left(\frac{\prod_{i=1}^{4}\left(8^3+(3-1)\gamma_i^3\right)^{\frac{T_i}{\sum_{i=1}^{4}T_i}}-\prod_{i=1}^{4}\left(8^3-\gamma_i^3\right)^{\frac{T_i}{\sum_{i=1}^{4}T_i}}}{\prod_{i=1}^{4}\left(8^3+(3-1)\gamma_i^3\right)^{\frac{T_i}{\sum_{i=1}^{4}T_i}}+(3-1)\prod_{i=1}^{4}\left(8^3-\gamma_i^3\right)^{\frac{T_i}{\sum_{i=1}^{4}T_i}}}\right)^{\frac{1}{3}}},\right.$$

$$\left.\mathfrak{S}_{8\left(\frac{3\prod_{i=1}^{4}\zeta_i^{3\frac{T_i}{\sum_{i=1}^{4}T_i}}}{\prod_{i=1}^{4}\left(8^3+(3-1)\left(8^3-\zeta_i^3\right)\right)^{\frac{T_i}{\sum_{i=1}^{4}T_i}}+(3-1)\prod_{i=1}^{4}\zeta_i^{3\frac{T_i}{\sum_{i=1}^{4}T_i}}}\right)^{\frac{1}{3}}}\right\rangle=\left\langle\mathfrak{S}_{5.6021},\ \mathfrak{S}_{2.7571}\right\rangle.$$

Furthermore, the proposed Lq-ROFHPWA operator meets certain important properties, which are stated as follows.

Theorem 2 (Idempotency) *Let $\widetilde{\beta}_i=\left\langle\mathfrak{S}_{\gamma_i},\mathfrak{S}_{\zeta_i}\right\rangle$ $(i=1,2,\ldots,n)$ be a collection of n Lq-ROFNs. If $\widetilde{\beta}_i=\widetilde{\beta}=\left\langle\mathfrak{S}_{\gamma},\mathfrak{S}_{\zeta}\right\rangle$ for all $i=1,2,\ldots,n$, then Lq-ROFHPWA$\left(\widetilde{\beta}_1,\widetilde{\beta}_2,\ldots,\widetilde{\beta}_n\right)=\widetilde{\beta}$.*

Proof. Since $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle = \left\langle \mathfrak{S}_{\gamma}, \mathfrak{S}_{\zeta} \right\rangle = \widetilde{\beta}$ for all $i = 1, 2, \ldots, n$;
Then,

$$\text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) = \text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}, \widetilde{\beta}, \ldots, \widetilde{\beta}\right)$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{n} (t^q+(\varsigma-1)\gamma^q)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} (t^q-\gamma^q)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} (t^q+(\varsigma-1)\gamma^q)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1) \prod_{i=1}^{n} (t^q-\gamma^q)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma \prod_{i=1}^{n} \zeta^{\frac{T_i}{\sum_{i=1}^{n} T_i} q}}{\prod_{i=1}^{n} (t^q+(\varsigma-1)(t^q-\zeta^q))^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1) \prod_{i=1}^{n} \zeta^{\frac{T_i}{\sum_{i=1}^{n} T_i} q}} \right)^{\frac{1}{q}}} \right\rangle$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{(t^q+(\varsigma-1)\gamma^q)^{\frac{T_1}{\sum_{i=1}^{n} T_i} + \frac{T_2}{\sum_{i=1}^{n} T_i} + \ldots + \frac{T_n}{\sum_{i=1}^{n} T_i}} - (t^q-\gamma^q)^{\frac{T_1}{\sum_{i=1}^{n} T_i} + \frac{T_2}{\sum_{i=1}^{n} T_i} + \ldots + \frac{T_n}{\sum_{i=1}^{n} T_i}}}{(t^q+(\varsigma-1)\gamma^q)^{\frac{T_1}{\sum_{i=1}^{n} T_i} + \frac{T_2}{\sum_{i=1}^{n} T_i} + \ldots + \frac{T_n}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)(t^q-\gamma^q)^{\frac{T_1}{\sum_{i=1}^{n} T_i} + \frac{T_2}{\sum_{i=1}^{n} T_i} + \ldots + \frac{T_n}{\sum_{i=1}^{n} T_i}}} \right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma \zeta^{q\left(\frac{T_1}{\sum_{i=1}^{n} T_i} + \frac{T_2}{\sum_{i=1}^{n} T_i} + \ldots + \frac{T_n}{\sum_{i=1}^{n} T_i}\right)}}{(t^q+(\varsigma-1)(t^q-\zeta^q))^{\frac{T_1}{\sum_{i=1}^{n} T_i} + \frac{T_2}{\sum_{i=1}^{n} T_i} + \ldots + \frac{T_n}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\zeta^{q\left(\frac{T_1}{\sum_{i=1}^{n} T_i} + \frac{T_2}{\sum_{i=1}^{n} T_i} + \ldots + \frac{T_n}{\sum_{i=1}^{n} T_i}\right)}} \right)^{\frac{1}{q}}} \right\rangle$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{(t^q+(\varsigma-1)\gamma^q)-(t^q-\gamma^q)}{(t^q+(\varsigma-1)\gamma^q)+(\varsigma-1)(t^q-\gamma^q)} \right)^{\frac{1}{q}}}, \ \mathfrak{S}_{t\left(\frac{\varsigma\zeta^q}{(t^q+(\varsigma-1)(t^q-\zeta^q))+(\varsigma-1)\zeta^q} \right)^{\frac{1}{q}}} \right\rangle = \left\langle \mathfrak{S}_{\gamma}, \ \mathfrak{S}_{\zeta} \right\rangle.$$

Hence the theorem is proved. □

Theorem 3 (Boundedness) *Let $\widetilde{\beta}_i = \left\langle S_{\gamma_i}, S_{\zeta_i} \right\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of Lq-ROFNs, and $\gamma^- = \{\gamma_i\}$, $\gamma^+ = \{\gamma_i\}$, $\zeta^- = \{\zeta_i\}$, $\zeta^+ = \{\zeta_i\}$ then*

$$\widetilde{\beta}^- \leqslant Lq\text{-}ROFHPWA\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) \leqslant \widetilde{\beta}^+,$$

where $\widetilde{\beta}^- = \left\langle S_{\gamma^-}, S_{\zeta^+} \right\rangle$ and $\widetilde{\beta}^+ = \left\langle S_{\gamma^+}, S_{\zeta^-} \right\rangle$.

Proof. Let $f(x) = \dfrac{t^q + (\varsigma - 1)x}{t^q - x}$, $x \in [0, t)$, then $f'(x) = \dfrac{t^q \varsigma}{(t^q - x)^2} > 0$, thus f is an increasing function. Since $\gamma^- \leqslant \gamma_i \leqslant \gamma^+$, for all $i = 1, 2, \ldots, n$,

$$\frac{\left(t^q + (\varsigma - 1)\left(\gamma^-\right)^q\right)}{\left(t^q - \left(\gamma^-\right)^q\right)} \leqslant \frac{\left(t^q + (\varsigma - 1)\gamma_i^q\right)}{\left(t^q - \gamma_i^q\right)} \leqslant \frac{\left(t^q + (\varsigma - 1)\left(\gamma^+\right)^q\right)}{\left(t^q - \left(\gamma^+\right)^q\right)}.$$

So,

$$\left(\frac{\left(t^q + (\varsigma-1)\left(\gamma^-\right)^q\right)}{\left(t^q - \left(\gamma^-\right)^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} \leqslant \left(\frac{\left(t^q + (\varsigma-1)\gamma_i^q\right)}{\left(t^q - \gamma_i^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} \leqslant \left(\frac{\left(t^q + (\varsigma-1)\left(\gamma^+\right)^q\right)}{\left(t^q - \left(\gamma^+\right)^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}}$$

$$\Leftrightarrow \prod_{i=1}^n \left(\frac{\left(t^q + (\varsigma - 1)\left(\gamma^-\right)^q\right)}{\left(t^q - \left(\gamma^-\right)^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} \leqslant \prod_{i=1}^n \left(\frac{\left(t^q + (\varsigma - 1)\gamma_i^q\right)}{\left(t^q - \gamma_i^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}}$$

$$\leqslant \prod_{i=1}^n \left(\frac{\left(t^q + (\varsigma - 1)\left(\gamma^+\right)^q\right)}{\left(t^q - \left(\gamma^+\right)^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} \Leftrightarrow \frac{\left(t^q + (\varsigma - 1)\left(\gamma^-\right)^q\right)}{\left(t^q - \left(\gamma^-\right)^q\right)} + (\varsigma - 1)$$

$$\leqslant \prod_{i=1}^n \left(\frac{\left(t^q + (\varsigma - 1)\gamma_i^q\right)}{\left(t^q - \gamma_i^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} + (\varsigma - 1) \leqslant \frac{\left(t^q + (\varsigma - 1)\left(\gamma^+\right)^q\right)}{\left(t^q - \left(\gamma^+\right)^q\right)} + (\varsigma - 1)$$

$$\Leftrightarrow \frac{1}{\frac{\left(t^q + (\varsigma-1)\left(\gamma^-\right)^q\right)}{\left(t^q - \left(\gamma^-\right)^q\right)} + (\varsigma - 1)} \geqslant \frac{1}{\prod_{i=1}^n \left(\frac{\left(t^q + (\varsigma-1)\gamma_i^q\right)}{\left(t^q - \gamma_i^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} + (\varsigma - 1)}$$

$$\geqslant \frac{1}{\frac{\left(t^q + (\varsigma-1)\left(\gamma^+\right)^q\right)}{\left(t^q - \left(\gamma^+\right)^q\right)} + (\varsigma - 1)} \Leftrightarrow \frac{\varsigma\left(t^q - \left(\gamma^-\right)^q\right)}{\varsigma t^q}$$

$$\geqslant \frac{\varsigma \prod_{i=1}^n \left(t^q - \gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod_{i=1}^n \left(t^q + (\varsigma - 1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} + (\varsigma - 1)\prod_{i=1}^n \left(t^q - \gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^n T_i}}} \geqslant \frac{\varsigma\left(t^q - \left(\gamma^+\right)^q\right)}{\varsigma t^q}$$

$$\Leftrightarrow 1-\frac{\varsigma\left(t^q-(\gamma^-)^q\right)}{\varsigma t^q}\leqslant 1-\frac{\varsigma\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}$$

$$\leqslant 1-\frac{\varsigma\left(t^q-(\gamma^-)^q\right)}{\varsigma t^q}\Leftrightarrow\frac{(\gamma^-)^q}{t^q}$$

$$\leqslant 1-\frac{\varsigma\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\leqslant\frac{(\gamma^+)^q}{t^q}$$

i.e.,

$$\gamma^-\leqslant t\left(\frac{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)^{\frac{1}{q}}\leqslant\gamma^+. \tag{5}$$

Again let $g(y)=\frac{(t^q+(\varsigma-1)(t^q-y))}{y}$, $y\in(0,t]$, $\varsigma>0$, then $g'(y)=-\frac{\varsigma t^q}{y^2}<0$, thus $g(y)$ is a decreasing function.

Since for all i, $\zeta^+\geqslant\zeta_i\geqslant\zeta^-$, then

$$\frac{\left(t^q+(\varsigma-1)\left(t^q-(\zeta^+)^q\right)\right)}{(\zeta^+)^q}\leqslant\frac{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\leqslant\frac{(t^q+(\varsigma-1)(t^q-(\zeta^-)^q))}{(\zeta^-)^q},$$

thus,

$$\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-(\zeta^+)^q\right)\right)}{(\zeta^+)^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\leqslant\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}$$

$$\leqslant\left(\frac{(t^q+(\varsigma-1)(t^q-(\zeta^-)^q))}{(\zeta^-)^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\Leftrightarrow\prod_{i=1}^{n}\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-(\zeta^+)^q\right)\right)}{(\zeta^+)^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}$$

$$\leqslant \prod_{i=1}^{n}\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-(\zeta^-)^q\right)\right)}{(\zeta^-)^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}$$

$$\Leftrightarrow \frac{\left(t^q+(\varsigma-1)\left(t^q-(\zeta^+)^q\right)\right)}{(\zeta^+)^q} \leqslant \prod_{i=1}^{n}\left(\frac{\left(t^q+t(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}$$

$$\leqslant \frac{\left(t^q+(\varsigma-1)\left(t^q-(\zeta^-)^q\right)\right)}{(\zeta^-)^q} \Leftrightarrow \frac{\varsigma t^q-(\varsigma-1)(\zeta^+)^q}{(\zeta^+)^q}+(\varsigma-1)$$

$$\leqslant \prod_{i=1}^{n}\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1) \leqslant \frac{\varsigma t^q-(\varsigma-1)(\zeta^-)^q}{(\zeta^-)^q}+(\varsigma-1)$$

$$\Leftrightarrow \frac{1}{\frac{\varsigma t^q}{(\zeta^+)^q}} \geqslant \frac{1}{\prod_{i=1}^{n}\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)} \geqslant \frac{1}{\frac{\varsigma t^q}{(\zeta^-)^q}}$$

$$\Leftrightarrow \zeta^+ \geqslant t\left(\frac{\varsigma\prod_{i=1}^{n}\zeta_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)\prod_{i=1}^{n}\zeta_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}\right)^{\frac{1}{q}} \geqslant \zeta^-. \quad (6)$$

From (5) and (6), it is clear that

$$S\left(\widetilde{\beta}^-\right) \leqslant S\left(\text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\right)\right) \leqslant S\left(\widetilde{\beta}^+\right).$$

Therefore, $\widetilde{\beta}^- \leqslant \text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\right) \leqslant \widetilde{\beta}^+$.

Definition 8 *Let* $\left\{\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\right\}$ *be a set of Lq-ROFNs, where* $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i}\right\rangle$ *(*$i = 1,2,\ldots,n$*) and* $q \geqslant 1$. *Then Lq-ROFHPWG operator is defined as*

$$Lq\text{-}ROFHPWG\left(\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\right) = \bigotimes_{i=1}^{n}{}_H\left(\widetilde{\beta}_i\right)^{\omega_i}, \quad (7)$$

where $\omega = (\omega_1,\omega_2,\ldots,\omega_n)$ *is the weight vectors of* $\widetilde{\beta}_i$ *with* $\omega_i \in [0,1]$ *and* $\omega_i = \frac{T_i}{\sum_{i=1}^{n}T_i}$, $T_i = \prod_{k=1}^{i-1}\frac{S\left(\widetilde{\beta}_k\right)}{t}$ *(*$i = 2,\ldots,n$*),* $T_1 = 1$ *and* $S\left(\widetilde{\beta}_i\right)$ *is the score of* $\widetilde{\beta}_i$.

Theorem 4 *Let $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle$ $(i = 1, 2, \ldots, n)$ be a set of Lq-ROFNs. Then, the aggregated result from the Lq-ROFHPWG operator is also a Lq-ROFN, where*

$$Lq\text{-}ROFHPWG\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) = \bigotimes_{i=1}^{n}{}_H \left(\widetilde{\beta}_i\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\varsigma \prod_{i=1}^{n} \gamma_i^{\frac{T_i}{\sum_{i=1}^{n} T_i} q}}{\prod_{i=1}^{n}\left(t^q + (\varsigma-1)\left(t^q - \gamma_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\prod_{i=1}^{n} \gamma_i^{\frac{T_i}{\sum_{i=1}^{n} T_i} q}}\right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{n}\left(t^q + (\varsigma-1)\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n}\left(t^q - \zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n}\left(t^q + (\varsigma-1)\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\prod_{i=1}^{n}\left(t^q - \zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right)^{\frac{1}{q}}} \right\rangle.$$

Proof. Proof of this theorem is similar to the proof of Theorem 1.

Now, some particular cases of the Lq-ROFHPWG operator are discussed based on parameter ς.

- When $\varsigma = 1$, Lq-ROFHPWG operator reduces to the Lq-ROF prioritized weighted geometric (Lq-ROFPWG) operator as follows:

$$\text{L}q\text{-ROFPWG}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) = \left\langle \mathfrak{S}_{t\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i} q}\right)^{\frac{1}{q}}}, \mathfrak{S}_{t\left(\frac{t^q - \prod_{i=1}^{n}\left(t^q - \zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{t^q}\right)^{\frac{1}{q}}} \right\rangle.$$

- When $\varsigma = 2$, Lq-ROFHPWG operator reduces to the Lq-ROF Einstein prioritized weighted geometric (Lq-ROFEPWG) operator as follows:

$$\text{L}q\text{-ROFEPWG}\left(\widetilde{\beta}_1,\widetilde{\beta}_2, ldots,\widetilde{\beta}_n\right)$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{2\prod_{i=1}^{n}\gamma_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}{\prod_{i=1}^{n}\left(t^q+\left(t^q-\gamma_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\gamma_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}\right)^{\frac{1}{q}}}, \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{n}\left(t^q+\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(t^q-\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(t^q+\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(t^q-\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)^{\frac{1}{q}}} \right\rangle.$$

Theorem 5 (Idempotency) *Let* $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle$ *$(i = 1,2,\dots,n)$ be a collection of n Lq-ROFNs. If* $\widetilde{\beta}_i = \widetilde{\beta} = \left\langle \mathfrak{S}_{\gamma}, \mathfrak{S}_{\zeta} \right\rangle$ *for all* $i = 1,2,\dots,n$, *then* $Lq\text{-}ROFHPWG\left(\widetilde{\beta}_1,\widetilde{\beta}_2,\dots,\widetilde{\beta}_n\right) = \widetilde{\beta}$.

Theorem 6 (Boundedness) *Let* $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle$ *$(i = 1,2,\dots,n)$ be a collection of Lq-ROFNs, and* $\gamma^- = \min_i \{\gamma_i\}$, $\gamma^+ = \max_i \{\gamma_i\}$, $\zeta^- = \min_i \{\zeta_i\}$, $\zeta^+ = \max_i \{\zeta_i\}$ *then*

$$\widetilde{\beta}^- \leqslant Lq\text{-}ROFHPWG\left(\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\right) \leqslant \widetilde{\beta}^+.$$

The proofs of Theorem 5 and 6 are analogous to the previous.

5. An MCGDM approach based on Lq-ROF prioritized aggregation operators

In this section, a novel MCGDM approach have been propounded, in which the evaluation information is in the form of Lq-ROFNs.

For a group decision making problem, let $\mathcal{E} = \left\{\mathcal{E}^{(1)}, \mathcal{E}^{(2)}, \dots, \mathcal{E}^{(u)}\right\}$ be the set of the DMs and the linear ordering $\mathcal{E}^{(1)} \succ \mathcal{E}^{(2)} \succ \dots \succ \mathcal{E}^{(u)}$ represents the prioritization relationship among the DMs' in such a manner that DM, $\mathcal{E}^{(k)}$, has a higher priority than DM, $\mathcal{E}^{(l)}$, if $k < l$. Suppose $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be a discrete collection of alternatives. $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n\}$ represents the set of criteria with their prioritization as $\mathcal{G}_1 \succ \mathcal{G}_2 \succ \dots \succ \mathcal{G}_n$, so that criteria $\mathcal{G}_j$ has a higher priority than $\mathcal{G}_i$, for $j < i$. DMs provide their evaluation values in terms of Lq-ROFNs based on LTS: $\mathfrak{S} = \{\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}_2, \dots, \mathfrak{S}_t\}$. A L$q$-ROF decision matrix (L$q$-ROFDM) $\widetilde{X}^{(l)} = \left[\widetilde{\beta}_{ij}^{(l)}\right]_{m\times n} = \left[\left\langle \mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}}, \mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}} \right\rangle\right]_{m\times n}$, where $\left\langle \mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}}, \mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}} \right\rangle$ denotes a Lq-ROFN given by the DM $\mathcal{E}^{(l)}$ for the alternative A_i

under the criteria $\mathcal{G}_j$. Here corresponding to the DM $\mathcal{E}^{(l)}$, $\mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}}$ indicates the satisfaction degree of the alternative A_i concerning the criteria $\mathcal{G}_j$; whereas $\mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}}$ indicates that of dissatisfaction degree.

The purpose is to find the best suitable alternative(s) in light of the presented approach. The computational process is summarized step-by-step as follows.

Step 1. Normalize $\widetilde{\mathcal{X}}^{(l)}$, if required, into $\widetilde{R}^{(l)} = \left[\widetilde{r}^{(l)}_{ij}\right]_{m\times n}$ as follows:

$$\widetilde{r}^{(l)}_{ij} = \begin{cases} \left\langle \mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}},\ \mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}} \right\rangle & \text{if } \mathcal{G}_j \text{ is type of benefit criteria;} \\ \left\langle \mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}},\ \mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}} \right\rangle & \text{if } \mathcal{G}_j \text{ is type of cost criteria.} \end{cases}$$

Step 2. Calculate the value of $T^{(l)}_{ij}$ $(l = 1, 2, \dots, u)$ with the following equations.

$$T^{(l)}_{ij} = \begin{cases} 1 & \text{for } l = 1, \\ \prod_{k=1}^{l-1} \dfrac{S\left(\widetilde{r}^{(k)}_{ij}\right)}{t} & \text{for } l = 2, 3, \dots, u. \end{cases} \tag{8}$$

Step 3. To aggregate all the individual Lq-ROFDM $\widetilde{R}^{(l)} = \left[\widetilde{r}^{(l)}_{ij}\right]_{m\times n}$ $(l = 1, 2, \dots, u)$, using the Lq-ROFHPWA operator and obtain overall DM $\widetilde{R} = \left[\widetilde{r}_{ij}\right]_{m\times n}$ as

$$\widetilde{r}_{ij} = \text{L}q\text{-ROFHPWA}\left(\widetilde{r}^{(1)}_{ij}, \widetilde{r}^{(2)}_{ij}, \dots, \widetilde{r}^{(u)}_{ij}\right)$$

$$= \left\langle \mathfrak{S}_{t\left(\dfrac{\prod_{l=1}^{u}\left(t^q+(\varsigma-1)\left(\gamma^{(l)}_{ij}\right)^q\right)^{\frac{T^{(l)}_{ij}}{\sum_{l=1}^{u} T^{(l)}_{ij}}} - \prod_{l=1}^{u}\left(t^q-\left(\gamma^{(l)}_{ij}\right)^q\right)^{\frac{T^{(l)}_{ij}}{\sum_{l=1}^{u} T^{(l)}_{ij}}}}{\prod_{l=1}^{u}\left(t^q+(\varsigma-1)\left(\gamma^{(l)}_{ij}\right)^q\right)^{\frac{T^{(l)}_{ij}}{\sum_{l=1}^{u} T^{(l)}_{ij}}} + (\varsigma-1)\prod_{l=1}^{u}\left(t^q-\left(\gamma^{(l)}_{ij}\right)^q\right)^{\frac{T^{(l)}_{ij}}{\sum_{l=1}^{u} T^{(l)}_{ij}}}}\right)^{\frac{1}{q}}}, \right.$$

$$\mathfrak{S}_{t\left(\frac{\varsigma\prod_{l=1}^{u}\left(\zeta_{ij}^{(l)}\right)^{q\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}}{\prod_{l=1}^{u}\left(t^{q}+(\varsigma-1)\left(t^{q}-\left(\zeta_{ij}^{(l)}\right)^{q}\right)\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}+(\varsigma-1)\prod_{l=1}^{u}\left(\zeta_{ij}^{(l)}\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}q}}\right)^{\frac{1}{q}}}\Bigg\rangle. \tag{9}$$

or, using the Lq-ROFHPWG operator

$$\widetilde{r}'_{ij}=\text{L}q\text{-ROFHPWG}\left(\widetilde{r}_{ij}^{(1)},\widetilde{r}_{ij}^{(2)},\ldots,\widetilde{r}_{ij}^{(u)}\right)$$

$$=\Bigg\langle\mathfrak{S}_{t\left(\frac{\varsigma\prod_{l=1}^{u}\left(\gamma_{ij}^{(l)}\right)^{q\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}}{\prod_{l=1}^{u}\left(t^{q}+(\varsigma-1)\left(t^{q}-\left(\gamma_{ij}^{(l)}\right)^{q}\right)\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}+(\varsigma-1)\prod_{l=1}^{u}\left(\gamma_{ij}^{(l)}\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}q}}\right)^{\frac{1}{q}}},$$

$$\mathfrak{S}_{t\left(\frac{\prod_{l=1}^{u}\left(t^{q}+(\varsigma-1)\left(\zeta_{ij}^{(l)}\right)^{q}\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}-\prod_{l=1}^{u}\left(t^{q}-\left(\zeta_{ij}^{(l)}\right)^{q}\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}}{\prod_{l=1}^{u}\left(t^{q}+(\varsigma-1)\left(\zeta_{ij}^{(l)}\right)^{q}\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}+(\varsigma-1)\prod_{l=1}^{u}\left(t^{q}-\left(\zeta_{ij}^{(l)}\right)^{q}\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}}\right)^{\frac{1}{q}}}\Bigg\rangle. \tag{10}$$

Step 4. Calculate the values of T_{ij} as

$$T_{ij}=\begin{cases}1 & \text{for } j=1\\ \prod_{k=1}^{j-1}\dfrac{S\left(\widetilde{r}_{ik}\right)}{t} & \text{for } j=2,3,\ldots,n.\end{cases} \tag{11}$$

Step 5. Aggregate the Lq-ROFNs $\widetilde{r}_{ij}$ for each alternative A_i using the Lq-ROFHPWA (or Lq-ROFHPWG) operators as follows:

$$\widetilde{r}_i = \text{L}q\text{-ROFHPWA}\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \dots, \widetilde{r}_{in}\right)$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\prod_{j=1}^{n}\left(t^q+(\varsigma-1)\gamma_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}} - \prod_{j=1}^{n}\left(t^q-\gamma_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\prod_{j=1}^{n}\left(t^q+(\varsigma-1)\gamma_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}} + (\varsigma-1)\prod_{j=1}^{n}\left(t^q-\gamma_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}\right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma\prod_{j=1}^{n}\left(\zeta_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}q}}{\prod_{j=1}^{n}\left(t^q+(\varsigma-1)\left(t^q-\zeta_{ij}^q\right)\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}} + (\varsigma-1)\prod_{j=1}^{n}\left(\zeta_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}q}}\right)^{\frac{1}{q}}} \right\rangle, \tag{12}$$

or

$$\widetilde{r}_i' = \text{L}q\text{-ROFHPWG}\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \dots, \widetilde{r}_{in}\right)$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\varsigma\prod_{j=1}^{n}\left(\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}q}}{\prod_{j=1}^{n}\left(t^q+(\varsigma-1)\left(t^q-\gamma_{ij}^q\right)\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}} + (\varsigma-1)\prod_{j=1}^{n}\left(\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}q}}\right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\prod_{j=1}^{n}\left(t^q+(\varsigma-1)\zeta_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}} - \prod_{j=1}^{n}\left(t^q-\zeta_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\prod_{j=1}^{n}\left(t^q+(\varsigma-1)\zeta_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}} + (\varsigma-1)\prod_{j=1}^{n}\left(t^q-\zeta_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}\right)^{\frac{1}{q}}} \right\rangle. \tag{13}$$

Step 6. Calculate the score values for each $\widetilde{r}_i$ (or $\widetilde{r}'_i$) $(i = 1, 2, \ldots, m)$ using Eq. (2).

Step 7. Rank the alternatives A_i $(i = 1, 2, \ldots, m)$ based on the comparison rule presented in Definition 4.

Based on the methodology developed in this paper, the following illustrative example is considered and solved.

6. Illustrative example

In this section, a numerical example, previously studied by Arora and Garg [50], has been illustrated from the field of global suppliers with Lq-ROF context.

Following notations are used to represent the MCGDM problem relating to the selection of the best global suppliers by a manufacturing company to utilize in their assembling process.

Suppose there are four alternatives A_1, A_2, A_3 and A_4 which are considered for evaluating over the five criteria $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5\}$. The prioritization relationship for the criterion is $\mathcal{G}_1 \succ \mathcal{G}_2 \succ \mathcal{G}_3 \succ \mathcal{G}_4 \succ \mathcal{G}_5$. The different alternatives A_i $(i = 1, 2, 3, 4)$ are evaluated by the four DMs, $\mathcal{E}^{(l)}$ $(l = 1, 2, 3, 4)$ with priority levels $\mathcal{E}^{(1)} \succ \mathcal{E}^{(2)} \succ \mathcal{E}^{(3)} \succ \mathcal{E}^{(4)}$ on the basis of the criteria $\mathcal{G}_i$ $(i = 1, 2, 3, 4, 5)$. DMs $\mathcal{E}^{(l)}$ $(l = 1, 2, 3, 4)$ provide his/her decision preferences in terms of Lq-ROFNs using the linguistic term set: $\mathfrak{S} = \{\mathfrak{S}_0$ = extremely poor, $\mathfrak{S}_1$ = very poor, $\mathfrak{S}_2$ = poor, $\mathfrak{S}_3$ = slightly poor, $\mathfrak{S}_4$ = fair, $\mathfrak{S}_5$ = slightly good, $\mathfrak{S}_6$ = good, $\mathfrak{S}_7$ = very good, $\mathfrak{S}_8$ = extremely good$\}$. In Tables 1, 2, 3 and 4, the decision information provided by the four DMs, $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$, $\mathcal{E}^{(3)}$ and $\mathcal{E}^{(4)}$ are presented in terms of Lq-ROFNs, respectively.

Table 1: Lq-ROFDM $\widetilde{X}^{(1)}$ provided by the DM $\mathcal{E}^{(1)}$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$
A_2	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$
A_3	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_3)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_3, \mathfrak{S}_4)$
A_4	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$

The procedure of selecting the most desirable alternative(s) utilizing the above-proposed operators are presented in the following steps.

Step 1. Since all the criteria are of the same type, the normalization process is not needed for this problem, i.e., $\widetilde{X}^{(l)} = \widetilde{R}^{(l)} = \left[\widetilde{r}_{ij}^{(l)}\right]_{m\times n}$ $(l = 1, 2, 3, 4)$.

Table 2: Lq-ROFDM $\widetilde{\mathcal{X}}^{(2)}$ provided by the DM $\mathcal{E}^{(2)}$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_3, \mathfrak{S}_5)$
A_2	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$
A_3	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_3)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$
A_4	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_3)$

Table 3: Lq-ROFDM $\widetilde{\mathcal{X}}^{(3)}$ provided by the DM $\mathcal{E}^{(3)}$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_3, \mathfrak{S}_4)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$
A_2	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$
A_3	$(\mathfrak{S}_5, \mathfrak{S}_3)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_3, \mathfrak{S}_1)$
A_4	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$

Table 4: Lq-ROFDM $\widetilde{\mathcal{X}}^{(4)}$ provided by the DM $\mathcal{E}^{(4)}$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_5, \mathfrak{S}_3)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_2)$
A_2	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$
A_3	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_3, \mathfrak{S}_4)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_3, \mathfrak{S}_3)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$
A_4	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$

Step 2. Utilizing Eq. (8), the values of T_{ij} are obtained as:

$$T_{ij}^1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad T_{ij}^2 = \begin{bmatrix} 0.9413 & 0.8892 & 0.8124 & 0.9413 & 0.8501 \\ 0.8892 & 0.8501 & 0.8921 & 0.8892 & 0.9413 \\ 0.8921 & 0.8414 & 0.9413 & 0.8532 & 0.7741 \\ 0.8501 & 0.9413 & 0.8124 & 0.8921 & 0.7937 \end{bmatrix},$$

$$T_{ij}^3 = \begin{bmatrix} 0.8860 & 0.7058 & 0.7224 & 0.8001 & 0.6286 \\ 0.8370 & 0.7253 & 0.7958 & 0.7559 & 0.7647 \\ 0.7583 & 0.7506 & 0.8860 & 0.7179 & 0.6144 \\ 0.7559 & 0.7647 & 0.6906 & 0.8397 & 0.6678 \end{bmatrix},$$

$$T_{ij}^4 = \begin{bmatrix} 0.8860 & 0.7058 & 0.7224 & 0.8001 & 0.6286 \\ 0.8370 & 0.7253 & 0.7958 & 0.7559 & 0.7647 \\ 0.7583 & 0.7506 & 0.8860 & 0.7179 & 0.6144 \\ 0.7559 & 0.7647 & 0.6906 & 0.8397 & 0.6678 \end{bmatrix}.$$

Step 3. Based on the DMs' information provided in Tables 1, 2, 3 and 4, the proposed Lq-ROFHPWA operator, presented in Eq. (9), is utilized to aggregate them into a collective matrix. The result obtained is summarized in Table 5.

Table 5: Collective Lq-ROFDM $\widetilde{R}$ based on Lq-ROFHPWA operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_{6.4811}, \mathfrak{S}_{1.2739})$	$(\mathfrak{S}_{5.0036}, \mathfrak{S}_{2.7792})$	$(\mathfrak{S}_{5.3178}, \mathfrak{S}_{2.3785})$	$(\mathfrak{S}_{6.2793}, \mathfrak{S}_{1.2059})$	$(\mathfrak{S}_{4.4125}, \mathfrak{S}_{2.6195})$
A_2	$(\mathfrak{S}_{6.5691}, \mathfrak{S}_{1.2187})$	$(\mathfrak{S}_{5.7933}, \mathfrak{S}_{1.4508})$	$(\mathfrak{S}_{6.2900}, \mathfrak{S}_{1.0000})$	$(\mathfrak{S}_{5.5805}, \mathfrak{S}_{2.0000})$	$(\mathfrak{S}_{5.8300}, \mathfrak{S}_{1.3641})$
A_3	$(\mathfrak{S}_{5.3499}, \mathfrak{S}_{1.7823})$	$(\mathfrak{S}_{5.0564}, \mathfrak{S}_{2.1833})$	$(\mathfrak{S}_{6.6134}, \mathfrak{S}_{1.1640})$	$(\mathfrak{S}_{4.5227}, \mathfrak{S}_{2.1264})$	$(\mathfrak{S}_{3.7615}, \mathfrak{S}_{2.6698})$
A_4	$(\mathfrak{S}_{5.4024}, \mathfrak{S}_{2.1750})$	$(\mathfrak{S}_{6.1455}, \mathfrak{S}_{1.3558})$	$(\mathfrak{S}_{4.5377}, \mathfrak{S}_{1.9576})$	$(\mathfrak{S}_{6.1452}, \mathfrak{S}_{1.3678})$	$(\mathfrak{S}_{4.4942}, \mathfrak{S}_{3.2912})$

Step 4. Using Eq. (11), the values of T_{ij} are calculated as:

$$T_{ij} = \begin{bmatrix} 1.0000 & 0.9141 & 0.7716 & 0.6627 & 0.5995 \\ 1.0000 & 0.9186 & 0.8105 & 0.7338 & 0.6395 \\ 1.0000 & 0.8636 & 0.7348 & 0.6767 & 0.5646 \\ 1.0000 & 0.8635 & 0.7755 & 0.6482 & 0.5820 \end{bmatrix}.$$

Step 5. The collective value $\widetilde{r}_i$ of each alternative A_i is obtained based on Lq-ROFHPWA operator using Eq. (12).

$$\widetilde{r}_1 = (\mathfrak{S}_{5.6888}, \mathfrak{S}_{1.9095}), \quad \widetilde{r}_2 = (\mathfrak{S}6.0830, \mathfrak{S}_{1.3555}),$$
$$\widetilde{r}_3 = (\mathfrak{S}_{5.3168}, \mathfrak{S}_{1.8840}), \quad \widetilde{r}_4 = (\mathfrak{S}_{5.4816}, \mathfrak{S}_{1.8903}).$$

Step 6. The score values for each $\widetilde{r}_i$ $(i = 1, 2, 3, 4)$ are calculated based on Eq. (2) as:

$$S(\widetilde{r}_1) = 7.0107, \quad S(\widetilde{r}_2) = 7.1615, \quad S(\widetilde{r}_3) = 6.8951, \quad S(\widetilde{r}_4) = 6.9450.$$

Step 7. The rank of the alternatives A_i $(i = 1, 2, 3, 4)$ based on the comparison rule presented in Definition 4 is found as $A_2 \succ A_1 \succ A_4 \succ A_3$.

On the other hand, if the above MCGDM problem is solved with Lq-ROFHPWG operator, the score values of four different alternatives are obtained as:

$$S(\widetilde{r}'_1) = 6.8368, \quad S(\widetilde{r}'_2) = 7.0992, \quad S(\widetilde{r}'_3) = 6.7596, \quad S(\widetilde{r}'_4) = 6.8349.$$

Thus the ordering of the alternatives are found as $A_2 \succ A_1 \succ A_4 \succ A_3$.

6.1. Influence of rung parameter q on decision making results

The proposed methodology allows DMs to flexibly change their range of evaluation information with the use of rung parameter q. The parameter q plays a significant role in the decision results. In solving the above numerical problem, the parameter $q = 3$ is considered. To investigate the impact of rung parameter q on the decision result, the above problem is further solved based on different values of the parameter q from 1 to 10. For convenience, the Hamacher parameter is kept fixed at $\varsigma = 3$ in the computational process.

Table 6: Influence of rung parameter q with Lq-ROFHPWA operator on ranking results

Parameter q	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking
$q = 1$	5.878	6.3438	5.7277	5.7425	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 2$	6.7813	7.0339	6.6403	6.703	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 3$	7.0107	7.1615	6.8951	6.945	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 4$	7.1366	7.2364	7.0445	7.0817	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 5$	7.2323	7.3004	7.1590	7.1871	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 6$	7.3109	7.3578	7.2523	7.274	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 7$	7.3767	7.4091	7.3296	7.3466	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 8$	7.4322	7.4546	7.3941	7.4077	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 9$	7.4794	7.4949	7.4485	7.4594	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 10$	7.5199	7.5307	7.4947	7.5035	$A_2 \succ A_1 \succ A_4 \succ A_3$

The obtained score values for each alternative are listed in Tables 6 and 7 using Lq-ROFHPWA and Lq-ROFHPWG operators, respectively. From the ranking results as viewed from Table 7, it is inferred that slight differences in the ranking results using Lq-ROFHPWG operator are found when parameter q changes. Whereas, based on using Lq-ROFHPWA operator in Table 6, the ranking of alternatives is consistent with the rung parameter q. However, in all the cases, A_2 is the optimal choice. This indicates that the parameter q has a steadiness in the decision results in terms of generating the best choice.

Further, in Figs. 1 and 2, a clear view of the impact of rung parameters utilizing Lq-ROFHPWA and Lq-ROFHPWG operators, respectively, have been depicted. From Figs.1 and2, it is observed that when the parameter $q \in [1, 10]$ changes, the score values for the alternatives changes accordingly. It reveals from Fig. 1 that different alternatives do not change their ordered positions. Thus for Lq-ROFHPWA operator, the ranking of alternatives is stable. On the other hand, in Fig. 2, there is a change in the ordered position of the alternatives A_1 and A_4 is noticed. As a consequence, the ranking of alternatives slightly differs based on the Lq-ROFHPWG operator.

Finally, it is important to note that DMs can change the value of q according to their preferences for expressing their evaluation values in a wider range, which makes the proposed methodology a flexible method.

Table 7: Influence of rung parameter q with Lq-ROFHPWG operator on ranking results

Parameter q	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking
$q = 1$	5.6188	6.2362	5.5312	5.5679	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 2$	6.5615	6.9549	6.4661	6.5596	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 3$	6.8368	7.0992	6.7596	6.8349	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 4$	6.9933	7.1799	6.9371	6.9914	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 5$	7.1117	7.2468	7.0731	7.1104	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 6$	7.2096	7.3073	7.1840	7.2089	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 7$	7.2923	7.3624	7.2757	7.2920	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 8$	7.3625	7.4124	7.3519	7.3624	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 9$	7.4222	7.4575	7.4155	7.4223	$A_2 \succ A_4 \succ A_1 \succ A_3$
$q = 10$	7.4730	7.4979	7.4689	7.4731	$A_2 \succ A_4 \succ A_1 \succ A_3$

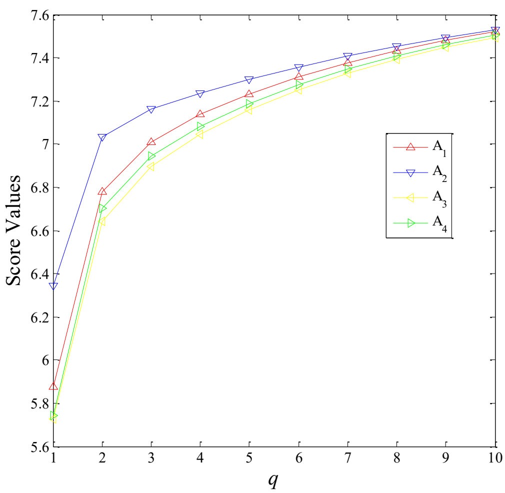

Figure 1: Score values of alternative for $q \in [1, 10]$ based on Lq-ROFHPWA operator ($\varsigma = 3$)

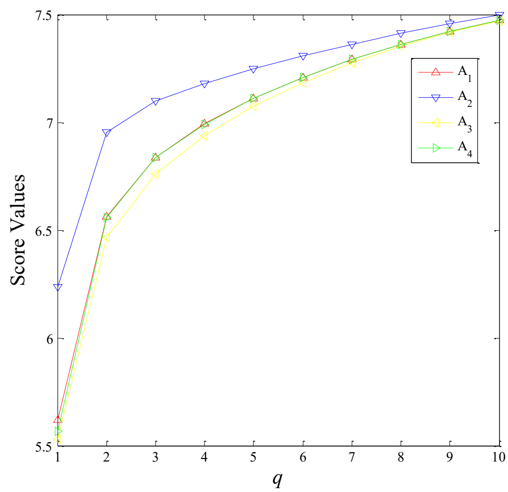

Figure 2: Score values of alternative for $q \in [1, 10]$ based on Lq-ROFHPWG operator ($\varsigma = 3$)

6.2. Influence of Hamacher parameter on decision making results

The proposed method carries the robustness of the Hamacher parameter ς. Varying the Hamacher parameter ς in (0, 10] the impact of the parameter on decision results is investigated. For convenience, the rung parameter is kept fixed at $q = 3$ in the computational process.

In Tables 8 and 9, the achieved results based on Lq-ROFHPWA and Lq-ROFHPWG operators are presented, respectively. The score of the alternatives varies accordingly with different parameters ς using Lq-ROFHPWA and Lq-ROFHPWG operators.

Table 8: Ranking results for varying ς by using Lq-ROFHPWA operator

Parameter ς	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking
$\varsigma = 1$	7.0628	7.1870	6.9333	6.9859	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 2$	7.0299	7.1705	6.9094	6.9598	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 3$	7.0107	7.1615	6.8951	6.9450	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 4$	6.9976	7.1558	6.8852	6.9352	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 5$	6.9880	7.1518	6.8778	6.9282	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 6$	6.9807	7.1488	6.8720	6.9229	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 7$	6.9748	7.1465	6.8673	6.9187	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 8$	6.9699	7.1446	6.8634	6.9153	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 9$	6.9659	7.1431	6.8601	6.9125	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 10$	6.9624	7.1418	6.8572	6.9101	$A_2 \succ A_1 \succ A_4 \succ A_3$

Table 9: Ranking results for varying ς by using Lq-ROFHPWG operator

Parameter ς	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking
$\varsigma = 1$	6.7963	7.0769	6.7349	6.8109	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\varsigma = 2$	6.8232	7.0920	6.7516	6.8270	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\varsigma = 3$	6.8368	7.0992	6.7596	6.8349	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 4$	6.8455	7.1036	6.7646	6.8399	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 5$	6.8516	7.1065	6.7681	6.8433	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 6$	6.8563	7.1086	6.7708	6.8459	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 7$	6.8600	7.1102	6.7730	6.8480	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 8$	6.8631	7.1115	6.7748	6.8497	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 9$	6.8657	7.1125	6.7763	6.8511	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 10$	6.8679	7.1133	6.7777	6.8523	$A_2 \succ A_1 \succ A_4 \succ A_3$

To visualize in effect in a better way, Figs. 3 and 4 are provided based on different values of $\varsigma \in (0,\ 10]$. In light of Fig. 3, the presented results reveal that no change in ranking order is found while using Lq-ROFHPWA operator. On the other hand, from Fig. 4, it is perceived that $\varsigma \in (0,\ 2.6050)$ the ranking is $A_2 \succ A_4 \succ A_1 \succ A_3$ and for $\varsigma \in [2.605,\ 10]$ the ranking is $A_2 \succ A_1 \succ A_4 \succ A_3$ based on Lq-ROFHPWG operator. But it is interesting to mention here that the optimal choice remains the same as A_2 for each case.

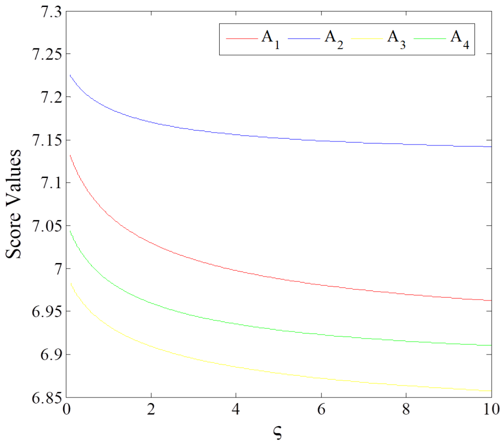

Figure 3: Score values of alternative for $\varsigma \in (0,\ 10]$ based on Lq-ROFHPWA operator ($q = 3$)

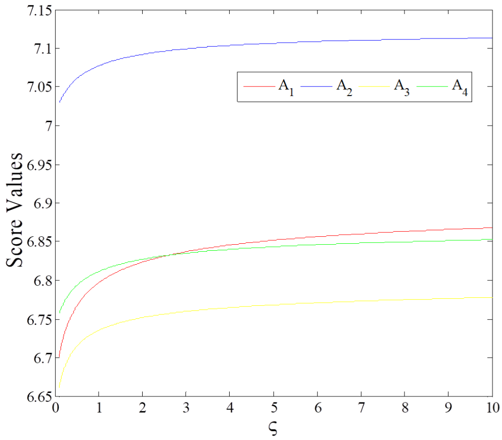

Figure 4: Score values of alternative for $\varsigma \in (0,\ 10]$ based on Lq-ROFHPWG operator ($q = 3$)

Moreover, an optimistic or pessimistic view of DMs can be reflected through the achieved outcomes. Because when the parameter ς becomes larger, the fused results based on Lq-ROFHPWA operator become smaller, while using Lq-ROFHPWG operator, the fused results become larger. Hence DMs can select appropriate Hamacher parameter values according to their needs while making decisions.

7. Comparative analysis

Arora and Garg [50] investigated MCGDM problems under LIF environment. They solved the problem presented in Section 6 using LIF prioritized WA operator, and a similar ranking result is found in the present paper. This shows the validity of the proposed method in dealing with MCGDM problems. However, the present method is more general and flexible than that of Arora and Garg [50]. Since the proposed MCGDM method is based on Lq-ROF environment, it can capture more fuzzy assessment information provided by the DMs. Also, Hamacher operations are considered in the present method that can easily replace the traditional algebraic operations by taking exact parameter values. So, the method proposed by Arora and Garg [50], which is basically developed on the basis of algebraic operations, becomes a particular case of the proposed method.

To prove the effectiveness of the developed operators more significantly, another comparative analysis by applying some existing operators, viz., LIFWA and LIFWG [31], LIFEWA and LIFEWG [51], LIFHWA and LIFHWG [52], LPFWA and LPFWG [32], LPFEWA and LPFEWG [53], LPFHWA and LPFHWG [54], and Lq-ROFWA and Lq-ROFWG [37] operators on the same numerical example considering the equal importance of the DMs and as well as for the criteria. The overall score values and the ranking of the alternatives by means of those existing operators are collected in Table 10.

Table 10: Score values and ranking results compared with existing methods

Operators	Score values	Ranking
LIFWA [31]	$S(A_1) = 5.8554,\ S(A_2) = 6.3633,$ $S(A_3) = 5.6240,\ S(A_4) = 5.7772$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LIFWG [31]	$S(A_1) = 5.3657,\ S(A_2) = 6.1800,$ $S(A_3) = 5.2144,\ S(A_4) = 5.4596$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LIFEWA [51]	$S(A_1) = 5.8012,\ S(A_2) = 6.3463,$ $S(A_3) = 5.5745,\ S(A_4) = 5.7396$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LIFEWG [51]	$S(A_1) = 5.4446,\ S(A_2) = 6.2069,$ $S(A_3) = 5.2784,\ S(A_4) = 5.5066$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LIFHWA ($\varsigma = 3$) [52]	$S(A_1) = 5.7773,\ S(A_2) = 6.3394,$ $S(A_3) = 5.5526,\ S(A_4) = 5.7237$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LIFHWG ($\varsigma = 3$) [52]	$S(A_1) = 5.4922,\ S(A_2) = 6.2248,$ $S(A_3) = 5.3169,\ S(A_4) = 5.5362$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LPFWA [32]	$S(A_1) = 6.8143,\ S(A_2) = 7.0517,$ $S(A_3) = 6.6335,\ S(A_4) = 6.7371$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LPFWG [32]	$S(A_1) = 6.4446,\ S(A_2) = 6.9138,$ $S(A_3) = 6.3316,\ S(A_4) = 6.5013$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LPFEWA [53]	$S(A_1) = 6.7718,\ S(A_2) = 7.0345,$ $S(A_3) = 6.5980,\ S(A_4) = 6.7062$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LPFEWG [53]	$S(A_1) = 6.4929,\ S(A_2) = 6.9336,$ $S(A_3) = 6.3662,\ S(A_4) = 6.5289$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LPFHWA ($\varsigma = 3$) [54]	$S(A_1) = 6.7498,\ S(A_2) = 7.0262,$ $S(A_3) = 6.5789,\ S(A_4) = 6.6909$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LPFHWG ($\varsigma = 3$) [54]	$S(A_1) = 6.5203,\ S(A_2) = 6.9446,$ $S(A_3) = 6.3852,\ S(A_4) = 6.5448$	$A_2 \succ A_4 \succ A_1 \succ A_3$
Lq-ROFWA ($q = 3$) [37]	$S(A_1) = 7.0435,\ S(A_2) = 7.1812,$ $S(A_3) = 6.9049,\ S(A_4) = 6.9754$	$A_2 \succ A_1 \succ A_4 \succ A_3$
Lq-ROFWG ($q = 3$) [37]	$S(A_1) = 6.7687,\ S(A_2) = 7.0681,$ $S(A_3) = 6.6906,\ S(A_4) = 6.8007$	$A_2 \succ A_4 \succ A_1 \succ A_3$
Lq-ROFHPWA operator	$S(A_1) = 7.0107,\ S(A_2) = 7.1615,$ $S(A_3) = 6.8951,\ S(A_4) = 6.9450$	$A_2 \succ A_1 \succ A_4 \succ A_3$
Lq-ROFHPWG operator	$S(A_1) = 6.8368,\ S(A_2) = 7.0992,$ $S(A_3) = 6.7596,\ S(A_4) = 6.8349$	$A_2 \succ A_1 \succ A_4 \succ A_3$

Abbreviations: LIF WA (LIFWA), LIF WG (LIFWG), LIF Einstein WA (LIFEWA), LIF Einstein WG (LIFEWG), LIF Hamacher WA (LIFHWA), LIF Hamacher WG (LIFHWG), LPF WA (LPFWA), LPF WG (LPFWG), LPF Einstein WA (LPFEWA) and LPF Einstein WG (LPFEWG), LPF Hamacher WA (LPFHWA), LPF Hamacher WG (LPFHWG), Lq-ROF WA (Lq-ROFWA), Lq-ROF WA (Lq-ROFWG).

From the above analysis, it is seen that all the operators have the same optimal alternatives. Nevertheless, ranking results differ using averaging and geometric operators for the existing methods. However, in the case of the present method, the ranking is consistent for both the Lq-ROFHPWA and Lq-ROFHPWG operators. The possible reason for this is the fact that method proposed operators can consider the priority over criteria, but all the existing methods [31,32,37,51–54]] fail to incorporate this important characteristic. Hence the proposed method is more reasonable and effective in dealing with real-life MCGDM problems.

8. Conclusion

This paper investigates MCGDM under Lq-ROF environment. For this purpose, two novel Lq-ROFHPWA and Lq-ROFHPWG operators are proposed in this paper. The proposed Lq-ROF operators combine Hamacher operations with prioritized aggregation functions. For this, the proposed operators can consider the prioritized relationship between the input arguments as well as they have the ability to make the aggregation process flexible and general by incorporating Hamacher parameter. Further, the newly developed operators are utilized to develop an MCGDM approach with Lq-ROF context. Subsequently, a numerical example is provided to verify the practicality and effectiveness of the developed approach. Figures and tables have also been delivered to describe the influences of rung parameter q and Hamacher parameter ς on the decision results in detail. In addition, a comparative analysis is also presented to analyze the superiority of the proposed method. In the future research, it would be meaningful to apply the proposed method to other decision-making fields, viz., fuzzy cluster analysis, image pattern recognition, supplier selection, pattern recognition and so forth.

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Interval-valued hesitant Pythagorean fuzzy Archimedean aggregation operators and their application to multicriteria decision-making

Arun Sarkar [a,*], Animesh Biswas [b]

[a] *Department of Mathematics, Heramba Chandra College, Kolkata 700029, India*
[b] *Department of Mathematics, University of Kalyani, Kalyani 741235, India*

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ABSTRACT

In the context of multicriteria decision making (MCDM), it is frequently observed that the representation of cognitive information may not always be sufficient using hesitant Pythagorean fuzzy set (HPFS). From this perspective, the interval-valued hesitant Pythagorean fuzzy (IVHPF) set is more flexible than HPFS to capture the cognitive versatility of decision makers. Again, Archimedean t-conorm and t-norms (At-CN&t-Ns) possess the efficient capability to generate versatile and flexible operational rules for fuzzy numbers. Utilizing the benefit of At-CN&t-Ns some basic operations are introduced to aggregate IVHPF elements so that many other kinds of t-conorm and t-norms (t-CN&t-Ns), such as algebraic, Einstein, Hamacher, Dombi, Frank and other classes of t-CN&t-Ns can be generated. Based on those concepts, several forms of aggregation operators, viz., weighted averaging, weighted geometric, ordered weighted averaging and ordered weighted geometric aggregation operators, are updated in the IVHPF environment. Conversion processes from At-CN&t-Ns based aggregation operators to other variants are also discussed. Using the proposed operators, a methodology for solving MCDM problems with IVHPF cognitive information is proposed. To explore the applicability of the proposed approach, two examples are considered and solved. Results obtained from this method are compared with the existing approaches to establish the efficiency of the proposed method.

1. Introduction

In this modern era of information technology, multicriteria decision making (MCDM) has appeared as an active area of research for its applicability in information processing and other allied fields. Through MCDM, a finite number of alternatives are ranked according to their attribute values. Due to complexities raised to evaluate attribute values of the alternatives, several kinds of uncertainties are frequently observed. Pythagorean fuzzy set (PFS) [1,2] has become an effective tool in recent times to tackle uncertainties associated with the attributes of the alternatives. PFS concept is extended by deriving detailed mathematical expressions of Pythagorean fuzzy numbers (PFNs) by Zhang and Xu [3]. Based on Einstein t-conorm (t-CN) and t-norm (t-N) [4], Garg [5] proposed some Pythagorean fuzzy weighted geometric (WG) and ordered WG (OWG) aggregation operators viz., Pythagorean fuzzy Einstein WG, and Pythagorean fuzzy Einstein OWG operators. Based on Hamacher t-CN and t-N, Wu and Wei [6] applied weighted averaging (WA) and ordered WA (OWA) operator on PFS and introduced Pythagorean fuzzy Hamacher WA, Pythagorean fuzzy Hamacher OWA and Pythagorean fuzzy Hamacher hybrid averaging operators and also their corresponding geometric operators. Biswas and Sarkar [7,8] defined point operator-based similarity measure of PFNs and extended TOPSIS to solve multicriteria group decision making problems under Pythagorean fuzzy environment. Jana et al. [9] utilized Dombi t-CN and t-N [10,11] to develop some Pythagorean fuzzy aggregation operators, viz., Pythagorean fuzzy Dombi WA, Pythagorean fuzzy Dombi OWA, Pythagorean fuzzy Dombi hybrid WA, and corresponding geometric operators. Furthermore, Peng and Yang [12] introduced the idea of interval-valued Pythagorean fuzzy (IVPF) sets (IVPFSs), generalization of PFS and interval-valued intuitionistic fuzzy set [13], where the membership and non-membership values of an element to a given set are represented by the subintervals of $[0,1]$. Under the IVPF environment, Rahman et al. [14] introduced three IVPF geometric aggregation operators to aggregate IVPF numbers (IVPFNs) such as IVPF WG, IVPF OWG, and IVPF hybrid geometric operators. Garg [15] introduced an accuracy function for ranking the IVPFNs by considering the hesitancy degree of IVFNs for solving MCDM problems. Again Garg [16] presented TOPSIS method on IVPF environment. Biswas and Sarkar [17] introduced point operator based similarity measures on IVPFSs and applied them to solve MCDM problems using TODIM.

* Corresponding author.
E-mail addresses: asarkarmth@gmail.com (A. Sarkar), abiswaskln@rediffmail.com (A. Biswas).

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Sometimes in practical situations, decision makers (DMs) face difficulty estimating the degree of membership by a single value; instead, they are interested in assigning a set of possible values. To adopt such situations, Torra and Narukawa [18] and Torra [19] introduced hesitant fuzzy set (HFS) by considering a set of possible membership values within 0 and 1. Later on, generalizing the concept of IFS, Zhu et al. [20] defined dual hesitant fuzzy (DHF) set (DHFSs) by simultaneous consideration of HFS and IFS. Garg and Arora [21] introduced DHF soft WA and WG aggregation operators. Biswas and Sarkar [22] proposed DHF prioritized WA and WG aggregation operators based on Einstein t-CN and t-N.

Afterwards, inspired by the idea of DHFS [20] and PFS [1,2], Liang and Xu [23] proposed the concept of the hesitant Pythagorean fuzzy (HPF) set (HPFS) and defined distance measures of HPFS and applied them in MCDM by providing TOPSIS. Based on Hamacher operations, Lu et al. [24] developed a series of aggregation operators, viz., HPF Hamacher (HPFH) WA, HPFH WG, HPFH OWA, HPFH OWG, HPFH hybrid average and HPFH hybrid geometric operators. Garg [25] introduced HPF- WA, OWA, hybrid average and its geometric aggregation operators. To capture the interrelationship arguments with HPF context, Garg [26] further developed HPF Maclaurin symmetric mean (MSM) operator for aggregating HPF information. Yang et al. [27] proposed several HPF interaction aggregation operators based on Bonferroni mean (BM), viz., HPF interaction BM, weighted BM, geometric BM and geometric weight BM operators. Recently Sarkar and Biswas [28] introduced some operational rules of HPF elements based on Archimedean t-CN and t-N (At-CN&t-N). They proposed some Archimedean operations-based MSM operators under HPF environments using those defined operations.

However, in several real-life MCDM models, due to insufficiency in available information, DMs are unable to exert their opinion with a crisp number but are comfortable putting the decision values by interval numbers within $[0, 1]$. To overcome such situations, Wang et al. [29] introduced the concept of interval-valued HPFS (IVHPFS), which takes the hesitant membership and non-membership degrees in the form of IVPFNs. It should be noted that when both the membership degree and non-membership degree of each element to a given set have a single interval value, the IVHPFS reduces to the IVPFS [12] and when the upper and lower limits of interval values are identical, IVHPFS becomes HPFS [23]. Thus, it is clear that IVHPFS is a more generalized form than other extensions of PFSs.

Different classes of t-CNs and t-Ns are derived from At-CN&t-N [30,31], viz., Algebraic, Einstein, Hamacher and Frank classes of t-CNs and t-Ns. Based on At-CN&t-N, Xia et al. [32] introduced At-CN&t-N based intuitionistic fuzzy WA and WG operators. Zhang and Wu [33] developed several At-CN&t-N-based interval-valued hesitant fuzzy WA and WG aggregation operators. On DHF environment, Yu [34] proposed DHF WA and WG aggregation operators based on At-CN&t-N operations. Recently, Sarkar and Biswas [35] introduced At-CN&t-N operations on Pythagorean hesitant fuzzy (PHF) sets and defined a class of At-CN&t-N-based PHF WA and WG operators. Again Sarkar and Biswas [36] applied At-CN&t-N on the interval-valued DHF (IVDHF) information and introduced a class of aggregation operators.

Motivated by the work of Sarkar and Biswas [35,36], this paper proposes At-CN&t-N based operational laws on IVHPFSs and investigates their properties. Based on those operational laws on IVHPFSs, At-CN&t-N based IVHPF WA (AIVHPFWA), WG (AIVHPFWG), OWA (AIVHPFOWA) and OWG (AIVHPFOWG) operators are developed. From the developed operators, different forms of other aggregation operators, viz., IVHPF WA (IVHPFWA), IVHPF Einstein WA (IVHPFEWA), IVHPF Hamacher WA (IVHPFHWA), IVHPF Dombi WA (IVHPFDWA), IVHPF Frank WA (IVHPFFWA) operators, also ordered weighted aggregation analogous operators and their geometric operators can be derived.

This article is organized as follows. At first, some preliminary concepts on HPFS, IVHPFS and At-CN&t-N are studied. Then At-CN&t-N-based operations on IVHPF elements (IVHPFEs) are defined. To aggregate the IVHPFEs, based on At-CN&t-N, IVHPF WA and OWA aggregation operators and their geometric form, viz., IVHPF WG and OWG operators are proposed. After that, the classification of the proposed operators is made for different types of decreasing functions. Some valuable properties and exceptional cases of the developed operators are also studied. In the sequel, an approach to MCDM under IVHPF environment is developed. Two numerical illustrations support the proposed method, and the sensitive nature of the model is checked by varying the parameter. A comparative study with the existing methods is presented by solving several previously considered invariants of fuzzy environments. Finally, conclusions and scope for future studies have been described.

2. Preliminaries

In this section, some elementary concepts relating to HPFS, IVHPFS and At-CN&t-N are briefly explained to introduce the proposed method.

2.1. Hesitant Pythagorean fuzzy set

Liang and Xu [23] extended the notion of PFSs by taking the degrees of membership and non-membership of a PFN through some possible degrees and termed as HPFSs. It is defined as follows:

Definition 1 (*[23]*)**.** Let X be a universe of discourse. Then an HPFS K on X is described as:

$$K = \left(\left(x, \mu_K(x), \nu_K(x) \right) \middle| x \in X \right)$$

where $\mu_K(x)$ and $\nu_K(x)$ represents two sets $\bigcup_{\alpha \in \mu_K(x)} \{\alpha\}$ and $\bigcup_{\beta \in \nu_K(x)} \{\beta\}$ in which α, β belongs to the closed unit interval, indicating the probable Pythagorean membership values and Pythagorean non-membership values, respectively, of the component $x \in X$ to the set K satisfying the conditions:

$$0 \le \alpha, \beta \le 1 \text{ and } 0 \le \left(\max_{\alpha \in \mu_K(x)} \{\alpha\} \right)^2 + \left(\max_{\beta \in \nu_K(x)} \{\beta\} \right)^2 \le 1 \text{ for all } x \in X.$$

For convenience, Liang and Xu [23] called the pair $K = \left(\mu_K(x), \nu_K(x) \right)$ as an HPF element (HPFE) symbolically written as $k = (\mu, \nu)$. Furthermore, Liang and Xu [23] defined score and accuracy functions for developing the comparison laws between HPFEs.

Definition 2 (*[23]*)**.** Let $k = (\mu, \nu)$ be an HPFE, then the score function $S(k)$ and accuracy function $A(k)$ of k is defined as follows:

$$S(k) = \frac{1}{2} \left(1 + \frac{1}{|\mu|} \sum_{\alpha \in \mu} \alpha^2 - \frac{1}{|\nu|} \sum_{\beta \in \nu} \beta^2 \right),$$

and

$$A(k) = \frac{1}{|\mu|}\sum_{\alpha\in\mu}\alpha^2 - \frac{1}{|\nu|}\sum_{\beta\in\nu}\beta^2,$$

where $|\mu|$ and $|\nu|$ are the numbers of the elements in μ and ν, respectively.

Definition 3 ([23]). Let k_1 and k_2 be any two HPFEs, then the ordering of those HPFEs is done by the following principles.

- If $S(k_1) > S(k_2)$, then $k_1 \succ k_2$;
- If $S(k_1) = S(k_2)$, then
  (1) If $A(k_1) > A(k_2)$, then $k_1 \succ k_2$;
  (2) If $A(k_1) = A(k_2)$, then $k_1 \approx k_2$.

2.2. IVHPFS

Sometimes, it becomes inadequate to describe an uncertain situation by HPF information. To tackle that situation, Wang et al. [29] introduced the concept of IVHPFS.

Definition 4. Let X be a fixed set and the power set of $[0,1]$ is denoted by $I([0,1])$. An IVHPFS $\widetilde{D}$ on X is presented as

$$\widetilde{D} = \left\{\left\langle x, \tilde{h}_D(x), \tilde{g}_D(x)\right\rangle \middle| x\in X\right\} \tag{1}$$

where $\tilde{h}_D(x) = \bigcup_{[\gamma^l,\gamma^u]\in\tilde{h}_D(x)}\left\{[\gamma^l,\gamma^u]\right\}$ and $\tilde{g}_D(x) = \bigcup_{[\delta^l,\delta^u]\in\tilde{g}_D(x)}\left\{[\delta^l,\delta^u]\right\}$ represent two sets of some interval values belonging to $I([0,1])$, denoting the possible membership and non-membership interval-values, respectively, corresponding to the element $x\in X$ satisfying the condition: $0 \le \left((\gamma^u)^+\right)^2 + \left((\delta^u)^+\right)^2 \le 1$, in which $(\gamma^u)^+ = \max_{[\gamma^l,\gamma^u]\in\tilde{h}_D(x)}\{\gamma^u\}$ and $(\delta^u)^+ = \max_{[\delta^l,\delta^u]\in\tilde{g}_D(x)}\{\delta^u\}$. For convenience, the pair $\left(\tilde{h}_D(x), \tilde{g}_D(x)\right)$ is called an IVHPF element (IVHPFE) denoted by $\tilde{d} = \left(\tilde{h}, \tilde{g}\right)$.

For example, if a DM provides possible membership degrees of $x \in X$ to the set $\widetilde{D}$ as $[0.1, 0.3], [0.4, 0.6]$ and $[0.7, 0.8]$, and possible non-membership degrees as $[0.3, 0.4]$ and $[0.4, 0.6]$ simultaneously, then the IVHPFE can be represented as
$\tilde{d} = (\{[0.1, 0.3], [0.4, 0.6], [0.7, 0.8]\}, \{[0.3, 0.4], [0.4, 0.6]\})$ in which $(\gamma^u)^+ = 0.8$, $(\delta^u)^+ = 0.6$ and $0 \le (0.8)^2 + (0.6)^2 \le 1$.

To compare the IVHPFEs, Wang et al. [29] defined the score and accuracy functions as follows.

Definition 5. Let $\tilde{d} = \left(\tilde{h}, \tilde{g}\right)$ be an IVHPFE. Then the score function of $\tilde{d}$ is defined as

$$S\left(\tilde{d}\right) = \frac{1}{2}\left(1 + \frac{1}{2\left|\tilde{h}\right|}\sum_{[\gamma^l,\gamma^u]\in\tilde{h}}\left(\left(\gamma^l\right)^2 + (\gamma^u)^2\right) - \frac{1}{2|\tilde{g}|}\sum_{[\delta^l,\delta^u]\in\tilde{g}}\left(\left(\delta^l\right)^2 + (\delta^u)^2\right)\right) \tag{2}$$

and the accuracy function of $\tilde{d}$ is defined as

$$A\left(\tilde{d}\right) = \frac{1}{2\left|\tilde{h}\right|}\sum_{[\gamma^l,\gamma^u]\in\tilde{h}}\left(\left(\gamma^l\right)^2 + (\gamma^u)^2\right) + \frac{1}{2|\tilde{g}|}\sum_{[\delta^l,\delta^u]\in\tilde{g}}\left(\left(\delta^l\right)^2 + (\delta^u)^2\right) \tag{3}$$

where $\left|\tilde{h}\right|$ and $|\tilde{g}|$ defined the number of intervals in $\tilde{h}$ and $\tilde{g}$, respectively.

Let $\tilde{d}_i$ $(i = 1, 2)$ be any two IVHPFEs, then the ordering of IVHPFEs is done in the following manner:

(i) If $S\left(\tilde{d}_1\right) > S\left(\tilde{d}_2\right)$ then $\tilde{d}_1 \succ \tilde{d}_2$;
(ii) If $S\left(\tilde{d}_1\right) = S\left(\tilde{d}_2\right)$ then
- if $A\left(\tilde{d}_1\right) > A\left(\tilde{d}_2\right)$ then $\tilde{d}_1 \succ \tilde{d}_2$; if $A\left(\tilde{d}_1\right) = A\left(\tilde{d}_2\right)$ then $\tilde{d}_1 \approx \tilde{d}_2$.

2.3. Archimedean t-norm and t-conorm

Definition 6 ([30,31]). A function $\mathrm{V}: [0,1]\times[0,1] \to [0,1]$ is called a t-CN if it satisfied associativity, symmetricity, non-decreasing and $\mathrm{V}(x, 0) = x$ for all x. If a binary operation $\Lambda: [0,1]\times[0,1] \to [0,1]$ is satisfies associativity, symmetricity, non-decreasing and $\Lambda(x, 1) = x$ for all x then Λ is known as a t-N.

Archimedean t-CN (At-CN) and Archimedean t-N (At-N) operations are expressed as follows:

Definition 7 ([37]). An At-CN V is formulated using increasing function g as

$$\mathrm{V}(a, b) = \mathrm{g}^{-1}(\mathrm{g}(a) + \mathrm{g}(b)) \tag{4}$$

Similarly, using decreasing function f, an At-N Λ is represented as

$$\Lambda(a, b) = f^{-1}(f(a) + f(b)) \text{ with } \mathrm{g}(t) = f(1-t) \quad \text{for all} \quad a, b, t \in [0,1]. \tag{5}$$

3. At-CN&t-N-based operations on IVHPFEs

Considering the concept of At-CN&t-N-based operational laws, some operations on IVHPFEs are proposed here.

Definition 8. Let $\tilde{d} = (\tilde{h}, \tilde{g})$, $\tilde{d}_1 = (\tilde{h}_1, \tilde{g}_1)$ and $\tilde{d}_2 = (\tilde{h}_2, \tilde{g}_2)$ be any three IVHPFEs. The arithmetic operations on IVHPFEs based on At-CN&t-N are defined as:

(1)

$$\tilde{d}_1 \oplus_A \tilde{d}_2 = \left(\left\{ \left[\sqrt{\mathrm{V}\left(\left(\gamma_1^l\right)^2, \left(\gamma_2^l\right)^2\right)}, \sqrt{\mathrm{V}\left(\left(\gamma_1^u\right)^2, \left(\gamma_2^u\right)^2\right)} \right] \middle| \begin{matrix} \left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1,2 \end{matrix} \right\}, \right.$$
$$\left. \left\{ \left[\sqrt{\Lambda\left(\left(\delta_1^l\right)^2, \left(\delta_2^l\right)^2\right)}, \sqrt{\Lambda\left(\left(\delta_1^u\right)^2, \left(\delta_2^u\right)^2\right)} \right] \middle| \begin{matrix} \left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i \\ i = 1,2 \end{matrix} \right\} \right)$$
$$= \left(\left\{ \left[\sqrt{\mathrm{g}^{-1}\left(\mathrm{g}\left(\left(\gamma_1^l\right)^2\right) + \mathrm{g}\left(\left(\gamma_2^l\right)^2\right)\right)}, \sqrt{\mathrm{g}^{-1}\left(\mathrm{g}\left(\left(\gamma_1^u\right)^2\right) + \mathrm{g}\left(\left(\gamma_2^u\right)^2\right)\right)} \right] \middle| \begin{matrix} \left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1,2 \end{matrix} \right\}, \right.$$
$$\left. \left\{ \left[\sqrt{f^{-1}\left(f\left(\left(\delta_1^l\right)^2\right) + f\left(\left(\delta_2^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(f\left(\left(\delta_1^u\right)^2\right) + f\left(\left(\delta_2^u\right)^2\right)\right)} \right] \middle| \begin{matrix} \left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i \\ i = 1,2 \end{matrix} \right\} \right);$$

(2)

$$\tilde{d}_1 \otimes_A \tilde{d}_2 = \left(\left\{ \left[\sqrt{\Lambda\left(\left(\gamma_1^l\right)^2, \left(\gamma_2^l\right)^2\right)}, \sqrt{\Lambda\left(\left(\gamma_1^u\right)^2, \left(\gamma_2^u\right)^2\right)} \right] \middle| \begin{matrix} \left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1,2 \end{matrix} \right\}, \right.$$
$$\left. \left\{ \left[\sqrt{\mathrm{V}\left(\left(\delta_1^l\right)^2, \left(\delta_2^l\right)^2\right)}, \sqrt{\mathrm{V}\left(\left(\delta_1^u\right)^2, \left(\delta_2^u\right)^2\right)} \right] \middle| \begin{matrix} \left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i \\ i = 1,2 \end{matrix} \right\} \right)$$
$$= \left(\left\{ \left[\sqrt{f^{-1}\left(f\left(\left(\gamma_1^l\right)^2\right) + f\left(\left(\gamma_2^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(f\left(\left(\gamma_1^u\right)^2\right) + f\left(\left(\gamma_2^u\right)^2\right)\right)} \right] \middle| \begin{matrix} \left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1,2 \end{matrix} \right\}, \right.$$
$$\left. \left\{ \left[\sqrt{\mathrm{g}^{-1}\left(\mathrm{g}\left(\left(\delta_1^l\right)^2\right) + \mathrm{g}\left(\left(\delta_2^l\right)^2\right)\right)}, \sqrt{\mathrm{g}^{-1}\left(\mathrm{g}\left(\left(\delta_1^u\right)^2\right) + \mathrm{g}\left(\left(\delta_2^u\right)^2\right)\right)} \right] \middle| \begin{matrix} \left[\delta_i^l, \delta_i^u\right] \in g_i \\ i = 1,2 \end{matrix} \right\} \right);$$

(3)

$$\lambda\tilde{d} = \left(\left\{ \left[\sqrt{\mathrm{g}^{-1}\left(\lambda \mathrm{g}\left(\left(\gamma^l\right)^2\right)\right)}, \sqrt{\mathrm{g}^{-1}\left(\lambda \mathrm{g}\left(\left(\gamma^u\right)^2\right)\right)} \right] \middle| \left[\gamma^l, \gamma^u\right] \in \tilde{h} \right\}, \right.$$
$$\left. \left\{ \left[\sqrt{f^{-1}\left(\lambda f\left(\left(\delta^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\lambda f\left(\left(\delta^u\right)^2\right)\right)} \right] \middle| \left[\delta^l, \delta^u\right] \in \tilde{g} \right\} \right) \quad \lambda > 0,$$

(4)

$$\tilde{d}^\lambda = \left(\left\{ \left[\sqrt{f^{-1}\left(\lambda f\left(\left(\gamma^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\lambda f\left(\left(\gamma^u\right)^2\right)\right)} \right] \middle| \left[\gamma^l, \gamma^u\right] \in \tilde{h} \right\}, \right.$$
$$\left. \left\{ \left[\sqrt{\mathrm{g}^{-1}\left(\lambda \mathrm{g}\left(\left(\delta^l\right)^2\right)\right)}, \sqrt{\mathrm{g}}^{-1}\left(\lambda \mathrm{g}\left(\left(\delta^u\right)^2\right)\right) \right] \middle| \left[\delta^l, \delta^u\right] \in \tilde{g} \right\} \right) \quad \lambda > 0,$$

Several t-CNs and t-Ns are derived using different forms of increasing and decreasing functions [37] and using these functions in different forms, viz., algebraic, Einstein, Hamacher, Dombi and Frank classes of t-CNs and t-Ns of IVHPFEs are derived and is presented in Tables 1 and 2, respectively.

4. IVHPF weighted aggregation operators and their properties

In this section, At-CN&t-N-based WA and OWA aggregation and their geometric operators with IVHPEs, viz., AIVHPFWA, AIVHPFOWA, AIVHPFWG and AIVHPFOWG operators are proposed. Several forms of aggregation operators derived from those operators are presented, and their properties are also discussed.

4.1. AIVHPFWA operator

By incorporating the importance of the experts and parameters during the decision analysis, AIVHPFWA operators are presented as follows:

Definition 9. Let $\{\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n\}$ be a set of IVHPFEs, $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector where $\sum_{i=1}^n \omega_i = 1$ and $\omega_i \in [0, 1]$. Then AIVHPFWA operator is a function

$$\widetilde{D}^n \to \widetilde{D}, \text{ given by } AIVHPFWA\left(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n\right) = \oplus_{A_{i=1}}^{\ n} \left(\omega_i \tilde{d}_i\right)$$

where $\oplus_A$ represents the Archimedean sum.

Conforming with the operations of IVHPFEs shown in Definition 8, the following theorem is derived.

Table 1
Forms of At-CN for IVHPFEs.

Name	Forms of At-CN	Functions
Algebraic t-CN	$\left(\left\{\left[\sqrt{\gamma_1^{l^2}+\gamma_2^{l^2}-\left(\gamma_1^l\gamma_2^l\right)^2}, \sqrt{\gamma_1^{u^2}+\gamma_2^{u^2}-\left(\gamma_1^u\gamma_2^u\right)^2}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i \\ i=1,2\end{matrix}\right\}, \left\{\left[\delta_1^l\delta_2^l, \delta_1^u\delta_2^u\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i \\ i=1,2\end{matrix}\right\}\right)$	$f(x) = -\log x,\ g(x) = -\log(1-x)$
Einstein t-CN	$\left(\left\{\left[\sqrt{\frac{(\gamma_1^l)^2+(\gamma_2^l)^2}{1+(\gamma_1^l\gamma_2^l)^2}}, \sqrt{\frac{(\gamma_1^u)^2+(\gamma_2^u)^2}{1+(\gamma_1^u\gamma_2^u)^2}}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i \\ i=1,2\end{matrix}\right\}, \left\{\left[\frac{\delta_1^l\delta_2^l}{\sqrt{1+\left(1-(\delta_1^l)^2\right)\left(1-(\delta_2^l)^2\right)}}, \frac{\delta_1^u\delta_2^u}{\sqrt{1+\left(1-(\delta_1^u)^2\right)\left(1-(\delta_2^u)^2\right)}}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i \\ i=1,2\end{matrix}\right\}\right)$	$f(x) = \log\left(\frac{2-x}{x}\right)$, $g(x) = \log\left(\frac{1+x}{1-x}\right)$
Hamacher t-CN	$\left(\left\{\left[\sqrt{\frac{\gamma_1^{l^2}+\gamma_2^{l^2}-(\gamma_1^l\gamma_2^l)^2-(1-\theta)(\gamma_1^l\gamma_2^l)^2}{1-(1-\theta)(\gamma_1^l\gamma_2^l)^2}}, \sqrt{\frac{\gamma_1^{u^2}+\gamma_2^{u^2}-(\gamma_1^u\gamma_2^u)^2-(1-\theta)(\gamma_1^u\gamma_2^u)^2}{1-(1-\theta)(\gamma_1^u\gamma_2^u)^2}}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i \\ i=1,2\end{matrix}\right\}, \left\{\left[\frac{\delta_1^l\delta_2^l}{\sqrt{\theta+(1-\theta)\left(\delta_1^{l^2}+\delta_2^{l^2}-(\delta_1^l\delta_2^l)^2\right)}}, \frac{\delta_1^u\delta_2^u}{\sqrt{\theta+(1-\theta)\left(\delta_1^{u^2}+\delta_2^{u^2}-(\delta_1^u\delta_2^u)^2\right)}}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i \\ i=1,2\end{matrix}\right\}\right)$	$f(x) = \log\left(\frac{\theta+(1-\theta)x}{x}\right)$, $g(x) = \log\left(\frac{\theta+(1-\theta)(1-x)}{(1-x)}\right),\ \theta > 0$
Dombi t-CN	$\left(\left\{\left[\sqrt{1-1/\left(1+\left(\left(\frac{\gamma_1^{l^2}}{1-\gamma_1^{l^2}}\right)^\rho+\left(\frac{\gamma_2^{l^2}}{1-\gamma_2^{l^2}}\right)^\rho\right)^{\frac{1}{\rho}}\right)}, \sqrt{1-1/\left(1+\left(\left(\frac{\gamma_1^{u^2}}{1-\gamma_1^{u^2}}\right)^\rho+\left(\frac{\gamma_2^{u^2}}{1-\gamma_2^{u^2}}\right)^\rho\right)^{\frac{1}{\rho}}\right)}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i \\ i=1,2\end{matrix}\right\}, \left\{\left[1/\sqrt{1+\left(\left(\frac{1-\delta_1^{l^2}}{\delta_1^{l^2}}\right)^\rho+\left(\frac{1-\delta_2^{l^2}}{\delta_2^{l^2}}\right)^\rho\right)^{\frac{1}{\rho}}}, 1/\sqrt{1+\left(\left(\frac{1-\delta_1^{u^2}}{\delta_1^{u^2}}\right)^\rho+\left(\frac{1-\delta_2^{u^2}}{\delta_2^{u^2}}\right)^\rho\right)^{\frac{1}{\rho}}}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i \\ i=1,2\end{matrix}\right\}\right)$	$f(x) = \left(\frac{1}{x}-1\right)^\rho$, $g(x) = \left(\frac{x}{1-x}\right)^\rho,\ \rho > 0$
Frank t-CN	$\left(\left\{\left[\sqrt{1-\log_\psi\left(1+\frac{\left(\psi^{1-\gamma_1^{l^2}}-1\right)\left(\psi^{1-\gamma_2^{l^2}}-1\right)}{\psi-1}\right)}, \sqrt{1-\log_\psi\left(1+\frac{\left(\psi^{1-\gamma_1^{u^2}}-1\right)\left(\psi^{1-\gamma_2^{u^2}}-1\right)}{\psi-1}\right)}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i \\ i=1,2\end{matrix}\right\}, \left\{\left[\sqrt{\log_\psi\left(1+\frac{\left(\psi^{\delta_1^{l^2}}-1\right)\left(\psi^{\delta_2^{l^2}}-1\right)}{\psi-1}\right)}, \sqrt{\log_\psi\left(1+\frac{\left(\psi^{\delta_1^{u^2}}-1\right)\left(\psi^{\delta_2^{u^2}}-1\right)}{\psi-1}\right)}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i \\ i=1,2\end{matrix}\right\}\right)$	$f(x) = \log\left(\frac{\psi-1}{\psi^x-1}\right)$, $g(x) = \log\left(\frac{\psi-1}{\psi^{1-x}-1}\right),\ \psi > 1$

Theorem 1. *Let $\tilde{d}_i = \left(\tilde{h}_i, \tilde{g}_i\right)$ $(i = 1, 2, \ldots, n)$ be a set of IVHPFEs, then the aggregating value using AIVHPFWA operator is also an IVHPFE and*

$$AIVHPFWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right)$$

$$= \left(\left\{\left[\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i \\ i=1,2,\ldots,n\end{matrix}\right\},\right.$$

$$\left.\left\{\left[\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\delta_i^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\delta_i^u\right)^2\right)\right)}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i \\ i=1,2,\ldots,n\end{matrix}\right\}\right) \tag{6}$$

Proof. For $n = 2$,

$$\omega_1\tilde{d}_1 = \left(\left\{\left[\sqrt{g^{-1}\left(\omega_1 g\left(\left(\gamma_1^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\omega_1 g\left(\left(\gamma_1^u\right)^2\right)\right)}\right] \mid \left[\gamma_1^l, \gamma_1^u\right] \in \tilde{h}_1\right\},\right.$$

$$\left.\left\{\left[\sqrt{f^{-1}\left(\omega_1 f\left(\left(\delta_1^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\omega_1 f\left(\left(\delta_1^u\right)^2\right)\right)}\right] \mid \left[\delta_1^l, \delta_1^u\right] \in \tilde{g}_1\right\}\right)$$

Table 2
Forms of At-N for IVHPFEs.

Name	Forms of At-Ns	Functions
Algebraic t-N	$\left(\left\{\left[\gamma_1^l\gamma_2^l, \gamma_1^u\gamma_2^u\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i\\ i=1,2\end{matrix}\right\}, \left\{\left[\sqrt{\delta_1^{l^2}+\delta_1^{l^2}-\left(\delta_1^l\delta_2^l\right)^2}, \sqrt{\delta_1^{u^2}+\delta_1^{u^2}-\left(\delta_1^u\delta_2^u\right)^2}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i\\ i=1,2\end{matrix}\right\}\right)$	$f(x)=-\log x,\ g(x)=-\log(1-x)$
Einstein t-N	$\left(\left\{\left[\frac{\gamma_1^l\gamma_2^l}{\sqrt{1+\left(1-(\gamma_1^l)^2\right)\left(1-(\gamma_2^l)^2\right)}}, \frac{\gamma_1^u\gamma_2^u}{\sqrt{1+\left(1-(\gamma_1^u)^2\right)\left(1-(\gamma_2^u)^2\right)}}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i\\ i=1,2\end{matrix}\right\}, \left\{\left[\sqrt{\frac{(\delta_1^l)^2+(\delta_2^l)^2}{1+(\delta_1^l\delta_2^l)^2}}, \sqrt{\frac{(\delta_1^u)^2+(\delta_2^u)^2}{1+(\delta_1^u\delta_2^u)^2}}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i\\ i=1,2\end{matrix}\right\}\right)$	$f(x)=\log\left(\frac{2-x}{x}\right)$, $g(x)=\log\left(\frac{1+x}{1-x}\right)$
Hamacher t-N	$\left(\left\{\left[\frac{\gamma_1^l\gamma_2^l}{\sqrt{\theta+(1-\theta)\left(\gamma_1^{l^2}+\gamma_2^{l^2}-(\gamma_1^l\gamma_2^l)^2\right)}}, \frac{\gamma_1^u\gamma_2^u}{\sqrt{\theta+(1-\theta)\left(\gamma_1^{u^2}+\gamma_2^{u^2}-(\gamma_1^u\gamma_2^u)^2\right)}}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i\\ i=1,2\end{matrix}\right\}, \left\{\left[\sqrt{\frac{\delta_1^{l^2}+\delta_2^{l^2}-(\delta_1^l\delta_2^l)^2-(1-\theta)(\delta_1^l\delta_2^l)^2}{1-(1-\theta)(\delta_1^l\delta_2^l)^2}}, \sqrt{\frac{\delta_1^{l^2}+\delta_2^{l^2}-(\delta_1^l\delta_2^l)^2-(1-\theta)(\delta_1^l\delta_2^l)^2}{1-(1-\theta)(\delta_1^l\delta_2^l)^2}}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i\\ i=1,2\end{matrix}\right\}\right)$	$f(x)=\log\left(\frac{\theta+(1-\theta)x}{x}\right)$, $g(x)=\log\left(\frac{\theta+(1-\theta)(1-x)}{(1-x)}\right),\ \theta>0$
Dombi t-N	$\left(\left\{\left[1/\sqrt{1+\left(\left(\frac{1-\gamma_1^{l^2}}{\gamma_1^{l^2}}\right)^\rho+\left(\frac{1-\gamma_2^{l^2}}{(\gamma_2^l)^2}\right)^\rho\right)^{\frac{1}{\rho}}}, 1/\sqrt{1+\left(\left(\frac{1-\gamma_1^{u^2}}{\gamma_1^{u^2}}\right)^\rho+\left(\frac{1-\gamma_2^{u^2}}{\gamma_2^{u^2}}\right)^\rho\right)^{\frac{1}{\rho}}}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i\\ i=1,2\end{matrix}\right\}, \left\{\left[\sqrt{1-1/\left(1+\left(\left(\frac{\delta_1^{l^2}}{1-\delta_1^{l^2}}\right)^\rho+\left(\frac{\delta_2^{l^2}}{1-\delta_2^{l^2}}\right)^\rho\right)^{\frac{1}{\rho}}\right)}, \sqrt{1-1/\left(1+\left(\left(\frac{\delta_1^{u^2}}{1-\delta_1^{u^2}}\right)^\rho+\left(\frac{\delta_2^{u^2}}{1-\delta_2^{u^2}}\right)^\rho\right)^{\frac{1}{\rho}}\right)}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i\\ i=1,2\end{matrix}\right\}\right)$	$f(x)=\left(\frac{1}{x}-1\right)^\rho$, $g(x)=\left(\frac{x}{1-x}\right)^\rho,\ \rho>0$
Frank t-N	$\left(\left\{\left[\sqrt{\log_\theta\left(1+\frac{\left(\psi^{\gamma_1^{l^2}}-1\right)\left(\psi^{\gamma_2^{l^2}}-1\right)}{\psi-1}\right)}, \sqrt{\log_\theta\left(1+\frac{\left(\psi^{\gamma_1^{u^2}}-1\right)\left(\psi^{\gamma_2^{u^2}}-1\right)}{\psi-1}\right)}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i\\ i=1,2\end{matrix}\right\}, \left\{\left[\sqrt{1-\log_\theta\left(1+\frac{\left(\psi^{1-(\delta_1^l)^2}-1\right)\left(\psi^{1-(\delta_2^l)^2}-1\right)}{\psi-1}\right)}, \sqrt{1-\log_\theta\left(1+\frac{\left(\psi^{1-(\delta_1^u)^2}-1\right)\left(\psi^{1-(\delta_2^u)^2}-1\right)}{\psi-1}\right)}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i\\ i=1,2\end{matrix}\right\}\right)$	$f(x)=\log\left(\frac{\psi-1}{\psi^x-1}\right)$, $g(x)=\log\left(\frac{\psi-1}{\psi^{1-x}-1}\right),\ \psi>1$

and

$$\omega_2\tilde{d}_2=\left(\left\{\left[\sqrt{g^{-1}\left(\omega_2 g\left(\left(\gamma_2^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\omega_2 g\left(\left(\gamma_2^u\right)^2\right)\right)}\right] \mid \left[\gamma_2^l,\gamma_2^u\right]\in\tilde{h}_2\right\},\right.$$
$$\left.\left\{\left[\sqrt{f^{-1}\left(\omega_2 f\left(\left(\delta_2^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\omega_2 f\left(\left(\delta_2^u\right)^2\right)\right)}\right] \mid \left[\delta_2^l,\delta_2^u\right]\in\tilde{g}_2\right\}\right)$$

now, $\omega_1\tilde{d}_1\oplus_A\omega_2\tilde{d}_2=$

$$\left(\left\{\left[\sqrt{g^{-1}\left(\sum_{i=1}^{2}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{2}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i\\ i=1,2\end{matrix}\right\},\right.$$
$$\left.\left\{\left[\sqrt{f^{-1}\left(\sum_{i=1}^{2}\omega_i f\left(\left(\delta_i^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{2}\omega_i f\left(\left(\delta_i^u\right)^2\right)\right)}\right] \mid \begin{matrix}[\delta_i^l,\delta_i^u]\in\tilde{g}_i\\ i=1,2\end{matrix}\right\}\right)$$

i.e., the theorem holds for $n=2$. Assume now that the theorem holds for $n=p$, i.e.,

$$AIVHPFWA\left(\tilde{d}_1,\tilde{d}_2,\dots,\tilde{d}_p\right)$$
$$=\left(\left\{\left[\sqrt{g^{-1}\left(\sum_{i=1}^{p}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{p}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right] \mid \begin{matrix}[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i\\ i=1,2,\dots,p\end{matrix}\right\},\right.$$

$$\left\{\left[\sqrt{f^{-1}\left(\sum_{i=1}^{p}\omega_i f\left(\left(\delta_i^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{p}\omega_i f\left(\left(\delta_i^u\right)^2\right)\right)}\right] \Bigg| \begin{matrix}\left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i \\ i = 1, 2, \ldots, p\end{matrix}\right\}\right)$$

Then for $n = p + 1$,

$$AIVHPFWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_p, \tilde{d}_{p+1}\right) = AIVHPFWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_p\right) \oplus_A \omega_{p+1}\tilde{d}_{p+1}$$

$$= \left(\left\{\left[\sqrt{g^{-1}\left(\sum_{i=1}^{p}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{p}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right] \Bigg| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, p\end{matrix}\right\},\right.$$

$$\left.\left\{\left[\sqrt{f^{-1}\left(\sum_{i=1}^{p}\omega_i f\left(\left(\delta_i^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{p}\omega_i f\left(\left(\delta_i^u\right)^2\right)\right)}\right] \Bigg| \left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i, i = 1, 2, \ldots, p\right\}\right)$$

$$\oplus_A \left(\left\{\left[\sqrt{g^{-1}\left(\omega_{p+1} g\left(\left(\gamma_{p+1}^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\omega_{p+1} g\left(\left(\gamma_{p+1}^u\right)^2\right)\right)}\right] \Bigg| \left[\gamma_{p+1}^l, \gamma_{p+1}^u\right] \in \tilde{h}_{p+1}\right\},\right.$$

$$\left.\left\{\left[\sqrt{f^{-1}\left(\omega_{p+1} f\left(\left(\delta_{p+1}^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\omega_{p+1} f\left(\left(\delta_{p+1}^u\right)^2\right)\right)}\right] \Bigg| \left[\delta_{p+1}^l, \delta_{p+1}^u\right] \in \tilde{g}_{p+1}\right\}\right)$$

$$= \left(\left\{\left[\sqrt{g^{-1}\left(\sum_{i=1}^{p}\omega_i g\left(\left(\gamma_i^l\right)^2\right) + \omega_{p+1} g\left(\left(\gamma_{p+1}^l\right)^2\right)\right)},\right.\right.\right.$$

$$\left.\left.\sqrt{g^{-1}\left(\sum_{i=1}^{p}\omega_i g\left(\left(\gamma_i^u\right)^2\right) + \omega_{p+1} g\left(\left(\gamma_{p+1}^u\right)^2\right)\right)}\right] \Bigg| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, p, p+1\end{matrix}\right\},$$

$$\left\{\left[\sqrt{f^{-1}\left(\sum_{i=1}^{p}\omega_i f\left(\left(\delta_i^l\right)^2\right) + \omega_{p+1} f\left(\left(\delta_{p+1}^l\right)^2\right)\right)},\right.\right.$$

$$\left.\left.\left.\sqrt{f^{-1}\left(\sum_{i=1}^{p}\omega_i f\left(\left(\delta_i^u\right)^2\right) + \omega_{p+1} f\left(\left(\delta_{p+1}^u\right)^2\right)\right)}\right] \Bigg| \left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i, i = 1, 2, \ldots, p, p+1\right\}\right)$$

$$= \left(\left\{\left[\sqrt{g^{-1}\left(\sum_{i=1}^{p+1}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{p+1}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right] \Bigg| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, p, p+1\end{matrix}\right\},\right.$$

$$\left.\left\{\left[\sqrt{f^{-1}\left(\sum_{i=1}^{p+1}\omega_i f\left(\left(\delta_i^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{p+1}\omega_i f\left(\left(\delta_i^u\right)^2\right)\right)}\right] \Bigg| \left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i, i = 1, \ldots, p+1\right\}\right)$$

Therefore, the theorem is true for $n = p + 1$ also; and hence is true for all n.
This completes the proof.

Now the developed aggregation operator is classified in various forms by choosing several decreasing generators, f.

4.1.1. IVHPF WA aggregation operator

For taking $f(x) = -\log x$, the AIVHPFWA operator reduces to IVHPFWA operator and is defined by

$$IVDHPFWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) =$$

$$\left(\left\{\left[\sqrt{1 - \prod_{i=1}^{n}\left(1 - \left(\gamma_i^l\right)^2\right)^{\omega_i}}, \sqrt{1 - \prod_{i=1}^{n}\left(1 - \left(\gamma_i^u\right)^2\right)^{\omega_i}}\right] \Bigg| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\},\right.$$

$$\left.\left\{\left[\prod_{i=1}^{n}\left(\delta_i^l\right)^{\omega_i}, \prod_{i=1}^{n}\left(\delta_i^u\right)^{\omega_i}\right] \Bigg| \left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i, i = 1, 2, \ldots, n\right\}\right)$$

4.1.2. IVHPF Einstein WA (IVHPFEWA) operator

If $f(x) = \log\left(\frac{2-x}{x}\right)$, the AIVHPFWA operator converted to IVHPFEWA operator, which is presented as:

$$IVHPFEWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \left(\left\{\left[\sqrt{\frac{\prod_{i=1}^n \left(1+\left(\gamma_i^l\right)^2\right)^{\omega_i} - \prod_{i=1}^n \left(1-\left(\gamma_i^l\right)^2\right)^{\omega_i}}{\prod_{i=1}^n \left(1+\left(\gamma_i^l\right)^2\right)^{\omega_i} + \prod_{i=1}^n \left(1-\left(\gamma_i^l\right)^2\right)^{\omega_i}}}, \sqrt{\frac{\prod_{i=1}^n \left(1+\left(\gamma_i^u\right)^2\right)^{\omega_i} - \prod_{i=1}^n \left(1-\left(\gamma_i^u\right)^2\right)^{\omega_i}}{\prod_{i=1}^n \left(1+\left(\gamma_i^u\right)^2\right)^{\omega_i} + \prod_{i=1}^n \left(1-\left(\gamma_i^u\right)^2\right)^{\omega_i}}}\right] \middle| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\}, \left\{\left[\frac{\sqrt{2}\prod_{i=1}^n \left(\delta_i^l\right)^{\omega_i}}{\sqrt{\prod_{i=1}^n \left(1+\left(1-\left(\delta_i^l\right)^2\right)\right)^{\omega_i} + \prod_{i=1}^n \left(\left(\delta_i^l\right)^2\right)^{\omega_i}}}, \frac{\sqrt{2}\prod_{i=1}^n \left(\delta_i^u\right)^{\omega_i}}{\sqrt{\prod_{i=1}^n \left(1+\left(1-\left(\delta_i^u\right)^2\right)\right)^{\omega_i} + \prod_{i=1}^n \left(\left(\delta_i^u\right)^2\right)^{\omega_i}}}\right] \middle| \begin{matrix}\left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\}\right)$$

4.1.3. IVHPF Hamacher WA (IVHPFHWA) operator

When the decreasing generator $f(x) = \log\left(\frac{\theta+(1-\theta)x}{x}\right)$, $\theta > 0$ is taken, AIVHPFWA operator is converted to IVHPFHWA operator which is given as:

$$IVHPFHWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \left(\left\{\left[\frac{\sqrt{\prod_{i=1}^n \left(1+(\theta-1)\gamma_i^{l2}\right)^{\omega_i} - \prod_{i=1}^n \left(1-\gamma_i^{l2}\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^n \left(1+(\theta-1)\gamma_i^{l2}\right)^{\omega_i} + (\theta-1)\prod_{i=1}^n \left(1-\gamma_i^{l2}\right)^{\omega_i}}}, \frac{\sqrt{\prod_{i=1}^n \left(1+(\theta-1)\gamma_i^{u2}\right)^{\omega_i} - \prod_{i=1}^n \left(1-\gamma_i^{u2}\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^n \left(1+(\theta-1)\gamma_i^{u2}\right)^{\omega_i} + (\theta-1)\prod_{i=1}^n \left(1-\gamma_i^{u2}\right)^{\omega_i}}}\right] \middle| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\}, \left\{\left[\frac{\sqrt{\theta}\prod_{i=1}^n \left(\delta_i^l\right)^{\omega_i}}{\sqrt{\prod_{i=1}^n \left(1+(\theta-1)\left(1-\delta_i^{l2}\right)\right)^{\omega_i} + (\theta-1)\prod_{i=1}^n \delta_i^{l2\omega_i}}}, \frac{\sqrt{\theta}\prod_{i=1}^n \left(\delta_i^u\right)^{\omega_i}}{\sqrt{\prod_{i=1}^n \left(1+(\theta-1)\left(1-\delta_i^{u2}\right)\right)^{\omega_i} + (\theta-1)\prod_{i=1}^n \delta_i^{u2\omega_i}}}\right] \middle| \begin{matrix}\left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\}\right) \quad (7)$$

Now, if the value of the parameter θ is considered as 1 and 2, then the IVHPFHWA operator reduces to IVHPFWA and IVHPFEWA operators, respectively.

4.1.4. IVHPF Dombi WA (IVHPFDWA)

When $f(x) = \left(\frac{1}{x} - 1\right)^{\rho}$, $\rho > 0$ is considered, the AIVHPFWA operator reduces to the IVHPFDWA operator as follows:

$$IVHPFDWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \left(\left\{\left[\sqrt{1 - \frac{1}{1+\left(\sum_{i=1}^n \omega_i \left(\frac{1}{1-\gamma_i^{l2}} - 1\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{1 - \frac{1}{1+\left(\sum_{i=1}^n \omega_i \left(\frac{1}{1-\gamma_i^{u2}} - 1\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right] \middle| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\}, \left\{\left[\sqrt{\frac{1}{1+\left(\sum_{i=1}^n \omega_i \left(\frac{1}{\delta_i^{l2}} - 1\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1+\left(\sum_{i=1}^n \omega_i \left(\frac{1}{\delta_i^{u2}} - 1\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right] \middle| \begin{matrix}\left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\}\right) \quad (8)$$

4.1.5. IVHPF Frank WA (IVHPFFWA) operator

When $f(x) = \log\left(\frac{\psi-1}{\psi^x-1}\right)$, $\psi > 1$, the AIVHPFWA operator converted to IVHPFFWA operator presented as:

$$IVHPFFWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \left(\left\{\left[\sqrt{1 - \frac{\log\left(1+\prod_{i=1}^n \left(\psi^{1-\gamma_i^{l2}} - 1\right)^{\omega_i}\right)}{\log\psi}}, \sqrt{1 - \frac{\log\left(1+\prod_{i=1}^n \left(\psi^{1-\gamma_i^{u2}} - 1\right)^{\omega_i}\right)}{\log\psi}}\right] \middle| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\}, \left\{\left[\sqrt{\frac{\log\left(1+\prod_{i=1}^n \left(\psi^{\delta_i^{l2}} - 1\right)^{\omega_i}\right)}{\log\psi}}, \sqrt{\frac{\log\left(1+\prod_{i=1}^n \left(\psi^{\delta_i^{u2}} - 1\right)^{\omega_i}\right)}{\log\psi}}\right] \middle| \begin{matrix}\left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i, \\ i = 1, 2, \ldots, n\end{matrix}\right\}\right) \quad (9)$$

Theorem 2 (*Idempotency*). *Suppose* $\{\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\}$ *is a collection of IVHPFEs, if all* $\tilde{d}_i$ *are equal, i.e.,* $\tilde{d}_i = \tilde{d} = (\{[\gamma^l, \gamma^u]\}, \{[\delta^l, \delta^u]\})$ *for all i, then*

$$AIVHPFWA(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n) = (\{[\gamma^l, \gamma^u]\}, \{[\delta^l, \delta^u]\}) \tag{10}$$

Proof.

$$AIVHPFWA(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n)$$
$$= \left(\left\{ \left[\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\gamma_i^{l2}\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\gamma_i^{u2}\right)\right)} \right] \Big| \begin{matrix}[\gamma_i^l, \gamma_i^u] \in \tilde{h}_i, \\ i = 1, 2, \ldots, n\end{matrix} \right\}, \right.$$
$$\left. \left\{ \left[\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\delta_i^{l2}\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\delta_i^{u2}\right)\right)} \right] \Big| \begin{matrix}[\delta_i^l, \delta_i^u] \in \tilde{g}_i \\ i = 1, 2, \ldots, n\end{matrix} \right\} \right)$$

now since $\tilde{d}_i = (\{[\gamma^l, \gamma^u]\}, \{[\delta^l, \delta^u]\})$ for all $(i = 1, 2, \ldots, n)$, then $\gamma_i^l = \gamma^l$, $\gamma_i^u = \gamma^u$, $\delta_i^l = \delta^l$ and $\delta_i^u = \delta^u$ for all $(i = 1, 2, \ldots, n)$.

Therefore, $AIVHPFWA(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n)$

$$= \left(\left\{ \left[\sqrt{g^{-1}\left(g\left(\gamma^{l2}\right)\sum_{i=1}^n \omega_i\right)}, \sqrt{g^{-1}\left(g\left(\gamma^{u2}\right)\sum_{i=1}^n \omega_i\right)} \right] \Big| \begin{matrix}[\gamma_i^l, \gamma_i^u] \in \tilde{h}_i, \\ i = 1, 2, \ldots, n\end{matrix} \right\}, \right.$$
$$\left. \left\{ \left[\sqrt{f^{-1}\left(f\left(\delta^{l2}\right)\sum_{i=1}^n \omega_i\right)}, \sqrt{f^{-1}\left(f\left(\delta^{u2}\right)\sum_{i=1}^n \omega_i\right)} \right] \Big| \begin{matrix}[\delta_i^l, \delta_i^u] \in \tilde{g}_i \\ i = 1, 2, \ldots, n\end{matrix} \right\} \right)$$
$$= \left(\left\{ [\gamma^l, \gamma^u] \Big| \begin{matrix}[\gamma_i^l, \gamma_i^u] \in \tilde{h}_i, \\ i = 1, 2, \ldots, n\end{matrix} \right\}, \left\{ [\delta^l, \delta^u] \Big| \begin{matrix}[\delta_i^l, \delta_i^u] \in \tilde{g}_i \\ i = 1, 2, \ldots, n\end{matrix} \right\} \right)$$
$$= (\{[\gamma^l, \gamma^u]\}, \{[\delta^l, \delta^u]\})$$

Hence the Theorem.

Theorem 3 (*Monotonicity*). *Let* $\tilde{d}_{\theta_i} = \left(\tilde{h}_{\theta_i}, \tilde{g}_{\theta_i}\right)$ *and* $\tilde{d}_{\phi_i} = \left(\tilde{h}_{\phi_i}, \tilde{g}_{\phi_i}\right)$ $(i = 1, \ldots, n)$ *be two sets of IVHPFEs, if* $\tilde{d}_{\theta_i} \le \tilde{d}_{\phi_i}$ *for all i; i.e., if* $\left[\gamma_{\theta_i}^l, \gamma_{\theta_i}^u\right] \in \tilde{h}_{\theta_i}$, $\left[\gamma_{\phi_i}^l, \gamma_{\phi_i}^u\right] \in \tilde{h}_{\phi_i}$, $\left[\delta_{\theta_i}^l, \delta_{\theta_i}^u\right] \in \tilde{g}_{\theta_i}$ *and* $\left[\delta_{\phi_i}^l, \delta_{\phi_i}^u\right] \in \tilde{g}_{\phi_i}$ *where* $\gamma_{\theta_i}^l \le \gamma_{\phi_i}^l$, $\gamma_{\theta_i}^u \le \gamma_{\phi_i}^u$, $\delta_{\theta_i}^l \ge \delta_{\phi_i}^l$ *and* $\delta_{\theta_i}^u \ge \delta_{\phi_i}^u$ *then*

$$AIVHPFWA\left(\tilde{d}_{\theta_1}, \tilde{d}_{\theta_2}, \ldots, \tilde{d}_{\theta_n}\right) \le AIVHPFWA\left(\tilde{d}_{\phi_1}, \tilde{d}_{\phi_2}, \ldots, \tilde{d}_{\phi_n}\right).$$

Proof. From the given inequality $\gamma_{\theta_i}^l \le \gamma_{\phi_i}^l$ and using the characteristic of g,

$$\omega_i g\left(\left(\gamma_i^l\right)^2\right) \le \omega_i g\left(\left(\gamma_{\phi_i}^l\right)^2\right) \text{ for all } i$$

then $\sum_{i=1}^n \omega_i g\left(\left(\gamma_{\theta_i}^l\right)^2\right) \le \sum_{i=1}^n \omega_i g\left(\left(\gamma_{\phi_i}^l\right)^2\right)$ for all i;

being g is increasing function, g^{-1} is also an increasing function, and therefore

$$\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_{\theta_i}^l\right)^2\right)\right)} \le \sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_{\phi_i}^l\right)^2\right)\right)} \text{ for all } i; \tag{11}$$

Similarly, the following inequality holds

$$\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_{\theta_i}^u\right)^2\right)\right)} \le \sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_{\phi_i}^u\right)^2\right)\right)} \text{ for all } i. \tag{12}$$

Again from the assumption of $\delta_{\theta_i}^l \ge \delta_{\phi_i}^l$ for all i and since f is decreasing function
$f\left(\delta_{\theta_i}^l\right)^2 \le f\left(\delta_{\phi_i}^l\right)^2$ which implies that $\sum_{i=1}^n \omega_i f\left(\left(\delta_{\theta_i}^l\right)^2\right) \le \sum_{i=1}^n \omega_i f\left(\left(\delta_{\phi_i}^l\right)^2\right)$ for all i;
therefore, since every inverse function of a decreasing function is also a decreasing function, then it follows that

$$\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\delta_{\theta_i}^l\right)^2\right)\right)} \ge \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\delta_{\phi_i}^l\right)^2\right)\right)} \text{ for all } i; \tag{13}$$

and similarly, it also can be shown that

$$\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\delta_{\theta_i}^u\right)^2\right)\right)} \ge \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\delta_{\phi_i}^u\right)^2\right)\right)} \text{ for all } i. \tag{14}$$

According to the formulas (11)–(14) and by comparison laws between two IVHPFEs, obtained

$$AIVHPFWA\left(\tilde{d}_{\theta_1}, \tilde{d}_{\theta_2}, \ldots, \tilde{d}_{\theta_n}\right) \leq AIVHPFWA\left(\tilde{d}_{\phi_1}, \tilde{d}_{\phi_2}, \ldots, \tilde{d}_{\phi_n}\right).$$

Hence the theorem is proved.

Theorem 4 (*Boundary*). *Let $\tilde{d}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of IVHPFEs. Also, let for all $i = 1, 2, \ldots, n$;*

$$\gamma^l_{min} = min\left\{\gamma^l_{i_{min}}\right\} \text{ where } \gamma^l_{i_{min}} = min_{[\gamma^l_i, \gamma^u_i]\in\tilde{h}_i}\left\{\gamma^l_i\right\};$$

$$\gamma^u_{min} = min\left\{\gamma^u_{i_{min}}\right\} \text{ where } \gamma^u_{i_{min}} = min_{[\gamma^l_i, \gamma^u_i]\in\tilde{h}_i}\left\{\gamma^u_i\right\};$$

$$\gamma^l_{max} = max\left\{\gamma^l_{i_{max}}\right\} \text{ where } \gamma^l_{i_{max}} = max_{[\gamma^l_i, \gamma^u_i]\in\tilde{h}_i}\left\{\gamma^l_i\right\};$$

$$\gamma^u_{max} = max\left\{\gamma^u_{i_{max}}\right\} \text{ where } \gamma^u_{i_{max}} = max_{[\gamma^l_i, \gamma^u_i]\in\tilde{h}_i}\left\{\gamma^u_i\right\};$$

$$\text{again, let } \delta^l_{min} = min\left\{\delta^l_{i_{min}}\right\} \text{ where } \delta^l_{i_{min}} = min_{[\delta^l_i, \delta^u_i]\in\tilde{h}_i}\left\{\delta^l_i\right\};$$

$$\delta^u_{min} = min\left\{\delta^u_{i_{min}}\right\} \text{ where } \delta^u_{i_{min}} = min_{[\delta^l_i, \delta^u_i]\in\tilde{h}_i}\left\{\delta^u_i\right\};$$

$$\delta^l_{max} = max\left\{\delta^l_{i_{max}}\right\} \text{ where } \delta^l_{i_{max}} = max_{[\delta^l_i, \delta^u_i]\in\tilde{h}_i}\left\{\delta^l_i\right\};$$

$$\delta^u_{max} = max\left\{\delta^u_{i_{max}}\right\} \text{ where } \delta^u_{i_{max}} = max_{[\delta^l_i, \delta^u_i]\in\tilde{h}_i}\left\{\delta^u_i\right\};$$

$$\text{also, if} \tilde{d}_- = \left(\left[\gamma^l_{min}, \gamma^u_{min}\right], \left[\delta^l_{max}, \delta^u_{max}\right]\right) \text{ and } \tilde{d}_+ = \left(\left[\gamma^l_{max}, \gamma^u_{max}\right], \left[\delta^l_{min}, \delta^u_{min}\right]\right),$$
$$\text{then } \tilde{d}_- \leq AIVHPFWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) \leq \tilde{d}_+. \tag{15}$$

Proof. According to the theorem of idempotency as well as monotonicity, it can be achieved that

$$AIVHPFWA\left(\tilde{d}_-, \tilde{d}_-, \ldots, \tilde{d}_-\right) \leq AIVHPFWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right),$$

and

and $AIVHPFWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) \leq AIVHPFWA\left(\tilde{d}_+, \tilde{d}_+, \ldots, \tilde{d}_+\right)$..

Subsequently, it is obtained that

$$\tilde{d}_- \leq AIVHPFWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) \leq \tilde{d}_+.$$

4.2. AIVHPFOWA aggregating operators

Definition 10. Let $\tilde{d}_i$ $(i = 1, 2, \ldots, n)$ be a collection of IVHPFEs, and then the AIVHPFOWA operator is defined as follows:

$$AIVHPFOWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \oplus_{A_{i=1}}^{n}\left(\omega_i \tilde{d}_{\sigma(i)}\right)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ represents a permutation of $\{1, 2, \ldots, n\}$, such that $\tilde{d}_{\sigma(i-1)} \geq \tilde{d}_{\sigma(i)}$ for all $i = 2, \ldots, n$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the aggregation-associated weight vector such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n}\omega_i = 1$.

Theorem 5. *Let $\tilde{d}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a set of IVHPFEs, then the aggregating value of IVHPFEs by using AIVHPFOWA operator is also an IVHPFE and is given by*

$$AIVHPFOWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \oplus_{A_{i=1}}^{n}\left(\omega_i \tilde{d}_{\sigma(i)}\right)$$
$$= \left(\left\{\left[\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\gamma^l_{\sigma(i)}\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\gamma^u_{\sigma(i)}\right)^2\right)\right)}\right] \middle| \begin{array}{c}\left[\gamma^l_{\sigma(i)}, \gamma^u_{\sigma(i)}\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{array}\right\},\right.$$
$$\left.\left\{\left[\sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\delta^l_{\sigma(i)}\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\delta^u_{\sigma(i)}\right)^2\right)\right)}\right] \middle| \begin{array}{c}\left[\delta^l_{\sigma(i)}, \delta^u_{\sigma(i)}\right] \in \tilde{g}_i \\ i = 1, 2, \ldots, n\end{array}\right\}\right) \tag{16}$$

Proof. Similar to the proof of Theorem 1.

It can be easily proved, similar to Theorem 1, 2 and 3, respectively, that the AIVHPFOWA operator possesses the following idempotency, monotonicity and boundedness properties.

Theorem 6 (*Idempotency*). *If all $\tilde{d}_i$ $(i = 1, 2, \ldots, n)$ are equal i.e., $\tilde{d}_i = \tilde{d} = \left(\left\{\left[\gamma^l, \gamma^u\right]\right\}, \left\{\left[\delta^l, \delta^u\right]\right\}\right)$ for all i, then*

$$AIVHPFOWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \tilde{d}.$$

Theorem 7 (*Monotonicity*). *Let $\tilde{d}_{\theta_i} = \left(\tilde{h}_{\theta_i}, \tilde{g}_{\theta_i}\right)$ and $\tilde{d}_{\phi_i} = \left(\tilde{h}_{\phi_i}, \tilde{g}_{\phi_i}\right)$ $(i = 1, \dots, n)$ be two sets of IVHPFEs, if $\tilde{d}_{\theta_i} \leq \tilde{d}_{\phi_i}$ for all i; then*

$$AIVHPFOWA\left(\tilde{d}_{\theta_1}, \tilde{d}_{\theta_2}, \dots, \tilde{d}_{\theta_n}\right) \leq AIVHPFOWA\left(\tilde{d}_{\phi_1}, \tilde{d}_{\phi_2}, \dots, \tilde{d}_{\phi_n}\right).$$

Theorem 8 (*Boundedness*). *Let $\tilde{d}_i = \left(\tilde{h}_i, \tilde{g}_i\right)$ $(i = 1, 2, \dots, n)$ be a collections of IVHPFEs, let*

$$\tilde{d}_- = \left(\left[\gamma^l_{min}, \gamma^u_{min}\right], \left[\delta^l_{max}, \delta^u_{max}\right]\right) \text{ and } \tilde{d}_+ = \left(\left[\gamma^l_{max}, \gamma^u_{max}\right], \left[\delta^l_{min}, \delta^u_{min}\right]\right), \text{ then}$$

$$\tilde{d}_- \leq AIVHPFWA\left(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n\right) \leq \tilde{d}_+,$$

where the notations $\tilde{d}_-$ and $\tilde{d}_+$ are already shown above in Theorem 3.

Next, some particular cases of the AIVHPFOWA operator are presented with respect to the decreasing generator f.

4.2.1. Algebraic t-CN&t-N-based OWA aggregation operator

When using the decreasing generator as $f(x) = -\log x$, IVHPF OWA (IVHPFOWA) operator is found from reduced AIVHPFOWA operator which is presented as follows:

$$IVHPFOWA\left(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n\right)$$

$$= \left(\left\{\left[\sqrt{1 - \prod_{i=1}^{n}\left(1 - \left(\gamma^l_{\sigma(i)}\right)^2\right)^{\omega_i}}, \sqrt{1 - \prod_{i=1}^{n}\left(1 - \left(\gamma^u_{\sigma(i)}\right)^2\right)^{\omega_i}}\right] \middle| \begin{matrix}\left[\gamma^l_{\sigma(i)}, \gamma^u_{\sigma(i)}\right] \in \tilde{h}_{\sigma(i)} \\ i = 1, 2, \dots, n\end{matrix}\right\},\right.$$
$$\left.\left\{\left[\prod_{i=1}^{n}\left(\delta^l_{\sigma(i)}\right)^{\omega_j}, \prod_{i=1}^{n}\left(\delta^l_{\sigma(i)}\right)^{\omega_j}\right] \middle| \left[\delta^l_{\sigma(i)}, \delta^u_{\sigma(i)}\right] \in \tilde{g}_{\sigma(i)}, i = 1, 2, \dots, n\right\}\right)$$

4.2.2. Einstein t-CN&t-N-based OWA aggregation operator

If $f(x) = \log\left(\frac{2-x}{x}\right)$, the AIVHPFOWA operator converted to IVHPF Einstein OWA (IVHPFEOWA) operator, which is presented as:

$$IVHPFEOWA\left(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n\right) =$$

$$\left(\left\{\left[\sqrt{\frac{\prod_{i=1}^{n}\left(1 + \gamma^{l2}_{\sigma(i)}\right)^{\omega_i} - \prod_{i=1}^{n}\left(1 - \gamma^{l2}_{\sigma(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1 + \gamma^{l2}_{\sigma(i)}\right)^{\omega_i} + \prod_{i=1}^{n}\left(1 - \gamma^{l2}_{\sigma(i)}\right)^{\omega_i}}}, \sqrt{\frac{\prod_{i=1}^{n}\left(1 + \gamma^{u2}_{\sigma(i)}\right)^{\omega_i} - \prod_{i=1}^{n}\left(1 - \gamma^{u2}_{\sigma(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1 + \gamma^{u2}_{\sigma(i)}\right)^{\omega_i} + \prod_{i=1}^{n}\left(1 - \gamma^{u2}_{\sigma(i)}\right)^{\omega_i}}}\right] \middle| \begin{matrix}\left[\gamma^l_{\sigma(i)}, \gamma^u_{\sigma(i)}\right] \in \tilde{h}_{\sigma(i)} \\ i = 1, 2, \dots, n\end{matrix}\right\},\right.$$
$$\left.\left\{\left[\frac{\sqrt{2}\prod_{i=1}^{n}\delta^{l\omega_i}_{\sigma(i)}}{\sqrt{\prod_{i=1}^{n}\left(1 + \left(1 - \delta^{l2}_{\sigma(i)}\right)\right)^{\omega_i} + \prod_{i=1}^{n}\delta^{l2\omega_i}_{\sigma(i)}}}, \frac{\sqrt{2}\prod_{i=1}^{n}\delta^{u\omega_i}_{\sigma(i)}}{\sqrt{\prod_{i=1}^{n}\left(1 + \left(1 - \delta^{u2}_{\sigma(i)}\right)\right)^{\omega_i} + \prod_{i=1}^{n}\delta^{u2\omega_i}_{\sigma(i)}}}\right] \middle| \begin{matrix}\left[\delta^l_{\sigma(i)}, \delta^u_{\sigma(i)}\right] \in \tilde{g}_{\sigma(i)} \\ i = 1, 2, \dots, n\end{matrix}\right\}\right)$$

4.2.3. OWA aggregation operators based on Hamacher t-CN&t-N

When taking the decreasing generator $f(x) = \log\left(\frac{\theta+(1-\theta)x}{x}\right)$, $\theta > 0$, the AIVHPFOWA operator transformed to IVHPF Hamacher OWA (IVHPFHOWA) operator which is given as follows:

$$IVHPFHOWA\left(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n\right) = \left(\left\{\left[\frac{\sqrt{\prod_{i=1}^{n}\left(1 + (\theta - 1)\gamma^{l2}_{\sigma(i)}\right)^{\omega_i} - \prod_{i=1}^{n}\left(1 - \gamma^{l2}_{\sigma(i)}\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1 + (\theta - 1)\gamma^{l2}_{\sigma(i)}\right)^{\omega_i} + (\theta - 1)\prod_{i=1}^{n}\left(1 - \gamma^{l2}_{\sigma(i)}\right)^{\omega_i}}},\right.\right.\right.$$
$$\left.\left.\frac{\sqrt{\prod_{i=1}^{n}\left(1 + (\theta - 1)\left(\gamma^u_{\sigma(i)}\right)^2\right)^{\omega_i} - \prod_{i=1}^{n}\left(1 - \left(\gamma^u_{\sigma(i)}\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1 + (\theta - 1)\left(\gamma^u_{\sigma(i)}\right)^2\right)^{\omega_i} + (\theta - 1)\prod_{i=1}^{n}\left(1 - \left(\gamma^u_{\sigma(i)}\right)^2\right)^{\omega_i}}}\right] \middle| \begin{matrix}\left[\gamma^l_{\sigma(i)}, \gamma^u_{\sigma(i)}\right] \in \tilde{h}_{\sigma(i)} \\ i = 1, 2, \dots, n\end{matrix}\right\},$$
$$\left\{\left[\frac{\sqrt{\theta}\prod_{i=1}^{n}\left(\delta^l_{\sigma(i)}\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1 + (\theta - 1)\left(1 - \left(\delta^l_{\sigma(i)}\right)^2\right)\right)^{\omega_i} + (\theta - 1)\prod_{i=1}^{n}\left(\left(\delta^l_{\sigma(i)}\right)^2\right)^{\omega_i}}},\right.\right.$$
$$\left.\left.\left.\frac{\sqrt{\theta}\prod_{i=1}^{n}\left(\delta^u_{\sigma(i)}\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1 + (\theta - 1)\left(1 - \left(\delta^u_{\sigma(i)}\right)^2\right)\right)^{\omega_i} + (\theta - 1)\prod_{i=1}^{n}\left(\left(\delta^u_{\sigma(i)}\right)^2\right)^{\omega_i}}}\right] \middle| \begin{matrix}\left[\delta^l_{\sigma(i)}, \delta^u_{\sigma(i)}\right] \in \tilde{g}_{\sigma(i)} \\ i = 1, 2, \dots, n\end{matrix}\right\}\right) \tag{17}$$

As like previous discussions, it is to be noted here that several operators can be derived from Hamacher operations, viz., IVHPFOWA and IVHPFEOWA operators, which can be framed by considering $\theta = 1$ and 2, respectively.

4.2.4. OWA aggregation operators based on Dombi t-CN&t-N

When $f(x) = \left(\frac{1}{x} - 1\right)^{\rho}$, $\rho > 0$, then the AIVHPFOWA operator reduces to the IVHPF Dombi OWA (IVHPFDOWA) operator as follows:

$$IVHPFDOWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) =$$
$$\left(\left\{\left[\sqrt{1 - \frac{1}{1 + \left(\sum_{i=1}^{n} \omega_i \left(\frac{\gamma^{l2}_{\sigma(i)}}{1-\gamma^{l2}_{\sigma(i)}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{1 - \frac{1}{1 + \left(\sum_{i=1}^{n} \omega_i \left(\frac{\gamma^{l2}_{\sigma(i)}}{1-\gamma^{l2}_{\sigma(i)}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right] \middle| \begin{array}{c}\left[\gamma^{l}_{\sigma(i)}, \gamma^{u}_{\sigma(i)}\right] \in \tilde{h}_{\sigma(i)} \\ i = 1, 2, \ldots, n\end{array}\right\},\right.$$

$$\left.\left\{\left[\sqrt{\frac{1}{1 + \left(\sum_{i=1}^{n} \omega_j \left(\frac{1-\delta^{l2}_{\sigma(i)}}{\delta^{l2}_{\sigma(i)}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1 + \left(\sum_{i=1}^{n} \omega_i \left(\frac{1-\delta^{l2}_{\sigma(i)}}{\delta^{l2}_{\sigma(i)}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right] \middle| \begin{array}{c}\left[\delta^{l}_{\sigma(i)}, \delta^{u}_{\sigma(i)}\right] \in \tilde{g}_{\sigma(i)} \\ i = 1, 2, \ldots, n\end{array}\right\}\right) \quad (18)$$

4.2.5. OWA aggregation operator based on Frank t-CN&t-N

When $f(x) = \log\left(\frac{\psi-1}{\psi^{x}-1}\right)$, $\psi > 1$, the AIVHPFOWA operator converted to IVHPF Frank OWA (IVHPFFOWA) operator presented as:

$$IVHPFFOWA\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) =$$
$$\left(\left\{\left[\sqrt{1 - \frac{\log\left(1 + \prod_{i=1}^{n}\left(\psi^{1-\gamma^{l2}_{\sigma(i)}} - 1\right)^{\omega_i}\right)}{\log\psi}}, \sqrt{1 - \frac{\log\left(1 + \prod_{i=1}^{n}\left(\psi^{1-\gamma^{l2}_{\sigma(i)}} - 1\right)^{\omega_i}\right)}{\log\psi}}\right] \middle| \begin{array}{c}\left[\gamma^{l}_{\sigma(i)}, \gamma^{u}_{\sigma(i)}\right] \in \tilde{h}_{\sigma(i)} \\ i = 1, 2, \ldots, n\end{array}\right\},\right.$$

$$\left.\left\{\left[\sqrt{\frac{\log\left(1 + \prod_{i=1}^{n}\left(\psi^{\delta^{l2}_{\sigma(i)}} - 1\right)^{\omega_i}\right)}{\log\psi}}, \sqrt{\frac{\log\left(1 + \prod_{i=1}^{n}\left(\psi^{\delta^{u2}_{\sigma(i)}} - 1\right)^{\omega_i}\right)}{\log\psi}}\right] \middle| \begin{array}{c}\left[\delta^{l}_{\sigma(i)}, \delta^{u}_{\sigma(i)}\right] \in \tilde{g}_{\sigma(i)}, \\ i = 1, 2, \ldots, n\end{array}\right\}\right) \quad (19)$$

Like averaging aggregation operators, the above processes may be analogously extended for geometric aggregation operators. However, a brief discussion on geometric mean based aggregation operators is presented below.

4.3. AIVHPFWG aggregation operators

This part describes aggregation operator AIVHPFWG with some of their desirable properties having IVHPF information.

Definition 11. Let $\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right)$ be a set of IVHPFEs and $\omega = \left(\omega_1, \omega_2, \ldots, \omega_n\right)^T$ denote the weight vector of IVHPFEs satisfying the conditions $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then, an AIVHPFWG operator is defined by a function

$: \widetilde{D}^n \to \widetilde{D}$ and is given by $\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \otimes_{A}{}_{i=1}^{n} \left(\tilde{d}_i^{\omega_i}\right)$,

Theorem 9. *Let $\tilde{d}_i = \left(\tilde{h}_i, \tilde{g}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IVHPFEs, then the aggregating value utilizing AIVHPFWG operator is also an IVHPFE and defined by*

$$AIVHPFWG\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right)$$
$$= \left(\left\{\left[\sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(\left(\gamma^{l}_i\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(\left(\gamma^{u}_i\right)^2\right)\right)}\right] \middle| \begin{array}{c}\left[\gamma^{l}_i, \gamma^{u}_i\right] \in \tilde{h}_i, \\ i = 1, 2, \ldots, n\end{array}\right\},\right.$$

$$\left.\left\{\left[\sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g\left(\left(\delta^{l}_i\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g\left(\left(\delta^{u}_i\right)^2\right)\right)}\right] \middle| \begin{array}{c}\left[\delta^{l}_i, \delta^{u}_i\right] \in \tilde{g}_i \\ i = 1, 2, \ldots, n\end{array}\right\}\right) \quad (20)$$

Proof. Similar to Theorem 1.

Considering different forms of the decreasing function, f, several particular types of AIVHPFWG operators can be achieved like averaging operators.

4.3.1. Algebraic operation-based aggregation operator

If $f(x) = -\log(x)$, the AIVHPFWG operator reduces to IVHPF WG (IVHPFWG) operator defined as:

$$IVHPFWG\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) =$$

$$\left(\left\{\left[\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\omega_i}, \prod_{i=1}^{n}\left(\gamma_i^u\right)^{\omega_i}\right] \Big| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\}, \left\{\left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\delta_i^l\right)^2\right)^{\omega_i}}, \sqrt{1-\prod_{i=1}^{n}\left(1-\left(\delta_i^u\right)^2\right)^{\omega_i}}\right] \Big| \left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i, i = 1, 2, \ldots, n\right\}\right)$$

4.3.2. Einstein operation-based aggregation operator

When $f(x) = \log\left(\frac{2-x}{x}\right)$, the AIVHPFWG operator changes to IVHPF Einstein WG (IVHPFEWG) operator, which is defined as:

$$IVHPFEWG\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) =$$

$$\left(\left\{\left[\frac{\sqrt{2}\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1+\left(1-\gamma_i^{l2}\right)\right)^{\omega_i}+\prod_{i=1}^{n}\left(\gamma_i^{l2}\right)^{\omega_i}}}, \frac{\sqrt{2}\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1+\left(1-\gamma_i^{u2}\right)\right)^{\omega_i}+\prod_{i=1}^{n}\left(\gamma_i^{u2}\right)^{\omega_i}}}\right] \Big| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\},\right.$$
$$\left.\left\{\left[\sqrt{\frac{\prod_{i=1}^{n}\left(1+\left(\delta_i^l\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\delta_i^l\right)^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\left(\delta_i^l\right)^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\left(\delta_i^l\right)^2\right)^{\omega_i}}}, \sqrt{\frac{\prod_{i=1}^{n}\left(1+\left(\delta_i^u\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\delta_i^u\right)^2\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\left(\delta_i^u\right)^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\left(\delta_i^u\right)^2\right)^{\omega_i}}}\right] \Big| \begin{matrix}\left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\}\right)$$

4.3.3. Aggregation operator based on Hamacher operation

When the decreasing generator $f(x) = \log\left(\frac{\theta+(1-\rho)\theta}{x}\right)$, $\theta > 0$ is taken, the AIVHPFWG operator converted to IVHPF Hamacher WG (IVHPFHWG) operator which is given as follows:

$$IVHPFHWG\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right)$$

$$= \left(\left\{\left[\sqrt{\theta}\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\omega_i} \Big/ \sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\left(1-\left(\gamma_i^l\right)^2\right)\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\left(\left(\gamma_i^l\right)^2\right)^{\omega_i}},\right.\right.\right.$$
$$\left.\left.\sqrt{\theta}\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\omega_i} \Big/ \sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\left(1-\gamma_i^{u2}\right)\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\left(\gamma_i^{u2}\right)^{\omega_i}}\right] \Big| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\},$$
$$\left\{\left[\frac{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\left(\delta_i^l\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\delta_i^l\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\left(\delta_i^l\right)^2\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\left(1-\left(\delta_i^l\right)^2\right)^{\omega_i}}},\right.\right.$$
$$\left.\left.\left.\frac{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\left(\delta_i^u\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\delta_i^u\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\left(\delta_i^u\right)^2\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\left(1-\left(\delta_i^u\right)^2\right)^{\omega_i}}}\right] \Big| \left[\delta_i^l, \delta_i^u\right] \in \tilde{g}_i, i = 1, 2, \ldots, n\right\}\right) \tag{21}$$

Putting the value $\theta = 1$ and 2 in (21), the AIVHPFHWG operator reduces to IVHPFWG and IVHPFEWG operators, respectively.

4.3.4. Aggregation operators based on Dombi operation

When $f(x) = \left(\frac{1}{x} - 1\right)^\rho$, $\rho > 0$, then the AIVHPFWG operator reduces to the IVHPF Dombi WG (IVHPFDWG) operator as follows:

$$IVHPFDWG\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right)$$

$$= \left(\left\{\left[\sqrt{\frac{1}{1+\left(\sum_{i=1}^{n}\omega_i\left(\frac{1}{\left(\gamma_i^l\right)^2}-1\right)^\rho\right)^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1+\left(\sum_{i=1}^{n}\omega_i\left(\frac{1}{\left(\gamma_i^u\right)^2}-1\right)^\rho\right)^{\frac{1}{\rho}}}}\right] \Big| \begin{matrix}\left[\gamma_i^l, \gamma_i^u\right] \in \tilde{h}_i \\ i = 1, 2, \ldots, n\end{matrix}\right\},\right.$$

$$\left\{\left[\sqrt{1-\frac{1}{1+\left(\sum_{i=1}^{n}\omega_i\left(\frac{1}{1-\left(\delta_i^l\right)^2}-1\right)^{\rho}\right)^{\frac{1}{\rho}}}},\sqrt{1-\frac{1}{1+\left(\sum_{i=1}^{n}\omega_i\left(\frac{1}{1-\left(\delta_i^u\right)^2}-1\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right]\Big|\begin{matrix}\left[\delta_i^l,\delta_i^u\right]\in\tilde{g}_i\\ i=1,2,\ldots,n\end{matrix}\right\}\Bigg) \tag{22}$$

4.3.5. Aggregation operator based on Frank operation

When $f(x)=\log\left(\frac{\psi-1}{\psi^x-1}\right)$, $\psi>1$, the AIVHPFWG operator converted to IVHPF Frank WG (IVHPFFWG) operator presented as:

$$IVHPFFWG\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)$$

$$=\Bigg(\left\{\left[\sqrt{\frac{\log\left(1+\prod_{i=1}^{n}\left(\psi^{\left(\gamma_i^l\right)^2}-1\right)^{\omega_i}\right)}{\log\psi}},\sqrt{\frac{\log\left(1+\prod_{i=1}^{n}\left(\psi^{\left(\gamma_i^u\right)^2}-1\right)^{\omega_i}\right)}{\log\psi}}\right]\Big|\begin{matrix}\left[\gamma_i^l,\gamma_i^u\right]\in\tilde{h}_i\\ i=1,2,\ldots,n\end{matrix}\right\},$$

$$\left\{\left[\sqrt{1-\frac{\log\left(1+\prod_{i=1}^{n}\left(\psi^{1-\left(\delta_i^l\right)^2}-1\right)^{\omega_i}\right)}{\log\psi}},\sqrt{1-\frac{\log\left(1+\prod_{i=1}^{n}\left(\psi^{1-\left(\delta_i^u\right)^2}-1\right)^{\omega_i}\right)}{\log\psi}}\right]\Big|\begin{matrix}\left[\delta_i^l,\delta_i^u\right]\in\tilde{g}_i,\\ i=1,2,\ldots,n\end{matrix}\right\}\Bigg) \tag{23}$$

4.4. AIVHPFOWG aggregation operators

Definition 12. Let $\tilde{d}_i$ $(i=1,2,\ldots,n)$ be a collection of IVHPFEs, then AIVHPFOWG operator is defined as follows:

$$AIVHPFOWG\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)=\otimes_{A_{i=1}}^{n}\left(\omega_i\tilde{d}_{\sigma(i)}\right)$$

where $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ represents a permutation of $(1,2,\ldots,n)$, such that $\tilde{d}_{\sigma(i-1)}\geq\tilde{d}_{\sigma(i)}$ for all $i=2,\ldots,n$, and $\omega=\left(\omega_1,\omega_2,\ldots,\omega_n\right)^T$ is the aggregation-associated weight vector such that $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$.

Theorem 10. *Let $\tilde{d}_i=\left(\tilde{h}_i,\tilde{g}_i\right)$ $(i=1,2,\ldots,n)$ be a set of IVHPFEs, then the aggregating value using AIVHPFOWG operator is also an IVHPFE and is given by*

$$AIVHPFOWG\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)=\otimes_{A_{i=1}}^{n}\left(\omega_i\tilde{d}_{\sigma(i)}\right)$$

$$=\Bigg(\left\{\left[\sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\gamma_{\sigma(i)}^l\right)^2\right)\right)},\sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\gamma_{\sigma(i)}^u\right)^2\right)\right)}\right]\Big|\begin{matrix}\left[\gamma_{\sigma(i)}^l,\gamma_{\sigma(i)}^u\right]\in\tilde{h}_i\\ i=1,2,\ldots,n\end{matrix}\right\},$$

$$\left\{\left[\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\delta_{\sigma(i)}^l\right)^2\right)\right)},\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\delta_{\sigma(i)}^u\right)^2\right)\right)}\right]\Big|\begin{matrix}\left[\delta_{\sigma(i)}^l,\delta_{\sigma(i)}^u\right]\in\tilde{g}_i\\ i=1,2,\ldots,n\end{matrix}\right\}\Bigg) \tag{24}$$

Proof. The proof is similar to the first theorem.

Considering different forms of the decreasing function, f, several particular types of AIVHPFOWG operators can be achieved like OWA operators.

4.4.1. Algebraic t-CN&t-N-based OWG aggregation operator

When use $f(x)=-\log x$, the IVHPF OWG (IVHPFOWG) operator is found from reduced AIVHPFOWG operator which is presented as follows:

$$IVHPFOWG\left(\tilde{d}_1,\tilde{d}_2,\ldots,\tilde{d}_n\right)=\Bigg(\left\{\left[\left[\prod_{i=1}^{n}\left(\gamma_{\sigma(i)}^l\right)^{\omega_j},\prod_{i=1}^{n}\left(\gamma_{\sigma(i)}^u\right)^{\omega_i}\right]\Big|\begin{matrix}\left[\gamma_{\sigma(i)}^l,\gamma_{\sigma(i)}^u\right]\in\tilde{h}_{\sigma(i)}\\ i=1,2,\ldots,n\end{matrix}\right]\right\},$$

$$\left\{\left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\delta_{\sigma(i)}^l\right)^2\right)^{\omega_i}},\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\delta_{\sigma(i)}^u\right)^2\right)^{\omega_i}}\right]\Big|\begin{matrix}\left[\delta_{\sigma(i)}^l,\delta_{\sigma(i)}^u\right]\in\tilde{g}_{\sigma(i)}\\ i=1,2,\ldots,n\end{matrix}\right\}\Bigg)$$

4.4.2. Einstein t-CN&t-N-based OWG aggregation operator

If $f(x) = \log\left(\frac{2-x}{x}\right)$, the AIVHPFOWG operator converted to IVHPF Einstein OWG (IVHPFEOWG) operator, which is presented as:

$$IVHPFEOWG\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) =$$

$$\left(\left\{\left[\frac{\sqrt{2}\prod_{i=1}^{n}\left(\gamma^{l}_{\sigma(i)}\right)^{\omega_j}}{\sqrt{\prod_{i=1}^{n}\left(1+\left(1-\gamma^{l2}_{\sigma(i)}\right)\right)^{\omega_i}+\prod_{i=1}^{n}\gamma^{l2\omega_i}_{\sigma(i)}}}, \frac{\sqrt{2}\prod_{i=1}^{n}\left(\gamma^{u}_{\sigma(i)}\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1+\left(1-\gamma^{u2}_{\sigma(i)}\right)\right)^{\omega_i}+\prod_{i=1}^{n}\gamma^{u2\omega_i}_{\sigma(i)}}}\right] \middle| \begin{array}{c}\left[\gamma^{l}_{\sigma(i)}, \gamma^{u}_{\sigma(i)}\right] \in \tilde{h}_{\sigma(i)}\\ i = 1, 2, \ldots, n\end{array}\right\},\right.$$

$$\left.\left\{\left[\sqrt{\frac{\prod_{i=1}^{n}\left(1+\delta^{l2}_{\sigma(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\delta^{l2}_{\sigma(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\delta^{l2}_{\sigma(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\delta^{l2}_{\sigma(i)}\right)^{\omega_i}}}, \sqrt{\frac{\prod_{i=1}^{n}\left(1+\delta^{u2}_{\sigma(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\delta^{u2}_{\sigma(i)}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\delta^{u2}_{\sigma(i)}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\delta^{u2}_{\sigma(i)}\right)^{\omega_i}}}\right] \middle| \begin{array}{c}\left[\delta^{l}_{\sigma(i)}, \delta^{u}_{\sigma(i)}\right] \in \tilde{g}_{\sigma(i)}\\ i = 1, 2, \ldots, n\end{array}\right\}\right)$$

4.4.3. Hamacher t-CN&t-N-based OWG aggregation operator

When taking the decreasing generator $f(x) = \log\left(\frac{\theta+(1-\theta)x}{x}\right)$, $\theta > 0$, the AIVHPFOWG operator converted to IVHPF Hamacher OWG (IVHPFHOWG) operator which is given as follows:

$$IVHPFHOWG\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \left(\left\{\left[\frac{\sqrt{\theta}\prod_{i=1}^{n}\left(\gamma^{l}_{\sigma(i)}\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\left(1-\gamma^{l2}_{\sigma(i)}\right)\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\left(\gamma^{l2}_{\sigma(i)}\right)^{\omega_i}}},\right.\right.\right.$$

$$\left.\left.\frac{\sqrt{\theta}\prod_{i=1}^{n}\left(\gamma^{u}_{\sigma(i)}\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\left(1-\gamma^{u2}_{\sigma(i)}\right)\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\left(\gamma^{u2}_{\sigma(i)}\right)^{\omega_i}}}\right] \middle| \begin{array}{c}\left[\gamma^{l}_{\sigma(i)}, \gamma^{u}_{\sigma(i)}\right] \in \tilde{h}_{\sigma(i)}\\ i = 1, 2, \ldots, n\end{array}\right\},$$

$$\left\{\left[\frac{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\delta^{l2}_{\sigma(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\delta^{l2}_{\sigma(i)}\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\delta^{l2}_{\sigma(i)}\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\left(1-\delta^{l2}_{\sigma(i)}\right)^{\omega_i}}},\right.\right.$$

$$\left.\left.\frac{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\delta^{u2}_{\sigma(i)}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\delta^{u2}_{\sigma(i)}\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+(\theta-1)\delta^{u2}_{\sigma(i)}\right)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}\left(1-\delta^{u2}_{\sigma(i)}\right)^{\omega_i}}}\right] \middle| \begin{array}{c}\left[\delta^{l}_{\sigma(i)}, \delta^{u}_{\sigma(i)}\right] \in \tilde{g}_{\sigma(i)}\\ i = 1, 2, \ldots, n\end{array}\right\}\right) \quad (25)$$

As like OWA aggregation operators, the concept of algebraic and Einstein-based OWG aggregation operators can also be constructed from Hamacher-based OWG aggregation operators for $\theta = 1$ and 2, respectively.

4.4.4. OWG aggregation operators based on Dombi operation

When $f(x) = \left(\frac{1}{x} - 1\right)^{\rho}$, $\rho > 0$, then the AIVHPFOWG operator reduces to the IVHPF Dombi OWG (IVHPFDOWG) operator as follows:

$$IVHPFDOWG\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) =$$

$$\left(\left\{\left[\sqrt{\frac{1}{1+\left(\sum_{i=1}^{n}\omega_i\left(\frac{1-\gamma^{l2}_{\sigma(i)}}{\gamma^{l2}_{\sigma(i)}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{\frac{1}{1+\left(\sum_{i=1}^{n}\omega_i\left(\frac{1-\gamma^{u2}_{\sigma(i)}}{\gamma^{u2}_{\sigma(i)}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right] \middle| \begin{array}{c}\left[\gamma^{l}_{\sigma(i)}, \gamma^{u}_{\sigma(i)}\right] \in \tilde{h}_{\sigma(i)}\\ i = 1, 2, \ldots, n\end{array}\right\},\right.$$

$$\left.\left\{\left[\sqrt{1-\frac{1}{1+\left(\sum_{i=1}^{n}\omega_i\left(\frac{\delta^{l2}_{\sigma(i)}}{1-\delta^{l2}_{\sigma(i)}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}, \sqrt{1-\frac{1}{1+\left(\sum_{i=1}^{n}\omega_i\left(\frac{\delta^{u2}_{\sigma(i)}}{1-\delta^{u2}_{\sigma(i)}}\right)^{\rho}\right)^{\frac{1}{\rho}}}}\right] \middle| \begin{array}{c}\left[\delta^{l}_{\sigma(i)}, \delta^{u}_{\sigma(i)}\right] \in \tilde{g}_{\sigma(i)}\\ i = 1, 2, \ldots, n\end{array}\right\}\right) \quad (26)$$

4.4.5. OWG aggregation operator based on Frank operation

When $f(x) = \log\left(\frac{\psi-1}{\psi^x-1}\right)$, $\psi > 1$, the AIVHPFOWG operator converted to IVHPF Frank OWG (IVHPFFOWG) operator presented as:

$$IVHPFFOWG\left(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n\right) = \left(\left\{ \left[\sqrt{\frac{\log\left(1+\prod_{i=1}^{n}\left(\psi^{\left(\gamma^l_{\sigma(i)}\right)^2}-1\right)^{\omega_i}\right)}{\log\psi}}, \sqrt{\frac{\log\left(1+\prod_{i=1}^{n}\left(\psi^{\left(\gamma^u_{\sigma(i)}\right)^2}-1\right)^{\omega_i}\right)}{\log\psi}} \right] \middle| \begin{matrix} \left[\gamma^l_{\sigma(i)}, \gamma^u_{\sigma(i)}\right] \in \tilde{h}_{\sigma(i)} \\ i = 1, 2, \ldots, n \end{matrix} \right\}, \left\{ \left[\sqrt{1-\frac{\log\left(1+\prod_{i=1}^{n}\left(\psi^{1-\delta^{l2}_{\sigma(i)}}-1\right)^{\omega_i}\right)}{\log\psi}}, \sqrt{1-\frac{\log\left(1+\prod_{i=1}^{n}\left(\psi^{1-\delta^{u2}_{\sigma(i)}}-1\right)^{\omega_i}\right)}{\log\psi}} \right] \middle| \begin{matrix} \left[\delta^l_{\sigma(i)}, \delta^u_{\sigma(i)}\right] \in \tilde{g}_{\sigma(i)}, \\ i = 1, 2, \ldots, n \end{matrix} \right\} \right) \tag{27}$$

5. Approach to MCDM with A t-CN&t-N-based IVHPF information

For an MCDM, let $Z = \{Z_1, Z_2, \ldots, Z_m\}$ be a collection of alternatives to be selected under IVHPF information. The proposed operators AIVHPFWA and AIVHPFWG are applied to develop an approach to solve MCDM problems under IVHPF environment. Let $C = \{C_1, C_2, \ldots, C_n\}$ be a collection of criteria on which the alternatives are evaluated. Also, let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the weight vector of C_j $(j = 1, 2, \ldots, n)$ satisfying $\omega_j > 0$ for $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} \omega_j = 1$. To evaluate the performance of the alternatives, a DM provides decision values in the forms of IVHPFEs as $\tilde{h}_{ij}$ and $\tilde{g}_{ij}$ which denote the degrees that the alternative Z_i satisfies the criterion C_j and does not satisfy the criterion C_j, respectively. After providing all the decision values corresponding to each alternative with respect to their satisfying criteria, the IVHPF decision matrix (IVHPFDM) is found $\widetilde{D} = \left[\tilde{d}_{ij}\right]_{m\times n} = \left[\tilde{h}_{ij}, \tilde{g}_{ij}\right]_{m\times n}$ as

$$\widetilde{D} = \begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} & \cdots & \tilde{d}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{d}_{m1} & \tilde{d}_{m2} & \cdots & \tilde{d}_{mn} \end{pmatrix}_{m\times n} \tag{28}$$

In MCDM, criteria are categorized into two types: one is benefit criteria, and the other one is cost criteria. If the IVHPFDM possesses cost type criteria, the matrix $\widetilde{D} = \left[\tilde{d}_{ij}\right]_{m\times n}$ can be converted into the normalized IVHPFDM form as $\tilde{R} = \left(\tilde{r}_{ij}\right)_{m\times n}$ in the following way,

$$\tilde{r}_{ij} = \begin{cases} \tilde{d}_{ij} & for\ benefit\ attribute\ C_j \\ \tilde{d}^c_{ij} & for\ cost\ attribute\ C_j \end{cases} \tag{29}$$

$(i = 1, \ldots, m\ \&\ j = 1, \ldots, n)$ Where $\tilde{d}^c_{ij}$ is the complement $\tilde{d}_{ij}$.

Suppose now that $\tilde{R} = \left(\tilde{r}_{ij}\right)_{m\times n}$ be a normalized IVHPFDM. Then, the proposed AIVHPFWA (or AIVHPFWG) operators are utilized to improve an approach for solving MCDM problems in IVHPF information. The proposed approach is explained through the following steps

***Step* 1.** Transform the IVHPFDM $\widetilde{D} = \left(\tilde{d}_{ij}\right)_{m\times n}$ into the normalized IVHPFDM $\tilde{R} = \left(\tilde{r}_{ij}\right)_{m\times n}$, if required, using Eq. (29).

***Step* 2.** In this step, two cases may arise.

Case 1: If the DMs want to weigh only the IVHPF arguments.

Aggregate the IVHPFNs $\tilde{r}_{ij}$ for each alternative Z_i using the AIVHPFWA (or AIVHPFWG) operator as follows:

$$\tilde{r}^A_i = AIVHPFWA\left(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}\right) = \left(\left\{ \left[\sqrt{g^{-1}\left(\sum_{j=1}^{n}\omega_j g\left(\left(\gamma^l_{ij}\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{j=1}^{n}\omega_j g\left(\left(\gamma^u_{ij}\right)^2\right)\right)} \right] \middle| \begin{matrix} \left[\gamma^l_{ij}, \gamma^u_{ij}\right] \in \tilde{h}_{ij}, \\ j = 1, 2, \ldots, n \end{matrix} \right\}, \left\{ \left[\sqrt{f^{-1}\left(\sum_{j=1}^{n}\omega_j f\left(\left(\delta^l_{ij}\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_j f\left(\left(\delta^u_{ij}\right)^2\right)\right)} \right] \middle| \begin{matrix} \left[\delta^l_{ij}, \delta^u_{ij}\right] \in \tilde{g}_{ij} \\ j = 1, 2, \ldots, n \end{matrix} \right\} \right) \tag{30}$$

or,

$$\tilde{r}^G_i = AIVHPFWG\left(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}\right) = \left(\left\{ \left[\sqrt{f^{-1}\left(\sum_{j=1}^{n}\omega_j f\left(\left(\gamma^l_{ij}\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{j=1}^{n}\omega_j f\left(\left(\gamma^u_{ij}\right)^2\right)\right)} \right] \middle| \begin{matrix} \left[\gamma^l_{ij}, \gamma^u_{ij}\right] \in \tilde{h}_{ij}, \\ j = 1, 2, \ldots, n \end{matrix} \right\}, \left\{ \left[\sqrt{g^{-1}\left(\sum_{j=1}^{n}\omega_j g\left(\left(\delta^l_{ij}\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_j g\left(\left(\delta^u_{ij}\right)^2\right)\right)} \right] \middle| \begin{matrix} \left[\delta^l_{ij}, \delta^u_{ij}\right] \in \tilde{g}_{ij} \\ j = 1, 2, \ldots, n \end{matrix} \right\} \right) \tag{31}$$

$(i = 1, \ldots, m\ \&\ j = 1, \ldots, n)$

Table 3
IVHPFDM given by DM.

	C_1	C_2	C_3	C_4
A_1	$\begin{pmatrix}\{[0.3,0.4],[0.4,0.5]\},\\ \{[0.3,0.4],[0.2,0.3]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.1,0.4],[0.4,0.5]\},\\ \{[0.2,0.3],[0.2,0.4]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.3,0.5],[0.2,0.4]\},\\ \{[0.2,0.4],[0.3,0.4]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.3,0.4],[0.4,0.5]\},\\ \{[0.3,0.5],[0.1,0.3]\}\end{pmatrix}$
A_2	$\begin{pmatrix}\{[0.1,0.4],[0.3,0.5]\},\\ \{[0.3,0.4],[0.1,0.5]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.2,0.4],[0.3,0.5]\},\\ \{[0.2,0.4],[0.3,0.5]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.1,0.3],[0.2,0.4]\},\\ \{[0.5,0.7],[0.4,0.6]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.1,0.4],[0.2,0.5]\},\\ \{[0.3,0.5],[0.2,0.4]\}\end{pmatrix}$
A_3	$\begin{pmatrix}\{[0.5,0.6],[0.4,0.7]\},\\ \{[0.2,0.3],[0.3,0.5]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.2,0.5],[0.3,0.6]\},\\ \{[0.3,0.5],[0.2,0.5]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.2,0.3],[0.1,0.3]\},\\ \{[0.2,0.4],[0.3,0.5]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.2,0.5],[0.3,0.5]\},\\ \{[0.3,0.5],[0.2,0.5]\}\end{pmatrix}$
A_4	$\begin{pmatrix}\{[0.3,0.5],[0.1,0.4]\},\\ \{[0.3,0.5],[0.4,0.6]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.4,0.6],[0.3,0.5]\},\\ \{[0.3,0.4],[0.2,0.4]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.1,0.3],[0.2,0.4]\},\\ \{[0.3,0.5],[0.2,0.5]\}\end{pmatrix}$	$\begin{pmatrix}\{[0.2,0.4],[0.2,0.3]\},\\ \{[0.4,0.6],[0.3,0.5]\}\end{pmatrix}$

Case 2: If the DMs want to weigh the ordered position of the IVHPF arguments.
Aggregate the IVHPFNs $\tilde{r}_{ij}$ for each alternative Z_i using the AIVHPFOWA (or AIVHPFOWG) operator as follows:

$$\tilde{r}_i^A = AIVHPFOWA\left(\tilde{r}_{i1},\tilde{r}_{i2},\dots,\tilde{r}_{in}\right) = \left(\left\{\left[\sqrt{g^{-1}\left(\sum_{j=1}^{n}\omega_j g\left(\left(\gamma^l_{i\sigma(j)}\right)^2\right)\right)},\sqrt{g^{-1}\left(\sum_{j=1}^{n}\omega_j g\left(\left(\gamma^u_{i\sigma(j)}\right)^2\right)\right)}\right]\Bigg|\begin{matrix}\left[\gamma^l_{ij},\gamma^u_{ij}\right]\in\tilde{h}_{ij},\\ j=1,2,\dots,n\end{matrix}\right\},\left\{\left[\sqrt{f^{-1}\left(\sum_{j=1}^{n}\omega_j f\left(\left(\delta^l_{i\sigma(j)}\right)^2\right)\right)},\sqrt{f^{-1}\left(\sum_{j=1}^{n}\omega_j f\left(\left(\delta^u_{i\sigma(j)}\right)^2\right)\right)}\right]\Bigg|\begin{matrix}\left[\delta^l_{ij},\delta^u_{ij}\right]\in\tilde{g}_{ij}\\ j=1,2,\dots,n\end{matrix}\right\}\right) \tag{32}$$

or,

$$\tilde{r}_i^G = AIVHPFOWG\left(\tilde{r}_{i1},\tilde{r}_{i2},\dots,\tilde{r}_{in}\right) = \left(\left\{\left[\sqrt{f^{-1}\left(\sum_{j=1}^{n}\omega_j f\left(\left(\gamma^l_{i\sigma(j)}\right)^2\right)\right)},\sqrt{f^{-1}\left(\sum_{j=1}^{n}\omega_j f\left(\left(\gamma^u_{i\sigma(j)}\right)^2\right)\right)}\right]\Bigg|\begin{matrix}\left[\gamma^l_{ij},\gamma^u_{ij}\right]\in\tilde{h}_{ij},\\ j=1,2,\dots,n\end{matrix}\right\},\left\{\left[\sqrt{g^{-1}\left(\sum_{j=1}^{n}\omega_j g\left(\left(\delta^l_{i\sigma(j)}\right)^2\right)\right)},\sqrt{g^{-1}\left(\sum_{j=1}^{n}\omega_j g\left(\left(\delta^u_{i\sigma(j)}\right)^2\right)\right)}\right]\Bigg|\begin{matrix}\left[\delta^l_{ij},\delta^u_{ij}\right]\in\tilde{g}_{ij}\\ j=1,2,\dots,n\end{matrix}\right\}\right) \tag{33}$$

where $(\sigma(1),\sigma(2),\dots,\sigma(n))$ represents a permutation of $\{1,2,\dots,n\}$, such that $\tilde{r}_{i\sigma(j-1)} \geq \tilde{r}_{i\sigma(j)}$ for all $i = 1,\dots,m \& j = 2,3,\dots,n$.

***Step* 3.** Utilizing the score function as mentioned in Definition 5.

***Step* 4.** The ordering of alternatives is evaluated according to Definition 5.

6. Illustrative examples

In this section, two examples, studied previously [23,29], are considered and solved to explore the applicability of the proposed method.

Example 1. In this section, an MCDM problem related to rail transit projects is adapted from an article previously studied by Wang et al. [29] to illustrate the application of the proposed MCDM method. The problem consists of selecting the most suitable private partner for a rail transit project. The attributes that are considered for the selection process are C_1: economy; C_2: experience; C_3: technology; and C_4: reputation. Accordingly, the weight vector of the criteria is given as $W = (0.35, 0.3, 0.1, 0.25)^T$. Here it is assumed that four eligible partners applied for the project, viz., $A = \{A_1, A_2, A_3, A_4\}$. Under the IVHPF environment, experts need to evaluate these alternatives with IVHPFEs. The IVHPF decision matrix is shown in Table 3.

***Step* 1.** Considering all the criteria C_j $(j = 1,2,3,4)$ are the benefit criteria, the performance values of the alternatives A_i $(i = 1,2,3,4,)$ do not need normalization.

***Step* 2.** The DMs want to weigh only the IVHPF arguments i.e, have to follow case 1. So for aggregating all the preference values $\tilde{r}_{ij}$ for each alternative A_i, IVHPFHWA aggregation operator as described in Eq. (7) is utilized to get $\tilde{r}_i^A$ (i = 1,2,3,4). Without loss of generality, take $\theta = 5$.

$$\begin{aligned}\tilde{r}_1^A = (\{&[0.2539, 0.4107], [0.2836, 0.4366], [0.2439, 0.4000], [0.2744, 0.4263], [0.3320,\\ &0.4417], [0.3572, 0.4665], [0.3237, 0.4315], [0.3493, 0.4567], [0.2949, 0.4467], [0.3220,\\ &0.4714], [0.2860, 0.4366], [0.3136, 0.4616], [0.3670, 0.4762], [0.3907, 0.5000], [0.3592,\\ &0.4665], [0.3833, 0.4906]\}, \{[0.2556, 0.3899], [0.1948, 0.3422], [0.2661, 0.3899], [0.2030,\\ &0.3422], [0.2556, 0.4237], [0.1948, 0.3729], [0.2661, 0.4237], [0.2030, 0.3729], [0.2217,\\ &0.3531], [0.1685, 0.3090], [0.2309, 0.3531], [0.1756, 0.3090], [0.2217, 0.3845], [0.1685,\\ &0.3373], [0.2309, 0.3845], [0.1756, 0.3373]\}),\end{aligned}$$

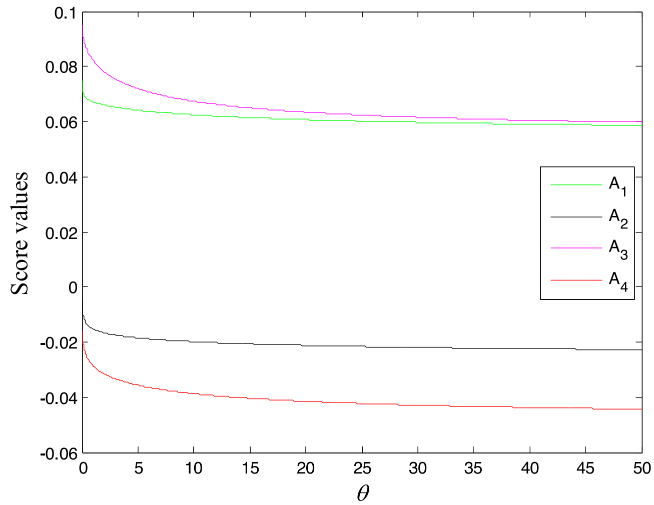

Fig. 1. Score values by IVHPFHWA operator.

$\tilde{r}_2^A = (\{[0.1369, 0.3907], [0.1619, 0.4174], [0.1474, 0.4000], [0.1710, 0.4263], [0.1801, 0.4226], [0.2004, 0.4481], [0.1885, 0.4315], [0.2081, 0.4567], [0.2138, 0.4278], [0.2317, 0.4531], [0.2211, 0.4366] [0.2385, 0.4616], [0.2456, 0.4581], [0.2619, 0.4824], [0.2522, 0.4665], [0.2681, 0.4906]\} \{[0.2812, 0.4511], [0.2543, 0.4264], [0.2742, 0.4423], [0.2479, 0.4180], [0.3167, 0.4818], [0.2869, 0.4561], [0.3090, 0.4727], [0.2798, 0.4473], [0.1926, 0.4871], [0.1736, 0.4612], [0.1877, 0.4779], [0.1691, 0.4523], [0.2179, 0.5191], [0.1966, 0.4924], [0.2124, 0.5097], [0.1916, 0.4831]\}),$

$\tilde{r}_3^A = (\{[0.3259, 0.5199], [0.3462, 0.5199], [0.3204, 0.5199], [0.3410, 0.5199], [0.3502, 0.5506], [0.3697, 0.5506], [0.3450, 0.5506], [0.3647, 0.5506], [0.2812, 0.5616], [0.3035, 0.5616], [0.2751, 0.5616] [0.2977, 0.5616], [0.3078, 0.5906], [0.3289, 0.5906], [0.3021, 0.5906], [0.3234, 0.5906]\}, \{[0.2505, 0.4123], [0.2263, 0.4123], [0.2608, 0.4218], [0.2357, 0.4218], [0.2217, 0.4123], [0.2000, 0.4123], [0.2309, 0.4218], [0.2084, 0.4218], [0.2883, 0.4894], [0.2608, 0.4894], [0.3000, 0.5000], [0.2715, 0.5000], [0.2556, 0.4894], [0.2309, 0.4894], [0.2661, 0.5000], [0.2405, 0.5000]\}),$

$\tilde{r}_4^A = (\{[0.2966, 0.4914], [0.2966, 0.4708], [0.3024, 0.4994], [0.3024, 0.4791], [0.2619, 0.4581], [0.2619, 0.4366], [0.2681, 0.4665], [0.2681, 0.4453], [0.2417, 0.4575], [0.2417, 0.4360], [0.2484, 0.4659], [0.2484, 0.4447], [0.2004, 0.4226], [0.2004, 0.4000], [0.2081, 0.4315], [0.2081, 0.4091]\}, \{[0.3229, 0.4921], [0.3000, 0.4686], [0.3105, 0.4921], [0.2883, 0.4686], [0.2868, 0.4921], [0.2661, 0.4686], [0.2756, 0.4921], [0.2556, 0.4686], [0.3573, 0.5261], [0.3324, 0.5017], [0.3438, 0.5261], [0.3197, 0.5017], [0.3181, 0.5261], [0.2955, 0.5017], [0.3058, 0.5261], [0.2839, 0.5017]\}).$

***Step* 3.** By Definition 5, calculate the score values of $\tilde{r}_i^A$ $(i = 1, 2, 3, 4)$ for each partner and are found as $S\left(\tilde{r}_1^A\right) = 0.06419$, $S\left(\tilde{r}_2^A\right) = -0.01845$, $S\left(\tilde{r}_3^A\right) = 0.07206$, $S\left(\tilde{r}_4^A\right) = -0.03559$.

***Step* 4.** Find out the ranking of the alternatives as $A_3 \succ A_1 \succ A_2 \succ A_4$.

6.1. The influence of the hamacher parameter θ, Dombi parameter ρ and Frank parameter ψ on the ranking results

Evidently, the parameter plays a significant role in the ranking orders. According to the score functions and the comparison formula with score functions, the ranking results differ with the changes in the parameter value between 0 and 50. The changes are observed as follows:

• For different values of the Hamacher parameter θ, ranging from 0 to 50, the score values obtained by the IVHPFHWA operator are depicted in Fig. 1. It is worthy to note here that the score values decrease with the increase in the value of θ by keeping the ordering of alternatives as to the same as $A_3 \succ A_1 \succ A_2 \succ A_4$.

• On the other hand, if the IVHPFHWG operator is utilized, the score value of each alternative A_i, $(i = 1, 2, 3, 4)$ is presented in Fig. 2. It is worth mentioning here that the score values increase with the increase of θ. However, some changes in the ordering of the alternatives are observed. When $\theta \in (0, 6.5080)$, the ranking of the four alternatives is $A_1 \succ A_3 \succ A_2 \succ A_4$ and the best choice is A_1. Whereas, when $\theta \in (6.5080,\ \infty)$, the ranking of the alternatives become $A_3 \succ A_1 \succ A_2 \succ A_4$, and the best choice is A_3. Thus for $\theta = 6.5080$ an alternative ranking of alternatives is found as $A_3 \approx A_1 \succ A_2 \succ A_4$.

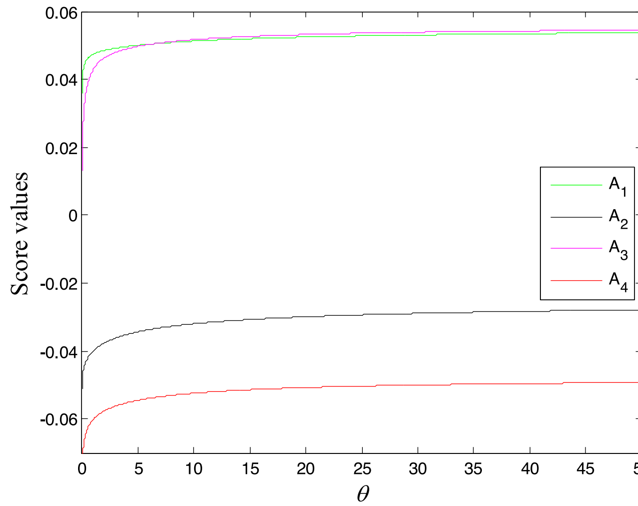

Fig. 2. Score values by IVHPFHWG operator.

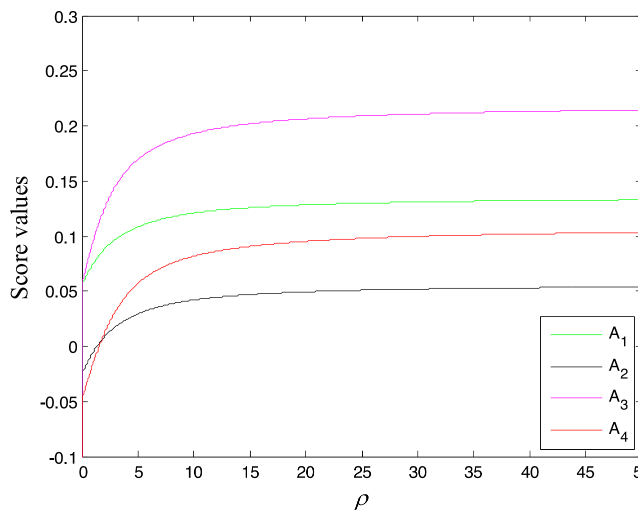

Fig. 3. Score values by IVHPFDWA operator.

• Again, when the alternatives are aggregated using Dombi operation, i.e., the IVHPFDWA and IVHPFDWG aggregation operators, the ranking results are changed with the parameter ρ. However, it differs from the Hamacher aggregation operators in that if the IVHPFDWA operator is utilized, the score values of the alternatives increase. In contrast, the score values of the alternatives decrease with the increase of the parameter ρ if IVHPFDWG operator is used. Further ordering of the alternatives is changed from $A_3 \succ A_1 \succ A_2 \succ A_4$ to $A_3 \succ A_1 \succ A_4 \succ A_2$ at $\rho = 1.7220$. In the case of the IVHPFDWG operator, the ranking of alternatives changes as $A_1 \succ A_3 \succ A_2 \succ A_4$ for $\rho \in (0, 3.5770)$ and $A_1 \succ A_3 \succ A_4 \succ A_2$ for $\rho \in (3.5770, \infty)$. These facts are described through Fig. 3 and Fig. 4 by varying the parameters in $[0, 50]$.

• If the aggregation operator IVHPFFWA is utilized, the ordering of alternatives is found as $A_3 \succ A_1 \succ A_2 \succ A_4$.

• Whenever IVHPFFWG operator is utilized, the ranking results are obtained as $A_1 \succ A_3 \succ A_2 \succ A_4$ for $\psi \in (1, 13.92)$ and $A_3 \succ A_1 \succ A_2 \succ A_4$ for $\psi \in (13.92, \infty)$.

These variations of score values using IVHPFFWA and IVHPFFWG operators are shown, in Fig. 5 and Fig. 6, respectively.

The score values and the ranking of alternatives for all the aggregation operators mentioned earlier are summarized in Table 4.

Example 2. An MCDM problem of energy development strategy In this section, an MCDM problem related to energy projects, adapted from an example previously studied by Liang and Xu [23], illustrates the application of the proposed method and demonstrates its feasibility and effectiveness in a realistic scenario. One energy development company wants to select the best energy project among five alternatives, $A = \{A_1, A_2, A_3, A_4, A_5\}$, by considering four attributes: C_1: economic; C_2: technological; C_3: environmental; and C_4: socio-political. Accordingly, the weight vector of the attributes is provided as $W = (0.15, 0.3, 0.2, 0.35)^T$. Experts evaluate the alternatives on the basis of the attributes under the IVHPF environment, and the IVHPF decision matrix is constructed as given in Table 5.

To obtain the ranking results among the alternative(s), the developed AIVHPFWA and AIVHPFWG operators are used, and the proposed method's step-by-step execution is described below. In this context, it is to be noted here that three types of At-CN&t-N, viz., Hamacher, Dombi and Frank Classes are considered. Algebraic and Einstein classes can be derived as particular cases of Hamacher class of t-CN&t-Ns.

Step 1. Considering all the attributes C_j $(j = 1, 2, 3, 4)$ as the benefit criteria, the performance values of the alternatives A_i $(i = 1, 2, 3, 4, 5)$ do not need normalization.

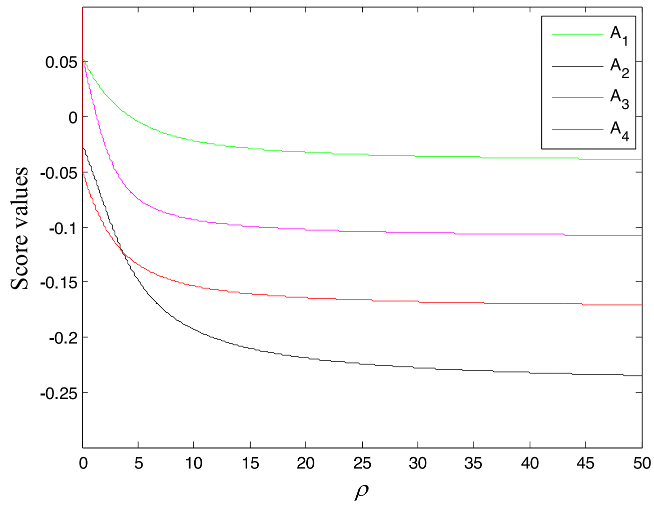

Fig. 4. Score values by IVHPFDWG operator.

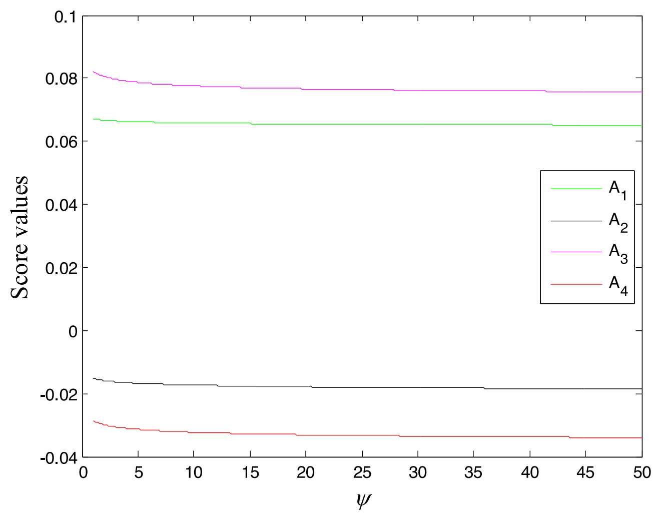

Fig. 5. Score values by IVHPFFWA operator.

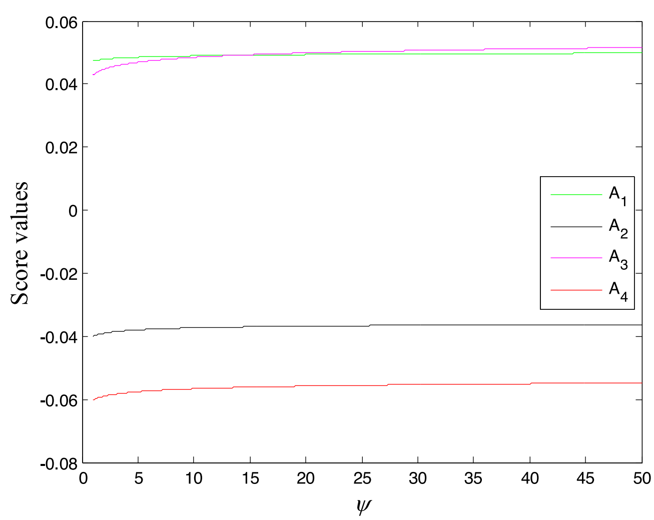

Fig. 6. Score values by IVHPFFWG operator.

Step 2. Case 1 is considered here. Without loss of generality, the value of the Hamacher parameter θ is considered as 5, and the aggregation operator IVHPFHWA as described in Eq. (7) is utilized to aggregate all the preference values $\tilde{r}_{ij}$ for each alternative A_i and $\tilde{r}_i^A$ $(i = 1, 2, \ldots, 5)$ are

Table 4
Ranking results of different parameters based on the proposed operators.

Operations	Aggregation operator	Parameter	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking results
Algebraic	IVHPFWA	$\theta = 1$	0.06723	−0.01516	0.0823	−0.02855	$A_3 \succ A_1 \succ A_2 \succ A_4$
Einstein	IVHPFEWA	$\theta = 2$	0.06608	−0.01661	0.07823	−0.03146	$A_3 \succ A_1 \succ A_2 \succ A_4$
Hamacher	IVHPFHWA	$\theta = 5$	0.06419	−0.01845	0.07206	−0.03559	$A_3 \succ A_1 \succ A_2 \succ A_4$
		$\theta = 10$	0.0625	−0.01985	0.06743	−0.03871	$A_3 \succ A_1 \succ A_2 \succ A_4$
		$\theta = 50$	0.05864	−0.02276	0.05978	−0.04432	$A_3 \succ A_1 \succ A_2 \succ A_4$
Algebraic	IVHPFWG	$\theta = 1$	0.04732	−0.04009	0.04279	−0.06025	$A_1 \succ A_3 \succ A_2 \succ A_4$
Einstein	IVHPFEWG	$\theta = 2$	0.04846	−0.03773	0.04623	−0.05772	$A_1 \succ A_3 \succ A_2 \succ A_4$
Hamacher	IVHPFHWG	$\theta = 5$	0.05004	−0.03438	0.04969	−0.05455	$A_1 \succ A_3 \succ A_2 \succ A_4$
		$\theta = 6.5080$	0.05051	−0.03344	0.05051	−0.05371	$A_3 \approx A_1 \succ A_2 \succ A_4$
		$\theta = 10$	0.05129	−0.03199	0.0517	−0.05244	$A_3 \succ A_1 \succ A_2 \succ A_4$
		$\theta = 50$	0.05373	−0.02784	0.05452	−0.04929	$A_3 \succ A_1 \succ A_2 \succ A_4$
Dombi	IVHPFDWA	$\rho = 1$	0.07479	−0.005229	0.09515	−0.01555	$A_3 \succ A_1 \succ A_2 \succ A_4$
		$\rho = 1.7220$	0.08467	0.004941	0.1178	0.004941	$A_3 \succ A_1 \succ A_2 \approx A_4$
		$\rho = 5$	0.1081	0.02915	0.1691	0.0568	$A_3 \succ A_1 \succ A_4 \succ A_2$
		$\rho = 10$	0.1209	0.04201	0.1930	0.08164	$A_3 \succ A_1 \succ A_4 \succ A_2$
	IVHPFDWG	$\rho = 1$	0.03603	−0.0513	0.0130	−0.07721	$A_1 \succ A_3 \succ A_2 \succ A_4$
		$\rho = 3.5770$	0.0060	−0.1215	−0.05855	−0.1215	$A_1 \succ A_3 \succ A_2 \approx A_4$
		$\rho = 5$	0.1081	−0.1480	−0.07402	−0.1338	$A_1 \succ A_3 \succ A_4 \succ A_2$
		$\rho = 50$	−0.03855	−0.2348	−0.1076	−0.1707	$A_1 \succ A_3 \succ A_4 \succ A_2$
Frank	IVHPFFWA	$\psi = 1.1$	0.06716	−0.01526	0.08204	−0.02874	$A_3 \succ A_1 \succ A_2 \succ A_4$
		$\psi = 5$	0.06617	−0.01674	0.07871	−0.03134	$A_3 \succ A_1 \succ A_2 \succ A_4$
		$\psi = 10$	0.06581	−0.01733	0.07757	−0.03228	$A_3 \succ A_1 \succ A_2 \succ A_4$
	IVHPFFWG	$\psi = 1.1$	0.0474	−0.03994	0.04306	−0.06007	$A_1 \succ A_3 \succ A_2 \succ A_4$
		$\psi = 5$	0.04853	−0.03797	0.04692	−0.05759	$A_1 \succ A_3 \succ A_2 \succ A_4$
		$\psi = 10$	0.04896	−0.03732	0.04846	−0.05664	$A_1 \succ A_3 \succ A_2 \succ A_4$
		$\psi = 13.92$	0.04916	−0.03705	0.04916	−0.05623	$A_3 \approx A_1 \succ A_2 \succ A_4$

Table 5
IVHPFDM.

	C_1	C_2	C_3	C_4
A_1	$\begin{pmatrix} \{[0.3,0.4],[0.4,0.5]\}, \\ \{[0.7,0.8],\} \end{pmatrix}$	$\begin{pmatrix} \{[0.1,0.2],[0.4,0.9]\}, \\ \{[0.2,0.4]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.2,0.5]\}, \\ \{[0.2,0.5],[0.6,0.8]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.3,0.5],[0.5,0.9]\}, \\ \{[0.2,0.3],[0.35,0.4]\} \end{pmatrix}$
A_2	$\begin{pmatrix} \{[0.2,0.5]\}, \\ \{[0.6,0.8]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.2,0.3],[0.5,0.7]\}, \\ \{[0.6,0.7]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.1,0.2],[0.5,0.8]\}, \\ \{[0.2,0.3],[0.4,0.6]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.3,0.7]\}, \\ \{[0.4,0.5],[0.5,0.7]\} \end{pmatrix}$
A_3	$\begin{pmatrix} \{[0.6,0.7]\}, \\ \{[[0.4,0.6]]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.6,0.9]\}, \\ \{[0.2,0.3],[0.3,0.4]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.3,0.5],[0.5,0.7]\}, \\ \{[0.4,0.6]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.4,0.8]\}, \\ \{[0.3,0.4],[0.5,0.6]\} \end{pmatrix}$
A_4	$\begin{pmatrix} \{[0.3,0.4],[0.6,0.7]\}, \\ \{[0.5,0.6],[0.6,0.7]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.2,0.4],[0.5,0.7]\}, \\ \{[0.4,0.6]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.1,0.2],[0.7,0.8]\}, \\ \{[0.3.0.6]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.5,0.9]\}, \\ \{[0.1.0.2],[0.2,0.4]\} \end{pmatrix}$
A_5	$\begin{pmatrix} \{[0.1,0.3],[0.4,0.6],[0.7,0.8]\}, \\ \{[0.4,0.6]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.4,0.6],[0.6,0.8]\}, \\ \{[0.1,0.2],[0.2,0.3]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.7,0.9]\}, \\ \{[0.1,0.2],[0.3,0.4]\} \end{pmatrix}$	$\begin{pmatrix} \{[0.3,0.7],[0.7,0.9]\}, \\ \{[0.3,0.4]\} \end{pmatrix}$

obtained as follows.

$\tilde{r}_1^A = (\{[0.0546, 0.1676], [0.1362, 0.3918], [0.1872, 0.4327], [0.2843, 0.6460], [0.0639, 0.1807], [0.1470, 0.4064], [0.1987, 0.4471], [0.2966, 0.6578]\}, \{[0.0326, 0.1703], [0.0435, 0.2059], [0.0529, 0.2368], [0.0702, 0.2826]\});$

$\tilde{r}_2^A = (\{[0.0505, 0.2322], [0.0950, 0.3610], [0.1079, 0.3568], [0.1574, 0.4870]\}, \{[0.1834, 0.3073], [0.2145, 0.3904], [0.2351, 0.3900], [0.2727, 0.4840]\});$

$\tilde{r}_3^A = (\{[0.2309, 0.6144], [0.2656, 0.6546]\}, \{[0.0873, 0.1842], [0.1270, 0.2472], [0.1105, 0.2163], [0.1593, 0.2873]\});$

$\tilde{r}_4^A = (\{[0.1072, 0.3873], [0.2003, 0.5162], [0.1707, 0.4882], [0.2707, 0.6089], [0.1441, 0.4382], [0.2415, 0.5637], [0.2108, 0.5368], [0.3139, 0.6514]\}, \{[0.0603, 0.1801], [0.0959, 0.2762], [0.0646, 0.1912], [0.1025, 0.2916]\});$

$\tilde{r}_5^A = (\{[0.1687, 0.4717], [0.3093, 0.6080], [0.2263, 0.5565], [0.3718, 0.6806], [0.1950, 0.5140], [0.3382, 0.6448], [0.2540, 0.5958], [0.4008, 0.7128], [0.2428, 0.5558], [0.3892, 0.6800], [0.3038, 0.6339], [0.4515, 0.7432]\}, \{[0.0335, 0.0938], [0.0519, 0.1231], [0.0504, 0.1185] [0.0775, 0.1546]\}).$

Step 3. Using Definition 5, calculate the score values of $\tilde{r}_i^A$ $(i = 1, 2, 3, 4, 5)$ for each alternative.
$S\left(\tilde{r}_1^A\right) = 0.09645,\ S\left(\tilde{r}_2^A\right) = -0.07871,\ S\left(\tilde{r}_3^A\right) = 0.2640,\ S\left(\tilde{r}_4^A\right) = 0.2078,\ S\left(\tilde{r}_5^A\right) = 0.3724.$

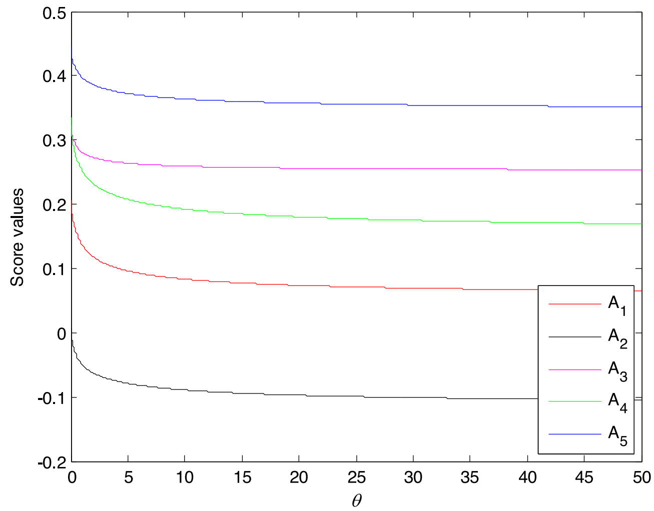

Fig. 7. Score values by IVHPFHWA operator.

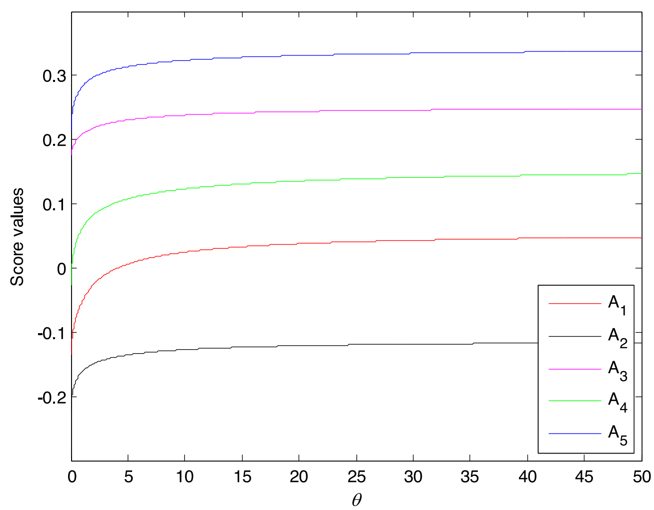

Fig. 8. Score values by IVHPFHWG operator.

Step 4. Find out the ranking of the alternatives as $A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$.

Alternatively, if IVHPFHWG operator is utilized whose expression is given in Eq. (21), the aggregated value of each alternative A_i, $(i = 1, 2, \ldots, 5)$ can be achieved as like IVHPFHWA operator.

6.2. Sensitivity analysis

6.2.1. For Hamacher parameter θ

The above example is solved by considering $\theta = 5$ as the value of Hamacher parameter. When θ is varied, the effect of changes is described as follows.

For different values of the parameter θ, ranging between 0 and 50, the change of score values obtained by the IVHPFHWA operator is shown in Fig. 7. It is to be mentioned here that the score values decrease with the increase of θ. Further, the changes in the ranking of alternatives are observed as follows.

(1) When $\theta \in (0, 0.0874)$, the ranking of the alternatives is $A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$ and the best choice is A_5.

(2) When $\theta \in (0.0874,\ 50]$, the ranking of the alternatives is $A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$, and the best choice is A_5.

(3) For $\theta = 0.0874$, alternative orderings are found as $A_5 \succ A_3 \approx A_4 \succ A_1 \succ A_2$.

Similarly, using IVDHPFHWG operator, the score values obtained depending on the parameter θ, ranging between 0 and 50 is depicted in Fig. 8. It is worthy to mention here that the score values increase with the increase of θ, but no change in the ranking of alternatives is observed in $(0,\ 50]$, and the ranking is found as $A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$.

6.2.2. For Dombi parameter ρ

On the other hand, when the ranking of alternatives is evaluated using Dombi operation, the ranking results are changed depending on the parameter ρ. If the aggregating operator IVHPFDWA is utilized, the orderings of the alternatives are found, which are presented in Fig. 9. The

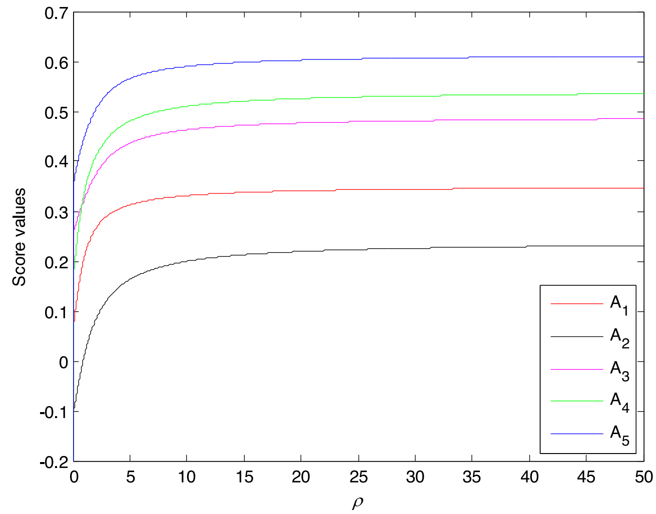

Fig. 9. Score values by IVHPFDWA operator.

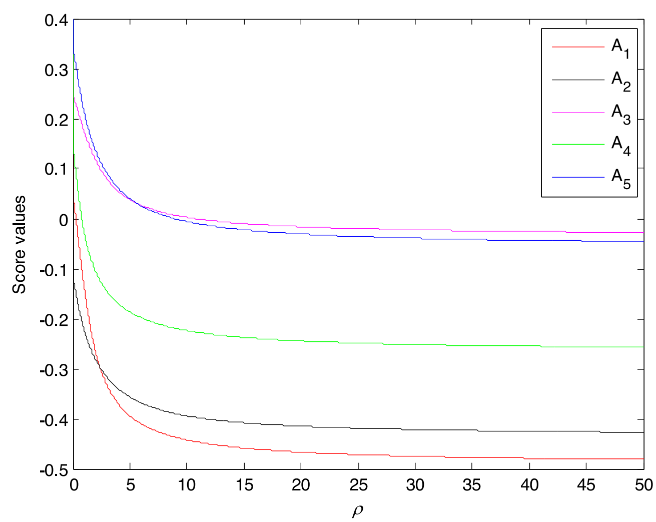

Fig. 10. Score values by IVHPFDWG operator.

ranking is changed by about $\rho = 0.8124$. In this case, the score values of alternatives are increased when the value of ρ is increased, unlike Hamacher operations.

Further, the ordering of alternatives is appeared as $A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$ when IVHPFDWG operator is used and the value of Dombi parameter $\rho \in (0, 2.3948)$. Again for $\rho \in (2.3948, 5.6139)$ the ranking order is changed and is found as $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$. But for the remaining span, i.e., for $\rho \in (5.6139, \infty)$ the ordering is appeared as $A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$. All the cases depending on the parameter ρ are summarized in Fig. 10.

6.2.3. *For Frank classes of operations parameter ψ*

Again when the aggregating operators IVHPFFWA and IVHPFFWG are utilized, the ranking of alternatives remains the same and is found as $A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$ which are shown in Fig. 11 and Fig. 12, respectively.

The overall score values and the ranking order of the alternatives using IVHPFHWA, IVHPFHWG, IVHPFDWA, IVHPFDWG, and IVHPFFWA IVHPFFWG operators, are summarized in Table 6.

• **Comparison with existing methods**

In this subsection, the proposed method is compared with the existing methods proposed by Wang et al. [29] and Liang and Xu [23]. Based on the proposed method, it is noted that the ranking of alternatives remains the same as using the method developed by Wang et al. [29]. However, the proposed method is superior in the sense that when the differences of the score values of the consecutive alternatives are calculated. The differences are much higher than the existing method [29] in each case, as presented in Fig. 13. On the other hand, the ranked alternatives are identified more easily than the existing method [23] as presented in Fig. 14, though the ranking remains the same. Additionally, in Ref. [23], Liang and Xu proposed the MCDM method under HPF environment. But the proposed approach extends the existing approach [23] to IVHPF environment by introducing interval numbers to HPF environment. So, the scope for describing uncertain information is also increased in the present study.

Further, it is found that the above aggregation operators are all based on different t-conorms and t-norms, which are more general and more versatile for aggregating fuzzy information. Also, it is observed from the sensitivity analysis, the proposed aggregation operators with parameters

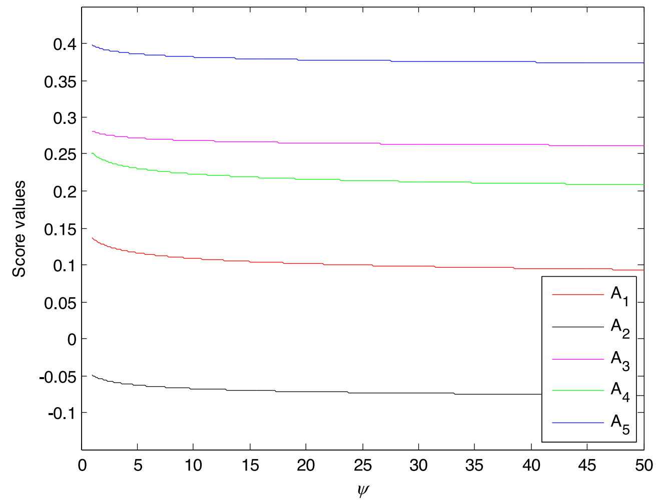

Fig. 11. Score values by IVHPFFWA operator.

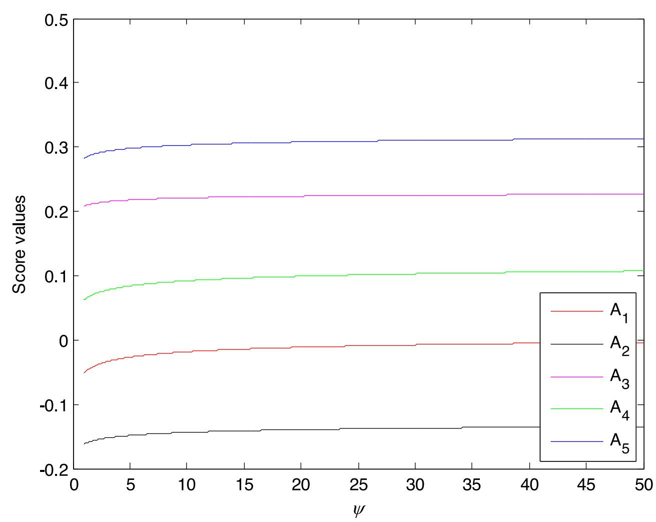

Fig. 12. Score values by IVHPFFWG operator.

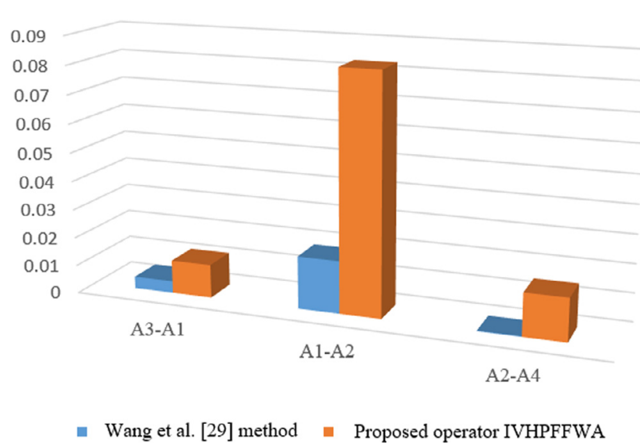

Fig. 13. Comparison with Wang et al. [29] method.

can provide the DMs with more choices and thus the proposed method is more flexible than the existing ones [23,29] because different values of the parameter can be selected according to the different situations, which is an interesting topic and is worthy to be further studied in the future.

Table 6
Ranking results of different parameters θ, ρ, ψ based on the proposed operators.

Operations	Operator	Parameter	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	Ranking results
Hamacher	IVHPFHWA	$\theta = 0.0874$	0.1897	−0.0081	0.3112	0.3112	0.4310	$A_5 \succ A_3 \approx A_4 \succ A_1 \succ A_2$
		$\theta = 1$	0.1372	−0.04902	0.2814	0.2523	0.3978	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
		$\theta = 2$	0.1183	−0.06238	0.2729	0.2322	0.3864	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
		$\theta = 5$	0.09645	−0.07871	0.2640	0.2078	0.3724	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
		$\theta = 10$	0.08349	−0.08913	0.2592	0.1924	0.3639	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
	IVHPFHWG	$\theta = 1$	−0.05209	−0.1620	0.2078	0.06183	0.2824	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
		$\theta = 5$	0.005817	−0.1355	0.2305	0.1075	0.3137	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
		$\theta = 10$	0.02464	−0.1272	0.2383	0.1233	0.3236	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
Dombi	IVHPFDWA	$\rho = 0.8124$	0.1863	−0.01119	0.3119	0.3119	0.4318	$A_5 \succ A_3 \approx A_4 \succ A_1 \succ A_2$
		$\rho = 5$	0.3121	0.1623	0.4356	0.4797	0.5649	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
		$\rho = 10$	0.3310	0.1995	0.4629	0.5099	0.5899	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
		$\rho = 20$	0.3408	0.2192	0.4766	0.5256	0.6026	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
		$\rho = 30$	0.3441	0.2257	0.4811	0.5308	0.6068	$A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$
	IVHPFDWG	$\rho = 2.3948$	−0.2972	−0.2972	0.1002	−0.1235	0.1158	$A_5 \succ A_3 \succ A_4 \succ A_1 \approx A_2$
		$\rho = 5$	−0.3936	−0.3558	0.03895	−0.1856	0.04091	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
		$\rho = 5.6139$	−0.4040	−0.3636	0.03136	−0.1933	0.03136	$A_5 \approx A_3 \succ A_4 \succ A_2 \succ A_1$
		$\rho = 10$	−0.4421	−0.3942	0.0027	−0.2235	−0.0056	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
		$\rho = 20$	−0.4668	−0.4148	−0.01619	−0.2444	−0.03065	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
		$\rho = 30$	−0.4750	−0.4216	−0.02248	−0.2514	−0.03914	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
Frank	IVHPFFWA	$\psi = 1.1$	0.1359	−0.04994	0.2808	0.2509	0.3971	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
		$\psi = 5$	0.1164	−0.0629	0.2719	0.2309	0.3859	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
		$\psi = 10$	0.1086	−0.06769	0.2684	0.2231	0.3816	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
	IVHPFFWG	$\psi = 1.1$	−0.05038	−0.1611	0.2085	0.06324	0.2834	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
		$\psi = 5$	−0.02691	−0.1486	0.2177	0.08379	0.2978	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
		$\psi = 10$	−0.01846	−0.1438	0.2209	0.09188	0.3032	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$

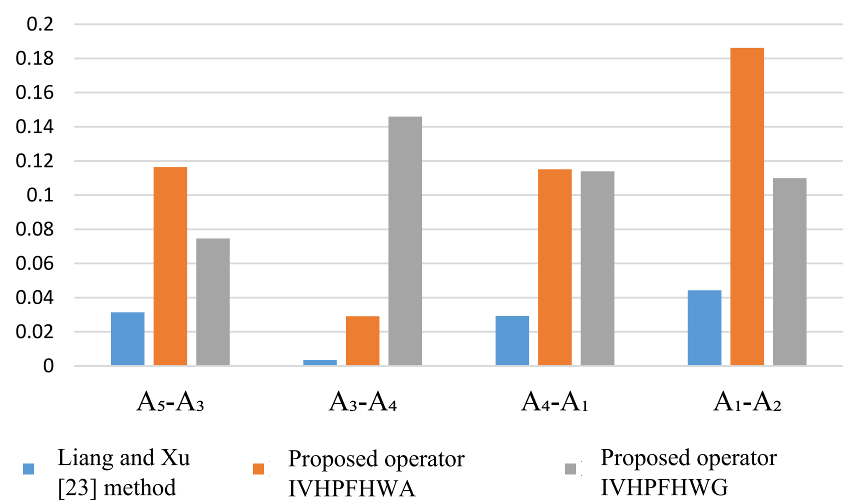

Fig. 14. Comparison between the proposed and Liang and Xu [23] methods.

7. Conclusions

The developed A*t*-CN&*t*-N based aggregation operators in the IVHPF environment can capture different kinds of existing classes of aggregation operators, viz., algebraic, Einstein, Hamacher, Dombi, Frank classes, and the like. Thus various kinds of aggregation operators, viz., IVHPFWA, IVHPFEWA, IVHPFHWA, IVHPFDWA, IVHPFFWA, IVHPFOWA, IVHPFEOWA, IVHPFHOWA, IVHPFDOWA, IVHPFFOWA and their corresponding geometric operators are derived. In the context of solving MCDM problems, it has been observed that the differences in score values of the ranked alternatives are much higher than the existing methods. As a consequence, the alternatives are ranked more prominently than other methods. Also, sensitivity analysis is performed to observe the ranking change with the change of various parameters associated with the aggregation operators. In future, some other types of IVHPF operators based on A*t*-N&*t*-CN, such as IVHPF power aggregation operators, IVHPF prioritized operators, and induced generalized IVHPF, IVHPF hybrid aggregation operators can be developed by following the proposed method. Also, the proposed approach can be applied to multi-objective location analytics problems [38].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Development of q-rung orthopair trapezoidal fuzzy Hamacher aggregation operators and its application in MCGDM problems

Souvik Gayen[1] · Arun Sarkar[2] · Animesh Biswas[1]

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Abstract
In comparison to intuitionistic and Pythagorean fuzzy sets, the q-rung orthopair fuzzy set is a more capable tool for dealing with uncertainties associated with the data. The objective of this research is to introduce a novel process to deal with multicriteria group decision making (MCGDM) issues in which the evaluation values take the form of q-rung orthopair trapezoidal fuzzy numbers (q-ROTrFNs) using the class of Hamacher t-norms and t-conorms. In doing so some fundamental operational laws of q-ROTrFNs are defined based on Hamacher operations. In view of these operations, different new aggregation operators, viz., q-ROTrF Hamacher weighted averaging, and q-ROTrF Hamacher weighted geometric operators, have been developed. In addition, certain key aspects of the proposed operators are investigated. Finally, the MCGDM method with q-ROTrF information is developed based on defined aggregation operators. To illustrate the proposed method, two real-life problems concerning hiring service selection and supplier selection are considered and solved. The effects of the Hamacher and rung parameters on decision results of those problems are also thoroughly examined to demonstrate the applicability and superiority of the developed method.

Keywords Multicriteria group decision making · q-rung orthopair trapezoidal fuzzy number · Hamacher operations · Weighted averaging · Weighted geometric aggregation operators

Mathematics Subject Classification 03E72 · 91B06 · 47S40

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✉ Arun Sarkar
asarkarmth@gmail.com

1 Department of Mathematics, University of Kalyani, Kalyani 741235, India

2 Department of Mathematics, Heramba Chandra College, Kolkata 700029, India

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1 Introduction

Multicriteria group decision making (MCGDM) is a technique for determining the most desired alternative from a collection of finite number of available options based on collective assessment values, provided by a group of decision makers (DMs), individually. Due to the complexity of human cognitive process, the MCGDM methods are involved with ambiguous information for the presence of evaluation values of multiple DMs. To cope with such issues, Atanassov (1986) proposed intuitionistic fuzzy set (IFS), which might be regarded as an attractive method for dealing with data having fuzziness and inaccuracy. The IFS is defined by membership and non-membership degrees with sum of degrees up to one. Apart from many benefits of IFS, some circumstances arise when the sum of membership and non-membership degrees exceeds 1. To overcome such situations, Yager (2013, 2014) developed Pythagorean fuzzy set (PFS), which satisfies the condition that the squared sum of its membership and non-membership degrees is less than or equal to 1. As a result, PFSs have a wider space to simulate real-life situations than IFSs. Since the inception of PFS, it has been extensively researched and used by scholars (Biswas and Deb 2021; Gayen and Biswas 2021; Sarkar and Biswas 2019, 2021b) in many domains.

However, real-life applications involve situations where the square sum of membership and non-membership degrees is higher than unity. In these cases, IFS and PFS are not suitable for describing DMs' evaluation information. To adopt this issue, Yager (2017) redefined the notion of q-rung orthopair fuzzy (q-ROF) set (q-ROFS), as a useful extension of IFS and PFS, in which sum of qth power of membership and non-membership degrees is less than or equals to 1. It is worth noting that if the rung parameter q increases, then the space of acceptable orthopairs is also increases. Thus q-ROFSs are more appropriate for tackling uncertain environments. Liu and Wang (2018) proposed q-ROF weighted averaging (WA) and q-ROF weighted geometric (WG) aggregation operators. Further, Liu and Wang (2020) introduced weighted generalized Maclaurin symmetric mean (MSM) and its geometric forms. Recently, Sarkar and Biswas (2021a) developed Bonferroni mean operator under dual hesitant q-ROF context and used it to solve MCGDM problems. Ever since q-ROFSs' appearance, many studies (Liu and Liu 2019a, 2019b; Wei et al. 2018, 2019; Yager et al. 2018; Jana et al. 2019; Shahzadi et al. 2021) have been conducted on decision making methods under q-ROF environments.

The uses of trapezoidal fuzzy numbers (TrFNs) are emerging as an important tool with the development of fuzzy sets. The trapezoidal fuzzy membership functions are firstly increased and then maintain the same membership value for a period of time before being decreased. If it keeps maximum membership level for a long duration, the system represents more stable realistic scenarios. If the alternative's uncertainty follows this pattern, the TrFN is the best choice for representing it. Motivated by the ideas of q-ROFS (Yager 2017) and TrFNs (De and Das 2014), Gupta et al. (2021) introduced the concept of q-rung orthopair TrFNs (q-ROTrFNs). Wan and Huang (2021) developed a new ranking method and Hamming distance measure for q-ROTrFNs. They also proposed a novel TODIM group decision making method with q-ROTrFNs.

The operational rules play significant roles in aggregating decision information. Hamacher operations (Hamacher 1978), a generalized form of algebraic and Einstein operations (Garg 2016), have significant importance in the aggregation process by means of a flexible parameter. The aggregation operator defined by Hamacher operations acts a major role in decision making. For example, in decision science, a DM may have optimistic or pessimistic nature, which makes a major impact on the results. The Hamacher parameter can consider all the risk levels of the DMs by varying it's parameter. Also, the aggregation operators are most commonly used to combine each individual preference with the overall preference data and provide a collective preference value for each alternative. In the literature, no such studies have been performed on aggregation processes for a collection of q-ROTrF data where the aggregation operator can consider the risk levels of the DMs. It is also to be noted here that the acceptability zone of q-ROTrFNs provides more flexibility in information evaluation due to the combination of the benefits of q-ROFNs and TrFNs. Therefore, it is important to investigate aggregation operators and their applications based on Hamacher operations in q-ROTrF environment.

On the basis of the above discussions, this article proposes a novel MCGDM technique using newly developed Hamacher operation-based aggregation operators in a q-ROTrF context. To demonstrate the usefulness and practicality of the presented approach, two decision making problems, viz., supplier selection and company job selection are considered.

The key contributions of this work are highlighted as follows:

1. From the viewpoint of capturing uncertainties in a better way, q-ROTrFNs are used. The q-ROTrFNs possess increasing acceptability with the increase of q, providing more freedom to the DMs in data capturing.
2. Based on Hamacher t-norms and t-conorms, basic operational laws (sum, product, scalar multiplication, and exponential) are introduced on q-ROTrFNs.
3. Based on those operational laws, q-ROTrF Hamacher WA (q-ROTrFHWA) and q-ROTrF Hamacher WG (q-ROTrFHWG) operators are proposed.
4. Particular situations of q-ROTrFHWA and q-ROTrFHWG operators by varying the associated parameter are presented, to reduce the limitations of the existing operators.
5. A novel MCGDM method has been proposed and applied to two real-life decision making situations. The validity of the proposed method is demonstrated by a comparative study with other existing methodologies.

The remainder of the paper is organized as: Sect. 2 briefly recalls fundamental concepts related to q-ROFS, q-ROTrFN, score and accuracy functions of q-ROTrFN and Hamacher t-norms and t-conorms. In Sect. 3, Hamacher operations-based basic operational laws for q-ROTrFNs are proposed. Further, using those laws, aggregation operators, viz., q-ROTrFHWA and q-ROTrFHWG, are also introduced to aggregate the q-ROTrFNs. Consequently, some characteristics of these developed operators are also exhibited here. Section 4 illustrates an MCGDM approach utilizing the proposed aggregation operators. Two numerical examples are solved by applying the developed operators and exhibiting the developed method's feasibility, and effectiveness is discussed in Sect. 5. Moreover, a comparative analysis with some existing methods and operators is presented in Sect. 6. Finally, an overall summarization of the present paper has been depicted in Sect. 7.

2 Preliminaries

In this section, several basic concepts are briefly studied which are required throughout the paper.

2.1 q-ROFS

Definition 1 (Yager 2017) A q-ROFS, $\mathcal{P}$ on a universal set X is presented by:

$$\mathcal{P} = \{(x,\ \mu_{\mathcal{P}}(x),\ \nu_{\mathcal{P}}(x))|x \in X\},$$

in which $\mu_{\mathcal{P}},\ \nu_{\mathcal{P}}\ \in\ [0,\ 1]$ represent the degree of membership and degree of non-membership, respectively, satisfying the condition that

$$\left((\mu_{\mathcal{P}}(x))^q + (\nu_{\mathcal{P}}(x))^q\right) \in [0,\ 1],$$

where $q \geq 1$.

For computational convenience, Yager (2017) called $(\mu_{\mathcal{P}}(x),\ \nu_{\mathcal{P}}(x))$ as a q-ROFN and denoted it by $\widetilde{\wp} = (\mu,\ \nu)$.

2.2 q-ROTrFN

Definition 2 (Gupta et al. 2021) Let X be a universe of discourse. A q-ROFN $\widetilde{\wp}$, is said to be q-ROTrFN defined on X, denoted by $\widetilde{R} = \langle\left([a,\ b,\ c,\ d]; \gamma_{\widetilde{R}}\right),\ \left([a_1,\ b,\ c,\ d_1]; \delta_{\widetilde{R}}\right)\rangle$ if:

$$\gamma_{\widetilde{R}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}\mu_{\widetilde{R}}, & \text{whenever } a \leq x \leq b \\ \mu_{\widetilde{R}}, & \text{when } b \leq x \leq c \\ \frac{(d-x)}{(d-c)}\mu_{\widetilde{R}}, & \text{for } c \leq x \leq d \\ 0 & \text{Otherwise} \end{cases},$$

$$\delta_{\widetilde{R}}(x) = \begin{cases} \frac{(b-x)+(x-a_1)}{(b-a_1)}\nu_{\widetilde{R}}, & \text{when } a_1 \leq x \leq b \\ \nu_{\widetilde{R}}, & \text{for } b \leq x \leq c \\ \frac{(x-c)+(d_1-x)}{(d_1-c)}\nu_{\widetilde{R}}, & \text{whenever } c \leq x \leq d_1 \\ 1 & \text{Otherwise} \end{cases},$$

where $\gamma_{\widetilde{R}}(x) \in [0,\ 1]$ represents the degree of membership and $\delta_{\widetilde{R}}(x) \in [0,\ 1]$ represents the degree of non-membership with the condition that $0 \leq (\gamma_{\widetilde{R}}(x))^q + (\delta_{\widetilde{R}}(x))^q \leq 1$ where $x \in X$, and $q \geq 1$.

For simplicity, take $a = a_1, d = d_1$, and so a q-ROTrFN is denoted by $\widetilde{r} = \langle[a,\ b,\ c,\ d]; \mu,\ \nu\rangle$. Diagrammatically, a q-ROTrFN is presented in Fig. 1.

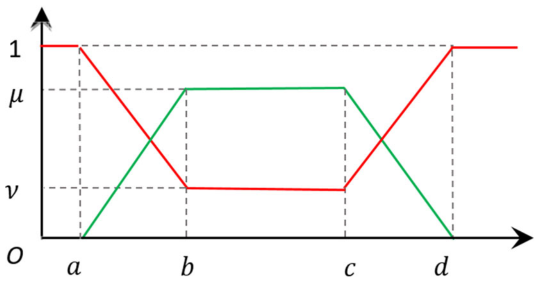

Fig. 1 Graphical representation of q-ROTrFN $\widetilde{r}=\langle[a,\ b,\ c,\ d]; \mu,\ \nu\rangle$

2.3 Score, accuracy functions and ranking method

Definition 3 (Wan and Huang 2021) Let $\widetilde{r} = \langle [a, b, c, d]; \mu, \nu\rangle$ be a q-ROTrFN, then the score function, $S(\widetilde{r})$ and the accuracy function, $A(\widetilde{r})$ are defined as follows:

$$S(\widetilde{r}) = \frac{a+b+c+d}{4}\left(\mu^q - \nu^q\right); \tag{1}$$

$$A(\widetilde{r}) = \frac{a+b+c+d}{4}\left(\mu^q + \nu^q\right). \tag{2}$$

Definition 4 (Wan and Huang 2021) Let $\widetilde{r}_1 = \langle [a_1, b_1, c_1, d_1]; \mu_1, \nu_1\rangle$ and $\widetilde{r}_2 = \langle [a_2, b_2, c_2, d_2]; \mu_2, \nu_2\rangle$ are any two q-ROTrFNs and $S(\widetilde{r}_1)$, $S(\widetilde{r}_2)$ are the score functions of $\widetilde{r}_1$ and $\widetilde{r}_2$ and $A(\widetilde{r}_1)$, $A(\widetilde{r}_2)$ are the accuracy functions of $\widetilde{r}_1$ and $\widetilde{r}_2$, respectively. Then the ordering of $\widetilde{r}_1$ and $\widetilde{r}_2$ is

(i) $\widetilde{r}_1 \prec \widetilde{r}_2$when $S(\widetilde{r}_1) < S(\widetilde{r}_2)$;
(ii) For $S(\widetilde{r}_1) = S(\widetilde{r}_2)$, then

- If $A(\widetilde{r}_1) < A(\widetilde{r}_2)$, then $\widetilde{r}_1 \prec \widetilde{r}_2$;
- If $A(\widetilde{r}_1) = A(\widetilde{r}_2)$, then $\widetilde{r}_1 \approx \widetilde{r}_2$.

2.4 Hamacher t-norms and t-conorms

In 1978, Hamacher (1978) introduced generalized t-norms and t-conorms, which are known as Hamacher t-norms and t-conorms, as follows ($\varrho > 0$):

- Hamacher t-norms: $T_{\varrho}^{H}(\S, \dagger) = \frac{\S\dagger}{\varrho+(1-\varrho)(\S+\dagger-\S\dagger)}$,
- Hamacher t-conorms: $S_{\varrho}^{H}(\S, \dagger) = \frac{\S+\dagger-\S\dagger-(1-\varrho)\S\dagger}{1-(1-\varrho)\S\dagger}$ $\forall \S, \dagger \in [0, 1]$.

3 Hamacher operations-based q-ROTrF aggregation operators

In this section, some basic operational laws of q-ROTrFNs are defined based on Hamacher operations and using those defined laws, two aggregation operators, viz., q-ROTrFHWA and q-ROTrFHWG operators are introduced successively along with their properties.

Definition 5 Let $\widetilde{r}_1 = \langle [a_1, b_1, c_1, d_1]; \mu_1, \nu_1\rangle$, $\widetilde{r}_2 = \langle [a_2, b_2, c_2, d_2]; \mu_2, \nu_2\rangle$ and $\widetilde{r} = \langle [a, b, c, d]; \mu, \nu\rangle$ be any three q-ROTrFNs, and $\lambda > 0$ be a scalar, then

(i)
$$\tilde{r}_1 \oplus_H \tilde{r}_2 = \left\langle [a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2]; \left(\frac{\mu_1^q+\mu_2^q-\mu_1^q\mu_2^q-(1-\varrho)\mu_1^q\mu_2^q}{1-(1-\varrho)\mu_1^q\mu_2^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_1^q\nu_2^q}{\varrho+(1-\varrho)\left(\nu_1^q+\nu_2^q-\nu_1^q\nu_2^q\right)}\right)^{\frac{1}{q}}\right\rangle;$$

(ii)
$$\tilde{r}_1 \otimes_H \tilde{r}_2 = \left\langle [a_1a_2, b_1b_2, c_1c_2, d_1d_2]; \left(\frac{\mu_1^q\mu_2^q}{\varrho+(1-\varrho)\left(\mu_1^q+\mu_2^q-\mu_1^q\mu_2^q\right)}\right)^{\frac{1}{q}}, \left(\frac{\nu_1^q+\nu_2^q-\nu_1^q\nu_2^q-(1-\varrho)\nu_1^q\nu_2^q}{1-(1-\varrho)\nu_1^q\nu_2^q}\right)^{\frac{1}{q}}\right\rangle;$$

(iii) $$\lambda \odot_H \widetilde{r} = \left\langle [\lambda a, \lambda b, \lambda c, \lambda d]; \left(\frac{(1+(\varrho-1)\mu^q)^\lambda - (1-\mu^q)^\lambda}{(1+(\varrho-1)\mu^q)^\lambda + (\varrho-1)(1-\mu^q)^\lambda} \right)^{\frac{1}{q}}, \left(\frac{\varrho \nu^{q\lambda}}{(1+(\varrho-1)(1-\nu^q))^\lambda + (\varrho-1)\nu^{q\lambda}} \right)^{\frac{1}{q}} \right\rangle;$$

(iv) $$\widetilde{r}^\lambda = \left\langle \left[a^\lambda, b^\lambda, c^\lambda, d^\lambda\right]; \left(\frac{\varrho \mu^{q\lambda}}{(1+(\varrho-1)(1-\mu^q))^\lambda + (\varrho-1)\mu^{q\lambda}} \right)^{\frac{1}{q}}, \left(\frac{(1+(\varrho-1)\nu^q)^\lambda - (1-\nu^q)^\lambda}{(1+(\varrho-1)\nu^q)^\lambda + (\varrho-1)(1-\nu^q)^\lambda} \right)^{\frac{1}{q}} \right\rangle.$$

3.1 q-ROTrFHWA operator

Definition 6 Let $\widetilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle$ $(j = 1, 2, \ldots, n)$ be a collection of q-ROTrFNs. The q-ROTrFHWA operator is defined based on q-ROTrFNs (Gupta et al. 2021) and Hamacher t-norms and t-conorms (Hamacher 1978) as follows:

$$q\text{-ROTrFHWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = \oplus_{H}{}_{j=1}^{n} \left(\omega_j \odot_H \widetilde{r}_j\right), \tag{3}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is a weight vector of q-ROTrFNs $\widetilde{r}_j$ $(j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 1 Let $\widetilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle$ $(j = 1, 2, \ldots, n)$ be a collection of q-ROTrFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be a weight vector of $\widetilde{r}_j$ where $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$. Then their aggregated value by the q-ROTrFHWA operator is also a q-ROTrFN and:

$$q - \text{ROTrFHWA}(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) = \left\langle \left[\sum_{j=1}^{n} \omega_j a_j, \sum_{j=1}^{n} \omega_j b_j, \sum_{j=1}^{n} \omega_j c_j, \sum_{j=1}^{n} \omega_j d_j \right]; \left(\frac{\prod_{j=1}^{n} \left(1+(\varrho-1)\mu_j^q\right)^{\omega_j} - \prod_{j=1}^{n} \left(1-\mu_j^q\right)^{\omega_j}}{\prod_{j=1}^{n} \left(1+(\varrho-1)\mu_j^q\right)^{\omega_j} + (\varrho-1)\prod_{j=1}^{n} \left(1-\mu_j^q\right)^{\omega_j}} \right)^{\frac{1}{q}}, \left(\frac{\varrho \prod_{j=1}^{n} \nu_j^{q\omega_j}}{\prod_{j=1}^{n} \left(1+(\varrho-1)\left(1-\nu_j^q\right)\right)^{\omega_j} + (\varrho-1)\prod_{j=1}^{n} \nu_j^{q\omega_j}} \right)^{\frac{1}{q}} \right\rangle. \tag{4}$$

Proof Based on Definition 5,

$$\omega_j \odot_H \tilde{r}_j = \left\langle [\omega_j a_j, \omega_j b_j, \omega_j c_j, \omega_j d_j]; \left(\frac{\left(1+(\varrho-1)\mu_j^q\right)^{\omega_j} - \left(1-\mu_j^q\right)^{\omega_j}}{\left(1+(\varrho-1)\mu_j^q\right)^{\omega_j} + (\varrho-1)\left(1-\mu_j^q\right)^{\omega_j}} \right)^{\frac{1}{q}}, \left(\frac{\varrho \nu_j^{q\omega_j}}{\left(1+(\varrho-1)\left(1-\nu_j^q\right)\right)^{\omega_j} + (\varrho-1)\nu_j^{q\omega_j}} \right)^{\frac{1}{q}} \right\rangle.$$

Now,

$$\omega_1\tilde{r}_1 \oplus_H \omega_2\tilde{r}_2 = \Bigg\langle [\omega_1 a_1 + \omega_2 a_2,\ \omega_1 b_1 + \omega_2 b_2,\ \omega_1 c_1 + \omega_2 c_2,\ \omega_1 d_1 + \omega_2 d_2];$$

$$\left(\frac{\prod_{j=1}^{2}\left(1+(\varrho-1)\mu_j^q\right)^{\omega_j} - \prod_{j=1}^{2}\left(1-\mu_j^q\right)^{\omega_j}}{\prod_{j=1}^{2}\left(1+(\varrho-1)\mu_j^q\right)^{\omega_j} + (\varrho-1)\prod_{j=1}^{2}\left(1-\mu_j^q\right)^{\omega_j}}\right)^{\frac{1}{q}},$$

$$\left(\frac{\varrho\prod_{j=1}^{2}\nu_j^{q\omega_j}}{\prod_{j=1}^{2}\left(1+(\varrho-1)\left(1-\nu_j^q\right)\right)^{\omega_j} + (\varrho-1)\prod_{j=1}^{2}\nu_j^{q\omega_j}}\right)^{\frac{1}{q}}\Bigg\rangle.$$

i.e., the theorem holds for $n = 2$. Assume now that the theorem holds for $n = k$. Hence,

$$q\text{-ROTrFHWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_k) = \left\langle \left[\sum_{j=1}^{k}\omega_j a_j, \sum_{j=1}^{k}\omega_j b_j, \sum_{j=1}^{k}\omega_j c_j, \sum_{j=1}^{k}\omega_j d_j\right];\right.$$

$$\left(\frac{\prod_{j=1}^{k}\left(1+(\varrho-1)\mu_j{}^q\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\mu_j{}^q\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+(\varrho-1)\mu_j{}^q\right)^{\omega_j} + (\varrho-1)\prod_{j=1}^{k}\left(1-\mu_j{}^q\right)^{\omega_j}}\right)^{\frac{1}{q}},$$

$$\left.\left(\frac{\varrho\prod_{j=1}^{k}\nu_j{}^{q\omega_j}}{\prod_{j=1}^{k}\left(1+(\varrho-1)\left(1-\nu_j{}^q\right)\right)^{\omega_j} + (\varrho-1)\prod_{j=1}^{k}\nu_j{}^{q\omega_j}}\right)^{\frac{1}{q}}\right\rangle.$$

Then for $n = k+1$,

$$q\text{-ROTrFHWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_k, \widetilde{r}_{k+1}) = q\text{-ROTrFHWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_k) \oplus_H (\omega_{k+1}\widetilde{r}_{k+1})$$

$$\left\langle \left[\sum_{j=1}^{k}\omega_j a_j, \sum_{j=1}^{k}\omega_j b_j, \sum_{j=1}^{k}\omega_j c_j, \sum_{j=1}^{k}\omega_j d_j\right];\right.$$

$$\left(\frac{\prod_{j=1}^{k}\left(1+(\varrho-1)\mu_j{}^q\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\mu_j{}^q\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+(\varrho-1)\mu_j{}^q\right)^{\omega_j} + (\varrho-1)\prod_{j=1}^{k}\left(1-\mu_j{}^q\right)^{\omega_j}}\right)^{\frac{1}{q}},$$

$$\left.\left(\frac{\varrho\prod_{j=1}^{k}\nu_j{}^{q\omega_j}}{\prod_{j=1}^{k}\left(1+(\varrho-1)\left(1-\nu_j{}^q\right)\right)^{\omega_j} + (\varrho-1)\prod_{j=1}^{k}\nu_j{}^{q\omega_j}}\right)^{\frac{1}{q}}\right\rangle$$

$$\oplus_H\Big\langle [\omega_{k+1}a_{k+1},\ \omega_{k+1}b_{k+1},\ \omega_{k+1}c_{k+1},\ \omega_{k+1}d_{k+1}];$$

$$\left(\frac{(1+(\varrho-1)\mu_{k+1}{}^q)^{\omega_{k+1}} - (1-\mu_{k+1}{}^q)^{\omega_j}}{(1+(\varrho-1)\mu_{k+1}{}^q)^{\omega_j} + (\varrho-1)(1-\mu_{k+1}{}^q)^{\omega_j}}\right)^{\frac{1}{q}},$$

$$\left(\frac{\varrho\nu_{k+1}{}^{q\omega_{k+1}}}{(1+(\varrho-1)(1-\nu_{k+1}{}^q))^{\omega_{k+1}} + (\varrho-1)\nu_{k+1}{}^{q\omega_{k+1}}}\right)^{\frac{1}{q}}\Big\rangle.$$

$$= \left\langle \left[\sum_{j=1}^{k+1}\omega_j a_j, \sum_{j=1}^{k+1}\omega_j b_j, \sum_{j=1}^{k+1}\omega_j c_j, \sum_{j=1}^{k+1}\omega_j d_j\right];\right.$$

$$\left(\frac{\prod_{j=1}^{k+1}\left(1+(\varrho-1)\mu_j{}^q\right)^{\omega_j}-\prod_{j=1}^{k+1}\left(1-\mu_j{}^q\right)^{\omega_j}}{\prod_{j=1}^{k+1}\left(1+(\varrho-1)\mu_j{}^q\right)^{\omega_j}+(\varrho-1)\prod_{j=1}^{k+1}\left(1-\mu_j{}^q\right)^{\omega_j}}\right)^{\frac{1}{q}},$$
$$\left.\left(\frac{\varrho\prod_{j=1}^{k+1}\nu_j{}^{q\omega_j}}{\prod_{j=1}^{k+1}\left(1+(\varrho-1)\left(1-\nu_j{}^q\right)\right)^{\omega_j}+(\varrho-1)\prod_{j=1}^{k+1}\nu_j{}^{q\omega_j}}\right)^{\frac{1}{q}}\right\rangle.$$

Therefore, the theorem is true for $n = k + 1$ also; and hence is true for all n. This completes the proof.

- Some special cases

Some specific variations of the developed q-ROTrFHWA operator can be established depending on the value of the Hamacher parameter ϱ.

If $\varrho = 1$, then q-ROTrFHWA operator is converted to q-ROTrF WA (q-ROTrFWA) operator as follows:

$$q\text{-ROTrFWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n)$$
$$= \left\langle \left[\sum_{j=1}^{n}\omega_j a_j, \sum_{j=1}^{n}\omega_j b_j, \sum_{j=1}^{n}\omega_j c_j, \sum_{j=1}^{n}\omega_j d_j\right]; \left(1-\prod_{i=1}^{n}\left(1-\mu_j{}^q\right)^{\omega_j}\right)^{\frac{1}{q}}, \prod_{j=1}^{n}\nu_j{}^{\omega_j}\right\rangle.$$

If $\varrho = 2$, the q-ROTrFHWA operator is converted to q-ROTrF Einstein WA (q-ROTrFEWA) operator as follows:

$$q\text{-ROTrFEWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = \left\langle \left[\sum_{j=1}^{n}\omega_j a_j, \sum_{j=1}^{n}\omega_j b_j, \sum_{j=1}^{n}\omega_j c_j, \sum_{j=1}^{n}\omega_j d_j\right];\right.$$
$$\left.\left(\frac{\prod_{j=1}^{n}\left(1+\mu_j{}^q\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-\mu_j{}^q\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\mu_j{}^q\right)^{\omega_j}+\prod_{j=1}^{n}\left(1-\mu_j{}^q\right)^{\omega_j}}\right)^{\frac{1}{q}}, \left(\frac{2\prod_{j=1}^{n}\nu_j{}^{q\omega_j}}{\prod_{j=1}^{n}\left(2-\nu_j{}^q\right)^{\omega_i}+\prod_{j=1}^{n}\nu_j{}^{q\omega_j}}\right)^{\frac{1}{q}}\right\rangle.$$

Now, some fundamental properties of the proposed q-ROTrFHWA operator are stated in follow up.

Theorem 2 (Idempotency) Let $\left\{\widetilde{r}_j = \langle\left[a_j, b_j, c_j, d_j\right]; \mu_j, \nu_j\rangle | j = 1, 2, \ldots n\right\}$ represents a collection of q-ROTrFNs. If $\widetilde{r}_j = \widetilde{r} = \langle[a, b, c, d]; \mu, \nu\rangle\ \forall j = 1, 2, \ldots n$, then:

$$q\text{-ROTrFHWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = \widetilde{r}.$$

Proof

$$q\text{-ROTrFHWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = \left\langle \left[\sum_{j=1}^{n}\omega_j a_j, \sum_{j=1}^{n}\omega_j b_j, \sum_{j=1}^{n}\omega_j c_j, \sum_{j=1}^{n}\omega_j d_j\right];\right.$$
$$\left(\frac{\prod_{j=1}^{n}\left(1+(\varrho-1)\mu_j{}^q\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-\mu_j{}^q\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\varrho-1)\mu_j{}^q\right)^{\omega_j}+(\varrho-1)\prod_{j=1}^{n}\left(1-\mu_j{}^q\right)^{\omega_j}}\right)^{\frac{1}{q}},$$
$$\left.\left(\frac{\varrho\prod_{j=1}^{n}\nu_j{}^{q\omega_j}}{\prod_{j=1}^{n}\left(1+(\varrho-1)\left(1-\nu_j{}^q\right)\right)^{\omega_j}+(\varrho-1)\prod_{j=1}^{n}\nu_j{}^{q\omega_j}}\right)^{\frac{1}{q}}\right\rangle.$$

Since $\widetilde{r}_j = \widetilde{r}\ \forall j = 1, 2, \ldots n$, then

$$
\begin{aligned}
&q\text{-ROTrFHWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = q\text{-ROTrFHWA}(\widetilde{r}, \widetilde{r}, \ldots, \widetilde{r}) \\
&= \left\langle \left[\left(\sum_{j=1}^{n} \omega_j\right) a, \left(\sum_{j=1}^{n} \omega_j\right) b, \left(\sum_{j=1}^{n} \omega_j\right) c, \left(\sum_{j=1}^{n} \omega_j\right) d \right]; \right. \\
&\quad \left(\frac{(1+(\varrho-1)\mu^q)^{\sum_j^n \omega_j} - (1-\mu^q)^{\sum_j^n \omega_j}}{(1+(\varrho-1)\mu^q)^{\sum_j^n \omega_j} + (\varrho-1)(1-\mu^q)^{\sum_j^n \omega_j}} \right)^{\frac{1}{q}}, \\
&\quad \left. \left(\frac{\varrho \nu^{q \sum_j^n \omega_j}}{(1+(\varrho-1)(1-\nu^q))^{\sum_j^n \omega_j} + (\varrho-1)\nu^{q\sum_j^n \omega_j}} \right)^{\frac{1}{q}} \right\rangle \\
&= \langle [a, b, c, d]; \mu, \nu \rangle \\
&= \widetilde{r}.
\end{aligned}
$$

Theorem 3 (Monotonicity) Let $\widetilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle$ and $\widetilde{r}'_j = \langle [a'_j, b'_j, c'_j, d'_j]; \mu'_j, \nu'_j \rangle$ $(j = 1, 2, \ldots, n)$ be two collections of q-ROTrFNs. If $a_j \le a'_j$, $b_j \le b'_j$, $c_j \le c'_j$, $d_j \le d'_j$, $\mu_j \le \mu'_j$ and $\nu_j \ge \nu'_j$ for all j, then,

$$
q\text{-ROTrFHWA}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) \preccurlyeq q\text{-ROTrFHWA}(\widetilde{r}'_1, \widetilde{r}'_2, \ldots, \widetilde{r}'_n). \tag{5}
$$

Proof Let $g(t) = \frac{1+(\varrho-1)t}{1-t}$, $t \in [0, 1)$, then $g'(t) = \frac{\varrho}{(1-t)^2} > 0$, thus g is an increasing function. Since for every $\widetilde{r}_j$ and $\widetilde{r}'_j$, $\mu_i \le \mu'_j$, $(i = 1, 2, \ldots, n)$

$$
\frac{(1+(\varrho-1)\mu_i{}^q)}{(1-\mu_i{}^q)} \le \frac{\left(1+(\varrho-1)\mu_i'^q\right)}{\left(1-\mu_i'^q\right)}
$$

thus,

$$
\left(\frac{(1+(\varrho-1)\mu_i{}^q)}{(1-\mu_i{}^q)}\right)^{\omega_i} \le \left(\frac{\left(1+(\varrho-1)\mu_i'^q\right)}{\left(1-\mu_i'^q\right)}\right)^{\omega_i}
$$

So,

$$
\begin{aligned}
&\prod_{i=1}^{n}\left(\frac{(1+(\varrho-1)\mu_i{}^q)}{(1-\mu_i{}^q)}\right)^{\omega_i} \le \prod_{i=1}^{n}\left(\frac{\left(1+(\varrho-1)\mu_i'^q\right)}{\left(1-\mu_i'^q\right)}\right)^{\omega_i} \\
&\iff \prod_{i=1}^{n}\left(\frac{(1+(\varrho-1)\mu_i{}^q)}{(1-\mu_i{}^q)}\right)^{\omega_i} + (\varrho-1) \le \prod_{i=1}^{n}\left(\frac{\left(1+(\varrho-1)\mu_i'^q\right)}{\left(1-\mu_i'^q\right)}\right)^{\omega_i} + (\varrho-1) \\
&\iff \frac{1}{\prod_{i=1}^{n}\left(\frac{(1+(\varrho-1)\mu_i{}^q)}{(1-\mu_i{}^q)}\right)^{\omega_i} + (\varrho-1)} \ge \frac{1}{\prod_{i=1}^{n}\left(\frac{\left(1+(\varrho-1)\mu_i'^q\right)}{\left(1-\mu_i'^q\right)}\right)^{\omega_i} + (\varrho-1)} \\
&\iff \frac{\prod_{i=1}^{n}\left(1-\mu_i{}^q\right)^{\omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\mu_i{}^q)^{\omega_i} + (\varrho-1)\prod_{i=1}^{n}(1-\mu_i{}^q)^{\omega_i}} \\
&\ge \frac{\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i'^q\right)^{\omega_i} + (\varrho-1)\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}
\end{aligned}
$$

$$\begin{aligned}
&\Longleftrightarrow \frac{\varrho\prod_{i=1}^{n}\left(1-\mu_i{}^q\right)^{\omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\mu_i{}^q)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}(1-\mu_i{}^q)^{\omega_i}}\\
&\geq \frac{\varrho\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i'^q\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}\\
&\Longleftrightarrow 1-\frac{\varrho\prod_{i=1}^{n}\left(1-\mu_i{}^q\right)^{\omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\mu_i{}^q)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}(1-\mu_i{}^q)^{\omega_i}}\\
&\leq 1-\frac{\varrho\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i'^q\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}\\
&\Longleftrightarrow \frac{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i{}^q\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\mu_i{}^q\right)^{\omega_i}-\varrho\prod_{i=1}^{n}\left(1-\mu_i{}^q\right)^{\omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\mu_i{}^q)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}(1-\mu_i{}^q)^{\omega_i}}\\
&\leq \frac{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i'^q\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}-\varrho\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i'^q\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}\\
&\Longleftrightarrow \frac{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i{}^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_i{}^q\right)^{\omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\mu_i{}^q)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}(1-\mu_i{}^q)^{\omega_i}}\\
&\leq \frac{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i'^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i'^q\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}\\
&\Longleftrightarrow \left(\frac{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i{}^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_i{}^q\right)^{\omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)\mu_i{}^q)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}(1-\mu_i{}^q)^{\omega_i}}\right)^{\frac{1}{q}}\\
&\leq \left(\frac{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i'^q\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\mu_i'^q\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}\left(1-\mu_i'^q\right)^{\omega_i}}\right)^{\frac{1}{q}}. \qquad (6)
\end{aligned}$$

Again let $f(u)=\frac{(1+(\varrho-1)(1-u))}{u}$, $u\in(0,1]$, $\varrho>0$, then $f'(u)=-\frac{\varrho}{u^2}<0$, thus $f(u)$ is a decreasing function.

Since for all i, $\nu_i{}^q \geq \nu_i'^q$, then

$$\frac{1+(\varrho-1)(1-\nu_i{}^q)}{\nu_i{}^q} \leq \frac{1+(\varrho-1)\left(1-\nu_i'^q\right)}{\nu_i'^q},$$

thus,

$$\left(\frac{1+(\varrho-1)(1-\nu_i{}^q)}{\nu_i{}^q}\right)^{\omega_i} \leq \left(\frac{1+(\varrho-1)\left(1-\nu_i'^q\right)}{\nu_i'^q}\right)^{\omega_i}$$

So,

$$\begin{aligned}
&\prod_{i=1}^{n}\left(\frac{1+(\varrho-1)(1-\nu_i{}^q)}{\nu_i{}^q}\right)^{\omega_i} \leq \prod_{i=1}^{n}\left(\frac{1+(\varrho-1)\left(1-\nu_i'^q\right)}{\nu_i'^q}\right)^{\omega_i}\\
&\Longleftrightarrow \prod_{i=1}^{n}\left(\frac{1+(\varrho-1)(1-\nu_i{}^q)}{\nu_i{}^q}\right)^{\omega_i}+(\varrho-1) \leq \prod_{i=1}^{n}\left(\frac{1+(\varrho-1)\left(1-\nu_i'^q\right)}{\nu_i'^q}\right)^{\omega_i}+(\varrho-1)\\
&\Longleftrightarrow \frac{1}{\prod_{i=1}^{n}\left(\frac{1+(\varrho-1)(1-\nu_i{}^q)}{\nu_i{}^q}\right)^{\omega_i}+(\varrho-1)} \geq \frac{1}{\prod_{i=1}^{n}\left(\frac{1+(\varrho-1)\left(1-\nu_i'^q\right)}{\nu_i'^q}\right)^{\omega_i}+(\varrho-1)}
\end{aligned}$$

$$\iff \frac{\prod_{i=1}^{n}{\nu_i}^{q\omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)(1-{\nu_i}^q))^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}{\nu_i}^{q\omega_i}}$$
$$\geq \frac{\prod_{i=1}^{n}{\nu_i'}^{q\omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\left(1-{\nu_i'}^{q}\right)\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}{\nu_i'}^{q\omega_i}}$$
$$\iff \frac{\varrho\prod_{i=1}^{n}{\nu_i}^{q\omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)(1-{\nu_i}^q))^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}{\nu_i}^{q\omega_i}}$$
$$\geq \frac{\varrho\prod_{i=1}^{n}{\nu_i'}^{q\omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\left(1-{\nu_i'}^{q}\right)\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}{\nu_i'}^{q\omega_i}}$$
$$\iff \left(\frac{\varrho\prod_{i=1}^{n}{\nu_i}^{q\omega_i}}{\prod_{i=1}^{n}(1+(\varrho-1)(1-{\nu_i}^q))^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}{\nu_i}^{q\omega_i}}\right)^{\frac{1}{q}}$$
$$\geq \left(\frac{\varrho\prod_{i=1}^{n}{\nu_i'}^{q\omega_i}}{\prod_{i=1}^{n}\left(1+(\varrho-1)\left(1-{\nu_i'}^{q}\right)\right)^{\omega_i}+(\varrho-1)\prod_{i=1}^{n}{\nu_i'}^{q\omega_i}}\right)^{\frac{1}{q}}. \tag{7}$$

From (6) and (7) and using the relations $\sum_{j=1}^{n}\omega_j a_j \leq \sum_{j=1}^{n}\omega_j a_j'$, $\sum_{j=1}^{n}\omega_j b_j \leq \sum_{j=1}^{n}\omega_j b_j'$, $\sum_{j=1}^{n}\omega_j c_j \leq \sum_{j=1}^{n}\omega_j c_j'$ and $\sum_{j=1}^{n}\omega_j d_j \leq \sum_{j=1}^{n}\omega_j d_j'$, it is clear that:

$$S(q\text{-ROTrFHWA}(\widetilde{r}_1,\widetilde{r}_2,\ldots,\widetilde{r}_n)) \leq S\left(q\text{-ROTrFHWA}\left(\widetilde{r}_1',\widetilde{r}_2',\ldots,\widetilde{r}_n'\right)\right).$$

Therefore,

$$q\text{-ROTrFHWA}(\widetilde{r}_1,\widetilde{r}_2,\ldots,\widetilde{r}_n) \preccurlyeq q\text{-ROTrFHWA}\left(\widetilde{r}_1',\widetilde{r}_2',\ldots,\widetilde{r}_n'\right).$$

Hence Eq. (5) follows.

Theorem 4 (Boundedness) Let $\widetilde{r}_j = \langle[a_j, b_j, c_j, d_j]; \mu_j, \nu_j\rangle$ $(j = 1, 2, \ldots, n)$ represents a collection of q-ROTrFNs, then,

$$\widetilde{r}^- \leq q\text{-ROTrFHWA}(\widetilde{r}_1,\widetilde{r}_2,\ldots,\widetilde{r}_n) \leq \widetilde{r}^+,$$

where $\widetilde{r}^- = \langle[\min\{a_j\}, \min\{b_j\}, \min\{c_j\}, \min\{d_j\}]; \min\{\mu_j\}, \max\{\nu_j\}\rangle$, and $\widetilde{r}^+ = \langle[\max\{a_j\}, \max\{b_j\}, \max\{c_j\}, \max\{d_j\}]; \max\{\mu_j\}, \min\{\nu_j\}\rangle$.

Proof Since $\min\{a_j\} \leq a_j \leq \max\{a_j\}$, $\min\{b_j\} \leq b_j \leq \max\{b_j\}$,

$\min\{c_j\} \leq c_j \leq \max\{c_j\}$, $\min\{d_j\} \leq d_j \leq \max\{d_j\}$, $\min\{\mu_j\} \leq \mu_j \leq \max\{\mu_j\}$ and $\min\{\nu_j\} \leq \nu_j \leq \max\{\nu_j\}$ $\forall j = 1, 2, \ldots, n$, then $\widetilde{r}^- \leq \widetilde{r}_j \leq \widetilde{r}^+$, $\forall j$.

Now, applying the property of monotonicity:

$$q\text{-ROTrFHWA}\left(\widetilde{r}^-,\widetilde{r}^-,\ldots,\widetilde{r}^-\right) \leq q\text{-ROTrFHWA}(\widetilde{r}_1,\widetilde{r}_2,\ldots,\widetilde{r}_n).$$

Therefore using the idempotency theorem, the above inequality takes the form as:

$$\widetilde{r}^- \leq q\text{-ROTrFHWA}(\widetilde{r}_1,\widetilde{r}_2,\ldots,\widetilde{r}_n). \tag{8}$$

Similarly, it can be revealed that:

$$q\text{-ROTrFHWA}(\widetilde{r}_1,\widetilde{r}_2,\ldots,\widetilde{r}_n) \leq \widetilde{r}^+. \tag{9}$$

So, merging (8) and (9) it follows that:

$$\widetilde{r}^- \leq q\text{-ROTrFHWA}(\widetilde{r}_1,\widetilde{r}_2,\ldots,\widetilde{r}_n) \leq \widetilde{r}^+.$$

3.2 q-ROTrFHWG operator

In this subsection, q-ROTrFHWG operator is developed based on Hamacher operational rules.

Definition 7 Let $\widetilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle$ $(j = 1, 2, \ldots, n)$ be a collection of q-ROTrFNs. The q-ROTrFHWG operator based on q-ROTrFNs (Gupta et al. 2021) and Hamacher t-norms and t-conorms (Hamacher 1978) is defined as follows:

$$q\text{-ROTrFHWG}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = \widetilde{r}_1^{\omega_1} \otimes_H \widetilde{r}_2^{\omega_2} \otimes_H \ldots \otimes_H \widetilde{r}_n^{\omega_n}, \tag{10}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is a vector of q-ROTrFNs $\widetilde{r}_j$ $(j = 1, 2, \ldots, n)$ $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 5 Let $\widetilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle$ $(j = 1, 2, \ldots, n)$ be a collection of q-ROTrFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ be a weight vector of $\widetilde{r}_j$ where $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$. Then their aggregated value by the q-ROTrFHWG operator is also a q-ROTrFN and:

$$q - \text{ROTrFHWG}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = \left\langle \left[\prod_{j=1}^{n} a_j^{\omega_j}, \prod_{j=1}^{n} b_j^{\omega_j}, \prod_{j=1}^{n} c_j^{\omega_j}, \prod_{j=1}^{n} d_j^{\omega_j} \right]; \right.$$
$$\left(\frac{\varrho \prod_{j=1}^{n} \mu_j^{q\omega_j}}{\prod_{j=1}^{n} \left(1 + (\varrho - 1)\left(1 - \mu_j^{q}\right)\right)^{\omega_j} + (\varrho - 1) \prod_{j=1}^{n} \mu_j^{q\omega_j}} \right)^{\frac{1}{q}},$$
$$\left. \left(\frac{\prod_{j=1}^{n} \left(1 + (\varrho - 1)\nu_j^{q}\right)^{\omega_j} - \prod_{j=1}^{n} \left(1 - \nu_j^{q}\right)^{\omega_j}}{\prod_{j=1}^{n} \left(1 + (\varrho - 1)\nu_j^{q}\right)^{\omega_j} + (\varrho - 1) \prod_{j=1}^{n} \left(1 - \nu_j^{q}\right)^{\omega_j}} \right)^{\frac{1}{q}} \right\rangle \tag{11}$$

Proof The proof is similar to Theorem 1.

- Some particular cases

As like q-ROTrFHWA operator, some particular cases of q-ROTrFHWG operator are discussed based on parameter ϱ.

If $\varrho = 1$, then q-ROTrFHWG operator is converted to q-ROTrF WG (q-ROTrFWG) operator as follows:

$$q\text{-ROTrFWG}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_k)$$
$$= \left\langle \left[\prod_{j=1}^{n} a_j^{\omega_j}, \prod_{j=1}^{n} b_j^{\omega_j}, \prod_{j=1}^{n} c_j^{\omega_j}, \prod_{j=1}^{n} d_j^{\omega_j} \right]; \prod_{j=1}^{n} \mu_j^{\omega_j}, \left(1 - \prod_{j=1}^{n} \left(1 - \nu_j^{q}\right)^{\omega_j}\right)^{\frac{1}{q}} \right\rangle.$$

If $\varrho = 2$, the q-ROTrFHWG operator is converted to q-ROTrF Einstein WG (q-ROTrFEWG) operator as follows:

$$q\text{-ROTrFEWG}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_k) = \left\langle \left[\prod_{j=1}^{n} a_j^{\omega_j}, \prod_{j=1}^{n} b_j^{\omega_j}, \prod_{j=1}^{n} c_j^{\omega_j}, \prod_{j=1}^{n} d_j^{\omega_j} \right]; \right.$$
$$\left. \left(\frac{2 \prod_{j=1}^{n} \mu_j^{q\omega_j}}{\prod_{j=1}^{n} \left(2 - \mu_j^{q}\right)^{\omega_i} + \prod_{j=1}^{n} \mu_j^{q\omega_j}} \right)^{\frac{1}{q}}, \left(\frac{\prod_{j=1}^{n} \left(1 + \nu_j^{q}\right)^{\omega_j} - \prod_{j=1}^{n} \left(1 - \nu_j^{q}\right)^{\omega_j}}{\prod_{j=1}^{n} \left(1 + \nu_j^{q}\right)^{\omega_j} + \prod_{j=1}^{n} \left(1 - \nu_j^{q}\right)^{\omega_j}} \right)^{\frac{1}{q}} \right\rangle.$$

Next, some desirable properties of q-ROTrFEWG operator are also investigated.

Theorem 6 (Idempotency) Let $\widetilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle$ $(j = 1, 2, \ldots, n)$ be a collection of n q-ROTrFNs. If $\widetilde{r}_i = \widetilde{r} = \langle [a, b, c, d]; \mu, \nu \rangle$ for all $j = 1, 2, \ldots, n$, then.

$$q\text{-ROTrFHWG}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) = \widetilde{r}.$$

Proof The proof is similar to Theorem 2.

Theorem 7 (Monotonicity) Let $\widetilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle$ and $\widetilde{r}'_j = \langle [a'_j, b'_j, c'_j, d'_j]; \mu'_j, \nu'_j \rangle$ $(j = 1, 2, \ldots, n)$ be two collections of q-ROTrFNs. If $a_j \le a'_j$, $b_j \le b'_j$, $c_j \le c'_j$, $d_j \le d'_j$, $\mu_j \le \mu'_j$ and $\nu_j \ge \nu'_j$ $\forall j$, then:

$$q - \text{ROTrFHWG}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) \le q - \text{ROTrFHWG}\left(\widetilde{r}'_1, \widetilde{r}'_2, \ldots, \widetilde{r}'_n\right). \tag{12}$$

Proof The proof is similar to Theorem 3.

Theorem 8 (Boundedness) Let $\widetilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle$ $(j = 1, 2, \ldots, n)$ be a collection of q-ROTrFNs, and let:

$$\widetilde{r}_j^- = \left\langle \left[\min_j \{a_j\}, \min_j \{b_j\}, \min_j \{c_j\}, \min_j \{d_j\} \right]; \min_j \{\mu_j\}, \max_j \{\nu_j\} \right\rangle$$

and $\widetilde{r}_j^+ = \langle [\max_j \{a_j\}, \max_j \{b_j\}, \max_j \{c_j\}, \max_j \{d_j\}]; \max_j \{\mu_j\}, \min_j \{\nu_j\} \rangle$, then

$$\widetilde{r}_j^- \le q - \text{ROTrFHWG}(\widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_n) \le \widetilde{r}_j^+. \tag{13}$$

Proof The proof is similar to Theorem 4.

4 A novel MCGDM method based on q-ROTrF environments

In this section, a novel MCGDM approach has been propounded in which evaluation data is presented in the form of q-ROTrFNs.

For a group decision making problem, let $E = \{e^{(1)}, e^{(2)}, \ldots, e^{(k)}\}$ be the set of the DMs with their associated weight vector $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_k)^T$ satisfying $\sum_{i=1}^{k} \Omega_i = 1$, where $\Omega_i \in [0, 1]$. Suppose $A = \{A_1, A_2, \ldots, A_m\}$ be the collection of m alternatives and $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\}$ represents the collection of n criteria along with their weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, satisfying $\sum_{i=1}^{n} \omega_i = 1$, where $\omega_i \in [0, 1]$. The DMs used q-ROTrFNs to express their judgement values and created the q-ROTrF decision matrices (q-ROTrFDM) as $\mathcal{D}^{(l)} = \left[\widetilde{r}_{ij}^{(l)}\right]_{m\times n} = \left[\langle \left[a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}\right]; \mu_{ij}^{(l)}, \nu_{ij}^{(l)} \rangle\right]_{m\times n}$ $(l = 1, 2, \ldots, k)$, where $\widetilde{r}_{ij}^{(l)} = \langle \left[a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}\right]; \mu_{ij}^{(l)}, \nu_{ij}^{(l)} \rangle$ denotes a q-ROTrFN assigned by the DM $e^{(l)}$ for the alternative A_i under the criteria $\mathcal{C}_j$.

The main goal is to choose the best alternative(s) based on the developed method. The computational process is summarized step-by-step as follows.

Step 1. Make decision matrices by identifying and determining the criteria and alternatives, $\mathcal{D}^{(l)} = \left[\widetilde{r}_{ij}^{(l)}\right]_{m\times n}$, where $\widetilde{r}_{ij}^{(l)} = \langle \left[a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}\right]; \mu_{ij}^{(l)}, \nu_{ij}^{(l)} \rangle$ $(l = 1, 2, \ldots, k)$.

Step 2. Normalize $\mathcal{D}^{(l)}$, if required, and obtain $\mathcal{N}^{(l)} = \left[\widetilde{\wp}_{ij}^{(l)}\right]_{m\times n}$ using the following rule:

$$\widetilde{\wp}_{ij}^{(l)} = \begin{cases} \left\langle \left[a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}\right]; \mu_{ij}^{(l)}, \nu_{ij}^{(l)} \right\rangle \text{ when } \mathcal{C}_j \text{ is benefit criteria} \\ \left\langle \left[a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}\right]; \nu_{ij}^{(l)}, \mu_{ij}^{(l)} \right\rangle \text{ when } \mathcal{C}_j \text{ is cost criteria} \end{cases},$$

$$i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n.$$

Step 3. The individual q-ROTrFDMs, $\mathcal{N}^{(l)} = \left[\widetilde{\wp}_{ij}^{(l)}\right]_{m\times n}$ $(l = 1, 2, \ldots, k)$ are aggregated to generate a single q-ROTrFDM, $\mathcal{N} = \left[\widetilde{\wp}_{ij}\right]_{m\times n}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ using q-ROTrFHWA (or q-ROTrFHWG) operator, on the basis of Eq. (3) (or Eq. (10)), as

$$\widetilde{\wp}_{ij} = \left\langle \left[\sum_{l=1}^{k} \Omega^{(l)} a_{ij}^{(l)}, \sum_{l=1}^{k} \Omega^{(l)} b_{ij}^{(l)}, \sum_{l=1}^{k} \Omega^{(l)} c_{ij}^{(l)}, \sum_{l=1}^{k} \Omega^{(l)} d_{ij}^{(l)}\right]; \right.$$

$$\left(\frac{\prod_{l=1}^{k}\left(1 + (\varrho - 1)\left(\mu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j} - \prod_{l=1}^{k}\left(1 - \left(\mu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}}{\prod_{l=1}^{k}\left(1 + (\varrho - 1)\left(\mu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j} + (\varrho - 1)\prod_{l=1}^{k}\left(1 - \left(\mu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}}\right)^{\frac{1}{q}},$$

$$\left.\left(\frac{\varrho\prod_{l=1}^{k}\left(\nu_{ij}^{(l)}\right)^{q\Omega_j}}{\prod_{l=1}^{k}\left(1 + (\varrho - 1)\left(1 - \left(\nu_{ij}^{(l)}\right)^{q}\right)\right)^{\Omega_j} + (\varrho - 1)\prod_{l=1}^{k}\left(\nu_{ij}^{(l)}\right)^{q\Omega_j}}\right)^{\frac{1}{q}} \right\rangle$$

Or

$$\widetilde{\wp}'_{ij} = \left\langle \left[\prod_{l=1}^{k}\left(a_{ij}^{(l)}\right)^{\Omega^{(l)}}, \prod_{l=1}^{k}\left(b_{ij}^{(l)}\right)^{\Omega^{(l)}}, \prod_{l=1}^{k}\left(c_{ij}^{(l)}\right)^{\Omega^{(l)}}, \prod_{l=1}^{k}\left(d_{ij}^{(l)}\right)^{\Omega^{(l)}}\right]; \right.$$

$$\left(\frac{\varrho\prod_{l=1}^{k}\left(\mu_{ij}^{(l)}\right)^{q\Omega_j}}{\prod_{l=1}^{k}\left(1 + (\varrho - 1)\left(1 - \left(\mu_{ij}^{(l)}\right)^{q}\right)\right)^{\Omega_j} + (\varrho - 1)\prod_{l=1}^{k}\left(\mu_{ij}^{(l)}\right)^{q\Omega_j}}\right)^{\frac{1}{q}},$$

$$\left.\left(\frac{\prod_{l=1}^{k}\left(1 + (\varrho - 1)\left(\nu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j} - \prod_{l=1}^{k}\left(1 - \left(\nu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}}{\prod_{l=1}^{k}\left(1 + (\varrho - 1)\left(\nu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j} + (\varrho - 1)\prod_{l=1}^{k}\left(1 - \left(\nu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}}\right)^{\frac{1}{q}} \right\rangle.$$

Step 4. Using the aggregation operator presented in Eq. (3) or (Eq. (10)), aggregate the q-ROTrFN $\widetilde{\wp}_{ij}$ (or $\widetilde{\wp}'_{ij}$) for each alternative A_i $(i = 1, 2, \ldots, m)$ based on criteria as:

$$\widetilde{\wp}_i = q\text{-ROTrFHWA}(\widetilde{\wp}_{i1}, \widetilde{\wp}_{i2}, \ldots, \widetilde{\wp}_{in}),$$
$$\text{or}\ \widetilde{\wp}'_i = q - ROTrFHWG(\widetilde{\wp}_{i1}, \widetilde{\wp}_{i2}, \ldots, \widetilde{\wp}_{in}).$$

Step 5. Determine the score values $S(\widetilde{\wp}_i)$ (or $S\left(\widetilde{\wp}'_i\right)$) of the $\widetilde{\wp}_i$ (or $\widetilde{\wp}'_i$) by Eq. (1) for obtaining ranking results of the alternatives.

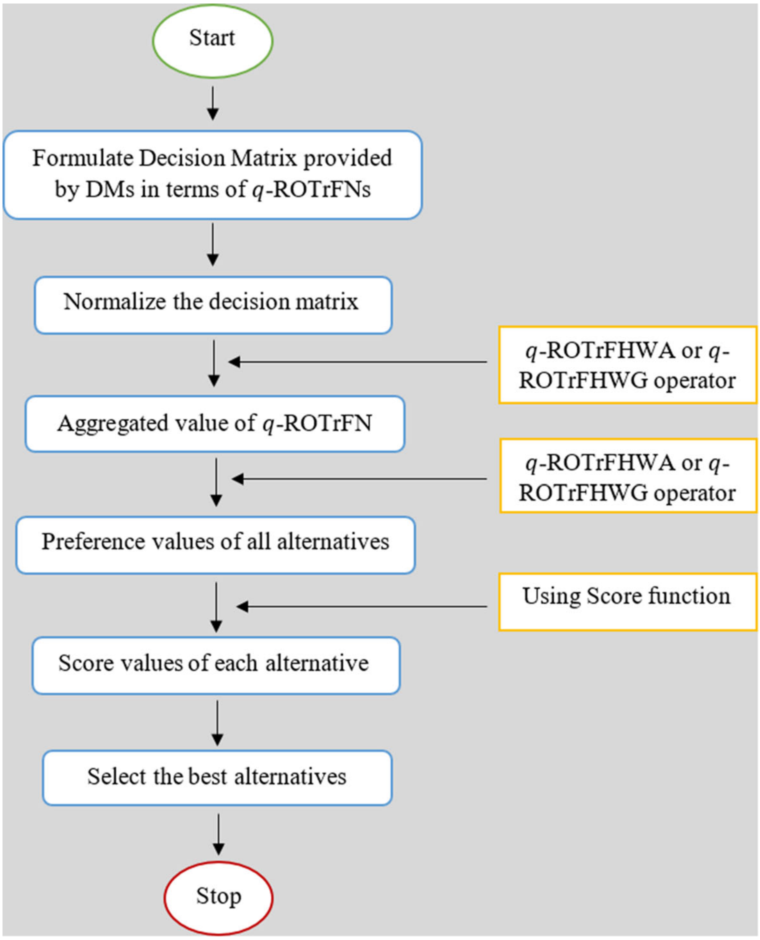

Fig. 2 Flowchart of the developed method

Step 6 Ranked the alternatives (from Definition 4) and select the best one with the highest score value.

The whole method is summarized through the following flow chart which is displayed in Fig. 2.

5 Illustrative examples

In this section, two real-life decision making examples, viz., job selection and supplier selection are investigated and solved to demonstrate the validity and benefits of the proposed MCGDM method.

5.1 Example 1

This example, previously studied by Aydin et al. (2020), is adopted in q-ROTrF environment to illustrate the application of the developed methodology. The human resources department of a company plans to recruit a sales consultant. Three human resource specialists $\left(e^{(1)}, e^{(2)}, e^{(3)}\right)$ evaluated the four candidates A_i $(i = 1, 2, 3, 4)$ based on four criteria: experiences ($\mathcal{C}_1$), competencies ($\mathcal{C}_2$), abilities in foreign language ($\mathcal{C}_3$) and human relationship management ($\mathcal{C}_4$). The weight vector of the criteria is provided as $\omega = (0.15, 0.25, 0.25, 0.35)^{\mathrm{T}}$ and weight vector of the experts is considered as $\Omega = (0.45, 0.25, 0.30)^{\mathrm{T}}$. It is to be noted that first three criteria are benefit type and the last one is cost type. After evaluation of the candidates, the DMs provided their judgement values in the form of q-ROTrFNs and constructed the decision matrices as presented in Table 1, 2 and 3.

To choose the best candidate, the q-ROTrFHWA or q-ROTrFHWG operators are used through the following algorithm.

Step 1 The decision matrices are constructed as discussed above.

Step 2 As experiences, competencies, and abilities in a foreign language are benefited type criteria and human relationship management is cost type criteria, the normalization process is needed. The normalization process is performed using the process as described in Step 2 of the developed methodology and are displayed in Tables 4, 5 and 6.

Step 3 Three DMs give their opinion by three individual q-ROTrFDMs, $\mathcal{N}^{(l)} = \left[\widetilde{\wp}_{ij}^{(l)}\right]_{4\times 4}$ $(l = 1, 2, 3)$. Utilized the q-ROTrFHWA operator based on Eq. (3) to aggregate all individual q-ROTrFDMs $\mathcal{N}^{(l)}$ $(l = 1, 2, 3)$ into the collective q-ROTrFDM, $\mathcal{N} = \left[\widetilde{\wp}_{ij}\right]_{4\times 4}$ where

$$\widetilde{\wp}_{ij} = \left\langle \left[\sum_{l=1}^{3}\Omega^{(l)}a_{ij}^{(l)}, \sum_{l=1}^{3}\Omega^{(l)}b_{ij}^{(l)}, \sum_{l=1}^{3}\Omega^{(l)}c_{ij}^{(l)}, \sum_{l=1}^{3}\Omega^{(l)}d_{ij}^{(l)}\right]; \right.$$

$$\left(\frac{\prod_{l=1}^{3}\left(1+(\varrho-1)\left(\mu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j} - \prod_{l=1}^{3}\left(1-\left(\mu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}}{\prod_{l=1}^{3}\left(1+(\varrho-1)\left(\mu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j} + (\varrho-1)\prod_{l=1}^{3}\left(1-\left(\mu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}}\right)^{\frac{1}{q}},$$

$$\left.\left(\frac{\varrho\prod_{l=1}^{3}\left(\nu_{ij}^{(l)}\right)^{q\Omega_j}}{\prod_{l=1}^{3}\left(1+(\varrho-1)\left(1-\left(\nu_{ij}^{(l)}\right)^{q}\right)\right)^{\Omega_j} + (\varrho-1)\prod_{l=1}^{3}\left(\nu_{ij}^{(l)}\right)^{q\Omega_j}}\right)^{\frac{1}{q}}\right\rangle$$

$$(l = 1, 2, 3; i = 1, 2, 3, 4; j = 1, 2, 3, 4)$$

and is presented in Table 7. It is to be noted that the rung parameter $q = 3$ and Hamacher parameter $\varrho = 3$ is considered.

Step 4 Utilizing q-ROTrFHWA operator which is formulated based on Eq. (3) to aggregate each candidate's collective evaluation values $\widetilde{\wp}_{ij}$ $(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$ w.r.t given criteria to obtain the comprehensive evaluation values $\widetilde{\wp}_i$ $(i = 1, 2, 3, 4)$ is performed.

$$\widetilde{\wp}_1 = \langle[0.4285, 0.5982, 0.7270, 0.8270]; 0.5951, 0.8402\rangle;$$

$$\widetilde{\wp}_2 = \langle[0.3420, 0.5125, 0.6840, 0.8327]; 0.6500, 0.4150\rangle;$$

$$\widetilde{\wp}_3 = \langle[0.3490, 0.4552, 0.6090, 0.7542]; 0.4870, 0.4116\rangle;$$

Table 1 Decision matrix by DM $e^{(1)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\left\langle [0.5, 0.6, 0.8, 0.9]; 0.5, 0.6 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.5, 0.2 \right\rangle$	$\left\langle [0.4, 0.7, 0.8, 0.9]; 0.7, 0.4 \right\rangle$	$\left\langle [0.2, 0.3, 0.4, 0.5]; 0.6, 0.9 \right\rangle$
A_2	$\left\langle [0.2, 0.3, 0.5, 0.6]; 0.3, 0.6 \right\rangle$	$\left\langle [0.1, 0.3, 0.6, 0.9]; 0.7, 0.2 \right\rangle$	$\left\langle [0.4, 0.6, 0.7, 0.9]; 0.3, 0.3 \right\rangle$	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.4, 0.8 \right\rangle$
A_3	$\left\langle [0.3, 0.4, 0.5, 0.9]; 0.4, 0.8 \right\rangle$	$\left\langle [0.2, 0.3, 0.5, 0.7]; 0.6, 0.1 \right\rangle$	$\left\langle [0.3, 0.4, 0.5, 0.6]; 0.4, 0.7 \right\rangle$	$\left\langle [0.4, 0.5, 0.7, 0.8]; 0.3, 0.6 \right\rangle$
A_4	$\left\langle [0.5, 0.7, 0.8, 0.9]; 0.8, 0.4 \right\rangle$	$\left\langle [0.2, 0.4, 0.6, 0.8]; 0.3, 0.8 \right\rangle$	$\left\langle [0.4, 0.5, 0.8, 0.9]; 0.8, 0.5 \right\rangle$	$\left\langle [0.3, 0.5, 0.6, 0.8]; 0.6, 0.4 \right\rangle$

Table 2 Decision matrix by DM $e^{(2)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\left\langle [0.4, 0.6, 0.7, 0.8]; 0.6, 0.7 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.1, 0.6 \right\rangle$	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.3, 0.6 \right\rangle$	$\left\langle [0.4, 0.5, 0.8, 0.9]; 0.4, 0.4 \right\rangle$
A_2	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.6, 0.7 \right\rangle$	$\left\langle [0.5, 0.7, 0.8, 0.9]; 0.3, 0.4 \right\rangle$	$\left\langle [0.1, 0.3, 0.5, 0.6]; 0.9, 0.5 \right\rangle$	$\left\langle [0.3, 0.6, 0.7, 0.8]; 0.5, 0.6 \right\rangle$
A_3	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.6, 0.9 \right\rangle$	$\left\langle [0.2, 0.4, 0.5, 0.7]; 0.4, 0.7 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.2, 0.6 \right\rangle$	$\left\langle [0.4, 0.5, 0.7, 0.8]; 0.2, 0.3 \right\rangle$
A_4	$\left\langle [0.5, 0.6, 0.8, 0.9]; 0.5, 0.6 \right\rangle$	$\left\langle [0.4, 0.5, 0.6, 0.7]; 0.3, 0.8 \right\rangle$	$\left\langle [0.4, 0.6, 0.8, 0.9]; 0.5, 0.6 \right\rangle$	$\left\langle [0.3, 0.4, 0.7, 0.9]; 0.6, 0.4 \right\rangle$

Table 3 Decision matrix by DM $e^{(3)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\left\langle [0.5, 0.6, 0.8, 0.9]; 0.3, 0.7 \right\rangle$	$\left\langle [0.5, 0.7, 0.8, 0.9]; 0.2, 0.4 \right\rangle$	$\left\langle [0.4, 0.7, 0.8, 0.9]; 0.2, 0.4 \right\rangle$	$\left\langle [0.4, 0.7, 0.8, 0.9]; 0.4, 0.4 \right\rangle$
A_2	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.4, 0.6 \right\rangle$	$\left\langle [0.3, 0.5, 0.7, 0.9]; 0.9, 0.5 \right\rangle$	$\left\langle [0.4, 0.5, 0.7, 0.9]; 0.4, 0.3 \right\rangle$	$\left\langle [0.3, 0.5, 0.8, 0.9]; 0.6, 0.3 \right\rangle$
A_3	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.1, 0.3 \right\rangle$	$\left\langle [0.2, 0.3, 0.5, 0.7]; 0.5, 0.8 \right\rangle$	$\left\langle [0.3, 0.4, 0.5, 0.6]; 0.6, 0.7 \right\rangle$	$\left\langle [0.4, 0.5, 0.7, 0.8]; 0.3, 0.4 \right\rangle$
A_4	$\left\langle [0.1, 0.3, 0.5, 0.7]; 0.2, 0.7 \right\rangle$	$\left\langle [0.2, 0.3, 0.4, 0.5]; 0.3, 0.2 \right\rangle$	$\left\langle [0.1, 0.2, 0.4, 0.5]; 0.3, 0.7 \right\rangle$	$\left\langle [0.3, 0.5, 0.6, 0.8]; 0.7, 0.2 \right\rangle$

Table 4 Normalized decision matrix $\mathcal{N}^{(1)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\left\langle [0.5, 0.6, 0.8, 0.9]; 0.5, 0.6 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.5, 0.2 \right\rangle$	$\left\langle [0.4, 0.7, 0.8, 0.9]; 0.7, 0.4 \right\rangle$	$\left\langle [0.2, 0.3, 0.4, 0.5]; 0.9, 0.6 \right\rangle$
A_2	$\left\langle [0.2, 0.3, 0.5, 0.6]; 0.3, 0.6 \right\rangle$	$\left\langle [0.1, 0.3, 0.6, 0.9]; 0.7, 0.2 \right\rangle$	$\left\langle [0.4, 0.6, 0.7, 0.9]; 0.3, 0.3 \right\rangle$	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.8, 0.4 \right\rangle$
A_3	$\left\langle [0.3, 0.4, 0.5, 0.9]; 0.4, 0.8 \right\rangle$	$\left\langle [0.2, 0.3, 0.5, 0.7]; 0.6, 0.1 \right\rangle$	$\left\langle [0.3, 0.4, 0.5, 0.6]; 0.4, 0.7 \right\rangle$	$\left\langle [0.4, 0.5, 0.7, 0.8]; 0.6, 0.3 \right\rangle$
A_4	$\left\langle [0.5, 0.7, 0.8, 0.9]; 0.8, 0.4 \right\rangle$	$\left\langle [0.2, 0.4, 0.6, 0.8]; 0.3, 0.8 \right\rangle$	$\left\langle [0.4, 0.5, 0.8, 0.9]; 0.8, 0.5 \right\rangle$	$\left\langle [0.3, 0.5, 0.6, 0.8]; 0.4, 0.6 \right\rangle$

Table 5 Normalized decision matrix $\mathcal{N}^{(2)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\left\langle [0.4, 0.6, 0.7, 0.8]; 0.6, 0.7 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.1, 0.6 \right\rangle$	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.3, 0.6 \right\rangle$	$\left\langle [0.4, 0.5, 0.8, 0.9]; 0.4, 0.4 \right\rangle$
A_2	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.6, 0.7 \right\rangle$	$\left\langle [0.5, 0.7, 0.8, 0.9]; 0.3, 0.4 \right\rangle$	$\left\langle [0.1, 0.3, 0.5, 0.6]; 0.9, 0.5 \right\rangle$	$\left\langle [0.3, 0.6, 0.7, 0.8]; 0.6, 0.5 \right\rangle$
A_3	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.6, 0.9 \right\rangle$	$\left\langle [0.2, 0.4, 0.5, 0.7]; 0.4, 0.7 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.2, 0.6 \right\rangle$	$\left\langle [0.4, 0.5, 0.7, 0.8]; 0.3, 0.2 \right\rangle$
A_4	$\left\langle [0.5, 0.6, 0.8, 0.9]; 0.5, 0.6 \right\rangle$	$\left\langle [0.4, 0.5, 0.6, 0.7]; 0.3, 0.8 \right\rangle$	$\left\langle [0.4, 0.6, 0.8, 0.9]; 0.5, 0.6 \right\rangle$	$\left\langle [0.3, 0.4, 0.7, 0.9]; 0.4, 0.6 \right\rangle$

Table 6 Normalized decision matrix $\mathcal{N}^{(3)}$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\left\langle [0.5, 0.6, 0.8, 0.9]; 0.3, 0.7 \right\rangle$	$\left\langle [0.5, 0.7, 0.8, 0.9]; 0.2, 0.4 \right\rangle$	$\left\langle [0.4, 0.7, 0.8, 0.9]; 0.2, 0.4 \right\rangle$	$\left\langle [0.4, 0.7, 0.8, 0.9]; 0.4, 0.4 \right\rangle$
A_2	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.4, 0.6 \right\rangle$	$\left\langle [0.3, 0.5, 0.7, 0.9]; 0.9, 0.5 \right\rangle$	$\left\langle [0.4, 0.5, 0.7, 0.9]; 0.4, 0.3 \right\rangle$	$\left\langle [0.3, 0.5, 0.8, 0.9]; 0.3, 0.6 \right\rangle$
A_3	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.1, 0.3 \right\rangle$	$\left\langle [0.2, 0.3, 0.5, 0.7]; 0.5, 0.8 \right\rangle$	$\left\langle [0.3, 0.4, 0.5, 0.6]; 0.6, 0.7 \right\rangle$	$\left\langle [0.4, 0.5, 0.7, 0.8]; 0.4, 0.3 \right\rangle$
A_4	$\left\langle [0.1, 0.3, 0.5, 0.7]; 0.2, 0.7 \right\rangle$	$\left\langle [0.2, 0.3, 0.4, 0.5]; 0.3, 0.2 \right\rangle$	$\left\langle [0.1, 0.2, 0.4, 0.5]; 0.3, 0.7 \right\rangle$	$\left\langle [0.3, 0.5, 0.6, 0.8]; 0.2, 0.7 \right\rangle$

Table 7 Aggregated q-ROTrFDM $\mathcal{N}$ (using q-ROTrFHWA operator)

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\langle$[0.4750, 0.6000, 0.7750, 0.8750]; 0.4893, 0.6544$\rangle$	$\langle$[0.5700, 0.7000, 0.8000, 0.9000]; 0.3862, 0.3274$\rangle$	$\langle$[0.4250, 0.6750, 0.7750, 0.8750]; 0.5426, 0.4443$\rangle$	$\langle$[0.3100, 0.4700, 0.6200, 0.7200]; 0.7360, 0.4826$\rangle$
A_2	$\langle$[0.3950, 0.4950, 0.6400, 0.7400]; 0.4377, 0.6245$\rangle$	$\langle$[0.2600, 0.4600, 0.6800, 0.9000]; 0.7383, 0.3150$\rangle$	$\langle$[0.3250, 0.4950, 0.6500, 0.8250]; 0.6138, 0.3418$\rangle$	$\langle$[0.3900, 0.5700, 0.7300, 0.8300]; 0.6661, 0.4795$\rangle$
A_3	$\langle$[0.4350, 0.5350, 0.6350, 0.8700]; 4332, 0.6475$\rangle$	$\langle$[0.2000, 0.3250, 0.5000, 0.7000]; 0.5313, 0.3210$\rangle$	$\langle$[0.3750, 0.4750, 0.5750, 0.6750]; 0.4545, 0.6746$\rangle$	$\langle$[0.4000, 0.5000, 0.7000, 0.8000]; 0.4953, 0.2712$\rangle$
A_4	$\langle$[0.3800, 0.5550, 0.7100, 0.8400]; 0.6433, 0.5293$\rangle$	$\langle$[0.2500, 0.3950, 0.5400, 0.6850]; 0.3000, 0.5586$\rangle$	$\langle$[0.3100, 0.4350, 0.6800, 0.7800]; 0.6484, 0.5818$\rangle$	$\langle$[0.3000, 0.4750, 0.6250, 0.8250]; 0.3607, 0.6294$\rangle$

Table 8 Aggregated q-ROTrFDM $\mathcal{N}$ (using q-ROTrFHWG operator)

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\langle [0.4729, 0.6000, 0.7737, 0.8739]; 0.4520, 0.6589 \rangle$	$\langle [0.5681, 0.7000, 0.8000, 0.9000]; 0.2565, 0.4215 \rangle$	$\langle [0.4229, 0.6735, 0.7737, 0.8739]; 0.3994, 0.4657 \rangle$	$\langle [0.2928, 0.4395, 0.5856, 0.6908]; 0.6080, 0.5082 \rangle$
A_2	$\langle [0.3497, 0.4600, 0.6262, 0.7281]; 0.3914, 0.6282 \rangle$	$\langle [0.2079, 0.4322, 0.6753, 0.9000]; 0.6380, 0.3831 \rangle$	$\langle [0.2828, 0.4777, 0.6435, 0.8132]; 0.4488, 0.3706 \rangle$	$\langle [0.3775, 0.5681, 0.7286, 0.8288]; 0.5731, 0.4986 \rangle$
A_3	$\langle [0.4158, 0.5196, 0.6221, 0.8688]; 0.2961, 0.7615 \rangle$	$\langle [0.2000, 0.3224, 0.5000, 0.7000]; 0.5152, 0.6205 \rangle$	$\langle [0.3568, 0.4601, 0.5623, 0.6640]; 0.3836, 0.6779 \rangle$	$\langle [0.4000, 0.5000, 0.7000, 0.8000]; 0.4504, 0.2811 \rangle$
A_4	$\langle [0.3085, 0.5224, 0.6948, 0.8346]; 0.4900, 0.5690 \rangle$	$\langle [0.2378, 0.3880, 0.5313, 0.6720]; 0.3000, 0.7148 \rangle$	$\langle [0.2639, 0.3975, 0.6498, 0.7545]; 0.5478, 0.5970 \rangle$	$\langle [0.3000, 0.4729, 0.6236, 0.8239]; 0.3259, 0.6335 \rangle$

$$\widetilde{\wp}_4 = \langle[0.3020, 0.4570, 0.6302, 0.7810]; 0.5050, 0.5842\rangle.$$

Step 5 The score value of each candidate is calculated from each $\widetilde{\wp}_i$ $(i = 1, 2, 3, 4)$ using Eq. (1) and is given as follows:

$$S(\widetilde{\wp}_1) = 0.0766,\ S(\widetilde{\wp}_2) = 0.1204,\ S(\widetilde{\wp}_3) = 0.0248,\ S(\widetilde{\wp}_4) = -0.0383.$$

Step 6 Based on the score values, the ranking of the candidates (using Definition 4) is achieved as follows:

$$A_2 \succ A_1 \succ A_3 \succ A_4$$

From the above ranking, the most suitable candidate for the job is found as A_2.

Further, if the problem is calculated by q-ROTrFHWG operator instead of q-ROTrFHWA operator then the obtained results are discussed through step by step. Since Step 1 and Step 2 are same, so it is not presented here.

Step 3 Utilizing the q-ROTrFHWG operator which is formulated based on Eq. (10) to aggregate all individual q-ROTrFDMs $\mathcal{N}^{(l)}$ $(l = 1, 2, 3)$ into the collective q-ROTrFDM, $\mathcal{N} = \left[\widetilde{\wp}'_{ij}\right]_{4\times4}$ where

$$\widetilde{\wp}'_{ij} = \langle\left[\prod_{l=1}^{3}\left(a_{ij}^{(l)}\right)^{\Omega^{(l)}}, \prod_{l=1}^{3}\left(b_{ij}^{(l)}\right)^{\Omega^{(l)}}, \prod_{l=1}^{3}\left(c_{ij}^{(l)}\right)^{\Omega^{(l)}}, \prod_{l=1}^{3}\left(d_{ij}^{(l)}\right)^{\Omega^{(l)}}\right];$$

$$\left(\frac{\varrho\prod_{l=1}^{3}\left(\mu_{ij}^{(l)}\right)^{q\Omega_j}}{\prod_{l=1}^{3}\left(1+(\varrho-1)\left(1-\left(\mu_{ij}^{(l)}\right)^{q}\right)\right)^{\Omega_j}+(\varrho-1)\prod_{l=1}^{3}\left(\mu_{ij}^{(l)}\right)^{q\Omega_j}}\right)^{\frac{1}{q}},$$

$$\left(\frac{\prod_{l=1}^{3}\left(1+(\varrho-1)\left(\nu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}-\prod_{l=1}^{3}\left(1-\left(\nu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}}{\prod_{l=1}^{3}\left(1+(\varrho-1)\left(\nu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}+(\varrho-1)\prod_{l=1}^{3}\left(1-\left(\nu_{ij}^{(l)}\right)^{q}\right)^{\Omega_j}}\right)^{\frac{1}{q}}\rangle.$$

$(l = 1, 2, 3; i = 1, 2, 3, 4; j = 1, 2, 3, 4)$ and is presented in Table 7. It is noted here that the rung parameter $q = 3$ and Hamacher parameter $\varrho = 3$ is considered.

Step 4 Utilize the q-ROTrFHWG operator which is formulated based on Eq. (10) to aggregate each candidate's collective evaluation values $\widetilde{\wp}'_{ij}$ $(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$ w.r.t given criteria to obtain the comprehensive evaluation values $\widetilde{\wp}'_i$ $(i = 1, 2, 3, 4)$.

Aggregate all the preference values $\widetilde{\wp}'_{ij}$ $(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$.

$$\widetilde{\wp}'_1 = \langle[0.4071, 0.5756, 0.7077, 0.8108]; 0.4258, 0.5093\rangle;$$

$$\widetilde{\wp}'_2 = \langle[0.2991, 0.4922, 0.6775, 0.8258]; 0.5255, 0.4738\rangle;$$

$$\widetilde{\wp}'_3 = \langle[0.3288, 0.4414, 0.5985, 0.7477]; 0.4212, 0.5941\rangle;$$

$$\widetilde{\wp}'_4 = \langle[0.2753, 0.4374, 0.6152, 0.7674]; 0.3882, 0.6393\rangle.$$

Step 5 The score value of each candidate is calculated from each $\widetilde{\wp}'_i$ $(i = 1, 2, 3, 4)$ using Eq. (1) and given as follows:

$$S(\widetilde{\wp}'_1) = -0.0344,\ S(\widetilde{\wp}'_2) = 0.0222,\ S(\widetilde{\wp}'_3) = -0.0714,\ S(\widetilde{\wp}'_4) = -0.1062.$$

Step 6 According to the score value, the ranking of the candidates (using Definition 4) is achieved as follows:

$$A_2 \succ A_1 \succ A_3 \succ A_4$$

Thus the most suitable candidate for the job is A_2.

So, from the above results, it is observed that the ranking of the candidates remains the same for both use of averaging and geometric operators and the ranking is found as $A_2 \succ A_1 \succ A_3 \succ A_4$. So, it is very clear that the candidate A_2 is the best choice over other candidates.

The above results are obtained from a particular value of q and ϱ. The ranking may change for various values of q and ϱ. So, a sensitivity analysis is needed to examine the robustness and stability of the ranking. In the next subsection, the impact of those parameters on decision making results is discussed.

5.1.1 The impact of several parameters on decision making results

Now, the effect of Hamacher parameter, ϱ, rung parameter, q on decision making results achieved through q-ROTrFHWA and q-ROTrFHWG operators are examined. Those parameters are crucial in determining the ranking of the alternatives. By changing the values of parameters, different score values are achieved for each candidate. Taking $q = 3$ and varying the Hamachar parameter ϱ the variation of the score values and ranking of the candidates are found by using proposed operators are presented in Tables 9 and 10 and Figs. 3 and 4.

When the q-ROTrFHWA operator is applied, the score values of the candidates are decreased when the parameter is increased from 0 to 20, as seen in Figs. 3 and 4. The situation is reversed when q-ROTrFHWG operator is used. But in both the cases, the ranking remains the same as $A_2 \succ A_1 \succ A_3 \succ A_4$.

Again, for fixed value of the Hamachar parameter $\varrho = 3$ and varying the rung parameter $q \in [2, 10]$ in q-ROTrFHWA and q-ROTrFHWG operators, the following results are figured out in Figs. 5 and 6, respectively. It is observed when q-ROTrFHWA operator is applied, two different ranking results are occurred i.e., when $q \in [2, 6.4123]$ the final ranking of the candidates is achieved as $A_2 \succ A_1 \succ A_3 \succ A_4$ and for $q \in [6.4123, 10]$ the final ranking of the candidates is achieved as $A_2 \succ A_1 \succ A_4 \succ A_3$.

Further, when q-ROTrFHWG operator is applied, two different ranking results is occurred i.e., when $q \in [2, 6.3671]$ the final ranking of the candidates is achieved as $A_2 \succ A_1 \succ$

Table 9 The effect of the parameter ϱ (fixing $q = 3$) utilising q-ROTrFHWA

Value of ϱ	A_1	A_2	A_3	A_4	Ordering
$\varrho = 1$	0.1112	0.1493	0.0320	−0.0180	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 2$	0.0891	0.1313	0.0274	−0.0310	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 3$	0.0766	0.1204	0.0248	−0.0383	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 4$	0.0681	0.1128	0.0230	−0.0432	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 8$	0.0498	0.0955	0.0187	−0.0537	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 12$	0.0406	0.0865	0.0163	−0.0589	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 16$	0.0348	0.0808	0.0146	−0.0620	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 20$	0.0307	0.0768	0.0134	−0.0642	$A_2 \succ A_1 \succ A_3 \succ A_4$

Table 10 The effect of the parameter ϱ (fixing $q = 3$) utilising q-ROTrFHWG

Value of ϱ	A_1	A_2	A_3	A_4	Ordering
$\varrho = 1$	−0.0445	0.0096	−0.0912	−0.1162	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 2$	−0.0381	0.0178	−0.0792	−0.1100	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 3$	−0.0344	0.0222	−0.0714	−0.1062	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 4$	−0.0317	0.0252	−0.0658	−0.1036	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 8$	−0.0254	0.0316	−0.0521	−0.0974	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 12$	−0.0221	0.0349	−0.0444	−0.0940	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 16$	−0.0199	0.0370	−0.0393	−0.0919	$A_2 \succ A_1 \succ A_3 \succ A_4$
$\varrho = 20$	−0.0183	0.0385	−0.0356	−0.0903	$A_2 \succ A_1 \succ A_3 \succ A_4$

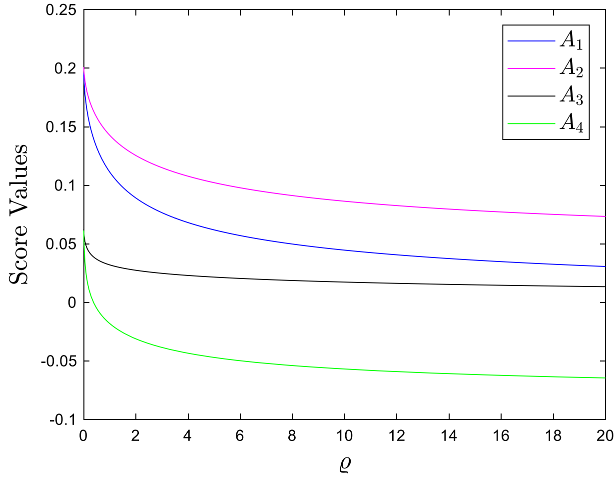

Fig. 3 Score value of the candidates utilizing q-ROTrFHWA operator ($q = 3$)

$A_3 \succ A_4$ and for $q \in [6.3671, 10]$. The final ranking of the candidates is achieved as $A_2 \succ A_1 \succ A_4 \succ A_3$.

It is found that both the operators identify the best candidate as A_2 for required vacancy.

So, it is clear that the Hamacher parameter and rung parameter plays an important role in ranking of alternatives. All parameters confirm the stability of the best candidate. The above analysis ensures that A_2 is the best candidate among the other three.

5.2 Example 2

Another MCGDM problem presented by Zhao et al. (2017) is further adopted here in q-ROTrF context. The problem is related to finding best green supplier for one of the critical components in the automobile manufacturing process. It is assumed that company establishes a panel with three DMs consisting of production department manager $\left(e^{(1)}\right)$, quality inspection department manager $\left(e^{(2)}\right)$ and purchasing department manager $\left(e^{(3)}\right)$ whose weight vector is $\Omega =$

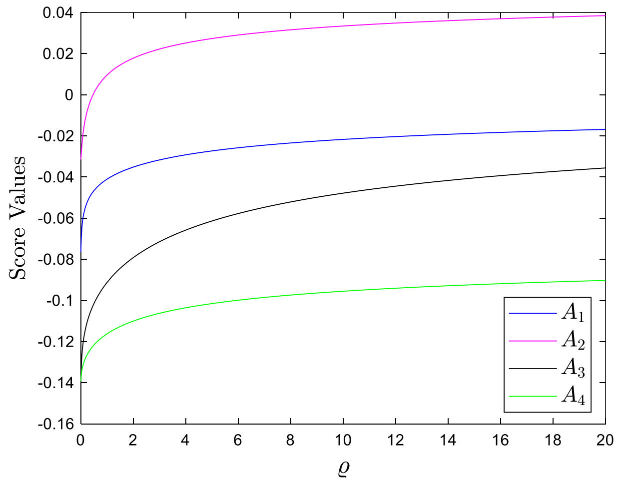

Fig. 4 Score value of the candidates utilizing q-ROTrFHWG operator ($q = 3$)

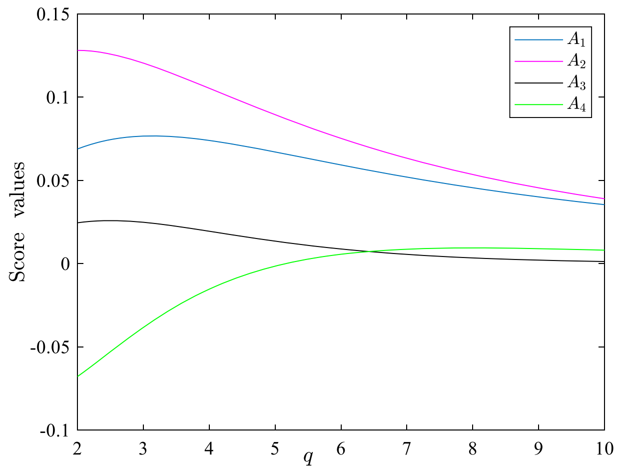

Fig. 5 Score value of the candidates utilizing q-ROTrFHWA operator ($\varrho = 3$)

$(0.35, 0.4, 0.25)^{\mathrm{T}}$. The company has to select the best supplier among five suppliers, A_i $(i = 1, 2, 3, 4, 5)$ for their company w.r.t. four criteria: product quality ($\mathcal{C}_1$), technology capability ($\mathcal{C}_2$), pollution control ($\mathcal{C}_3$), environment management ($\mathcal{C}_4$), whose weight vector is given by $\omega = (0.2, 0.1, 0.3, 0.4)^T$). Based on their judgement values, the normalized decision matrices, $\mathcal{N}^{(l)} = \left(\widetilde{\wp}_{ij}^{(l)}\right)_{5\times 4}$ $(l = 1, 2, 3)$ are given in Table 11, 12 and 13.

Table 14 shows the final score values of the various suppliers obtained through the proposed method under consideration of the rung parameter $q = 3$ and Hamacher parameter $\varrho = 3$.

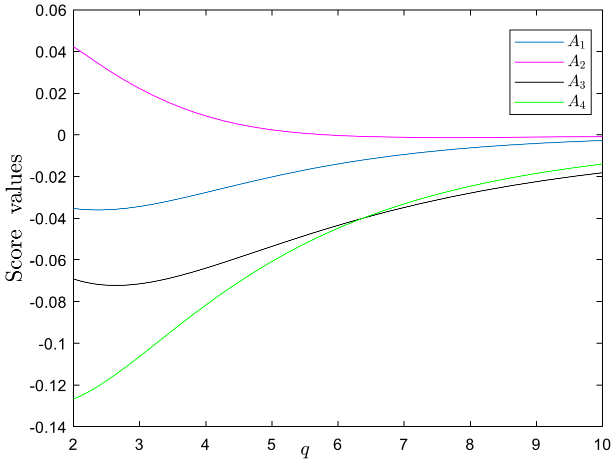

Fig. 6 Score value of the candidates utilizing q-ROTrFHWG operator ($\varrho = 3$)

5.2.1 Result and discussion

The achieved results are explained by adjusting the Hamacher parameter, ϱ, and the rung parameter, q, at the defined intervals using the q-ROTrFHWA and q-ROTrFHWG operators, as shown in Figs. 7, 8, 9 and 10.

Figure 7 shows the graphical representation of the score values of the different suppliers achieved by using q-ROTrFHWA operator with constant values of $\varrho = 3$ and varying the rung parameter, q, between 1 and 10. It is noticed that several ranking outcomes of the suppliers are achieved when q varies from 1 to 10.

When $q \in [1, 2.377]$ the ordering of the suppliers is achieved as $A_2 > A_5 > A_3 > A_4 > A_1$. Again for $q \in [2.377, 3.598]$ the ordering of the suppliers is achieved as $A_2 > A_5 > A_4 > A_3 > A_1$ and for $q \in [3.598, 10]$ the ordering of the suppliers is found as $A_2 > A_4 > A_5 > A_3 > A_1$.

Moreover, by changing the rung parameter, q, between 1 and 10, and employing the q-ROTrFHWG operator with constant values of $\varrho = 3$, Fig. 8 depicts the graphical interpretation of score values of the suppliers.

As q increases from 1 to 10, multiple ranking of the suppliers' outcomes is achieved, as shown in Fig. 8.

When $q \in [1, 4.15]$ the ordering of the suppliers is achieved as $A_2 > A_5 > A_3 > A_4 > A_1$.

And for $q \in [4.15, 10]$ the ordering of the suppliers is achieved as $A_2 > A_3 > A_5 > A_4 > A_1$.

Figure 9 signifies the graphical interpretation of score values of the suppliers by varying the Hamacher parameter, ϱ, between 0 and 10, using q-ROTrFHWA operator with fixed values of $q = 3$.

From Fig. 9, it is observed that the score values of the suppliers are decreasing and many ranking results are obtained, as ϱ changes from 0 to 10.

When $\varrho \in [0, 0.823]$, the ordering of the suppliers is achieved as $A_2 > A_4 > A_5 > A_3 > A_1$.

Table 11 Normalized decision matrix by DM e_1

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.5, 0.4 \right\rangle$	$\left\langle [0.1, 0.2, 0.3, 0.4]; 0.6, 0.3 \right\rangle$	$\left\langle [0.5, 0.6, 0.8, 0.9]; 0.3, 0.6 \right\rangle$	$\left\langle [0.4, 0.5, 0.6, 0.7]; 0.2, 0.7 \right\rangle$
A_2	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.7, 0.3 \right\rangle$	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.7, 0.2 \right\rangle$	$\left\langle [0.4, 0.5, 0.7, 0.8]; 0.7, 0.2 \right\rangle$	$\left\langle [0.5, 0.6, 0.7, 0.9]; 0.4, 0.5 \right\rangle$
A_3	$\left\langle [0.1, 0.2, 0.4, 0.5]; 0.6, 0.4 \right\rangle$	$\left\langle [0.2, 0.3, 0.5, 0.6]; 0.5, 0.4 \right\rangle$	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.5, 0.3 \right\rangle$	$\left\langle [0.3, 0.5, 0.7, 0.9]; 0.2, 0.3 \right\rangle$
A_4	$\left\langle [0.3, 0.4, 0.5, 0.6]; 0.8, 0.1 \right\rangle$	$\left\langle [0.1, 0.3, 0.4, 0.5]; 0.6, 0.3 \right\rangle$	$\left\langle [0.1, 0.3, 0.5, 0.7]; 0.3, 0.4 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.2, 0.6 \right\rangle$
A_5	$\left\langle [0.2, 0.3, 0.4, 0.5]; 0.6, 0.2 \right\rangle$	$\left\langle [0.3, 0.4, 0.5, 0.6]; 0.4, 0.3 \right\rangle$	$\left\langle [0.2, 0.3, 0.4, 0.5]; 0.7, 0.1 \right\rangle$	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.1, 0.3 \right\rangle$

Table 12 Normalized decision by DM e_2

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\langle [0.4, 0.5, 0.6, 0.7]; 0.4, 0.3 \rangle$	$\langle [0.1, 0.2, 0.3, 0.4]; 0.5, 0.2 \rangle$	$\langle [0.4, 0.5, 0.7, 0.8]; 0.2, 0.5 \rangle$	$\langle [0.3, 0.4, 0.5, 0.6]; 0.1, 0.6 \rangle$
A_2	$\langle [0.5, 0.6, 0.7, 0.8]; 0.6, 0.2 \rangle$	$\langle [0.4, 0.5, 0.6, 0.7]; 0.6, 0.1 \rangle$	$\langle [0.3, 0.4, 0.6, 0.7]; 0.6, 0.1 \rangle$	$\langle [0.3, 0.4, 0.6, 0.8]; 0.3, 0.4 \rangle$
A_3	$\langle [0.1, 0.2, 0.3, 0.4]; 0.5, 0.3 \rangle$	$\langle [0.1, 0.2, 0.4, 0.5]; 0.4, 0.3 \rangle$	$\langle [0.4, 0.5, 0.6, 0.7]; 0.4, 0.2 \rangle$	$\langle [0.2, 0.4, 0.6, 0.8]; 0.5, 0.2 \rangle$
A_4	$\langle [0.2, 0.3, 0.4, 0.5]; 0.7, 0.1 \rangle$	$\langle [0.1, 0.2, 0.3, 0.5]; 0.5, 0.2 \rangle$	$\langle [0.1, 0.2, 0.4, 0.6]; 0.2, 0.3 \rangle$	$\langle [0.5, 0.6, 0.7, 0.8]; 0.1, 0.5 \rangle$
A_5	$\langle [0.1, 0.2, 0.3, 0.4]; 0.5, 0.1 \rangle$	$\langle [0.2, 0.3, 0.4, 0.5]; 0.3, 0.2 \rangle$	$\langle [0.1, 0.2, 0.3, 0.4]; 0.6, 0.2 \rangle$	$\langle [0.4, 0.5, 0.6, 0.7]; 0.4, 0.2 \rangle$

Table 13 Normalized decision by DM e_3

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.4, 0.5 \right\rangle$	$\left\langle [0.2, 0.3, 0.4, 0.5]; 0.5, 0.4 \right\rangle$	$\left\langle [0.6, 0.7, 0.9, 1.0]; 0.2, 0.7 \right\rangle$	$\left\langle [0.5, 0.6, 0.7, 0.8]; 0.1, 0.8 \right\rangle$
A_2	$\left\langle [0.7, 0.8, 0.9, 1.0]; 0.6, 0.4 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.6, 0.3 \right\rangle$	$\left\langle [0.5, 0.6, 0.8, 0.9]; 0.6, 0.3 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 1.0]; 0.3, 0.6 \right\rangle$
A_3	$\left\langle [0.2, 0.3, 0.5, 0.6]; 0.5, 0.5 \right\rangle$	$\left\langle [0.3, 0.4, 0.6, 0.7]; 0.4, 0.5 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.4, 0.4 \right\rangle$	$\left\langle [0.4, 0.6, 0.8, 1.0]; 0.5, 0.4 \right\rangle$
A_4	$\left\langle [0.4, 0.5, 0.6, 0.7]; 0.7, 0.2 \right\rangle$	$\left\langle [0.2, 0.4, 0.5, 0.6]; 0.5, 0.4 \right\rangle$	$\left\langle [0.2, 0.4, 0.6, 0.8]; 0.2, 0.5 \right\rangle$	$\left\langle [0.7, 0.8, 0.9, 1.0]; 0.6, 0.3 \right\rangle$
A_5	$\left\langle [0.3, 0.4, 0.5, 0.6]; 0.5, 0.3 \right\rangle$	$\left\langle [0.4, 0.5, 0.6, 0.7]; 0.3, 0.4 \right\rangle$	$\left\langle [0.3, 0.4, 0.5, 0.6]; 0.6, 0.2 \right\rangle$	$\left\langle [0.6, 0.7, 0.8, 0.9]; 0.4, 0.4 \right\rangle$

Table 14 Score values obtained through of proposed method

Proposed method	Score values					Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	
q-ROTrFHWA	−0.0665	0.0958	0.0370	0.0466	0.0553	$A_2 > A_5 > A_4 > A_3 > A_1$
q-ROTrFHWG	−0.1108	0.0412	0.0181	−0.0224	0.0199	$A_2 > A_5 > A_3 > A_4 > A_1$

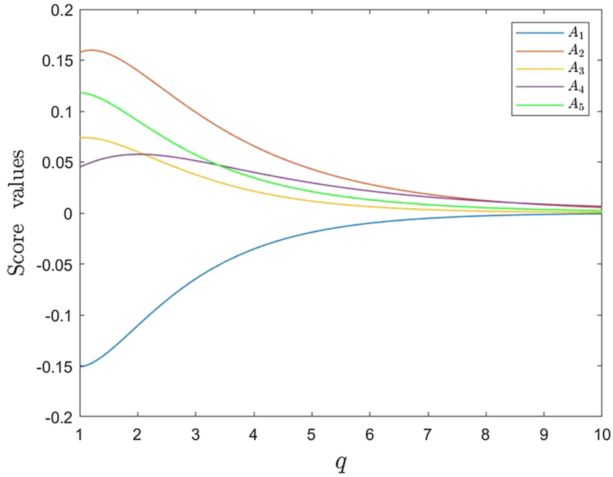

Fig. 7 Impact of rung parameter (q) on q-ROTrFHWA operator

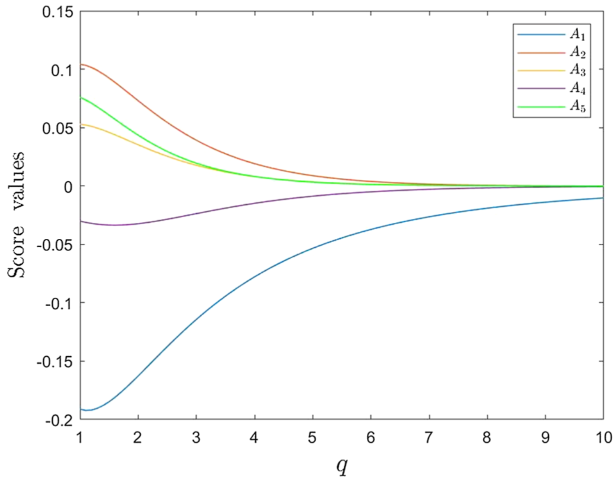

Fig. 8 Impact of rung parameter (q) on q-ROTrFHWG operator

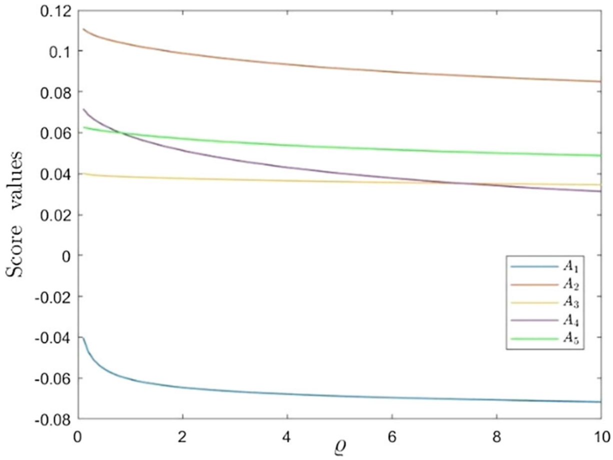

Fig. 9 Impact of Hamacher parameter (ϱ) on q-ROTrFHWA operator

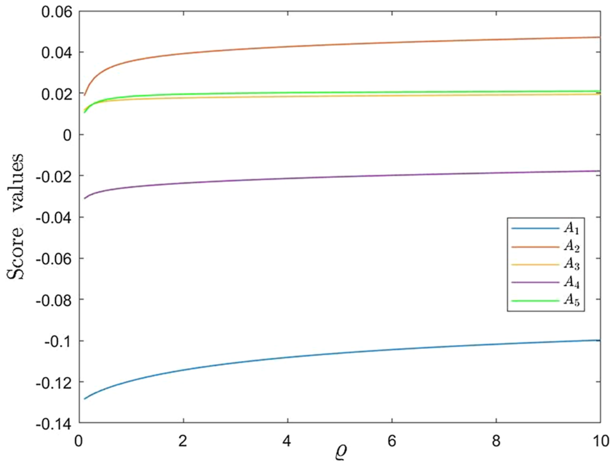

Fig. 10 Impact of Hamacher parameter (ϱ) on q-ROTrFHWG operator

Again for $\varrho \in [0.823, 7.316]$, the ordering of the suppliers is achieved as $A_2 > A_5 > A_4 > A_3 > A_1$. And when $\varrho \in [7.316, 10]$ the ordering of the suppliers is achieved as $A_2 > A_5 > A_3 > A_4 > A_1$.

Figure 10 signifies the graphical representation of score values of the suppliers by adjusting the Hamacher parameter, ϱ, between 0 and 10, using q-ROTrFHWG operator with fixed values of $q = 3$.

The score values of the suppliers are increasing in Fig. 10, and two ranking outcomes are achieved as ϱ changes from 0 to 10.

When $\varrho \in [0, 0.265]$ the ordering of the suppliers is achieved as $A_2 > A_3 > A_5 > A_4 > A_1$. Again for $\varrho \in [0.265, 10]$ the ordering of the suppliers is achieved as $A_2 > A_5 > A_3 > A_4 > A_1$.

So, it is observed that in all cases, A_2 is the best green supplier and A_1 is the worst green supplier.

6 Comparative analysis

In this section, the new approach is compared to certain existing ones.

Firstly, the results of Example 1 are compared with Aydin's (Aydin et al. 2020) methodologies. Table 15 represents the ranking using Aydin's Technique (Aydin et al. 2020) and proposed method. Both Aydin's (Aydin et al. 2020) and suggested approaches result in the same ranking. However, almost everywhere, the suggested method's difference between two successive candidates (rank wise) is larger than current Aydin's techniques (Aydin et al. 2020). This shows the betterment of the proposed method. The bar diagram of the score value difference is presented in Fig. 11.

Next, example 2 is compared with various operators, including ITFWAA (Jianqiang and Zhong 2009), ITFWG (Wu and Cao 2013), ITFEWA and ITFEWG (Zhao et al. 2017), PTFWA (Shakeel et al. 2019a, b), and PTFEWG (Shakeel et al. 2019a, b). Table 16 shows the score values and ranking of the suppliers.

Table 16 demonstrates that the ranks of the suppliers obtained by various operators are almost similar to the suggested operators, implying that the proposed ranking approach is

Table 15 Comparison of score value and ranking for Example 1

Method	Score values				Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	
Aydin et al. (2020) method	0.100	0.143	0.093	0.048	$A_2 > A_1 > A_3 > A_4$
q-ROTrFHWA	0.0766	0.1204	0.0248	−0.0383	$A_2 > A_1 > A_3 > A_4$
q-ROTrFHWG	−0.0344	0.0222	−0.0714	−0.1062	$A_2 > A_1 > A_3 > A_4$

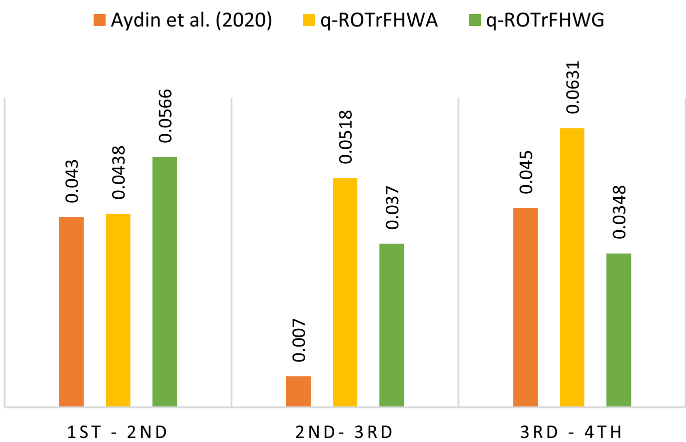

Fig. 11 Difference between ordered alternatives' score values

Table 16 Comparison based on Example 2

Operators	Score value					Ranking results
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	
ITFWAA (Jianqiang and Zhong 2009)	−0.1400	0.1674	0.0771	−0.0620	0.1240	$A_2 > A_5 > A_3 > A_4 > A_1$
ITFWG (Wu and Cao 2013)	−0.1987	0.0927	0.0489	−0.0421	0.0686	$A_2 > A_5 > A_3 > A_4 > A_1$
ITFEWA (Zhao et al. 2017)	−0.1512	0.1577	0.0738	0.0450	0.1180	$A_2 > A_5 > A_3 > A_4 > A_1$
ITFEWG (Zhao et al. 2017)	−0.1911	0.1042	0.0529	−0.0301	0.0759	$A_2 > A_5 > A_3 > A_4 > A_1$
PTFWA (Shakeel et al. 2019a, b)	−0.1026	0.1470	0.0620	0.0699	0.0949	$A_2 > A_5 > A_4 > A_3 > A_1$
PTFEWG (Shakeel et al. 2019a, b)	−0.1627	0.0730	0.0353	−0.0323	0.0435	$A_2 > A_5 > A_3 > A_4 > A_1$
q-ROTrFHWA	−0.0665	0.0958	0.0370	0.0466	0.0553	$A_2 > A_5 > A_4 > A_3 > A_1$
q-ROTrFHWG	−0.1108	0.0412	0.0181	−0.0224	0.0199	$A_2 > A_5 > A_3 > A_4 > A_1$

effective. Furthermore, all of the following operators may be generated from the suggested operator by taking into account the specific value of the Hamacher parameter, ϱ, and the rung parameter, q.

If $q = 1$ and $\varrho = 1$, then q-ROTrFHWA (or q-ROTrFHWA) reduces to ITFWAA (Jianqiang and Zhong 2009) (or ITFWG (Wu and Cao 2013)) operator. If $q = 1$ and $\varrho = 2$, then q-ROTrFHWA (or q-ROTrFHWA) reduces to ITFEWA (Zhao et al. 2017) (or ITFEWG (Zhao et al. 2017)) operator. If $q = 2$ and $\varrho = 1$, then q-ROTrFHWA reduces to PTFWA (Shakeel et al. 2019a, b) operator. If $q = 2$ and $\varrho = 2$, then q-ROTrFHWG reduces to PTFEWG (Shakeel et al. 2019a, b) operator. So, the proposed operator can cover all the above-mentioned operators.

Moreover, when this problem is executed with q-ROTrFHWA or q-ROTrFHWG operator, the suppliers are ranked in several ways. As a consequence, by modifying the associated parameters, the proposed approach may capture the idea of a large quantity of data.

7 Conclusion

The q-ROTrFNs is a combination of q-ROFN and TrFN which is a powerful tool to solve decision making problems involving uncertainty. Hamacher t-norms and t-conorms possess a more generalised structure that successfully integrates complicated data. In this paper Hamacher based new operational laws are introduced, using those laws two aggregation operators viz., Hamacher weighted averaging q-ROTrFHWA, q-rung orthopair trapezoidal fuzzy Hamacher weighted geometric q-ROTrFHWG operators are introduced in q-ROTrF context. The proposed operators satisfy three fundamental properties of an aggregation operator, viz., idempotency, monotonicity and boundedness. Further, a novel MCGDM technique is presented, with the DMs' assessment values in the form of q-ROTrFN. This is the first paper on Hamacher averaging and geometric operators with q-rung orthopair trapezoidal fuzzy numbers. These developments are due to two factors (1) In comparison to other forms of fuzzy sets, q-ROTrFSs include more information, and (2) If there are outliers in the data, Hamacher averaging and the Hamacher geometric operator can catch the value of the data. Therefore, combining Hamacher averaging or geometric operator and q-rung orthopair trapezoidal fuzzy number provides advantages in the MCGDM problem. A sales consultant selection problem is solved to demonstrate the applicability of the proposed methodology. Three DMs' opinions were evaluated in order to choose the best candidate among four candidates. It demonstrates that the proposed methodology is capable of dealing with the MCGDM problem. An empirical application validates the proposed approach to selecting the best green supplier in an automobile manufacturing company. Further, considering an example based on supplier section problems, the effectiveness and purpose of these studies have been shown. Brief comparative studies between the existing method and the proposed method are discussed and it is shown that the proposed method is more reliable and effective than the existing method. In future, the following topics may be explored: Hamachar operation based aggregation operators on probabilistic q-ROTrFSs, neutrosophic fuzzy sets (Jana et al. 2020), Linguistic q-ROTrF power aggregation operators, Archimedean t-norms and t-conorms based aggregation operator on q-ROTrFSs, Some correlation coefficients on q-ROTrFSs, Some similarity measure on q-ROTrFSs.

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Appendix

X	Universal set
$\mathcal{P}$	q-ROFS
q	Rung parameter
$\mu_{\mathcal{P}}$	Membership degree of $\mathcal{P}$
$\nu_{\mathcal{P}}$	Non membership degree of $\mathcal{P}$
$\widetilde{\wp}$	q-ROFN
μ	Membership value of q-ROFN
ν	Non-membership value of q-ROFN
$\widetilde{r}$	A q-ROTrFN
$\mu_{\widetilde{r}}$	Membership value of q-ROTrFN $\widetilde{r}$
$\nu_{\widetilde{r}}$	Non-membership value of q-ROTrFN $\widetilde{r}$

$S(\widetilde{r})$	Score function of q-ROTrFN $\widetilde{r}$
$A(\widetilde{r})$	Accuracy function of q-ROTrFN $\widetilde{r}$
ϱ	Hamacher parameter
T_{ϱ}^{H}	Hamacher t-norm
S_{ϱ}^{H}	Hamacher t-conorm
λ	A scaler
ω	Weight vector

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RESEARCH ARTICLE

Development of q-Rung Orthopair Trapezoidal Fuzzy Einstein Aggregation Operators and Their Application in MCGDM Problems

Arun Sarkar[1,*], Animesh Biswas[2] and Moitreyee Kundu[2]

[1]*Department of Mathematics, Heramba Chandra College, India*

[2]*Department of Mathematics, University of Kalyani, India*

Abstract: Compared to previous extensions, the q-rung orthopair fuzzy sets are superior to intuitionistic ones and Pythagorean ones because they allow decision-makers to use a more extensive domain to present judgment arguments. The purpose of this study is to explore the multicriteria group decision-making (MCGDM) problem with the q-rung orthopair trapezoidal fuzzy (q-ROTrF) context by employing Einstein t-conorms and t-norms. Firstly, some arithmetical operations for q-ROTrF numbers, such as Einstein-based sum, product, scalar multiplication, and exponentiation, are introduced based on Einstein t-conorms and t-norms. Then, Einstein operations-based averaging and geometric aggregation operators (AOs), viz., q-ROTrF Einstein weighted averaging and weighted geometric operators, are developed. Further, some prominent characteristics of the suggested operators are investigated. Then, based on defined AOs, a MCGDM model with q-ROTrF numbers is developed. In accordance with the proposed operators and the developed model, two numerical examples are illustrated. The impacts of the rung parameter on decision results are also analyzed in detail to reflect the suitability and supremacy of the developed approach.

Keywords: Multicriteria group decision-making, q-rung orthopair trapezoidal fuzzy number, Einstein operations, weighted averaging and weighted geometric aggregating operators

1. Introduction

Multicriteria group decision-making (MCGDM) is a technique for choosing the most desirable alternatives from a collection of finite alternatives based on a group of decision-makers' (DMs) aggregate assessment values. However, because it incorporates the complexity of human cognitive thinking, the MCGDM process tends to be vague and imprecise, making it difficult for DMs to provide precise evaluations or preference information during the evaluation process. To cope with such issues, Atanassov's intuitionistic fuzzy set (IFS) (Atanassov, 1986) might be considered an appealing method for dealing with data fuzziness and inaccuracy. IFS is characterized by membership and nonmembership degrees in which their sum is not beyond one. Despite numerous IFS's advantages, there may be situations in which the sum of membership and nonmembership degrees is greater than 1. Yager (2013a) and Yager (2013b) introduced the Pythagorean fuzzy set (PFS) to address these issues, ensuring that the squared sum of its degree of membership and degree of nonmembership is ≤1. As a result, PFS have a more extensive region to model real-life situations than IFSs. Wang and Garg (2021) introduced Archimedean t-conorm and t-norm-based Pythagorean fuzzy interactive weighted averaging (WA) and weighted geometric (WG) operators as novel interaction Pythagorean operators. After the inception of PFS, it has been broadly studied and employed by scholars (Fei & Deng, 2020; Zeng et al., 2016; Sarkar & Biswas, 2019).

However, in real-world situations, the square sum of the degrees of membership and degree of nonmembership is more than 1. In such situations, PFS and IFS are inadequate for describing DMs' evaluation information. To address this flaw, Yager (2016) redefined the notion of q-rung orthopair fuzzy (q-ROF) set (q-ROFS) as a generalization of PFS and IFS, wherein the sum of q^{th} power of membership and nonmembership degrees is less than or equal to unity. It is important to keep in mind that the space of admissible orthopairs expands as the rung q increases, making q-ROFs better suited to unpredictable environments. Based on q-ROF environment, Peng et al. (2021) defined entropy measure, distance measure, and similarity measure and solved decision-making problems utilizing those measures. Under q-ROF context, Riaz et al. (2021a) established numerous WA and WG aggregation operators (AOs),

*Corresponding author: Arun Sarkar, Department of Mathematics, Heramba Chandra College, India. Email: asarkarmth@gmail.com

viz., q-ROF fuzzy interaction-ordered and hybrid averaging AOs as well as geometric versions of these AOs. Zeng et al. (2021) defined induced weighted logarithmic-based two distance measures of q-ROFSs. Recently, Alkan and Kahraman (2021) developed two different TOPSIS methods under the q-ROF context and applied to determine the most appropriate strategy. Ever since q-ROFSs' appearance, many studies (Liu et al., 2018; Liu & Wang, 2020; Sarkar & Biswas, 2021) have been conducted on decision-making methods under q-ROF environment.

The use of trapezoidal fuzzy numbers (TrFNs) (Abbasbandy & Hajjari, 2009) has also become increasingly widespread as a starting point for developing fuzzy sets. TrFN is the best fit for conveying the uncertainty of the alternative. If the alternative's uncertainty is expressed as an interval, TrFN is the best choice for representing it. Gupta et al. (2021) presented the notion of q-rung orthopair TrFNs (q-ROTrFNs), which was inspired by the ideas of q-ROFS (Yager, 2016) and TrFN (Wang & Zhang, 2009). For q-ROTrFNs, Wan et al. (2021a) established a novel ranking algorithm and Hamming distance measure. They also recommended using q-ROTrFNs for developing a new TODIM group decision-making approach.

1.1 Motivations

It is worth noting that operational regulations play a crucial role in data integration. Gupta et al. (2021) proposed the basic operations laws and defined WA and WG AOs for q-ROTrFNs and moreover developed a TOPSIS approach for solving the MAGDM problem. As an alternative to algebraic sum and product, Einstein-based t-norm and t-conorm provide the best approximation for sum and product of q-ROTrFNs. The AOs are most typically employed to aggregate each individual preference into the overall preference information and generate a collective preference value for each alternative. There appear to be limited studies into aggregation approaches for aggregating a collection of q-ROTrF data in the literature. From the above motivation, the aim of this research is to design some information AOs using Einstein operations on q-ROTrFNs.

1.2 Contributions

In the present paper, we will research some Einstein-based operational laws of the q-ROTrNs. Moreover, as the applications, we give two novel AOs. As can be summarized from the motivations above, the contributions are shown in the following:

- Using Einstein t-conorm and t-norms, the current study prolonged the concept of aggregating distinct q-ROTrFNs. For this purpose, firstly Einstein operating laws for q-ROTrFNs have been devised.
- Using defined operational rules, a set of q-ROTrF Einstein WA (q-ROTrFEWA) and q-ROTrF Einstein WG (q-ROTrFEWG) operators have been proposed for integrating q-ROTrF information. Some desirable properties of these developed operators are also investigated in detail.
- A novel MCGDM method based on the proposed operators has been described under q-ROTrF context.
- By comparing the proposed approach to the existing method, it is determined that the method proposed in this study has proven to be useful in q-ROTrFNs research.

The following is the outline of the paper: Section 2 briefly recalls fundamental conceptions related to q-ROFS, q-ROTrFN, and Einstein operations. Based on Einstein operations, some basic operational rules for q-ROTrFN are defined in Section 3. To aggregate q-ROTrFNs, Section 4 introduces some operators based on Einstein operations, viz., q-ROTrFEWA and q-ROTrFEWG operators. Further, some characteristics of these developed operators are also exhibited in this section. Section 5 illustrates a MCGDM approach utilizing the developed AOs. Utilizing the proposed approach, two numerical examples have been solved in Section 6, and comparative and sensitivity analyses are also presented here. Finally, in Section 7, an overall summary of the current study is depicted.

2. Preliminaries

Several basic principles that will be used throughout the article are briefly reviewed in this section. In order to better understand this paper, we will introduce some basic and useful concepts of q-ROFSs (Yager, 2016), q-ROTrFN (Gupta et al., 2021), and Einstein operations (Klement et al., 2004) in this section.

2.1 q-ROFS

The notion of q-ROFS is introduced by Yager (2016). In the following, some basic notions pertaining to q-ROF sets are presented from Yager (2016).

Definition 2.1. (Yager, 2016) On a universal set X, a q-ROFS, $\mathcal{P}$ is presented by:

$$\mathcal{P} = \{(x, \mu_{\mathcal{P}}(x), \nu_{\mathcal{P}}(x))|x \in X\},$$

where the values of $\mu_{\mathcal{P}}$ and $\nu_{\mathcal{P}}$ that lie in the closed unit interval designate membership and nonmembership values, respectively, following the requirement that

$$((\mu_{\mathcal{P}}(x))^q + (\nu_{\mathcal{P}}(x))^q) \in [0, 1], \text{ where rung parameter } q \geq 1.$$

For convenience, Yager (2016) named the pair $(\mu_{\mathcal{P}}(x), \nu_{\mathcal{P}}(x))$ as a q-ROF number (q-ROFN) and symbolized it by $\tilde{\wp} = (\mu, \nu)$.

2.2 q-ROTrFN

The concept of q-ROTrFN suggested by Gupta et al. (2021) as a generalization of intuitionistic TrFN and Pythagorean fuzzy number is as follows:

Definition 2.2. (Gupta et al., 2021) Suppose X be a fixed set. A q-ROFN $\tilde{R}$ is said to be q-ROTrFN explained on [0,1], denoted by $\tilde{R} = \langle([a, b, c, d]; \gamma_{\tilde{R}}), ([a_1, b, c, d_1]; \delta_{\tilde{R}})\rangle$ if

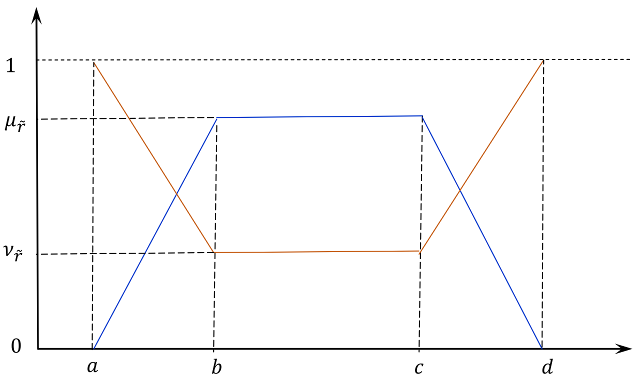
$$\gamma_{\tilde{R}}(x) = \begin{cases} \frac{(x-a)\mu_{\tilde{R}}}{(b-a)}, & a \leq x \leq b \\ \mu_{\tilde{R}}, & b \leq x \leq c \\ \frac{(d-x)\mu_{\tilde{R}}}{(d-c)}, & c \leq x \leq d \\ 0 & \textit{Otherwise} \end{cases} \quad (1)$$

$$\delta_{\tilde{R}}(x) = \begin{cases} \frac{(b-x)+(x-a_1)\nu_{\tilde{R}}}{(b-a_1)}, & a_1 \leq x \leq b \\ \nu_{\tilde{R}}, & b \leq x \leq c \\ \frac{(x-c)+(d_1-x)\nu_{\tilde{R}}}{(d_1-c)}, & c \leq x \leq d_1 \\ 1 & \textit{Otherwise} \end{cases} \quad (2)$$

where a, a_1, b, c, d, and d_1 are given numbers, and $\gamma_{\tilde{R}}(x) \in [0, 1]$ denotes the degree of membership and $\delta_{\tilde{R}}(x) \in [0, 1]$ denotes the degree of nonmembership with the condition that $0 \leq (\gamma_{\tilde{R}}(x))^q + (\delta_{\tilde{R}}(x))^q \leq 1$ where $x \in X$ and rung parameter $q \geq 1$.

For convenience, consider $a = a_1$ and $d = d_1$; therefore, the real numbers a, b, c, and d and $\mu_{\tilde{r}}$, $\nu_{\tilde{r}}$ define the q-ROTrFN $\tilde{r}$ which is denoted by $\langle[a, b, c, d]; \mu_{\tilde{r}}, \nu_{\tilde{r}}\rangle$. The membership function $\gamma_{\tilde{R}}(x)$ and nonmembership function $\delta_{\tilde{R}}(x)$ of a q-ROTrFN have a graphical representation, as shown in Figure 1, of a trapezoidal with $[a, d]$ being the base of the trapezoidal.

Figure 1
Graphical representation of q-ROTrFN $\tilde{r} = \langle [a, b, c, d]; \mu_{\tilde{r}}, \nu_{\tilde{r}} \rangle$
Note. Several fuzzy numbers can be generated from q-ROTrFN based on changing the rung parameter q



- When $q = 1$ is considered, q-ROTrFN reduces to an intuitionistic trapezoidal fuzzy number (Ye, 2011).
- For $q = 2$, q-ROTrFN reduces to the Pythagorean trapezoidal fuzzy number (Shakeel et al., 2018; Shakeel et al., 2019).
- If $q = 1$ and $b = c$ are considered, q-ROTrFN is converted to an intuitionistic triangular fuzzy number (Riaz et al., 2021a).
- The q-ROTrFN is converted to Pythagorean triangular fuzzy number (Zhang & Liu, 2010) for considering $q = 2$ and $b = c$.
- When $b = c$, q-ROTrFN changes in q-rung orthopair triangular fuzzy number (Fahmi & Aslam, 2021; Wan et al., 2021a).

There are many different t-conorms and t-norms families to choose from when modeling intersections and unions, and Einstein product and Einstein sum are good choices because they typically yield the same smooth approximation as algebraic product and algebraic sum, respectively.

2.3 Einstein operations

Klement et al. (2004) introduced one of generalized t-norm and t-conorm, which is known as Einstein t-norms and t-conorms and expressed as:

- Einstein t-norm: $T^E(x, y) = \frac{xy}{1+(1-x)(1-y)}$,
- Einstein t-conorm: $S^E(x, y) = \frac{x+y}{1+xy}$.

2.4 Score and accuracy functions

Wan et al. (2021b) proposed the definition of a score and accuracy functions for q-ROTrFNs in order to compare them.

Definition 2.3. (Wan et al., 2021b) Let $\tilde{r} = \langle [a, b, c, d]; \mu, \nu \rangle$ be a q-ROTrFN, then score function $S(\tilde{r})$ and accuracy function $A(\tilde{r})$ are presented as:

$$S(\tilde{r}) = \frac{a+b+c+d}{4}(\mu^q - \nu^q); \tag{3}$$

$$A(\tilde{r}) = \frac{a+b+c+d}{4}(\mu^q - \nu^q). \tag{4}$$

To effectively compare the two q-ROTrFNs, using the score $S(\tilde{r})$ and accuracy $A(\tilde{r})$ functions, Wan et al. (2021b) defined a comparison law presented as follows:

Definition 2.4. (Wan et al., 2021b) Let $\tilde{r}_1 = \langle [a_1, b_1, c_1, d_1]; \mu_1, \nu_1 \rangle$ and $\tilde{r}_2 = \langle [a_2, b_2, c_2, d_2]; \mu_2, \nu_2 \rangle$ are any two q-ROTrFNs, then comparison rule between $\tilde{r}_1$ and $\tilde{r}_2$ are presented in the following way:

(i) If $S(\tilde{r}_1) > S(\tilde{r}_2)$, then $\tilde{r}_1 \succ \tilde{r}_2$;
(ii) If $S(\tilde{r}_1) = S(\tilde{r}_2)$, then
- If $A(\tilde{r}_1) < A(\tilde{r}_2)$, then $\tilde{r}_1 \prec \tilde{r}_2$;
- If $A(\tilde{r}_1) = A(\tilde{r}_2)$, then $\tilde{r}_1 \approx \tilde{r}_2$.

3. Einstein Operations-Based q-ROTrF AOs

This section first introduces some basic operational laws for q-ROTrFNs based on Einstein t-norm and t-conorm, and then using defined operational rules, two new AOs were constructed.

3.1. Einstein operations for q-ROTrFNs

In this part, the Einstein t-conorm, S^E, and t-norm, T^E, are used to propose several q-ROTrF Einstein AOs.

The Einstein sum and product on two q-ROTrFNs $\tilde{r}_1$ and $\tilde{r}_2$ are also be a q-ROTrFN denoted by $\tilde{r}_1 \oplus_E \tilde{r}_2$ and $\tilde{r}_1 \otimes_E \tilde{r}_2$, respectively, as follows.

Definition 3.1. Let $\tilde{r}_i = \langle [a_i, b_i, c_i, d_i]; \mu_i, \nu_i \rangle$, $(i = 1, 2)$ and $\tilde{r} = \langle [a, b, c, d]; \mu, \nu \rangle$ be any three q-ROTrFNs, then their addition, $\tilde{r}_1 \oplus_E \tilde{r}_2$, multiplication, $\tilde{r}_1 \otimes_E \tilde{r}_2$, $(\lambda > 0)$

(i) $\tilde{r}_1 \oplus_E \tilde{r}_2 = \left\langle [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \left(\frac{\mu_1{}^q + \mu_2{}^q}{1+\mu_1{}^q \mu_2{}^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_1{}^q \nu_2{}^q}{1+(1-\nu_1{}^q)(1-\nu_2{}^q)}\right)^{\frac{1}{q}} \right\rangle$;

(ii) $\tilde{r}_1 \otimes_E \tilde{r}_2 = \left\langle [a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2]; \left(\frac{\mu_1{}^q \mu_2{}^q}{1+(1-\mu_1{}^q)(1-\mu_2{}^q)}\right)^{\frac{1}{q}}, \left(\frac{\nu_1{}^q + \nu_2{}^q}{1+\nu_1{}^q \nu_2{}^q}\right)^{\frac{1}{q}} \right\rangle$;

(iii) $\lambda \odot_E \tilde{r} = \left\langle [\lambda a, \lambda b, \lambda c, \lambda d]; \left(\frac{(1+\mu^q)^\lambda - (1-\mu^q)^\lambda}{(1+\mu^q)^\lambda + (1-\mu^q)^\lambda}\right)^{\frac{1}{q}}, \left(\frac{2\nu^{q\lambda}}{(2-\nu^q)^\lambda + \nu^{q\lambda}}\right)^{\frac{1}{q}} \right\rangle$;

(iv) $\tilde{r}^\lambda = \left\langle [a^\lambda, b^\lambda, c^\lambda, d^\lambda]; \left(\frac{2\mu^{q\lambda}}{(2-\mu^q)^\lambda + \mu^{q\lambda}}\right)^{\frac{1}{q}}, \left(\frac{(1+\nu^q)^\lambda - (1-\nu^q)^\lambda}{(1+\nu^q)^\lambda + (1-\nu^q)^\lambda}\right)^{\frac{1}{q}} \right\rangle$.

***Proof* (i).** Since $a_i, b_i, c_i, d_i \in \mathbb{R}$, then it is evident that $a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2 \in \mathbb{R}$.

We have to show that $\left(\left(\frac{\mu_1{}^q + \mu_2{}^q}{1+\mu_1{}^q \mu_2{}^q}\right)^{\frac{1}{q}}\right)^q + \left(\left(\frac{\nu_1{}^q + \nu_2{}^q}{1+(1-\nu_1{}^q)(1-\nu_2{}^q)}\right)^{\frac{1}{q}}\right)^q \leq 1$, i.e., $\frac{\mu_1{}^q + \mu_2{}^q}{1+\mu_1{}^q \mu_2{}^q} + \frac{\nu_1{}^q + \nu_2{}^q}{1+(1-\nu_1{}^q)(1-\nu_2{}^q)} \leq 1$.

From the definition of q-ROTrFNs, $\tilde{r}_i$, the membership and nonmembership degrees satisfy the condition that

$$\mu_1{}^q + \nu_1{}^q \leq 1; \tag{5}$$

and

$$\mu_2{}^q + \nu_2{}^q \leq 1; \tag{6}$$

i.e., $\mu_1{}^q \leq 1 - \nu_1{}^q$, $\mu_2{}^q \leq 1 - \nu_2{}^q$,

$$\begin{aligned} &\Rightarrow \mu_2{}^q \mu_2{}^q \leq (1 - \nu_1{}^q)(1 - \nu_2{}^q) (\because \mu_i, \nu_i \in [0, 1] \\ &\Rightarrow 1 + \mu_2{}^q \mu_2{}^q \leq 1 + (1 - \nu_1{}^q)(1 - \nu_2{}^q). \end{aligned} \tag{7}$$

Adding (5) and (6),

$$\mu_1{}^q + \nu_1{}^q + \mu_2{}^q + \nu_2{}^q \leq 2, \tag{8}$$

and since $\mu_i \in [0, 1]$

$$\mu_1{}^q \mu_2{}^q \leq 1, i.e., 1 + \mu_1{}^q \mu_2{}^q \leq 2. \quad (9)$$

From (8) and (9),

$$\frac{\mu_1{}^q + \mu_2{}^q + \nu_1{}^q + \nu_2{}^q}{1 + \mu_1{}^q \mu_2{}^q} \leq 1, \text{or}, \frac{\mu_1{}^q + \mu_2{}^q}{1 + \mu_1{}^q \mu_2{}^q} + \frac{\nu_1{}^q + \nu_2{}^q}{1 + \mu_1{}^q \mu_2{}^q} \leq 1,$$

Now using (3.3), $\frac{\mu_1{}^q + \mu_2{}^q}{1+\mu_1{}^q \mu_2{}^q} + \frac{\nu_1{}^q + \nu_2{}^q}{1+(1-\nu_1{}^q)(1-\nu_2{}^q)} \leq 1.$

So $\tilde{r}_1 \oplus_E \tilde{r}_2$ is an q-ROTrFN.

In a parallel way, it can be proven that each of $\tilde{r}_1 \otimes_E \tilde{r}_2$, $\lambda \odot_E \tilde{r}$, and $\tilde{r}^\lambda$ is a q-ROTrFN.

Example 1: Let $\tilde{r}_1 = \langle [0.4, 0.5, 0.6, 0.7]; 0.7, 0.3 \rangle$ and $\tilde{r}_2 = \langle [0.3, 0.4, 0.6, 0.8]; 0.8, 0.5 \rangle$ be any two q-ROTrFNs. Then for taking $q = 3$, some Einstein operations of $\tilde{r}_1$ and $\tilde{r}_2$ can be defined as follows:

$$\tilde{r}_1 \oplus_E \tilde{r}_2 = \left\langle [0.4 + 0.3, 0.5 + 0.4, 0.6 + 0.6, 0.7 + 0.8]; \left(\frac{0.7^3 + 0.8^3}{1 + 0.7^3 0.8^3}\right)^{\frac{1}{3}}, \left(\frac{0.3^3 0.5^3}{1 + (1 - 0.3^3)(1 - 0.5^3)}\right)^{\frac{1}{3}} \right\rangle; = \langle [0.7, 0.9, 1.2, 1.5]; 0.8993, 0.1222 \rangle,$$

$$\tilde{r}_1 \otimes_E \tilde{r}_2 = \left\langle [[0.4 \times 0.3, 0.5 \times 0.4, 0.6 \times 0.6, 0.7 \times 0.8]]; \left(\frac{0.7^3 0.8^3}{1 + (1 - 0.7^3)(1 - 0.8^3)}\right)^{\frac{1}{3}}, \left(\frac{0.3^3 + 0.5^3}{1 + 0.3^3 0.5^3}\right)^{\frac{1}{3}} \right\rangle; = \langle [0.12, 0.2, 0.36, 0.56]; 0.5104, 0.5331 \rangle,$$

$$2 \odot_E \tilde{r}_1 = \left\langle [2 \times 0.4, 2 \times 0.5, 2 \times 0.6, 2 \times 0.7]; \left(\frac{(1 + 0.7^3)^2 - (1 - 0.7^3)^2}{(1 + 0.7^3)^2 + (1 - 0.7^3)^2}\right)^{\frac{1}{3}}, \left(\frac{2 \times 0.3^{3 \times 2}}{(2 - 0.3^3)^2 + 0.3^{3 \times 2}}\right)^{\frac{1}{q}} \right\rangle; = \langle [0.8, 1.0, 1.2, 1.4]; 0.8498, 0.0721 \rangle,$$

$$\tilde{r}_1^2 = \left\langle [0.4^2, 0.5^2, 0.6^2, 0.7^2]; \left(\frac{2 \times 0.7^{3 \times 2}}{(2 - 0.7^3)^2 + 0.7^{3 \times 2}}\right)^{\frac{1}{3}}, \left(\frac{(1 + 0.3^3)^2 - (1 - 0.3^3)^2}{(1 + 0.3^3)^2 + (1 - 0.3^3)^2}\right)^{\frac{1}{3}} \right\rangle; = \langle [0.16, 0.25, 0.36, 0.49]; 0.4348, 0.3779 \rangle.$$

3.2 Einstein operations-based q-ROTrF AOs

With the help of Einstein operations, the q-ROTrF averaging and geometric AOs are introduced in the section.

- ***q*-ROTrFEWA operator**

Definition 3.2. Let $\{\tilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle | j = 1, 2, \ldots, n\}$ be a collection of q-ROTrFNs. The q-ROTrFEWA operator is defined as follows:

$$q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) = \oplus_{Ej=1}^n (\omega_j \odot_E \tilde{r}_j), \quad (10)$$

In which addition $\oplus_E$ and scalar multiplication $\odot_E$, laws are presented in Definition 3.1, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is a weight vector of q-ROTrFNs $\tilde{r}_j$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$.

Theorem 3.1. Let $\{\tilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j \rangle | j = 1, 2, \ldots, n\}$ be a group of q-ROTrFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be a weight vector of $\tilde{r}_j$ where $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$. Then aggregated value of $\{\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n\}$ by the q-ROTrFEWA operator is still a q-ROTrFN and

$$q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) = \left\langle \left[\sum_{j=1}^n \omega_j a_j, \sum_{j=1}^n \omega_j b_j, \sum_{j=1}^n \omega_j c_j, \sum_{j=1}^n \omega_j d_j\right]; \left(\frac{\prod_{j=1}^n (1 + \mu_j{}^q)^{\omega_j} - \prod_{j=1}^n (1 - \mu_j{}^q)^{\omega_j}}{\prod_{j=1}^n (1 + \mu_j{}^q)^{\omega_j} + \prod_{j=1}^n (1 - \mu_j{}^q)^{\omega_j}}\right)^{\frac{1}{q}}, \left(\frac{2\prod_{j=1}^n \nu_j^{q\omega_j}}{\prod_{j=1}^n (2 - \nu_j{}^q)^{\omega_j} + \prod_{j=1}^n \nu_j^{q\omega_j}}\right)^{\frac{1}{q}} \right\rangle. \quad (11)$$

Proof. Based on Definition 3.1,

$$\omega_j \odot_E \tilde{r}_j = \left\langle [\omega_j a_j, \omega_j b_j, \omega_j c_j, \omega_j d_j]; \left(\frac{(1 + \mu_j{}^q)^{\omega_j} - (1 - \mu_j{}^q)^{\omega_j}}{(1 + \mu_j{}^q)^{\omega_j} + (1 - \mu_j{}^q)^{\omega_j}}\right)^{\frac{1}{q}}, \left(\frac{2\nu_j^{q\omega_j}}{(2 - \nu_j{}^q)^{\omega_j} + \nu_j^{q\omega_j}}\right)^{\frac{1}{q}} \right\rangle; \text{now, } \omega_1 \tilde{r}_1 \oplus_E \omega_2 \tilde{r}_2$$

$$= \langle [\omega_1 a_1 + \omega_2 a_2, \omega_1 b_1 + \omega_2 b_2, \omega_1 c_1 + \omega_2 c_2, \omega_1 d_1 + \omega_2 d_2]; \left(\frac{\prod_{j=1}^2 (1 + \mu_j{}^q)^{\omega_j} - \prod_{j=1}^2 (1 - \mu_j{}^q)^{\omega_j}}{\prod_{j=1}^2 (1 + \mu_j{}^q)^{\omega_j} + \prod_{j=1}^2 (1 - \mu_j{}^q)^{\omega_j}}\right)^{\frac{1}{q}}, \left(\frac{2\prod_{j=1}^2 \nu_j^{q\omega_j}}{\prod_{j=1}^2 (2 - \nu_j{}^q)^{\omega_j} + \prod_{j=1}^2 \nu_j^{q\omega_j}}\right)^{\frac{1}{q}} \rangle$$

i.e., the theorem holds for $n = 2$. Now, assume that the theorem is valid for $n = k$.

Hence, $q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_k) = \left\langle \left[\sum_{j=1}^k \omega_j a_j, \sum_{j=1}^k \omega_j b_j, \sum_{j=1}^k \omega_j c_j, \sum_{j=1}^k \omega_j d_j\right]; \left(\frac{\prod_{j=1}^k (1 + \mu_j{}^q)^{\omega_j} - \prod_{j=1}^k (1 - \mu_j{}^q)^{\omega_j}}{\prod_{j=1}^k (1 + \mu_j{}^q)^{\omega_j} + \prod_{j=1}^k (1 - \mu_j{}^q)^{\omega_j}}\right)^{\frac{1}{q}}, \left(\frac{2\prod_{j=1}^k \nu_j^{q\omega_j}}{\prod_{j=1}^k (2 - \nu_j{}^q)^{\omega_j} + \prod_{j=1}^k \nu_j^{q\omega_j}}\right)^{\frac{1}{q}} \right\rangle.$

Then for $n = k + 1$,

$$q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_k, \tilde{r}_{k+1})$$
$$= q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_k) \oplus_E (\omega_{k+1} \tilde{r}_{k+1})$$
$$= \left\langle \left[\sum_{j=1}^k \omega_j a_j, \sum_{j=1}^k \omega_j b_j, \sum_{j=1}^k \omega_j c_j, \sum_{j=1}^k \omega_j d_j\right]; \left(\frac{\prod_{j=1}^k (1 + \mu_j{}^q)^{\omega_j} - \prod_{j=1}^k (1 - \mu_j{}^q)^{\omega_j}}{\prod_{j=1}^k (1 + \mu_j{}^q)^{\omega_j} + \prod_{j=1}^k (1 - \mu_j{}^q)^{\omega_j}}\right)^{\frac{1}{q}}, \left(\frac{2\prod_{j=1}^k \nu_j^{q\omega_j}}{\prod_{j=1}^k (2 - \nu_j{}^q)^{\omega_j} + \prod_{j=1}^k \nu_j^{q\omega_j}}\right)^{\frac{1}{q}} \right\rangle$$
$$\oplus_E \left\langle [\omega_{k+1} a_{k+1}, \omega_{k+1} b_{k+1}, \omega_{k+1} c_{k+1}, \omega_{k+1} d_{k+1}]; \left(\frac{(1 + \mu_{k+1}{}^q)^{\omega_{k+1}} - (1 - \mu_{k+1}{}^q)^{\omega_j}}{(1 + \mu_{k+1}{}^q)^{\omega_j} + (1 - \mu_{k+1}{}^q)^{\omega_j}}\right)^{\frac{1}{q}}, \left(\frac{2\nu_{k+1}{}^{q\omega_{k+1}}}{(2 - \nu_j{}^q)^{\omega_{k+1}} + \nu_{k+1}{}^{q\omega_{k+1}}}\right)^{\frac{1}{q}} \right\rangle,$$
$$= \left\langle \left[\sum_{j=1}^{k+1} \omega_j a_j, \sum_{j=1}^{k+1} \omega_j b_j, \sum_{j=1}^{k+1} \omega_j c_j, \sum_{j=1}^{k+1} \omega_j d_j\right]; \left(\frac{\prod_{j=1}^{k+1} (1 + \mu_j{}^q)^{\omega_j} - \prod_{j=1}^{k+1} (1 - \mu_j{}^q)^{\omega_j}}{\prod_{j=1}^{k+1} (1 + \mu_j{}^q)^{\omega_j} + \prod_{j=1}^{k+1} (1 - \mu_j{}^q)^{\omega_j}}\right)^{\frac{1}{q}}, \left(\frac{2\prod_{j=1}^{k+1} \nu_j^{q\omega_j}}{\prod_{j=1}^{k+1} (2 - \nu_j{}^q)^{\omega_j} + \prod_{j=1}^{k+1} \nu_j^{q\omega_j}}\right)^{\frac{1}{q}} \right\rangle.$$

Therefore, the theorem is true for $n = k + 1$ also and is valid $\forall n$.

Example 2: Let $\tilde{r}_1 = \langle [0.4, 0.5, 0.6, 0.7]; 0.4, 0.3 \rangle$, $\tilde{r}_2 = \langle [0.1, 0.2, 0.3, 0.4]; 0.5, 0.2 \rangle$, and $\tilde{r}_3 = \langle [0.4, 0.5, 0.7, 0.8]; 0.2, 0.5 \rangle$

be a collection of q-ROTrFNs. If the weights of three q-ROTrFNs are taken, respectively, such as 0.3, 0.25, and 0.35, then their aggregated value by using the q-ROTrFEWA operator is also a q-ROTrF and obtained as:

$$q-ROTrFEWA(\tilde{r}_1,\tilde{r}_2,\tilde{r}_3) = \left\langle\left[\sum_{j=1}^{3}\omega_j a_j,\sum_{j=1}^{3}\omega_j b_j,\sum_{j=1}^{3}\omega_j c_j,\sum_{j=1}^{3}\omega_j d_j\right];\right.$$
$$\left.\left(\frac{\prod_{j=1}^{3}(1+\mu_j{}^3)^{\omega_j}-\prod_{j=1}^{3}(1-\mu_j{}^3)^{\omega_j}}{\prod_{j=1}^{3}(1+\mu_j{}^3)^{\omega_j}+\prod_{j=1}^{3}(1-\mu_j{}^3)^{\omega_j}}\right)^{\frac{1}{3}},\left(\frac{2\prod_{j=1}^{3}\nu_j{}^{3\omega_j}}{\prod_{j=1}^{3}(2-\nu_j{}^3)^{\omega_j}+\prod_{j=1}^{3}\nu_j{}^{3\omega_j}}\right)^{\frac{1}{3}}\right\rangle;$$
$$= \langle[0.2850, 0.3750, 0.5000, 0.5900]; 0.3765, 0.5734\rangle.$$

Now, some fundamental characteristics of the proposed q-ROTrFEWA operator are stated in the following section.

Theorem 3.2. (*Idempotency*) Suppose $\{\tilde{r}_j = \langle[a_j,b_j,c_j,d_j];\mu_j,\nu_j\rangle \mid j=1,2,\ldots,n\}$ be a group of q-ROTrFNs. If $\tilde{r}_j=\tilde{r}=\langle[a,b,c,d];\mu,\nu\rangle\ \forall j$, then

$$q-ROTrFEWA(\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_n)=\tilde{r}.$$

.

Proof. $q-ROTrFEWA(\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_n)=$

$$\left\langle\left[\sum_{j=1}^{n}\omega_j a_j,\sum_{j=1}^{n}\omega_j b_j,\sum_{j=1}^{n}\omega_j c_j,\sum_{j=1}^{n}\omega_j d_j\right];\left(1-\prod_{j=1}^{n}(1-\mu_j{}^q)^{\omega_j}\right)^{\frac{1}{q}},\prod_{j=1}^{n}\nu_j{}^{\omega_j}\right\rangle.$$

Since $\tilde{r}_j=\tilde{r}\ \forall j$,

$$q-ROTrFEWA(\tilde{r},\tilde{r},\ldots,\tilde{r}) = \left\langle\left[\left(\sum_{j=1}^{n}\omega_j\right)a,\left(\sum_{j=1}^{n}\omega_j\right)b,\left(\sum_{j=1}^{n}\omega_j\right)c,\left(\sum_{j=1}^{n}\omega_j\right)d\right];\right.$$
$$\left.\left(1-(1-\mu^q)^{\sum_{j=1}^{n}\omega_j}\right)^{\frac{1}{q}},\nu^{\sum_{j=1}^{n}\omega_j}\right\rangle=\langle[a,b,c,d];\mu,\nu\rangle=\tilde{r}.$$

Theorem 3.3. (*Monotonicity*) Let $\{\tilde{r}_j=\langle[a_j,b_j,c_j,d_j];\mu_j,\nu_j\rangle\}$ and $\{\tilde{r}_j{}'=\langle[a_j',b_j',c_j',d_j'];\mu_j',\nu_j'\rangle\}$ be two collections of n q-ROTrFNs. If $a_j\le a_j', b_j\le b_j', c_j\le c_j', d_j\le d_j', \mu_j\le\mu_j'$ and $\nu_j\ge\nu_j'\ \forall j$, then,

$$q-ROTrFEWA(\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_n)\preccurlyeq q-ROTrFEWA(\tilde{r}_1{}',\tilde{r}_2{}',\ldots,\tilde{r}_n{}'). \tag{12}$$

Proof. Let $g(t)=\frac{1+t}{1-t}$, $t\in[0,1)$, then $g'(t)=\frac{2}{(1-t)^2}>0$, thus g is an increasing function. Since for every $\tilde{r}_j$ and $\tilde{r}_j{}'$, $\mu_j\le\mu_j'$,

$$\frac{(1+\mu_j{}^q)}{(1-\mu_j{}^q)}\le\frac{\left(1+\mu_j'^q\right)}{\left(1-\mu_j'^q\right)}.\ So,\ \left(\frac{1+\mu_j{}^q}{1-\mu_j{}^q}\right)^{\omega_j}\le\left(\frac{1+\mu_j'^q}{1-\mu_j'^q}\right)^{\omega_j},$$
$$\Leftrightarrow\prod_{j=1}^{n}\left(\frac{1+\mu_j{}^q}{1-\mu_j{}^q}\right)^{\omega_j}\le\prod_{j=1}^{n}\left(\frac{1+\mu_j'^q}{1-\mu_j'^q}\right)^{\omega_j},$$
$$\Leftrightarrow\prod_{j=1}^{n}\left(\frac{1+\mu_j{}^q}{1-\mu_j{}^q}\right)^{\omega_j}+1\le\prod_{j=1}^{n}\left(\frac{1+\mu_j'^q}{1-\mu_j'^q}\right)^{\omega_j}+1,$$
$$\Leftrightarrow\frac{1}{\prod_{j=1}^{n}\left(\frac{1+\mu_j{}^q}{1-\mu_j{}^q}\right)^{\omega_j}+1}\ge\frac{1}{\prod_{j=1}^{n}\left(\frac{1+\mu_j'^q}{1-\mu_j'^q}\right)^{\omega_j}+1},$$
$$\Leftrightarrow\frac{2\prod_{j=1}^{n}(1-\mu_j{}^q)^{\omega_j}}{\prod_{j=1}^{n}(1+\mu_j{}^q)^{\omega_j}+\prod_{j=1}^{n}(1-\mu_j{}^q)^{\omega_j}}$$
$$\ge\frac{2\prod_{j=1}^{n}\left(1-\mu_j'^q\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\mu_j'^q\right)^{\omega_j}+\prod_{j=1}^{n}\left(1-\mu_j'^q\right)^{\omega_j}},$$
$$\Leftrightarrow 1-\frac{2\prod_{j=1}^{n}(1-\mu_j{}^q)^{\omega_j}}{\prod_{j=1}^{n}(1+\mu_j{}^q)^{\omega_j}+\prod_{j=1}^{n}(1-\mu_j{}^q)^{\omega_j}}$$
$$\le 1-\frac{2\prod_{j=1}^{n}\left(1-\mu_j'^q\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\mu_j'^q\right)^{\omega_j}+\prod_{j=1}^{n}\left(1-\mu_j'^q\right)^{\omega_j}},$$
$$\Leftrightarrow\frac{\prod_{j=1}^{n}(1+\mu_j{}^q)^{\omega_i}-\prod_{j=1}^{n}(1-\mu_j{}^q)^{\omega_j}}{\prod_{j=1}^{n}(1+\mu_j{}^q)^{\omega_j}+\prod_{j=1}^{n}(1-\mu_j{}^q)^{\omega_j}}$$
$$\le\frac{\prod_{j=1}^{n}\left(1+\mu_j'^q\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-\mu_j'^q\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\mu_j'^q\right)^{\omega_j}+\prod_{j=1}^{n}\left(1-\mu_j'^q\right)^{\omega_j}}, \tag{13}$$

Again let $f(u)=\frac{2-u}{u}$, $u\in(0,1]$, then $f'(u)=-\frac{2}{u^2}<0$, thus $f(u)$ is a decreasing function.

Since, $\nu_j{}^q\ge\nu_j'^q\forall j$, then

$$\frac{2-\nu_j{}^q}{\nu_j{}^q}\le\frac{2-\nu_j'^q}{\nu_j'^q},\ \text{thus},\ \left(\frac{2-\nu_j{}^q}{\nu_j{}^q}\right)^{\omega}_j\le\left(\frac{2-\nu_j'^q}{\nu_j'^q}\right)^{\omega_j},$$
$$\Leftrightarrow\prod_{j=1}^{n}\left(\frac{2-\nu_j^q}{\nu_j^q}\right)^{\omega_j}\le\prod_{j=1}^{n}\left(\frac{2-\nu_j'^q}{\nu_j'^q}\right)^{\omega_j},\Leftrightarrow\prod_{j=1}^{n}\left(\frac{2-\nu_j^q}{\nu_j^q}\right)^{\omega_j}$$
$$+1\le\prod_{j=1}^{n}\left(\frac{2-\nu_j'^q}{\nu_j'^q}\right)^{\omega_j}+1,\Leftrightarrow\frac{1}{\prod_{j=1}^{n}\left(\frac{2-\nu_j{}^q}{\nu_j{}^q}\right)^{\omega_j}+1}$$
$$\ge\frac{1}{\prod_{j=1}^{n}\left(\frac{2-\nu_j'^q}{\nu_j'^q}\right)^{\omega_j}+1},\Leftrightarrow\frac{2\prod_{j=1}^{n}\nu_j{}^{q\omega_j}}{\prod_{j=1}^{n}(2-\nu_j{}^q)^{\omega_j}+\prod_{j=1}^{n}\nu_j{}^{q\omega_j}}$$
$$\ge\frac{2\prod_{j=1}^{n}\nu_j'^{q\omega_j}}{\prod_{j=1}^{n}\left(2-\nu_j'^q\right)^{\omega_j}+\prod_{j=1}^{n}\nu_j'^{q\omega_j}}. \tag{14}$$

From (13) and (14) and using the relations $\sum_{j=1}^{n}\omega_j a_j\le\sum_{j=1}^{n}\omega_j a_j'$, $\sum_{j=1}^{n}\omega_j b_j\le\sum_{j=1}^{n}\omega_j b_j'$, $\sum_{j=1}^{n}\omega_j c_j\le\sum_{j=1}^{n}\omega_j c_j'$ and $\sum_{j=1}^{n}\omega_j d_j\le\sum_{j=1}^{n}\omega_j d_j'$, it is clear that

$$S(q-ROTrFEWA(\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_n))\le S(q-ROTrFEWA(\tilde{r}_1{}',\tilde{r}_2{}',\ldots,\tilde{r}_n{}')).$$

Therefore, $q-ROTrFEWA(\tilde{r}_1,\tilde{r}_2,\ldots,\tilde{r}_n)\preccurlyeq q-ROTrFEWA(\tilde{r}_1{}',\tilde{r}_2{}',\ldots,\tilde{r}_n{}')$.

Hence, inequality (12) follows.

Theorem 3.4. (Boundedness) Let $\{\tilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j\rangle | j = 1, 2, \ldots, n\}$ be a group of q-ROTrFNs and assume

$\tilde{r}_j^- = \langle [min_j\{a_j\}, min_j\{b_j\}, min_j\{c_j\}, min_j\{d_j\}]; min_j\{\mu_j\}, max_j\{\nu_j\}\rangle$, and $\tilde{r}_j^+ = \langle [max_j\{a_j\}, max_j\{b_j\}, max_j\{c_j\}, max_j\{d_j\}]; max_j\{\mu_j\}, min_j\{\nu_j\}\rangle$,

then, $\tilde{r}_j^- \leq q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) \leq \tilde{r}_j^+$.

Proof. Since $min\{a_j\} \leq a_j \leq max\{a_j\}$, $min\{b_j\} \leq b_j \leq max\{b_j\}$, $min\{c_j\} \leq c_j \leq max\{c_j\}$, $min\{d_j\} \leq d_j \leq max\{d_j\}$, $min\{\mu_j\} \leq \mu_j \leq max\{\mu_j\}$ and $min\{\nu_j\} \leq \nu_j \leq max\{\nu_j\} \forall j$, then $\tilde{r}_j^- \leq \tilde{r}_j \forall j$.

Thus, from monotonicity

$$q - ROTrFEWA(\tilde{r}_j^-, \tilde{r}_j^-, \ldots, \tilde{r}_j^-) \leq q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n).$$

Now applying the idempotency theorem, the above inequality takes the form as:

$$\tilde{r}_j^- \leq q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n). \quad (15)$$

Similarly, it can be shown that

$$q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) \leq \tilde{r}_j^+. \quad (16)$$

So, by combining (15) and (16), it follows that

$$\tilde{r}_j^- \leq q - ROTrFEWA(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) \leq \tilde{r}_j^+.$$

- ***q*-ROTrFEWG operator**

In this subsection, q-ROTrFEWG operator is developed based on Einstein operational rules.

Definition 3.3. Let $\{\tilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j\rangle | j = 1, 2, \ldots, n\}$ be a collection of q-ROTrFNs. The q-ROTrFEWG operator is defined as follows:

$$q - ROTrFEWG(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) = \tilde{r}_1^{\omega_1} \otimes_E \tilde{r}_2^{\omega_2} \otimes_E \ldots \otimes_E \tilde{r}_n^{\omega_n}, \quad (17)$$

In which multiplication $\otimes_E$ and exponential laws are presented in Definition 3.1, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is a vector of q-ROTrFNs $\tilde{r}_j$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$.

Theorem 3.5. Let $\{\tilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j\rangle | j = 1, 2, \ldots, n\}$ be a set of q-ROTrFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ represent the weight vector of $\tilde{r}_j$ where $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$. Then their aggregated value using q-ROTrFEWG operator is furthermore a q-ROTrFN and

$$q - ROTrFEWG(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) = \left\langle \left[\prod_{j=1}^n a_j^{\omega_j}, \prod_{j=1}^n b_j^{\omega_j}, \prod_{j=1}^n c_j^{\omega_j}, \prod_{j=1}^n d_j^{\omega_j}\right]; \left(\frac{2\prod_{j=1}^n \mu_j^{q\omega_j}}{\prod_{j=1}^n (2 - \mu_j^q)^{\omega_j} + \prod_{j=1}^n \mu_j^{q\omega_j}}\right)^{\frac{1}{q}}, \left(\frac{\prod_{j=1}^n (1 + \nu_j^q)^{\omega_j} - \prod_{j=1}^n (1 - \nu_j^q)^{\omega_j}}{\prod_{j=1}^n (1 + \nu_j^q)^{\omega_j} + \prod_{j=1}^n (1 - \nu_j^q)^{\omega_j}}\right)^{\frac{1}{q}} \right\rangle. \quad (18)$$

Proof. The proof is same as Theorem 3.1.

Example 3:. In Example 2, if the geometric aggregation operator is used q-ROTrFEWG, then the aggregating values of the three q-ROTrFNs, $\tilde{r}_1$, $\tilde{r}_2$, and $\tilde{r}_3$, are computed as:

$$q - ROTrFEWG(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) = \left\langle \left[\prod_{j=1}^3 a_j^{\omega_j}, \prod_{j=1}^3 b_j^{\omega_j}, \prod_{j=1}^3 c_j^{\omega_j}, \prod_{j=1}^3 d_j^{\omega_j}\right]; \left(\frac{2\prod_{j=1}^3 \mu_j^{3\omega_j}}{\prod_{j=1}^3 (2 - \mu_j^3)^{\omega_j} + \prod_{j=1}^3 \mu_j^{3\omega_j}}\right)^{\frac{1}{3}}, \left(\frac{\prod_{j=1}^3 (1 + \nu_j^3)^{\omega_j} - \prod_{j=1}^3 (1 - \nu_j^3)^{\omega_j}}{\prod_{j=1}^3 (1 + \nu_j^3)^{\omega_j} + \prod_{j=1}^3 (1 - \nu_j^3)^{\omega_j}}\right)^{\frac{1}{3}} \right\rangle;$$

$$= \langle [0.3100, 0.4262, 0.5604, 0.6609]; 0.7112, 0.3780\rangle.$$

Next, the characteristics of the defined q-ROTrFEWG operator are presented.

Theorem 3.6. (Idempotency) Let $\tilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j\rangle$ $(j = 1, 2, \ldots, n)$ be a set of n Lq-ROFNs. If $\tilde{r}_j = \tilde{r} = \langle [a, b, c, d]; \mu, \nu\rangle$ $\forall j$, then

$$q - ROTrFEWG(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) = \tilde{r}$$

.

Proof. The proof is same as Theorem 3.2.

Theorem 3.7. (*Monotonicity*) Suppose $\tilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j\rangle$ and $\tilde{r}_j' = \langle [a_j', b_j', c_j', d_j']; \mu_j', \nu_j'\rangle$ be two set of n q-ROTrFNs. If $a_j \leq a_j'$, $b_j \leq b_j'$, $c_j \leq c_j'$, $d_j \leq d_j'$, $\mu_j \leq \mu_j'$ and $\nu_j \geq \nu_j'$ $\forall j$, then

$$q - ROTrFEWG(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) \leq q - ROTrFEWG(\tilde{r}_1', \tilde{r}_2', \ldots, \tilde{r}_n')$$

.

Proof. The proof is similar as Theorem 3.3.

Theorem 3.8. (*Boundedness*) If $\{\tilde{r}_j = \langle [a_j, b_j, c_j, d_j]; \mu_j, \nu_j\rangle\}$ represents a set of n q-ROTrFNs, and

$\tilde{r}_j^- = \langle [min_j\{a_j\}, min_j\{b_j\}, min_j\{c_j\}, min_j\{d_j\}]; min\{\mu_j\}, max_j\{\nu_j\}\rangle$ and $\tilde{r}_j^+ = \langle [max_j\{a_j\}, max_j\{b_j\}, max_j\{c_j\}, max_j\{d_j\}]; max_j\{\mu_j\}, min_j\{\nu_j\}\rangle$, then $\tilde{r}_j^- \leq q - ROTrFEWG(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n) \leq \tilde{r}_j^+$.

Proof. The proof is same as Theorem 3.4.

4. MCGDM Approach Based on the Proposed AOs under *q*-ROTrF Environment

In this part, a novel MCGDM method has been propounded in which the evaluation information is in the form of q-ROTrFNs.

For a MCGDM problem, let $E = \{e^{(1)}, e^{(2)}, \ldots, e^{(k)}\}$ be the group of the DMs with their associated weight vector $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_k)^T$. Suppose $A = \{A_i | i = 1, 2, \ldots, m\}$ be a set of m discrete alternatives and $\mathcal{C} = \{\mathcal{C}_j | j = 1, 2, \ldots, n\}$ represents the set of n criteria along with their weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. DMs give their assessment values in terms of q-ROTrFNs. The DMs use q-ROTrFNs to express their judgment values, and q-ROTrF decision matrix (q-ROTrFDM) is provided as $\mathcal{D}^{(l)} = [\tilde{r}_{ij}^{(l)}]_{m\times n} = [\langle [a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}]; \mu_{ij}^{(l)}, \nu_{ij}^{(l)}\rangle]_{m\times n}$ $(l = 1, 2, .., k)$, where $\tilde{r}_{ij}^{(l)} = \langle [a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}]; \mu_{ij}^{(l)}, \nu_{ij}^{(l)}\rangle$ denotes a q-ROTrFN given by the DM $e^{(l)}$ for the alternative A_i under the criteria $\mathcal{C}_j$.

The purpose is to find the best suitable alternative(s) in light of the presented approach. The following is a step-by-step breakdown of the computing procedure.

Step 1. Normalize $\mathcal{D}^{(l)}$, if required, into $\mathcal{N}^{(l)} = \left[\tilde{\wp}_{ij}{}^{(l)}\right]_{m\times n}$ as follows:

$$\tilde{\wp}_{ij}{}^{(l)} = \begin{cases} \left\langle \left[a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}\right]; \mu_{ij}^{(l)}, \nu_{ij}^{(l)} \right\rangle & \text{if } C_j \text{ is type of benefit criteria} \\ \left\langle \left[a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}\right]; \nu_{ij}^{(l)}, \mu_{ij}^{(l)} \right\rangle & \text{if } C_j \text{ is type of cost criteria,} \end{cases} \tag{19}$$

$i = 1, 2, \ldots, m, j = 1, 2, \ldots, n.$

Step 2. Utilize q-ROTrFEWA (or q-ROTrFEWG) operator to aggregate all the individual normalized q-ROTrFDMs, $\mathcal{N}^{(l)} = \left[\tilde{\wp}_{ij}{}^{(l)}\right]_{m\times n}$ $(l = 1, 2, \ldots, k)$ into a single q-ROTrFDM, $\mathcal{N} = \left[\tilde{\wp}_{ij}\right]_{m\times n}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ as:

$$\tilde{\wp}_{ij} = \left\langle \left[\sum_{l=1}^{3} \Omega^{(l)} a_{ij}^{(l)}, \sum_{l=1}^{3} \Omega^{(l)} b_{ij}^{(l)}, \sum_{l=1}^{3} \Omega^{(l)} c_{ij}^{(l)}, \sum_{l=1}^{3} \Omega^{(l)} d_{ij}^{(l)}\right]; \right.$$
$$\left(\frac{\prod_{l=1}^{3}\left(1+\left(\mu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}} - \prod_{l=1}^{3}\left(1-\left(\mu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}}}{\prod_{l=1}^{3}\left(1+\left(\mu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}} + \prod_{l=1}^{3}\left(1-\left(\mu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}}}\right)^{\frac{1}{q}},$$
$$\left.\left(\frac{2\prod_{l=1}^{3}\left(\nu_{ij}^{(l)}\right)^{q\Omega^{(l)}}}{\prod_{l=1}^{3}\left(2-\left(\nu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}} + \prod_{l=1}^{3}\left(\nu_{ij}^{(l)}\right)^{q\Omega^{(l)}}}\right)^{\frac{1}{q}}\right\rangle; \tag{20}$$

$$\text{or } \tilde{\wp}_{ij}{}' = \left\langle \left[\prod_{l=1}^{3}\left(a_{ij}^{(l)}\right)^{\Omega^{(l)}}, \prod_{l=1}^{3}\left(a_{ij}^{(l)}\right)^{\Omega^{(l)}}, \prod_{l=1}^{3}\left(a_{ij}^{(l)}\right)^{\Omega^{(l)}}, \prod_{l=1}^{3}\left(a_{ij}^{(l)}\right)^{\Omega^{(l)}}\right]; \right.$$
$$\left(\frac{2\prod_{l=1}^{3}\left(\mu_{ij}^{(l)}\right)^{q\Omega^{(l)}}}{\prod_{l=1}^{3}\left(2-\left(\mu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}} + \prod_{l=1}^{3}\left(\mu_{ij}^{(l)}\right)^{q\Omega^{(l)}}}\right)^{\frac{1}{q}},$$
$$\left.\left(\frac{\prod_{l=1}^{3}\left(1+\left(\nu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}} - \prod_{l=1}^{3}\left(1-\left(\nu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}}}{\prod_{l=1}^{3}\left(1+\left(\nu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}} + \prod_{l=1}^{3}\left(1-\left(\nu_{ij}^{(l)}\right)^q\right)^{\Omega^{(l)}}}\right)^{\frac{1}{q}}\right\rangle. \tag{21}$$

Step 3. Aggregate the q-ROTrFN $\tilde{\wp}_{ij}$ (or $\tilde{\wp}_{ij}{}'$) for each A_i $(i = 1, 2, \ldots, m)$ applying q-ROTrFEWA (or q-ROTrFEWG) operator as follows:

$$\tilde{\wp}_i = q - ROTrFEWA(\tilde{\wp}_{i1}, \tilde{\wp}_{i2}, \ldots, \tilde{\wp}_{in}); \tag{22}$$

or,

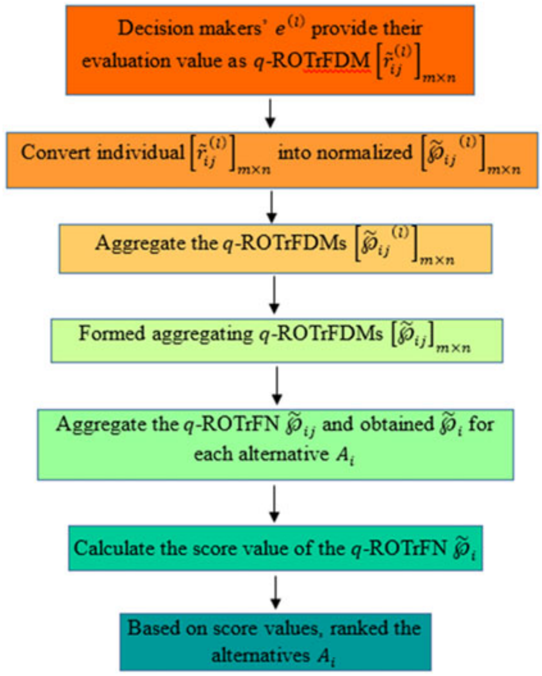
$$\tilde{\wp}_i{}' = q - ROTrFEWG(\tilde{\wp}_{i1}, \tilde{\wp}_{i2}, \ldots, \tilde{\wp}_{in}). \tag{23}$$

Step 4. Compute the score values $S(\tilde{\wp}_i)$ (or $S(\tilde{\wp}_i'))$ of the $\tilde{\wp}_i$ (or $\tilde{\wp}_i'$) for obtaining ordering among the alternatives, A_i.

Step 5. Sort the scores of all the alternatives in descending order, then choose the one with the highest score function.

The flowchart of the above methodology is presented through the Figure 2.

Figure 2
Flowchart of the proposed methodology



5. Illustrative Examples

In this part, two numerical examples, previously studied by Aydin et al. (2020) and Zhao et al. (2017), are given to illustrate the application of the proposed q-ROTrFEWA and q-ROTrFEWG operators.

5.1 Example 4

The human resources department of a corporation is looking to hire a sales consultant. Three human resource specialists will assess the four candidates based on the following criteria:

C_1: experience;
C_2: competencies;
C_3: foreign language skills;
C_4: human relationship management.

where C_1, C_2, and C_3, are benefit type, and last one is cost type. DMs evaluate four candidates $\{A_1, A_2, A_3, A_4\}$ with q-ROTrFNs presented in Tables 1, 2 and 3. Let $\omega = (0.15, 0.25, 0.25, 0.35)^T$ and $\Omega = (0.45, 0.25, 0.30)^T$ represent the weight vector of criteria and DMs, respectively.

Now q-ROTrFEWA and q-ROTrFEWG operators are implemented to choose the ideal candidate.

Step 1: The criteria are classified into two groups: criteria $C_1 - C_3$ are classified as benefit criteria. The cost criterion is C_4. So, by using Eq. (19), the normalized q-ROTrFDMs is obtained, which is shown in Tables 4, 5 and 6, respectively.

Step 2: Apply the q-ROTrFEWA operator, presented in Eq. (20), to aggregate all the normalized q-ROTrFDM $\mathcal{N}^{(l)} = \left[\tilde{\wp}_{ij}{}^{(l)}\right]_{m\times n}$ $(l = 1, 2, 3, 4)$. The integrated q-ROTrFDM, $\mathcal{N} = \left[\tilde{\wp}_{ij}\right]_{m\times n}$ is shown in Table 7.

Step 3: Again, by Eq. (22) and Table 7, the final aggregated values $\tilde{\wp}_i$ of A_i are found as:

Table 1
q-ROTrFDM $e^{(1)}$

	C_1	C_2	C_3	C_4
A_1	$\langle[0.5, 0.6, 0.8, 0.9]; 0.5, 0.6\rangle$	$\langle[0.6, 0.7, 0.8, 0.9]; 0.5, 0.2\rangle$	$\langle[0.4, 0.7, 0.8, 0.9]; 0.7, 0.4\rangle$	$\langle[0.2, 0.3, 0.4, 0.5]; 0.6, 0.9\rangle$
A_2	$\langle[0.2, 0.3, 0.5, 0.6]; 0.3, 0.6\rangle$	$\langle[0.1, 0.3, 0.6, 0.9]; 0.7, 0.2\rangle$	$\langle[0.4, 0.6, 0.7, 0.9]; 0.3, 0.3\rangle$	$\langle[0.5, 0.6, 0.7, 0.8]; 0.4, 0.8\rangle$
A_3	$\langle[0.3, 0.4, 0.5, 0.9]; 0.4, 0.8\rangle$	$\langle[0.2, 0.3, 0.5, 0.7]; 0.6, 0.1\rangle$	$\langle[0.3, 0.4, 0.5, 0.6]; 0.4, 0.7\rangle$	$\langle[0.4, 0.5, 0.7, 0.8]; 0.3, 0.6\rangle$
A_4	$\langle[0.5, 0.7, 0.8, 0.9]; 0.8, 0.4\rangle$	$\langle[0.2, 0.4, 0.6, 0.8]; 0.3, 0.8\rangle$	$\langle[0.4, 0.5, 0.8, 0.9]; 0.8, 0.5\rangle$	$\langle[0.3, 0.5, 0.6, 0.8]; 0.6, 0.4\rangle$

Table 2
q-ROTrFDM $e^{(2)}$

	C_1	C_2	C_3	C_4
A_1	$\langle[0.4, 0.6, 0.7, 0.8]; 0.6, 0.7\rangle$	$\langle[0.6, 0.7, 0.8, 0.9]; 0.1, 006\rangle$	$\langle[0.5, 0.6, 0.7, 0.8]; 0.3, 0.6\rangle$	$\langle[0.4, 0.5, 0.8, 0.9]; 0.4, 0.4\rangle$
A_2	$\langle[0.5, 0.6, 0.7, 0.8]; 0.6, 0.7\rangle$	$\langle[0.5, 0.7, 0.8, 0.9]; 0.3, 0.4\rangle$	$\langle[0.1, 0.3, 0.5, 0.6]; 0.9, 0.5\rangle$	$\langle[0.3, 0.6, 0.7, 0.8]; 0.5, 0.6\rangle$
A_3	$\langle[0.6, 0.7, 0.8, 0.9]; 0.6, 0.9\rangle$	$\langle[0.2, 0.4, 0.5, 0.7]; 0.4, 0.7\rangle$	$\langle[0.6, 0.7, 0.8, 0.9]; 0.2, 0.6\rangle$	$\langle[0.4, 0.5, 0.7, 0.8]; 0.2, 0.3\rangle$
A_4	$\langle[0.5, 0.6, 0.8, 0.9]; 0.5, 0.6\rangle$	$\langle[0.4, 0.5, 0.6, 0.7]; 0.3, 0.8\rangle$	$\langle[0.4, 0.6, 0.8, 0.9]; 0.5, 0.6\rangle$	$\langle[0.3, 0.4, 0.7, 0.9]; 0.6, 0.4\rangle$

Table 3
q-ROTrFDM $e^{(3)}$

	C_1	C_2	C_3	C_4
A_1	$\langle[0.5, 0.6, 0.8, 0.9]; 0.3, 0.7\rangle$	$\langle[0.5, 0.7, 0.8, 0.9]; 0.2, 0.4\rangle$	$\langle[0.4, 0.7, 0.8, 0.9]; 0.2, 0.4\rangle$	$\langle[0.4, 0.7, 0.8, 0.9]; 0.4, 0.4\rangle$
A_2	$\langle[0.6, 0.7, 0.8, 0.9]; 0.4, 0.6\rangle$	$\langle[0.3, 0.5, 0.7, 0.9]; 0.9, 0.5\rangle$	$\langle[0.4, 0.5, 0.7, 0.9]; 0.4, 0.3\rangle$	$\langle[0.3, 0.5, 0.8, 0.9]; 0.6, 0.3\rangle$
A_3	$\langle[0.5, 0.6, 0.7, 0.8]; 0.1, 0.3\rangle$	$\langle[0.2, 0.3, 0.5, 0.7]; 0.5, 0.8\rangle$	$\langle[0.3, 0.4, 0.5, 0.6]; 0.6, 00.7\rangle$	$\langle[0.4, 0.5, 0.7, 0.8]; 0.3, 0.4\rangle$
A_4	$\langle[0.1, 0.3, 0.5, 0.7]; 0.2, 0.7\rangle$	$\langle[0.2, 0.3, 0.4, 0.5]; 0.3, 0.2\rangle$	$\langle[0.1, 0.2, 0.4, 0.5]; 0.3, 0.7\rangle$	$\langle[0.3, 0.5, 0.6, 0.8]; 0.7, 0.2\rangle$

Table 4
Normalized q-ROTrFDM $\mathcal{N}^{(1)}$

	C_1	C_2	C_3	C_4
A_1	$\langle[0.5, 0.6, 0.8, 0.9]; 0.5, 0.6\rangle$	$\langle[0.6, 0.7, 0.8, 0.9]; 0.5, 0.2\rangle$	$\langle[0.4, 0.7, 0.8, 0.9]; 0.7, 0.4\rangle$	$\langle[0.2, 0.3, 0.4, 0.5]; 0.9, 0.6\rangle$
A_2	$\langle[0.2, 0.3, 0.5, 0.6]; 0.3, 0.6\rangle$	$\langle[0.1, 0.3, 0.6, 0.9]; 0.7, 0.2\rangle$	$\langle[0.4, 0.6, 0.7, 0.9]; 0.3, 0.3\rangle$	$\langle[0.5, 0.6, 0.7, 0.8]; 0.8, 0.4\rangle$
A_3	$\langle[0.3, 0.4, 0.5, 0.9]; 0.4, 0.8\rangle$	$\langle[0.2, 0.3, 0.5, 0.7]; 0.6, 0.1\rangle$	$\langle[0.3, 0.4, 0.5, 0.6]; 0.4, 0.7\rangle$	$\langle[0.4, 0.5, 0.7, 0.8]; 0.6, 0.3\rangle$
A_4	$\langle[0.5, 0.7, 0.8, 0.9]; 0.8, 0.4\rangle$	$\langle[0.2, 0.4, 0.6, 0.8]; 0.3, 0.8\rangle$	$\langle[0.4, 0.5, 0.8, 0.9]; 0.8, 0.5\rangle$	$\langle[0.3, 0.5, 0.6, 0.8]; 0.4, 0.6\rangle$

Table 5
Normalized q-ROTrFDM $\mathcal{N}^{(2)}$

	C_1	C_2	C_3	C_4
A_1	$\langle[0.4, 0.6, 0.7, 0.8]; 0.6, 0.7\rangle$	$\langle[0.6, 0.7, 0.8, 0.9]; 0.1, 0.6\rangle$	$\langle[0.5, 0.6, 0.7, 0.8]; 0.3, 0.6\rangle$	$\langle[0.4, 0.5, 0.8, 0.9]; 0.4, 0.4\rangle$
A_2	$\langle[0.5, 0.6, 0.7, 0.8]; 0.6, 0.7\rangle$	$\langle[0.5, 0.7, 0.8, 0.9]; 0.3, 0.4\rangle$	$\langle[0.1, 0.3, 0.5, 0.6]; 0.9, 0.5\rangle$	$\langle[0.3, 0.6, 0.7, 0.8]; 0.6, 0.5\rangle$
A_3	$\langle[0.6, 0.7, 0.8, 0.9]; 0.6, 0.9\rangle$	$\langle[0.2, 0.4, 0.5, 0.7]; 0.4, 0.7\rangle$	$\langle[0.6, 0.7, 0.8, 0.9]; 0.2, 0.6\rangle$	$\langle[0.4, 0.5, 0.7, 0.8]; 0.3, 0.2\rangle$
A_4	$\langle[0.5, 0.6, 0.8, 0.9]; 0.5, 0.6\rangle$	$\langle[0.4, 0.5, 0.6, 0.7]; 0.3, 0.8\rangle$	$\langle[0.4, 0.6, 0.8, 0.9]; 0.5, 0.6\rangle$	$\langle[0.3, 0.4, 0.7, 0.9]; 0.4, 0.6\rangle$

Table 6
Normalized q-ROTrFDM $\mathcal{N}^{(3)}$

	C_1	C_2	C_3	C_4
A_1	$\langle[0.5, 0.6, 0.8, 0.9]; 0.3, 0.7\rangle$	$\langle[0.5, 0.7, 0.8, 0.9]; 0.2, 0.4\rangle$	$\langle[0.4, 0.7, 0.8, 0.9]; 0.2, 0.4\rangle$	$\langle[0.4, 0.7, 0.8, 0.9]; 0.4, 0.4\rangle$
A_2	$\langle[0.6, 0.7, 0.8, 0.9]; 0.4, 0.6\rangle$	$\langle[0.3, 0.5, 0.7, 0.9]; 0.9, 0.5\rangle$	$\langle[0.4, 0.5, 0.7, 0.9]; 0.4, 0.3\rangle$	$\langle[0.3, 0.5, 0.8, 0.9]; 0.3, 0.6\rangle$
A_3	$\langle[0.5, 0.6, 0.7, 0.8]; 0.1, 0.3\rangle$	$\langle[0.2, 0.3, 0.5, 0.7]; 0.5, 0.8\rangle$	$\langle[0.3, 0.4, 0.5, 0.6]; 0.6, 0.7\rangle$	$\langle[0.4, 0.5, 0.7, 0.8]; 0.4, 0.3\rangle$
A_4	$\langle[0.1, 0.3, 0.5, 0.7]; 0.2, 0.7\rangle$	$\langle[0.2, 0.3, 0.4, 0.5]; 0.3, 0.2\rangle$	$\langle[0.1, 0.2, 0.4, 0.5]; 0.3, 0.7\rangle$	$\langle[0.3, 0.5, 0.6, 0.8]; 0.2, 0.7\rangle$

Table 7
Collective q-ROTrFDM using q-ROTrFEWA operator

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
A_1	$\langle[0.4750, 0.6000, 0.7750, 0.8750]; 0.4918, 0.6540\rangle$	$\langle[0.5700, 0.7000, 0.8000, 0.9000]; 0.3896, 0.3265\rangle$	$\langle[0.4250, 0.6750, 0.7750, 0.8750]; 0.5522, 0.4439\rangle$	$\langle[0.3100, 0.4700, 0.6200, 0.7200]; 0.7511, 0.4819\rangle$
A_2	$\langle[0.3950, 0.4950, 0.6400, 0.7400]; 0.4415, 0.6242\rangle$	$\langle[0.2600, 0.4600, 0.6800, 0.9000]; 0.7479, 0.3145\rangle$	$\langle[0.3250, 0.4950, 0.6500, 0.8250]; 0.6359, 0.3416\rangle$	$\langle[0.3900, 0.5700, 0.7300, 0.8300]; 0.6747, 0.4790\rangle$
A_3	$\langle[0.4350, 0.5350, 0.6350, 0.8700]; 0.4376, 0.6372\rangle$	$\langle[0.2000, 0.3250, 0.5000, 0.7000]; 0.5323, 0.3159\rangle$	$\langle[0.3750, 0.4750, 0.5750, 0.6750]; 0.4585, 0.6743\rangle$	$\langle[0.4000, 0.5000, 0.7000, 0.8000]; 0.4988, 0.2712\rangle$
A_4	$\langle[0.3800, 0.5550, 0.7100, 0.8400]; 0.6546, 0.5277\rangle$	$\langle[0.4350, 0.2500, 0.5400, 0.6850]; 0.3000, 0.5496\rangle$	$\langle[0.3100, 0.4350, 0.6800, 0.7800]; 0.6587, 0.5810\rangle$	$\langle[0.3000, 0.4750, 0.6250, 0.8250]; 0.3615, 0.6291\rangle$

$$\tilde{\wp}_1 = \langle[0.4285, 0.5982, 0.7270, 0.8270]; 0.6121, 0.4502\rangle,$$
$$\tilde{\wp}_2 = \langle[0.3420, 0.5125, 0.7270, 0.6840]; 0.6637, 0.4140\rangle,$$
$$\tilde{\wp}_3 = \langle[0.3490, 0.4552, 0.6090, 0.7542]; 0.4905, 0.4070\rangle,$$
$$\tilde{\wp}_4 = \langle[0.3483, 0.4207, 0.6302, 0.7810]; 0.5182, 0.5811\rangle.$$

Step 4: Utilizing Eq. (3), calculate the scores of $\tilde{\wp}_1$, $\tilde{\wp}_2$, $\tilde{\wp}_3$, and $\tilde{\wp}_4$ as $S(\tilde{\wp}_1) = 0.0891$, $S(\tilde{\wp}_2) = 0.1254$, $S(\tilde{\wp}_3) = 0.0274$, and $S(\tilde{\wp}_4) = -0.0312$.

Step 5: Conferring to the score function, using Definition 2.4, alternatives' ranking is achieved as follows:
$A_2 \succ A_1 \succ A_3 \succ A_4$. Thus, the best alternative is A_2.

Now, developed geometric operator, i.e., q-ROTrFEWG is used to aggregate the separable q-ROTrF data into a communal one.

Step 3: Apply the geometric operator, q-ROTrFEWG, to aggregate all the individual q-ROTrFDMs into a collective q-ROTrFDM $\mathcal{N}' = \left[\tilde{\wp}_{ij}'\right]_{m\times n}$, as shown in Table 8.

Step 4: For collecting overall values $\tilde{\wp}_i'$, aggregate all the preference values $\tilde{\wp}_{ij}'$ $(i = 1, 2, \ldots, 4; j = 1, 2, \ldots, 4)$.

$$\tilde{\wp}_1' = \langle[0.4071, 0.5125, 0.5756, 0.8108]; 0.4258, 0.5093\rangle,$$
$$\tilde{\wp}_2' = \langle[0.2991, 0.4922, 0.6775, 0.8258]; 0.5255, 0.4738\rangle,$$
$$\tilde{\wp}_3' = \langle[0.3288, 0.4414, 0.5985, 0.7477]; 0.4212, 0.5941\rangle,$$
$$\tilde{\wp}_4' = \langle[0.2753, 0.4374, 0.6152, 0.7674]; 0.3882, 0.6393\rangle.$$

Step 5: Use the score function, as displayed in Eq. (3), for finding the score value of $\tilde{\wp}_1'$, $\tilde{\wp}_2'$, $\tilde{\wp}_3'$, and $\tilde{\wp}_4'$. The score values are found as $S(\tilde{\wp}_1') = -0.0352$, $S(\tilde{\wp}_2') = 0.0178$, $S(\tilde{\wp}_3') = -0.0792$, and $S(\tilde{\wp}_4') = -0.1100$.

Step 6: Rank the alternatives based on the above score values, $S(\tilde{\wp}_i')$, using Definition 2.4. Alternatives' ordering is obtained as $A_2 \succ A_1 \succ A_3 \succ A_4$. So, the best alternative is identified as A_2.

We can see that the rankings are the same in two cases, viz., using q-ROTrFWA and q-ROTrFWG operators. Hence, the candidate A_2 is the most potential sales consultant over the other three candidates. As q is assigned different values, the developed approach provides more general and versatile properties when combined with Einstein operations. The proposed approach is superior to other recent research works in real practical decision-making situations.

5.2 Example 5

Another MCGDM problem is previously studied by Zhao et al. (2017) which is looking for the best green supplier for one of the essential components in the automobile production process. Suppose a company sets up a panel with three DMs, viz., e_1, e_2 and e_3, whose weighting vector is $\Omega = (0.35, 0.4, 0.25)^T$. Let there be five supplier A_i $(i = 1, 2, 3, 4, 5)$. We have to evaluate the most suitable alternative through the evaluation process on the basis of four criteria: product quality $\mathcal{C}_1$, technology capability $\mathcal{C}_2$, pollution control $\mathcal{C}_3$, and environment management $\mathcal{C}_4$, whose weighting vector is $\omega = (0.2, 0.1, 0.3, 0.4)^T$), and construct the following three normalized intuitionistic trapezoidal fuzzy decision matrices, $\mathcal{N}^{(l)} = \left(\tilde{\wp}_{ij}^{(l)}\right)_{5\times 4}$ $(l = 1, 2, 3)$ as shown in Tables 9, 10 and 11.

Table 8
Collective q-ROTrFDM using q-ROTrFEWG operator

	C_1	C_2	C_3	C_4
A_1	$\langle[0.4729, 0.6000, 0.7737, 0.8739]; 0.4512, 0.6597\rangle$	$\langle[0.5681, 0.7000, 0.8000, 0.9000]; 0.2559, 0.4264\rangle$	$\langle[0.4229, 0.6735, 0.7737, 0.8739]; 0.3965, 0.4681\rangle$	$\langle[0.2928, 0.4395, 0.5856, 0.6908]; 0.5975, 0.5107\rangle$
A_2	$\langle[0.3497, 0.4600, 0.6262, 0.7281]; 0.3908, 0.6288\rangle$	$\langle[0.2079, 0.4322, 0.6753, 0.9000]; 0.6295, 0.3854\rangle$	$\langle[0.2828, 0.4777, 0.6435, 0.8132]; 0.4428, 0.3723\rangle$	$\langle[0.3775, 0.5681, 0.7286, 0.8288]; 0.5678, 0.5006\rangle$
A_3	$\langle[0.4158, 0.5196, 0.6221, 0.8688]; 0.2950, 0.7706\rangle$	$\langle[0.2000, 0.3224, 0.5000, 0.7000]; 0.5147, 0.6331\rangle$	$\langle[0.3568, 0.4601, 0.5623, 0.6640]; 0.3826, 0.6784\rangle$	$\langle[0.4000, 0.5000, 0.7000, 0.8000]; 0.4495, 0.2813\rangle$
A_4	$\langle[0.3085, 0.5224, 0.6948, 0.8346]; 0.4839, 0.5735\rangle$	$\langle[0.2378, 0.3880, 0.5313, 0.6720]; 0.3000, 0.7233\rangle$	$\langle[0.2639, 0.3975, 0.6498, 0.7545]; 0.5425, 0.5993\rangle$	$\langle[0.3000, 0.4729, 0.6236, 0.8239]; 0.3256, 0.6342\rangle$

After evaluation, the final score values of alternatives are achieved by the proposed methodology as shown in Table 12.

5.2.1 Result and discussion

Using q-ROTrFEWA and q-ROTrFEWG operators, the achieved results are discussed by varying rung parameter, q, continuously in a specified interval as shown in Figures 3 and 4.

Using the q-ROTrFEWA operator and adjusting the rung parameter, q, between 1 and 10, Figure 2 provides the graphical clarification of the score values of the alternatives.

As the value of q changes from 1 to 10, it is noticed in Figure 3 that several ranking results are obtained.

When $q \in [1, 2.1061]$, the alternative's rank is achieved as $A_2 > A_5 > A_3 > A_4 > A_1$.

When $q \in [2.377, 3.3901]$, the alternative's rank is achieved as $A_2 > A_5 > A_4 > A_3 > A_1$. And when $q \in [3.3901, 10]$, the alternative's rank is achieved as $A_2 > A_4 > A_5 > A_3 > A_1$.

Further, Figure 4 signifies the graphical interpretation of score values of the alternatives by varying the rung parameter, q, between 1 and 10, using q-ROTrFEWG operator.

From Figure 4, it is experimental that many ordering results are obtained, as q changes from 1 to 10.

When $q \in (1, 4.1443)$, the ranking of alternatives is achieved as $A_2 > A_5 > A_3 > A_4 > A_1$.

And when $q \in (4.1443, 10)$, the ranking of alternative is achieved as $A_2 > A_3 > A_5 > A_4 > A_1$. So in all cases, we obtained that the A_2 is the best alternative and A_1 is the worst alternative.

5.3 Comparative analysis

The new method is compared to various existing methods in this section.

First, we have compared the results of Example 4 with Aydin's (Aydin et al., 2020) method. The rankings of the Aydin's method (Aydin et al., 2020) and our method are presented in Table 13. The rankings of both Aydin's (Aydin et al., 2020) and proposed methods are the same. However, the score value difference of two consecutive alternatives (rank-wise) in the proposed method is higher than existing Aydin's method (Aydin et al., 2020) almost everywhere.

Next, Example 5 is compared with some existing operators such as ITFWAA (Wang & Zhang, 2009), ITFWG (Wu & Cao, 2013), ITFEWA and ITFEWG (Zhao et al., 2017), PTFWA (Shakeel et al., 2019), and PTFEWG (Shakeel et al., 2019) operators. The score values and rankings of alternatives are described in Table 14.

Table 14 shows that the rankings of the alternatives acquired by different operators are almost identical to the proposed operators, indicating that the proposed ranking technique is effective.

The suggested MCDM strategy based on q-ROTrFN AOs is found to have two key advantages. On the one hand, the q-ROTrFNs used in this work can be used to represent assessment data in a variety of ways. They can also manage a variety of specific situations where a variety of alternative values can cause confusion about the best option while maintaining the accuracy of the original data. The proposed operators, on the other hand, are based on Einstein t-norms and t-conorms, which makes them more beneficial than regular algebraic operations. Furthermore, the proposed approach can provide a variety of options for implementing decision-making with q-ROTrF data. This circumstance can prevent the preferred information from being lost or distorted. As a result, the final outcomes are more closely related to real-world decision-making issues.

Table 9
Normalized decision matrix given by DM e_1

	C_1	C_2	C_3	C_4
A_1	⟨[0.5, 0.6, 0.7, 0.8]; 0.5, 0.4⟩	⟨[0.1,0.2,0.3,0.4]; 0.6,0.3⟩	⟨[0.5,0.6,0.8,0.9]; 0.3,0.6⟩	⟨[0.4,0.5,0.6,0.7]; 0.2,0.7⟩
A_2	⟨[0.6, 0.7, 0.8, 0.9]; 0.7, 0.3⟩	⟨[0.5,0.6,0.7,0.8]; 0.7,0.2⟩	⟨[0.4,0.5,0.7,0.8]; 0.7,0.2⟩	⟨[0.5,0.6,0.7,0.9]; 0.4,0.5⟩
A_3	⟨[0.1, 0.2, 0.4, 0.5]; 0.6, 0.4⟩	⟨[0.2,0.3,0.5,0.6]; 0.5,0.4⟩	⟨[0.5,0.6,0.7,0.8]; 0.5,0.3⟩	⟨[0.3,0.5,0.7,0.9]; 0.2,0.3⟩
A_4	⟨[0.3, 0.4, 0.5, 0.6]; 0.8, 0.1⟩	⟨[0.1,0.3,0.4,0.5]; 0.6,0.3⟩	⟨[0.1,0.3,0.5,0.7]; 0.3,0.4⟩	⟨[0.6,0.7,0.8,0.9]; 0.2,0.6⟩
A_5	⟨[0.2, 0.3, 0.4, 0.5]; 0.6, 0.2⟩	⟨[0.3,0.4,0.5,0.6]; 0.4,0.3⟩	⟨[0.2,0.3,0.4,0.5]; 0.7,0.1⟩	⟨[0.5,0.6,0.7,0.8]; 0.1,0.3⟩

Table 10
Normalized decision given by DM e_2

	C_1	C_2	C_3	C_4
A_1	⟨[0.4,0.5,0.6,0.7]; 0.4,0.3⟩	⟨[0.1,0.2,0.3,0.4]; 0.5,0.2⟩	⟨[0.4,0.5,0.7,0.8]; 0.2,0.5⟩	⟨[0.3,0.4,0.5,0.6]; 0.1,0.6⟩
A_2	⟨[0.5,0.6,0.7,0.8]; 0.6,0.2⟩	⟨[0.4,0.5,0.6,0.7]; 0.6,0.1⟩	⟨[0.3,0.4,0.6,0.7]; 0.6,0.1⟩	⟨[0.3,0.4,0.6,0.8]; 0.3,0.4⟩
A_3	⟨[0.1,0.2,0.3,0.4]; 0.5,0.3⟩	⟨[0.1,0.2,0.4,0.5]; 0.4,0.3⟩	⟨[0.4,0.5,0.6,0.7]; 0.4,0.2⟩	⟨[0.2,0.4,0.6,0.8]; 0.5,0.2⟩
A_4	⟨[0.2,0.3,0.4,0.5]; 0.7,0.1⟩	⟨[0.1,0.2,0.3,0.5]; 0.5,0.2⟩	⟨[0.1,0.2,0.4,0.6]; 0.2,0.3⟩	⟨[0.5,0.6,0.7,0.8]; 0.1,0.5⟩
A_5	⟨[0.1,0.2,0.3,0.4]; 0.5,0.1⟩	⟨[0.2,0.3,0.4,0.5]; 0.3,0.2⟩	⟨[0.1,0.2,0.3,0.4]; 0.6,0.2⟩	⟨[0.4,0.5,0.6,0.7]; 0.4,0.2⟩

Table 11
Normalized decision by DM e_3

	C_1	C_2	C_3	C_4
A_1	⟨[0.6,0.7,0.8,0.9]; 0.4,0.5⟩	⟨[0.2,0.3,0.4,0.5]; 0.5,0.4⟩	⟨[0.6,0.7,0.9,1.0]; 0.2,0.7⟩	⟨[O.5,0.6,0.7,0.8]; 0.1,0.8⟩
A_2	⟨[0.7,0.8,0.9,1.0]; 0.6,0.4⟩	⟨[0.6,0.7,0.8,0.9]; 0.6,0.3⟩	⟨[0.5,0.6,0.8,0.9]; 0.6,0.3⟩	⟨[0.6,0.7,0.8,1.0]; 0.3,0.6⟩
A_3	⟨[0.2,0.3,0.5,0.6]; 0.5,0.5⟩	⟨[0.3,0.4,0.6,0.7]; 0.4 0.5⟩	⟨[0.6,0.7,0.8,0.9]; 0.4,0.4⟩	⟨0.4,0.6,0.8,1.0]; 0.5,0.4⟩
A_4	⟨[0.4,0.5,0.6,0.7]; 0.7,0.2⟩	⟨[0.2,0.4,0.5,0.6]; 0.5,0.4⟩	⟨[0.2,0.4,0.6,0.8]; 0.2,0.5⟩	⟨[0.7,0.8,0.9,1.0]; 0.6,0.3⟩
A_5	⟨[0.3,0.4,0.5,0.6]; 0.5,0.3⟩	⟨[0.4,0.5,0.6,0.7]; 0.3,0.4⟩	⟨[0.3,0.4,0.5,0.6]; 0.6,0.2⟩	⟨[0.6,0.7,0.8,0.9]; 0.4,0.4⟩

Table 12
Score values obtained through the proposed method

Proposed method	Score values					Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	
q-ROTrFEWA	−0.0646	0.0989	0.0377	0.0513	0.0571	$A2 > A_5 > A_4 > A3 > A_1$
q-ROTrFEWG	−0.1143	0.0392	0.0177	−0.0237	0.0195	$A2 > A_5 > A3 > A_4 > A_1$

Figure 3
Effect of rung parameter (q) on q-ROTrFEWA operator

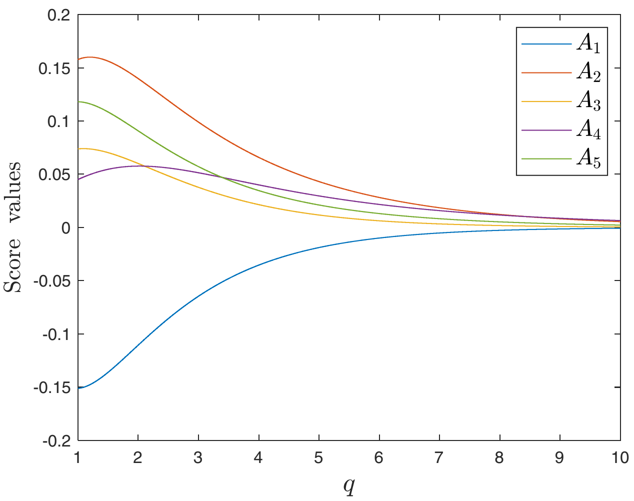


Figure 4
Effect of rung parameter (q) on q-ROTrFEWG operator

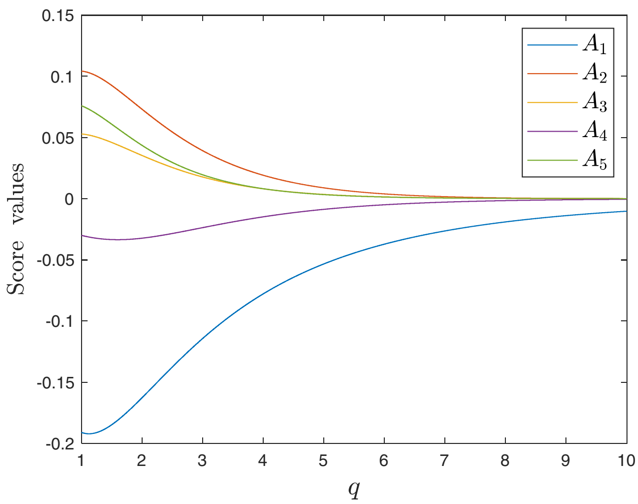


Table 13
Comparison of score value and ranking for Example 4

Method	Score values				Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	
Aydin et al. (2020) method	0.100	0.143	0.093	0.048	$A_2 > A_1 > A_3 > A_4$
q-ROTrFEWA	0.0891	0.1254	0.0274	−0.0312	$A_2 > A_1 > A_3 > A_4$
q-ROTrFEWG	−0.0352	0.0178	−0.0792	−0.1100	$A_2 > A_1 > A_3 > A_4$

Table 14
Comparison based on Example 5

Operators	Score values				Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	
ITFEWA (Shakeel et al., 2019)	−0.1512	0.1577	0.0738	0.0450	$A2 > A_5 > A_3 > A_4 > A_1$
PTFEWG (Shakeel et al., 2019)	−0.1627	0.0730	0.0353	−0.0323	$A2 > A_5 > A_3 > A_4 > A_1$
q-ROTrFEWA	−0.0665	0.0958	0.0370	0.0466	$A2 > A_5 > A_4 > A_3 > A_1$
q-ROTrFEWG	−0.1108	0.0412	0.0181	−0.0224	$A2 > A_5 > A_3 > A_4 > A_1$

6. Conclusion

This research looks into the MCGDM problem using assessment values in the form of q-ROTrFN and proposes a novel MCGDM approach. On the basis of Einstein t-conorm and t-norm, some basic operation laws for q-ROTrFNs are defined. Two AOs based on Einstein operations, q-ROTrFEWA and q-ROTrFEWG, are introduced in this study. Their appropriate characteristics, viz., idempotency, monotonicity, and boundedness, are also defined. Two motives for these expansions are as follows: (1) q-ROTrFN comprise more information than other kinds of fuzzy numbers and (2) Einstein averaging and Einstein geometric operators have the capability to catch the value if there are outliers of data. So, merging Einstein averaging and geometric operators and q-ROTrFN provides advantages in the MCGDM problem. This article tackles a personnel selection problem to demonstrate the applicability of the presented methodology. It proves that the proposed methodology can handle the MCGDM problem efficiently.

However, our study still has some limitations. Our methodology will be unable to determine the best alternative when DMs' and criteria weight are totally unknown. The developed AOs are insufficient to evaluate information when DMs hesitate to make the decision. Our proposed method neglects the preference information of DMs.

In the future, we will develop the concept of hesitant q-ROTrFN. Moreover, various decision-making methods will be extended to handle hesitant q-ROTrFNs. The proposed operators could be used in a variety of domains, viz., bipolar fuzzy (Poulik & Ghorai, 2021), cubic fuzzy (Riaz et al., 2021b), T-spherical fuzzy (Chen, 2021), and other environments. We will continue to work on expanding and applying the proposed operators to other disciplines, such as medical diagnostics (Šušteršič et al., 2021) and pattern recognition (Sánchez-Salgado et al., 2021) and, in the future.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

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Uncertainty evaluations through interval-valued Pythagorean hesitant fuzzy Archimedean aggregation operators in multicriteria decision making

Arun Sarkar[a], Nayana Deb[b] and Animesh Biswas[b,*]
[a]*Department of Mathematics, Heramba Chandra College, Kolkata, India*
[b]*Department of Mathematics, University of Kalyani, Kalyani, India*

Abstract. In many cases, use of Pythagorean hesitant fuzzy sets may not be sufficient to characterize uncertain information associated with decision making problems. From that view point the concept of interval-valued Pythagorean hesitant fuzzy sets are introduced in this paper. Considering the flexibility with the general parameters, Archimedean t-conorms and t-norms are applied to develop several operational laws in interval-valued Pythagorean hesitant fuzzy environment. Some characteristics of the developed operators are presented. The newly developed operators are used to derive a methodology for solving multicriteria decision making problems with interval-valued Pythagorean hesitant fuzzy information. Finally, two illustrative examples are provided to establish the validity of the proposed approach and are compared with the existing technique to exhibit its flexibility and effectiveness.

Keywords: Multicriteria decision making, interval-valued Pythagorean fuzzy set, Pythagorean hesitant fuzzy set, Archimedean t-conorm and t-norm, weighted averaging and geometric operators

1. Introduction

Human perceptions are frequently involved with indeterminacy and indecisiveness. Most of the time, it becomes difficult for the decision makers (DMs) to exert their opinion using a single crisp value. Under this situation, fuzzy sets [1] came into account. Several variants of fuzzy sets, viz., intuitionistic fuzzy sets (IFSs) [2–4], Pythagorean fuzzy sets (PFSs) [5,6], etc. appeared, thereafter, as the extensions of fuzzy sets and been implemented successfully in solving multi-criteria decision making (MCDM) [7–10] problems. PFS is more general than IFS due to the fact that PFS consists membership degree along with non-membership degree having their square sum less than or equals to 1; whereas, the sum of membership and non-membership degrees is less than or equals to 1 in IFS. So, in terms of flexibility and dealing with uncertainties, PFS can express uncertain information more effectively. For example, if a DM provides 0.7, as membership value and 0.6, as non-membership value, PFS can successfully deal with such values. But IFS fails to consider those values. For this useful characteristics, PFSs are applied to solve real life decision making problems, e.g., pattern recognition [11], supplier selection [12], transportation problem [13], risk evaluation [14,15], and other emerging areas. Following the concept of interval-valued fuzzy sets, Peng and Yang [16] introduced the idea of interval-valued Pythagorean fuzzy (IVPF) sets (IVPFSs), where the membership

*Corresponding author: Animesh Biswas, Department of Mathematics, University of Kalyani, Kalyani – 741235, India. E-mail: abiswaskln@rediffmail.com.

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and non-membership values of an element to a given set are represented by the subintervals of [0, 1]. With the use of IVPFS, DMs can evaluate the input data in a more convenient way than IFS as well as PFS.

Sometimes, in many MCDM situations, DMs face difficulties in assigning membership value corresponding to some element; rather they prefer to express their opinions using a set of possible values. To address such situations, Torra and Narukawa [17] and Torra [18] proposed another generalization of fuzzy sets, called hesitant fuzzy sets (HFSs). It offers the DMs flexibility to assign the membership degree using several possible crisp numbers lying in between [0, 1]. In many MCDM problems HFSs [19–24] are applied efficiently. Combining the idea of HFS and PFS, Wei et al. [25] defined Pythagorean hesitant fuzzy (PHF) set (PHFS). PHFS is found to be very much useful in modelling uncertainties associated with real life problems.

Inspired by the concept of IVPFS, PHFSs have been extended in this article to generate interval-valued PHF (IVPHF) set (IVPHFS). It possesses greater capability of capturing uncertainties than existing variants of fuzzy sets. It is to be noted here that, if lower and upper limits of the intervals coincide with each other, an IVPHFS becomes PHFS. Again, if each membership and non-membership degrees of the elements of IVPHFS are expressed by single intervals, IVPHFS reduces to IVPFS. Therefore, it can be concluded that IVPHFS is a more generalised version of the other variants of fuzzy sets as described above.

Moreover, Archimedean t-norm and t-conorm (At-N&t-CN) satisfy all the properties of t-norm (t-N) and t-conorm (t-CN) to aggregate input arguments. Algebraic, Einstein, Hamacher, Frank classes are widely used At-N&t-CNs for developing aggregation operators. Motivated by the aforementioned discussions, in this article various weighted averaging (WA) and weighted geometric (WG) aggregation operators are developed using At-N&t-CN based operations on IVPHF numbers (IVPHFNs). Following those operations, IVPHF WA (IVPHFWA), IVPHF Einstein WA (IVPHFEWA), IVPHF Hamacher WA (IVPHFHWA) and IVPHF Frank WA (IVPHFFWA) operators, and their WG variants are presented. Several desirable properties of those developed operators are also investigated. Using those operators, an approach for solving MCDM problems under the IVPHF environment is presented to show the efficiency and usefulness of the proposed operators. Literature review on the allied topics is performed in the next section.

2. Literature review

It is well known that PFS [5] is one of the most useful tools to resolve ambiguous information of MCDM problems. Zhang and Xu [26] provided standard arithmetic operations on PFSs. For aggregating PFSs, Yager [6] proposed a series of WA and WG aggregation operators. Several aggregation operators [27,28] and methods [29,30] for solving MCDM problems are developed on PFSs by numerous researchers.

After the development of IVPFS [16], an accuracy function for ranking the IVPF numbers (IVPFNs) was developed by Garg [31], considering the hesitancy degree of IVPFNs for solving MCDM problems. Rahman et al. [32] introduced a class of IVPF geometric aggregation operators for IVPFNs, viz. IVPF WG, ordered WG, hybrid geometric operators. Biswas and Sarkar [33] introduced similarity measures based on point operators for IVPFSs. Technique for order of preference by similarity to ideal solution (TOPSIS) method was introduced by Garg [34] under IVPF environment. Further, Garg [35] proposed some new IVPF aggregation operators on the basis of novel exponential operational laws of IVPFS. Generalised IVPF aggregation operators are introduced by Rahman and Abdullah [36] for IVPF multicriteria group decision making (MCGDM) problems. An IVPF extended TOPSIS method was developed by Yu et al. [37] for sustainable supplier selection under MCGDM context. Tang et al. [38] proposed IVPF MCDM approach based on Muirhead mean and applied it on green supplier selection. Some induced IVPF aggregation operators are developed by Rahman et al. [39] for tackling uncertainties in MCDM situation.

Following the concepts of HFS and PFSs, PHFS [25] has now become another emerging area of research. An application to PHFS with incomplete weight information was proposed by Khan et al. [40] for solving group decision making problems. Wei et al. [25] developed PHF Hamacher WA (PHFHWA) and PHF Hamacher WG (PHFHWG) aggregation functions to aggregate the PHF information. A new approach for PHF TOPSIS method was presented by Khan et al. [41] in the context of solving MCDM problems. Recently, Sarkar and Biswas [42] proposed At-N&t-CN based operations [43] on PHF numbers (PHFNs) to construct two types of aggregation operators such as At-N&t-CN-based PHF WA and WG operators. An extension of VIKOR method was developed by Khan et al. [44] for PHF MCDM context. Further, Khan et al. [45] introduced Choquet integral to PHFSs and generated a series of aggregation operators, viz., PHF Choquet integral averaging operator, PHF Choquet integral geometric operator, generalised PHF

Choquet integral averaging operator and generalised PHF Choquet integral geometric operator. Khan et al. [46] also developed some hybrid aggregation operators to aggregate PHF data for MCGDM. From the literature survey, it had been found that the combination of IVPFS and PHFS had not been done yet. Thus it would be significant to develop a series of aggregation operators, viz., IVPHFWA, IVPHFEWA, IVPHFHWA, IVPHFFWA, IVPHF WG (IVPHFWG), IVPHF Einstein WG (IVPHFEWG), IVPHF Hamacher WG (IVPHFHWG) and IVPHF Frank WG (IVPHFFWG) operators based on At-N&t-CN for evaluating IVPHF information. In addition, the novelty of this research work is to deliver an approach for solving MCDM problems under the IVPHF environment to show the applicability of the developed operators. Two illustrative examples are provided for the sake of application of the developed MCDM method.

3. Preliminaries

In this section some elementary definitions related to the development of At-N&t-CN based operators are briefly discussed.

3.1. IVPFS

Peng and Yang [16] introduced the concept of IVPFSs, which is presented as follows:

Definition 1. Let X be a fixed set. An IVPFS, $\tilde{P}$ in X is given by

$$\tilde{P} = \left\{ \left\langle x, \left[\mu^l_{\tilde{P}}(x), \mu^u_{\tilde{P}}(x)\right], \left[\nu^l_{\tilde{P}}(x), \nu^u_{\tilde{P}}(x)\right] \right\rangle | x \in X \right\},$$

where two closed intervals $[\mu^l_{\tilde{P}}(x), \mu^u_{\tilde{P}}(x)]$ and $[\nu^l_{\tilde{P}}(x), \nu^u_{\tilde{P}}(x)]$ are subintervals of $[0,1]$, denote the degree of membership and degree of non-membership of the element, $x \in X$ to the set $\tilde{P}$, respectively, with the condition $0 \leqslant \left(\mu^u_{\tilde{P}}(x)\right)^2 + \left(\nu^u_{\tilde{P}}(x)\right)^2 \leqslant 1$.

For computational convenience, Peng and Yang [16] used the notation, $\tilde{p} = \left(\left[\mu^l, \mu^u\right], \left[\nu^l, \nu^u\right]\right)$ to represent an IVPFN.

3.2. PHFS

Inspired by the idea of PFSs and HFSs, Wei et al. [25] elaborated PFS to PHFS, which is structured with a set of several possible PFNs, symbolically defined as follows:

Definition 2. Let X be a universe of discourse. A PHFS on X is defined as:

$$\hat{A} = \left\{ \left\langle x, h_{\hat{A}}(x) \right\rangle | x \in X \right\},$$

where $h_{\hat{A}}(x)$ is a set of possible PFNs.

For convenience, Wei et al. [25] called $\hat{a} = h_{\hat{A}}(x) = \bigcup_{(\gamma,\eta)\in h_{\hat{A}}(x)} \{(\gamma,\eta)\}$ as a PHFN, and it is denoted by $\hat{a} = h_{\hat{A}} = (\mu,\nu)$.

For ordering of PHFNs, the score and accuracy functions of PHFNs are presented [25] as follows:

Let $\hat{a} = (\mu,\nu)$ be any PHFN, the score function, $S(\hat{a})$ of $\hat{a}$ be presented as follows:

$$S(\hat{a}) = \frac{1}{2}\left(1 + \frac{1}{l_h}\sum_{(\gamma,\eta)\in(\mu,\nu)} \left(\gamma^2 - \eta^2\right)\right),$$

and the accuracy function, $A(\hat{a})$ of $\hat{a}$ be expressed as follows:

$$A(\hat{a}) = \frac{1}{l_h}\sum_{(\gamma,\eta)\in(\mu,\nu)} \left(\gamma^2 + \eta^2\right),$$

where l_h are the number of elements in h.

Definition 3. For two PHFNs, $\hat{a}_i = (\mu_i, \nu_i)$ $(i = 1, 2)$, the ranking can be done in the following manners [25]:

- If $S(\hat{a}_1) > S(\hat{a}_2)$, then $\hat{a}_1 > \hat{a}_2$;
- If $S(\hat{a}_1) = S(\hat{a}_2)$, then

(1) If $A(\hat{a}_1) > A(\hat{a}_2)$, then $\hat{a}_1 > \hat{a}_2$;
(2) If $A(\hat{a}_1) = A(\hat{a}_2)$, then $\hat{a}_1 = \hat{a}_2$.

3.3. At-N&t-CN operations

Definition 4. Satisfying the following properties, a function $T : [0,1] \times [0,1] \to [0,1]$ is known as a t-N [43]:

(i) $T(a,1) = a$ for all a.
(ii) $T(b,d) \leqslant T(b^*, d^*)$ for $b \leqslant b^*$ and $d \leqslant d^*$
(iii) $T(a,b) = T(b,a)$ for all a and b.
(iv) $T(a, T(b,d)) = T(T(a,b), d)$ for all a, b and d.
(v) T is a continuous function.
(vi) $T(a,a) < a$.

Definition 5. A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-CN if the following criteria are satisfied [43]:

(i) $S(a,0) = a$ for all a.
(ii) $S(b,d) \leqslant S(b^*, d^*)$ when $b \leqslant b^*$ and $d \leqslant d^*$
(iii) $S(a,b) = S(b,a)$ for all a and for all b.
(iv) $S(a, S(b,d)) = S(S(a,b), d)$ for all a, b and d.
(v) S is a continuous function.
(vi) $S(a,a) > a$.

It is well known that At-N&t-CN operations are expressed using increasing or decreasing generators as follows [47]:

Definition 6. An Archimedean t-N (At-N), T is formulated using a decreasing function, f as

$$T(a,b) = f^{(-1)}(f(a) + f(b)). \tag{1}$$

Similarly, using increasing function, g, an Archimedean t-CN (At-CN) S is represented as

$$S(a,b) = g^{(-1)}(g(a) + g(b)) \text{ for all } a, b \in [0,\ 1]. \tag{2}$$

The generators have the relationship $g(t) = f(1-t)$ for all $a, b, t \in [0,\ 1]$.

Several t-Ns and t-CNs are derived by Klement [47] using different forms of increasing and decreasing functions which are furnished below:

(1) For $f(t) = -\log t$, At-N&t-CN is reduced to algebraic t-N and t-CN [48], and are defined as $S^A(a,b) = a + b - ab$, $T^A(a,b) = ab$.
(2) When $f(t) = \log\left(\frac{2-t}{t}\right)$, $g(t) = \log\left(\frac{2-(1-t)}{1-t}\right)$, $f^{-1}(t) = \frac{2}{e^t+1}$, $g^{-1}(t) = 1 - \frac{2}{e^t+1}$, A$t$-N&$t$-CN is reduced to Einstein t-N and t-CN [48] and are defined as $S^E(a,b) = \frac{a+b}{1+ab}$, $T^E(a,b) = \frac{ab}{1+(1-a)(1-b)}$.
(3) Let $f(t) = \log\left(\frac{\psi+(1-\psi)t}{t}\right)$, $\psi > 0$, $g(t) = \log\left(\frac{\psi+(1-\psi)(1-t)}{1-t}\right)$, $f^{-1}(t) = \frac{\psi}{e^t+\psi-1}$, $g^{-1}(t) = 1 - \frac{\psi}{e^t+\psi-1}$, then A$t$-N&$t$-CN called Hamacher t-N and t-CN [48] and are presented as $S^H_\psi(a,b) = \frac{a+b-ab-(1-\psi)ab}{1-(1-\psi)ab}$, $T^H_\psi(a,b) = \frac{ab}{\psi+(1-\psi)(a+b-ab)}$, $\psi > 0$.
When $\psi = 1$, the Hamacher t-N and t-CN reduce to algebraic t-N and t-CN, respectively. Again, for $\psi = 2$, the Hamacher t-N and t-CN reduce to Einstein t-N and t-CN, respectively.
(4) For $f(t) = \log\left(\frac{\tau-1}{\tau^t-1}\right)$, $\tau > 1$, $g(t) = \log\left(\frac{\tau-1}{\tau^{1-t}-1}\right)$, $f^{-1}(t) = \frac{\log\left(\frac{\tau-1+e^t}{e^t}\right)}{\log\tau}$, $g^{-1}(t) = 1 - \frac{\log\left(\frac{\tau-1+e^t}{e^t}\right)}{\log\tau}$, an A$t$-N&$t$-CN are named as Frank t-N and t-CN [48], respectively, and are defined as $S^F_\tau(a,b) = 1 - \log_\tau\left(1 + \frac{(\tau^{1-a}-1)(\tau^{1-b}-1)}{\tau-1}\right)$, $T^F_\tau(a,b) = \log_\tau\left(1 + \frac{(\tau^a-1)(\tau^b-1)}{\tau-1}\right)$, $\tau > 1$. Especially, if $\tau \to 1$, $\lim_{\tau\to1} f(t) = -\log t$.

3.4. At-N&t-CN operations on PHF environment

Sarkar and Biswas [42] recently introduced At-N&t-CN operations on PHF environment, which are defined as follows:

Definition 7. Let $\hat{a} = (\mu, \nu)$ and $\hat{a}_i = (\mu_i, \nu_i)$ $(i = 1, 2)$ be three arbitrary PHFNs, and λ be any positive number.

Then the operational laws for the PHFNs based on At-N&t-CN are defined as follows:

Let $\hat{a} = (\mu, \nu)$ and $\hat{a}_i = (\mu_i, \nu_i)$ $(i = 1, 2)$ be three arbitrary PHFNs and λ be any positive number. Then the operational laws for the PHFNs based on At-N&t-CN are defined as follows:

(1) $\hat{a}_1 \oplus_A \hat{a}_2 = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2}} \left\{ \left(\sqrt{S({\gamma_1}^2, {\gamma_2}^2)}, \sqrt{T({\eta_1}^2, {\eta_2}^2)} \right) \right\}$

$$= \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2}} \left\{ \left(\sqrt{g^{-1}(g({\gamma_1}^2) + g({\gamma_2}^2))}, \sqrt{f^{-1}(f({\eta_1}^2) + f({\eta_2}^2))} \right) \right\};$$

(2) $\hat{a}_1 \otimes_A \hat{a}_2 = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2}} \left\{ \left(\sqrt{T({\gamma_1}^2, {\gamma_2}^2)}, \sqrt{S({\eta_1}^2, {\eta_2}^2)} \right) \right\}$

$$= \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2}} \left\{ \left(\sqrt{f^{-1}(f({\gamma_1}^2) + f({\gamma_2}^2))}, \sqrt{g^{-1}(g({\eta_1}^2) + g({\eta_1}^2))} \right) \right\};$$

(3) $\lambda\hat{a} = \bigcup_{(\gamma,\eta)\in(\mu,\nu)} \left\{ \left(\sqrt{g^{-1}(\lambda g(\gamma^2))}, \sqrt{f^{-1}(\lambda f(\eta^2))} \right) \right\};$

(4) $\hat{a}^\lambda = \bigcup_{(\gamma,\eta)\in(\mu,\nu)} \left\{ \left(\sqrt{f^{-1}(\lambda f(\gamma^2))}, \sqrt{g^{-1}(\lambda g(\eta^2))} \right) \right\}.$

Based on the above concepts, the notion of IVPHFSs is introduced in the following section.

4. Development of IVPHFS

Sometimes, it becomes inadequate to describe an uncertain situation using PHF information. To tackle this type of situation, the concept of IVPHFS is introduced. In this section, PHFS is extended with the use of interval numbers to construct IVPHFS. Also, score and accuracy functions on them are defined. Furthermore, some operations based on At-N&t-CN are proposed in this section.

4.1. IVPHFS

Definition 8. Let X be a universe of discourse, an IVPHFS, $\tilde{K}$ on X is expressed as:

$$\tilde{K} = \{\langle x, h_{\tilde{K}}(x)\rangle | x \in X\}, \tag{3}$$

in which $h_{\tilde{K}}(x) = \{([\gamma^l(x), \gamma^u(x)], [\eta^l(x), \eta^u(x)])\}$ is a set containing some IVPFNs, where $[\gamma^l x), \gamma^u(x)]$, $[\eta^l(x), \eta^u(x)] \subseteq [0,1]$ satisfies the condition $0 \leqslant ((\gamma^u(x))^+)^2 + ((\eta^u(x))^+)^2 \leqslant 1$, $(\gamma^u(x))^+ = max\{\gamma^u(x)\}$ and $(\eta^u(x))^+ = max\{\eta^u(x)\}$, for all $x \in X$.

Thus, IVPHFS represents a HFS whose membership degrees are expressed by several IVPFNs.

For convenience, $h_{\tilde{K}} = \{([\gamma^l, \gamma^u], [\eta^l, \eta^u])\}$ is named as an IVPHFN and is denoted by $\tilde{k}$. The set of all IVPHFNs is given by $\tilde{K}$.

To illustrate the above definition, the following example is presented.

Example 1. Let the possible assessment values of an element $x \in X$ are represented by the IVPFNs, $([0.1, 0.3]$, $[0.3, 0.4])$, $([0.5, 0.6]$, $[0.4, 0.5])$ and $([0.7, 0.8]$, $[0.4, 0.6])$, where $[0.1, 0.3]$, $[0.5, 0.6]$ and $[0.7, 0.8]$ indicate the possible degree of membership and $[0.3, 0.4]$, $[0.4, 0.5]$ and $[0.4, 0.6]$ indicate the possible degree of non-membership for the element $x \in X$. Then the IVPHFN is represented as an element of the set $\tilde{K}$ as $\tilde{k} = \{([0.1, 0.3], [0.3, 0.4]), ([0.5, 0.6], [0.4, 0.5]), ([0.7, 0.8], [0.4, 0.6])\}$ in which $(\gamma^u)^+ = \max\{\gamma^u(x)\} = \max\{0.3, 0.6, 0.8\} = 0.8$, $(\eta^u)^+ = \max\{\eta^u(x)\} = \max\{0.4, 0.5, 0.6\} = 0.6$, satisfying the condition $0 \leqslant (0.8)^2 + (0.6)^2 \leqslant 1$.

To compare any two IVPHFNs, the score and accuracy functions are presented as follows:

Definition 9. For an IVPHFN, $\tilde{k}$, the score function, $S\left(\tilde{k}\right)$ is described as:

$$S\left(\tilde{k}\right)=\frac{1}{2}\left(1+\frac{1}{2}\left(\frac{1}{\left|\tilde{k}\right|}\sum\nolimits_{([\gamma^l,\ \gamma^u],[\eta^l,\ \eta^u])\in\tilde{k}}\left(\left(\gamma^l\right)^2+\left(\gamma^u\right)^2-\left(\eta^l\right)^2-\left(\eta^u\right)^2\right)\right)\right), \tag{4}$$

and the accuracy function, $A\left(\tilde{k}\right)$ is expressed as:

$$A\left(\tilde{k}\right)=\frac{1}{2}\left(\frac{1}{\left|\tilde{k}\right|}\sum\nolimits_{([\gamma^l,\ \gamma^u],[\eta^l,\ \eta^u])\in\tilde{k}}\left(\left(\gamma^l\right)^2+\left(\gamma^u\right)^2+\left(\eta^l\right)^2+\left(\eta^u\right)^2\right)\right), \tag{5}$$

where $\left|\tilde{k}\right|$ is the number of IVPFNs in $\tilde{k}$.

Using the score and accuracy functions, S and A, respectively, any two IVPHFNs can be compared via the following rules as shown below:

Definition 10. Let $\tilde{k}_i\ (i=1,2)$ be any two IVPHFNs, then

(i) If $S\left(\tilde{k}_1\right)>S\left(\tilde{k}_2\right)$ then $\tilde{k}_1>\tilde{k}_2$;

(ii) If $S\left(\tilde{k}_1\right)=S\left(\tilde{k}_2\right)$ then

- $\tilde{k}_1>\tilde{k}_2$ for $A\left(\tilde{k}_1\right)>A\left(\tilde{k}_2\right)$;
- $\tilde{k}_1=\tilde{k}_2$ for $A\left(\tilde{k}_1\right)=A\left(\tilde{k}_2\right)$.

Thus from the above ranking process, it is clear that the rank of the IVPHFNs are first performed based on the score values. If those are found as equal, then the rank is made based on the value of the accuracy function.

4.2. IVPHF operations based on At-N&t-CN

On the basis of At-N&t-CN operations, several operations on IVPHFNs are proposed as follows:

Definition 11. Let $\tilde{k}_i\ (i=1,2)$ and $\tilde{k}$ be any three IVPHFNs, and $\lambda>0$ be any scalar. Now, utilising At-N&t-CNs, the basic algebraic operations of IVPHFNs, viz., addition $\oplus_A$ and multiplication $\otimes_A$, are defined as follows.

(1) **Addition:** $\tilde{k}_1\oplus_A\tilde{k}_2=\bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i\\ i=1,2}}\left\{\left(\left[\sqrt{S\left(\left(\gamma_1^l\right)^2,\left(\gamma_2^l\right)^2\right)},\sqrt{S\left(\left(\gamma_1^u\right)^2,\left(\gamma_2^u\right)^2\right)}\right],\right.\right.$

$$\left.\left.\left[\sqrt{T\left(\left(\eta_1^l\right)^2,\left(\eta_2^l\right)^2\right)},\sqrt{T\left(\left(\eta_1^u\right)^2,\left(\eta_2^u\right)^2\right)}\right]\right)\right\};$$

$$=\bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i\\ i=1,2}}\left\{\left(\left[\sqrt{g^{-1}\left(g\left(\left(\gamma_1^l\right)^2\right)+g\left(\gamma_2^l\right)^2\right)},\sqrt{g^{-1}\left(g\left(\left(\gamma_1^u\right)^2\right)+g\left(\left(\gamma_2^u\right)^2\right)\right)}\right],\right.\right.$$

$$\left.\left.\left[\sqrt{f^{-1}\left(f\left(\left(\eta_1^l\right)^2\right)+f\left(\left(\eta_2^l\right)^2\right)\right)},\sqrt{f^{-1}\left(f\left(\left(\eta_1^u\right)^2\right)+f\left(\left(\eta_2^u\right)^2\right)\right)}\right]\right)\right\};$$

(2) **Multiplication:** $\tilde{k}_1\otimes_A\tilde{k}_2=\bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\\ \in\tilde{k}_i,i=1,2}}\left\{\left(\left[\sqrt{T\left(\left(\gamma_1^l\right)^2,\left(\gamma_2^l\right)^2\right)},\sqrt{T\left(\left(\gamma_1^u\right)^2,\left(\gamma_2^u\right)^2\right)}\right],\right.\right.$

$$\left.\left.\left[\sqrt{S\left(\left(\eta_1^l\right)^2,\left(\eta_2^l\right)^2\right)},\sqrt{S\left(\left(\eta_1^u\right)^2,\left(\eta_2^u\right)^2\right)}\right]\right)\right\};$$

$$=\bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i,\\ i=1,2}}\left\{\left(\left[\sqrt{f^{-1}\left(f\left(\left(\gamma_1^l\right)^2\right)+f\left(\left(\gamma_2^l\right)^2\right)\right)},\sqrt{f^{-1}\left(f\left(\left(\gamma_1^u\right)^2\right)+f\left(\left(\gamma_2^u\right)^2\right)\right)}\right],\right.\right.$$

$$\left[\sqrt{g^{-1}\left(g\left(\left(\eta_1^l\right)^2\right)+g\left(\left(\eta_2^l\right)^2\right)\right)},\sqrt{g^{-1}\left(g\left(\left(\eta_1^u\right)^2\right)+g\left(\left(\eta_2^u\right)^2\right)\right)}\right]\Bigg)\Bigg\};$$

(3) **Scalar multiplication:** $\lambda\tilde{k}=\bigcup_{([\gamma^l,\ \gamma^u],[\eta^l,\ \eta^u])\in\tilde{k}}\Bigg\{\Bigg(\left[\sqrt{g^{-1}\left(\lambda g\left({\gamma^l}^2\right)\right)},\sqrt{g^{-1}\left(\lambda g\left({\gamma^u}^2\right)\right)}\right],$

$$\left[\sqrt{f^{-1}\left(\lambda f\left(\left(\eta^l\right)^2\right)\right)},\sqrt{f^{-1}\left(\lambda f\left(\left(\eta^u\right)^2\right)\right)}\right]\Bigg)\Bigg\};$$

(4) **Exponentiation:** $\tilde{k}^\lambda=\bigcup_{([\gamma^l,\ \gamma^u],[\eta^l,\ \eta^u])\in\tilde{k}}\Bigg\{\Bigg(\left[\sqrt{f^{-1}\left(\lambda f\left(\left(\gamma^l\right)^2\right)\right)},\sqrt{f^{-1}\left(\lambda f\left(\left(\gamma^u\right)^2\right)\right)}\right],$
$$\left[\sqrt{g^{-1}\left(\lambda g\left(\left(\eta^l\right)^2\right)\right)},\sqrt{g^{-1}\left(\lambda g\left(\left(\eta^u\right)^2\right)\right)}\right]\Bigg)\Bigg\}.$$

The above operations are defined based on several increasing and decreasing operators. Now, choosing specific decreasing generating functions, some special types of aggregation operators are obtained, which are presented in the following manners:

- **Algebraic t-N and t-CN:** Considering $f(t)=-\log t$, operational laws for IVPHFNs based on Algebraic t-N and t-CN, are described as follows:

(1) $\tilde{k}_1\oplus\tilde{k}_2=\bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i,\\ i=1,2}}$
$$\left\{\left(\left[\sqrt{\left(\gamma_1^l\right)^2+\left(\gamma_2^l\right)^2-\left(\gamma_1^l\right)^2\left(\gamma_2^l\right)^2},\sqrt{\left(\gamma_1^u\right)^2+\left(\gamma_2^u\right)^2-\left(\gamma_1^u\right)^2\left(\gamma_2^u\right)^2}\right],\left[\eta_1^l\eta_2^l,\eta_1^u\eta_2^u\right]\right)\right\}$$

(2) $\tilde{k}_1\otimes\tilde{k}_2=\bigcup_{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i,i=1,2}$
$$\left\{\left(\left[\gamma_1^l\gamma_2^l,\gamma_1^u\gamma_2^u\right],\left[\sqrt{\left(\eta_1^l\right)^2+\left(\eta_2^l\right)^2-\left(\eta_1^l\right)^2\left(\eta_2^l\right)^2},\sqrt{\left(\eta_1^u\right)^2+\left(\eta_2^u\right)^2-\left(\eta_1^u\right)^2\left(\eta_2^u\right)^2}\right]\right)\right\};$$

(3) $$\lambda\tilde{k}=\bigcup_{([\gamma^l,\gamma^u],[\eta^l,\eta^u])\in\tilde{k}}\left\{\left(\left[\sqrt{1-\left(1-\left(\gamma^l\right)^2\right)^\lambda},\sqrt{1-\left(1-\left(\gamma^u\right)^2\right)^\lambda}\right],\left[\left(\eta^l\right)^\lambda,\left(\eta^u\right)^\lambda\right]\right)\right\},$$

(4) $$\tilde{k}^\lambda=\bigcup_{([\gamma^l,\ \gamma^u],[\eta^l,\ \eta^u])\in\tilde{k}}\left\{\left(\left[\left(\gamma^l\right)^\lambda,\left(\gamma^u\right)^\lambda\right],\left[\sqrt{1-\left(1-\left(\eta^l\right)^2\right)^\lambda},\sqrt{1-\left(1-\left(\eta^u\right)^2\right)^\lambda}\right]\right)\right\}.$$

- **Einstein t-N and t-CN:** Considering $f(t)=\log\left(\frac{2-t}{t}\right)$, operational laws for IVPHFNs based on Einstein t-N and t-CN, are described as follows:

(1) $\tilde{k}_1\oplus_E\tilde{k}_2=\bigcup_{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i,i=1,2}$
$$\left\{\left(\left[\sqrt{\frac{{\gamma_1^l}^2+{\gamma_2^l}^2}{1+{\gamma_1^l}^2{\gamma_2^l}^2}},\sqrt{\frac{{\gamma_1^u}^2+{\gamma_2^u}^2}{1+{\gamma_1^u}^2{\gamma_2^u}^2}}\right],\left[\frac{\eta_1^l\eta_2^l}{\sqrt{1+\left(1-{\eta_1^l}^2\right)\left(1-{\eta_2^l}^2\right)}},\frac{\eta_1^u\eta_2^u}{\sqrt{1+\left(1-{\eta_1^u}^2\right)\left(1-{\eta_2^u}^2\right)}}\right]\right)\right\};$$

(2) $\tilde{k}_1\otimes_E\tilde{k}_2=\bigcup_{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i,i=1,2}\Bigg\{\Bigg(\left[\frac{\gamma_1^l\gamma_2^l}{\sqrt{1+\left(1-\left(\gamma_1^l\right)^2\right)\left(1-\left(\gamma_2^l\right)^2\right)}},\frac{\gamma_1^u\gamma_2^u}{\sqrt{1+\left(1-\left(\gamma_1^u\right)^2\right)\left(1-\left(\gamma_2^u\right)^2\right)}}\right],$
$$\left[\sqrt{\frac{\left(\eta_1^l\right)^2+\left(\eta_2^l\right)^2}{1+\left(\eta_1^l\right)^2\left(\eta_2^l\right)^2}},\sqrt{\frac{\left(\eta_1^u\right)^2+\left(\eta_2^u\right)^2}{1+\left(\eta_1^u\right)^2\left(\eta_2^u\right)^2}}\right]\Bigg)\Bigg\};$$

(3) $\lambda\tilde{k}=\bigcup_{([\gamma^l,\ \gamma^u],[\eta^l,\ \eta^u])\in\tilde{k}}\Bigg\{\Bigg(\left[\sqrt{\frac{\left(1+(\gamma^l)^2\right)^\lambda-\left(1-(\gamma^l)^2\right)^\lambda}{\left(1+(\gamma^l)^2\right)^\lambda+\left(1-(\gamma^l)^2\right)^\lambda}},\sqrt{\frac{\left(1+(\gamma^u)^2\right)^\lambda-\left(1-(\gamma^u)^2\right)^\lambda}{\left(1+(\gamma^u)^2\right)^\lambda+\left(1-(\gamma^u)^2\right)^\lambda}}\right],$

$$\left[\frac{\sqrt{2}\left(\eta^l\right)^\lambda}{\sqrt{\left(2-(\eta^l)^2\right)^\lambda+\left((\eta^l)^2\right)^\lambda}}, \frac{\sqrt{2}(\eta^u)^\lambda}{\sqrt{\left(2-(\eta^u)^2\right)^\lambda+\left((\eta^u)^2\right)^\lambda}}\right]\Bigg)\Bigg\},$$

(4) $$\tilde{k}^\lambda = \bigcup_{([\gamma^l,\ \gamma^u],[\eta^l,\ \eta^u])\in\tilde{k}} \Bigg\{\Bigg(\left[\frac{\sqrt{2}\left(\gamma^l\right)^\lambda}{\sqrt{\left(2-(\gamma^l)^2\right)^\lambda+\left((\gamma^l)^2\right)^\lambda}}, \frac{\sqrt{2}\left(\gamma^u\right)^\lambda}{\sqrt{\left(2-(\gamma^u)^2\right)^\lambda+\left((\gamma^u)^2\right)^\lambda}}\right],$$

$$\left[\sqrt{\frac{\left(1+(\eta^l)^2\right)^\lambda-\left(1-(\eta^l)^2\right)^\lambda}{\left(1+(\eta^l)^2\right)^\lambda+\left(1-(\eta^l)^2\right)^\lambda}}, \sqrt{\frac{\left(1+(\eta^u)^2\right)^\lambda-\left(1-(\eta^u)^2\right)^\lambda}{\left(1+(\eta^u)^2\right)^\lambda+\left(1-(\eta^u)^2\right)^\lambda}}\right]\Bigg)\Bigg\}.$$

- **Hamacher t-N and t-CN:** When $f(t) = \log\left(\frac{\psi+(1-\psi)t}{t}\right)$, $\psi > 0$, then the following operations for IVPHFNs, based on Hamacher t-N and t-CN, are defined as

(1) $$\tilde{k}_1\oplus_H\tilde{k}_2 = \bigcup_{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i, i=1,2} \Bigg\{\Bigg(\left[\sqrt{\frac{\left(\gamma_1^l\right)^2+\left(\gamma_2^l\right)^2-\left(\gamma_1^l\right)^2\left(\gamma_2^l\right)^2-(1-\psi)\left(\gamma_1^l\right)^2\left(\gamma_2^l\right)^2}{1-(1-\psi)\left(\gamma_1^l\right)^2\left(\gamma_2^l\right)^2}},\right.$$

$$\left.\sqrt{\frac{\left(\gamma_1^u\right)^2+\left(\gamma_2^u\right)^2-\left(\gamma_1^u\right)^2\left(\gamma_2^u\right)^2-(1-\psi)\left(\gamma_1^u\right)^2\left(\gamma_2^u\right)^2}{1-(1-\psi)\left(\gamma_1^u\right)^2\left(\gamma_2^u\right)^2}}\right],$$

$$\left[\frac{\eta_1^l\eta_2^l}{\sqrt{\psi+(1-\psi)\left(\left(\eta_1^l\right)^2+\left(\eta_2^l\right)^2-\left(\eta_1^l\right)^2\left(\eta_2^l\right)^2\right)}}, \frac{\eta_1^u\eta_2^u}{\sqrt{\psi+(1-\psi)\left(\left(\eta_1^u\right)^2+\left(\eta_2^u\right)^2-\left(\eta_1^u\right)^2\left(\eta_2^u\right)^2\right)}}\right]\Bigg)\Bigg\};$$

(2) $$\tilde{k}_1\otimes_H\tilde{k}_2 = \bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i\\ i=1,2}}$$

$$\Bigg\{\Bigg(\left[\frac{\gamma_1^l\gamma_2^l}{\sqrt{\psi+(1-\psi)\left(\left(\gamma_1^l\right)^2+\left(\gamma_2^l\right)^2-\left(\gamma_1^l\right)^2\left(\gamma_2^l\right)^2\right)}}, \frac{\gamma_1^u\gamma_2^u}{\sqrt{\psi+(1-\psi)\left(\left(\gamma_1^u\right)^2+\left(\gamma_2^u\right)^2-\left(\gamma_1^u\right)^2\left(\gamma_2^u\right)^2\right)}}\right],$$

$$\left[\sqrt{\frac{\left(\eta_1^l\right)^2+\left(\eta_2^l\right)^2-\left(\eta_1^l\right)^2\left(\eta_2^l\right)^2-(1-\psi)\left(\eta_1^l\right)^2\left(\eta_2^l\right)^2}{1-(1-\psi)\left(\eta_1^l\right)^2\left(\eta_2^l\right)^2}}, \sqrt{\frac{\left(\eta_1^u\right)^2+\left(\eta_2^u\right)^2-\left(\eta_1^u\right)^2\left(\eta_2^u\right)^2-(1-\psi)\left(\eta_1^u\right)^2\left(\eta_2^u\right)^2}{1-(1-\psi)\left(\eta_1^u\right)^2\left(\eta_2^u\right)^2}}\right]\Bigg)\Bigg\};$$

(3) $$\lambda\tilde{k} = \bigcup_{\substack{([\gamma^l,\ \gamma^u],\\ [\eta^l,\ \eta^u])\in\tilde{k}}} \Bigg\{\Bigg(\left[\sqrt{\frac{\left(1+(\psi-1)(\gamma^l)^2\right)^\lambda-\left(1-\gamma^{l2}\right)^\lambda}{\left(1+(\psi-1)\gamma^{l2}\right)^\lambda+(\psi-1)\left(1-\gamma^{l2}\right)^\lambda}}, \sqrt{\frac{\left(1+(\psi-1)(\gamma^u)^2\right)^\lambda-\left(1-(\gamma^u)^2\right)^\lambda}{\left(1+(\psi-1)(\gamma^u)^2\right)^\lambda+(\psi-1)\left(1-(\gamma^u)^2\right)^\lambda}}\right],$$

$$\left[\frac{\sqrt{\psi}\left(\eta^l\right)^\lambda}{\sqrt{\left(1+(\psi-1)\left(1-(\eta^l)^2\right)\right)^\lambda+(\psi-1)(\eta^l)^{2\lambda}}}, \frac{\sqrt{\psi}(\eta^u)^\lambda}{\sqrt{\left(1+(\psi-1)\left(1-(\eta^u)^2\right)\right)^\lambda+(\psi-1)(\eta^u)^{2\lambda}}}\right]\Bigg)\Bigg\};$$

(4) $$\tilde{k}^\lambda = \bigcup_{([\gamma^l,\ \gamma^u],[\eta^l,\ \eta^u])\in\tilde{k}} \Bigg\{\Bigg(\left[\frac{\sqrt{\psi}\left(\gamma^l\right)^\lambda}{\sqrt{\left(1+(\psi-1)\left(1-\gamma^{l2}\right)\right)^\lambda+(\psi-1)\gamma^{l2\lambda}}}, \frac{\sqrt{\psi}(\gamma^u)^\lambda}{\sqrt{(1+(\psi-1)(1-\gamma^{u2}))^\lambda+(\psi-1)\gamma^{u2\lambda}}}\right],$$

$$\left[\sqrt{\frac{\left(1+(\psi-1)(\eta^l)^2\right)^\lambda-\left(1-(\eta^l)^2\right)^\lambda}{\left(1+(\psi-1)(\eta^l)^2\right)^\lambda+(\psi-1)\left(1-(\eta^l)^2\right)^\lambda}}, \sqrt{\frac{\left(1+(\psi-1)(\eta^u)^2\right)^\lambda-\left(1-(\eta^u)^2\right)^\lambda}{\left(1+(\psi-1)(\eta^u)^2\right)^\lambda+(\psi-1)\left(1-(\eta^u)^2\right)^\lambda}}\right]\Bigg)\Bigg\}.$$

For $\psi = 1$ the above four Hamacher operational laws are reduced to algebraic operational laws and for $\psi = 2$ those four Hamacher operational laws are reduced into Einstein operational laws.

Example 2. Let $\tilde{k}_1 = \{([0.3, 0.5], [0.2, 0.4]), ([0.7, 0.8], [0.4, 0.6])\}$ and $\tilde{k}_2 = \{([0.3, 0.4], [0.2, 0.6]), ([0.1, 0.3], [0.2, 0.5]), ([0.4, 0.7], [0.3, 0.6])\}$ be two IVPHFNs, and consider $\psi = 2$ and $\lambda = 3$. The mathematical operations based on Hamacher t-N and t-CN are presented as follows:

(1) $$\tilde{k}_1 \oplus_H \tilde{k}_2 = \bigcup_{\substack{\left(\left[\gamma_1^l,\gamma_1^u\right],\left[\eta_1^l,\eta_1^u\right]\right)\in \\ \{([0.3,0.5],[0.2,0.4]),([0.7,0.8],[0.4,0.6])\}, \\ \left(\left[\gamma_2^l,\gamma_2^u\right],\left[\eta_2^l,\eta_2^u\right]\right)\in \\ \{([0.3,0.4],[0.2,0.6]),([0.1,0.3],[0.2,0.5]), \\ ([0.4,0.7],[0.3,0.6])\}}} \left\{\left(\left[\sqrt{\frac{{\gamma_1^l}^2+{\gamma_2^l}^2-{\gamma_1^l}^2{\gamma_2^l}^2+{\gamma_1^l}^2{\gamma_2^l}^2}{1+{\gamma_1^l}^2{\gamma_2^l}^2}}, \sqrt{\frac{{\gamma_1^u}^2+{\gamma_2^u}^2-{\gamma_1^u}^2{\gamma_2^u}^2+{\gamma_1^u}^2{\gamma_2^u}^2}{1+{\gamma_1^u}^2{\gamma_2^u}^2}}\right],\right.\right.$$
$$\left.\left.\left[\frac{\eta_1^l\eta_2^l}{\sqrt{2-\left({\eta_1^l}^2+{\eta_2^l}^2-{\eta_1^l}^2{\eta_2^l}^2\right)}}, \frac{\eta_1^u\eta_2^u}{\sqrt{2-\left({\eta_1^u}^2+{\eta_2^u}^2-{\eta_1^u}^2{\eta_2^u}^2\right)}}\right]\right)\right\}$$
$$= \{([0.4146, 0.6083], [0.0400, 0.2400]), ([0.3148, 0.5635], [0.0400, 0.2000]),$$
$$([0.4854, 0.7858], [0.0600, 0.2400]), ([0.7321, 0.8352], [0.0800, 0.3600]),$$
$$([0.7036, 0.8200], [0.0800, 0.3000]), ([0.7560, 0.9035], [0.1200, 0.3600])\}$$

(2) $$\tilde{k}_1 \otimes_H \tilde{k}_2 = \bigcup_{\substack{\left(\left[\gamma_1^l,\gamma_1^u\right],\left[\eta_1^l,\eta_1^u\right]\right)\in \\ \{([0.3,0.5],[0.2,0.4]),([0.7,0.8],[0.4,0.6])\}, \\ \left(\left[\gamma_2^l,\gamma_2^u\right],\left[\eta_2^l,\eta_2^u\right]\right)\in \\ \{([0.3,0.4],[0.2,0.6]),([0.1,0.3],[0.2,0.5]), \\ ([0.4,0.7],[0.3,0.6])\}}} \left\{\left(\left[\frac{\gamma_1^l\gamma_2^l}{\sqrt{2-\left({\gamma_1^l}^2+{\gamma_2^l}^2-{\gamma_1^l}^2{\gamma_2^l}^2\right)}}, \frac{\gamma_1^u\gamma_2^u}{\sqrt{2-\left({\gamma_1^u}^2+{\gamma_2^u}^2-{\gamma_1^u}^2{\gamma_2^u}^2\right)}}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{\frac{{\eta_1^l}^2+{\eta_2^l}^2-{\eta_1^l}^2{\eta_2^l}^2+{\eta_1^l}^2{\eta_2^l}^2}{1+{\eta_1^l}^2{\eta_2^l}^2}}, \sqrt{\frac{{\eta_1^u}^2+{\eta_2^u}^2-{\eta_1^u}^2{\eta_2^u}^2+{\eta_1^u}^2{\eta_2^u}^2}{1+{\eta_1^u}^2{\eta_2^u}^2}}\right]\right)\right\}$$
$$= \{([0.0900, 0.2000], [0.2800, 0.6800]), ([0.0300, 0.1500], [0.2800, 0.6083]),$$
$$([0.1200, 0.3500], [0.3555, 0.6800]), ([0.2100, 0.3200], [0.4400, 0.7684]),$$
$$([0.0700, 0.2400], [0.4400, 0.7211]), ([0.2800, 0.5600], [0.4854, 0.7684])\}$$

(3) $$\tilde{k}_1 = \bigcup_{\substack{\left(\left[\gamma_1^l,\gamma_1^u\right],\left[\eta_1^l,\eta_1^u\right]\right)\in \\ \{([0.3,0.5],[0.2,0.4]) \\ ([0.7,0.8],[0.4,0.6])\}}} \left\{\left(\left[\sqrt{\frac{\left(1+\gamma^{l2}\right)^3-\left(1-\gamma^{l2}\right)^3}{\left(1+\gamma^{l2}\right)^3+\left(1-\gamma^{l2}\right)^3}}, \sqrt{\frac{(1+\gamma^{u2})^3-(1-\gamma^{u2})^3}{(1+\gamma^{u2})^3+(1-\gamma^{u2})^3}}\right], \left[\frac{\sqrt{2}\eta^{l3}}{\sqrt{\left(1+\left(1-\eta^{l2}\right)\right)^3+\eta^{l6}}}, \frac{\sqrt{2}\eta^{u3}}{\sqrt{(1+(1-\eta^{u2}))^3+\eta^{u6}}}\right]\right)\right\}$$
$$= \{([0.4964, 0.7603], [0.0080, 0.0640]), ([0.9313, 0.9764], [0.0640, 0.2160])\}$$

(4) $$\tilde{k}_1^3 = \bigcup_{\substack{\left(\left[\gamma_1^l,\gamma_1^u\right],\left[\eta_1^l,\eta_1^u\right]\right)\in \\ \{([0.3,0.5],[0.2,0.4]) \\ ([0.7,0.8],[0.4,0.6])\}}} \left\{\left(\left[\frac{\sqrt{2}\gamma^{l3}}{\sqrt{\left(1+\left(1-\gamma^{l2}\right)\right)^3+\gamma^{l6}}}, \frac{\sqrt{2}\gamma^{u3}}{\sqrt{(1+(1-\gamma^{u2}))^3+\gamma^{u6}}}\right], \left[\sqrt{\frac{\left(1+\eta^{l2}\right)^3-\left(1-\eta^{l2}\right)^3}{\left(1+\eta^{l2}\right)^3+\left(1-\eta^{l2}\right)^3}}, \sqrt{\frac{(1+\eta^{u2})^3-(1-\eta^{u2})^3}{(1+\eta^{u2})^3+(1-\eta^{u2})^3}}\right]\right)\right\}$$
$$= \{([0.0270, 0.1250], [0.3395, 0.6382]), ([0.3430, 0.5120], [0.6382, 0.8590])\}.$$

- **Frank t-N and t-CN:** Considering $f(t) = \log\left(\frac{\tau-1}{\tau^t-1}\right)$, $\tau > 1$, operational laws for IVPHFNs based on Frank t-N and t-CN are described as follows:

(1) $$\tilde{k}_1 \oplus_F \tilde{k}_2 = \bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u]) \\ \in \tilde{k}_i \\ i=1,2}} \left\{\left(\left[\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\gamma_1^{l\,2}}-1\right)\left(\tau^{1-\gamma_2^{l\,2}}-1\right)}{\tau-1}\right)\right)^{\frac{1}{2}}, \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\gamma_1^{u\,2}}-1\right)\left(\tau^{1-\gamma_2^{u\,2}}-1\right)}{\tau-1}\right)\right)^{\frac{1}{2}}\right],\right.\right.$$
$$\left.\left.\left[\left(\log_\tau\left(1+\frac{\left(\tau^{\eta_1^{l\,2}}-1\right)\left(\tau^{\eta_2^{l\,2}}-1\right)}{\tau-1}\right)\right)^{\frac{1}{2}}, \left(\log_\tau\left(1+\frac{\left(\tau^{\eta_1^{u\,2}}-1\right)\left(\tau^{\eta_2^{u\,2}}-1\right)}{\tau-1}\right)\right)^{\frac{1}{2}}\right]\right)\right\};$$

(2) $$\tilde{k}_1 \otimes_F \tilde{k}_2 = \bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u]) \in \tilde{k}_i \\ i=1,2}} \left\{\left(\left[\sqrt{\log_\tau\left(1+\frac{\left(\tau^{\gamma_1^{l\,2}}-1\right)\left(\tau^{\gamma_2^{l\,2}}-1\right)}{\tau-1}\right)},\right.\right.\right.$$
$$\left.\sqrt{\log_\tau\left(1+\frac{\left(\tau^{(\gamma_1^u)^2}-1\right)\left(\tau^{(\gamma_2^u)^2}-1\right)}{\tau-1}\right)}\right], \left[\sqrt{1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(\eta_1^l\right)^2}-1\right)\left(\tau^{1-\left(\eta_2^l\right)^2}-1\right)}{\tau-1}\right)},\right.$$
$$\left.\left.\left.\sqrt{1-\log_\tau\left(1+\frac{\left(\tau^{1-(\eta_1^u)^2}-1\right)\left(\tau^{1-(\eta_2^u)^2}-1\right)}{\tau-1}\right)}\right]\right)\right\};$$

(3) $$\lambda\tilde{k} = \bigcup_{([\gamma^l,\,\gamma^u],[\eta^l,\,\eta^u]) \in \tilde{k}} \left\{\left(\left[\sqrt{1-\log_\tau\left(1+\frac{\left(\tau^{1-(\gamma^l)^2}-1\right)^\lambda}{(\tau-1)^{\lambda-1}}\right)}, \sqrt{1-\log_\tau\left(1+\frac{\left(\tau^{1-(\gamma^u)^2}-1\right)^\lambda}{(\tau-1)^{\lambda-1}}\right)}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{\log_\tau\left(1+\frac{\left(\tau^{(\eta^l)^2}-1\right)^\lambda}{(\tau-1)^{\lambda-1}}\right)}, \sqrt{\log_\tau\left(1+\frac{\left(\tau^{(\eta^u)^2}-1\right)^\lambda}{(\tau-1)^{\lambda-1}}\right)}\right]\right)\right\};$$

(4) $$\tilde{k}^\lambda = \bigcup_{([\gamma^l,\,\gamma^u],[\eta^l,\,\eta^u]) \in \tilde{k}} \left\{\left(\left[\sqrt{\log_\tau\left(1+\frac{\left(\tau^{(\gamma^l)^2}-1\right)^\lambda}{(\tau-1)^{\lambda-1}}\right)}, \sqrt{\log_\tau\left(1+\frac{\left(\tau^{(\gamma^u)^2}-1\right)^\lambda}{(\tau-1)^{\lambda-1}}\right)}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{1-\log_\tau\left(1+\frac{\left(\tau^{1-(\eta^l)^2}-1\right)^\lambda}{(\tau-1)^{\lambda-1}}\right)}, \sqrt{1-\log_\tau\left(1+\frac{\left(\tau^{1-(\eta^u)^2}-1\right)^\lambda}{(\tau-1)^{\lambda-1}}\right)}\right]\right)\right\}.$$

For $\tau \to 1$ the above four Frank operational laws are reduced to algebraic operations.

It is to be noted here that, for each of the cases (3) and (4), $\lambda > 0$ is considered.

Example 3. Let $\tilde{k}_1 = \{([0.3, 0.5], [0.2, 0.4]), ([0.7, 0.8], [0.4, 0.6])\}$ and $\tilde{k}_2 = \{([0.3, 0.4], [0.2, 0.6]), ([0.1, 0.3], [0.2, 0.5]), ([0.4, 0.7], [0.3, 0.6])\}$ be two IVPHFNs and consider the Frank parameter as $\tau = 2$ and the scalar value as $\lambda = 3$. The mathematical operations based on Frank t-N and t-CN are presented as follows:

(1) $$\tilde{k}_1 \oplus_F \tilde{k}_2 = \bigcup_{\substack{([\gamma_1^l,\gamma_1^u],[\eta_1^l,\eta_1^u]) \in \\ \{([0.3,0.5],[0.2,0.4]), \\ ([0.7,0.8],[0.4,0.6])\}, \\ ([\gamma_2^l,\gamma_2^u],[\eta_2^l,\eta_2^u]) \in \\ \{([0.3,0.4],[0.2,0.6]), \\ ([0.1,0.3],[0.2,0.5]), \\ ([0.4,0.7],[0.3,0.6])\}}} \left\{\left(\left[\left(1-\log_2\left(1+\left(2^{1-\gamma_1^{l\,2}}-1\right)\left(2^{1-\gamma_2^{l\,2}}-1\right)\right)\right)^{\frac{1}{2}},\right.\right.\right.$$

$$\left(1-\log_2\left(1+\left(2^{1-\gamma_1^{u\,2}}-1\right)\left(2^{1-\gamma_2^{u\,2}}-1\right)\right)\right)^{\frac{1}{2}}\Bigg],\Bigg[\left(\log_2\left(1+\left(2^{\eta_1^{l\,2}}-1\right)\left(2^{\eta_2^{l\,2}}-1\right)\right)\right)^{\frac{1}{2}},$$

$$\left(\log_2\left(1+\left(2^{\eta_1^{u\,2}}-1\right)\left(2^{\eta_2^{u\,2}}-1\right)\right)\right)^{\frac{1}{2}}\Bigg]\Bigg)\Bigg\}$$

$$=\{([0.4172,0.6151],[0.0338,0.2172]),([0.3152,0.5679],[0.0338,0.1779]),$$
$$([0.4890,0.7960],[0.0511,0.2172]),([0.7368,0.8417],[0.0689,0.3339]),$$
$$([0.7042,0.8241],[0.0689,0.2745]),([0.7636,0.9145],[0.1042,0.3339])\}$$

(2) $$\tilde{k}_1\otimes_F\tilde{k}_2=\bigcup_{\substack{([\gamma_1^l,\gamma_1^u],[\eta_1^l,\eta_1^u])\in\\ \{([0.3,0.5],[0.2,0.4]),([0.7,0.8],[0.4,0.6])\},\\ ([\gamma_2^l,\gamma_2^u],[\eta_2^l,\eta_2^u])\in\\ \{([0.3,0.4],[0.2,0.6]),([0.1,0.3],[0.2,0.5])\\ ([0.4,0.7],[0.3,0.6])\}}}\Bigg\{\Bigg(\Bigg[\sqrt{\log_2\left(1+\left(2^{\gamma_1^{l\,2}}-1\right)\left(2^{\gamma_2^{l\,2}}-1\right)\right)},$$

$$\sqrt{\log_2\left(1+\left(2^{\gamma_1^{u\,2}}-1\right)\left(2^{\gamma_2^{u\,2}}-1\right)\right)}\Bigg],\Bigg[\sqrt{1-\log_2\left(1+\left(2^{1-\eta_1^{l\,2}}-1\right)\left(2^{1-\eta_2^{l\,2}}-1\right)\right)},$$

$$\sqrt{1-\log_2\left(1+\left(2^{1-\eta_1^{u\,2}}-1\right)\left(2^{1-\eta_2^{u\,2}}-1\right)\right)}\Bigg]\Bigg)\Bigg\}$$

$$=\{([0.0772,0.1779],[0.2808,0.6876]),([0.0254,0.1322],[0.2808,0.6151]),$$
$$([0.1042,0.3262],[0.3569,0.6876]),([0.1926,0.3025],[0.4419,0.7801]),$$
$$([0.0637,0.2257],[0.4419,0.7312]),([0.2586,0.5420],[0.4890,0.7801])\}$$

(3) $$3\tilde{k}_1=\bigcup_{\substack{([\gamma_1^l,\gamma_1^u],[\eta_1^l,\eta_1^u])\in\\ \{([0.3,0.5],[0.2,0.4]),([0.7,0.8],[0.4,0.6])\}}}\Bigg\{\Bigg(\Bigg[\sqrt{1-\log_2\left(1+\frac{\left(2^{1-\gamma^{l\,2}}-1\right)^3}{(2-1)^{3-1}}\right)},$$

$$\sqrt{1-\log_2\left(1+\frac{\left(2^{1-\gamma^{u\,2}}-1\right)^3}{(2-1)^{3-1}}\right)}\Bigg],\Bigg[\sqrt{\log_2\left(1+\frac{\left(2^{\eta^{l\,2}}-1\right)^3}{(2-1)^{3-1}}\right)},\sqrt{\log_2\left(1+\frac{\left(2^{\eta^{u\,2}}-1\right)^3}{(2-1)^{3-1}}\right)}\Bigg]\Bigg)\Bigg\}$$

$$=\{([0.6511,0.7312],[0.1277,0.2586]),([0.8327,0.8877],[0.2586,0.3957])\}$$

(4) $$\tilde{k}_1^3=\bigcup_{\substack{([\gamma_1^l,\gamma_1^u],[\eta_1^l,\eta_1^u])\in\\ \{([0.3,0.5],[0.2,0.4]),([0.7,0.8],[0.4,0.6])\}}}\Bigg\{\Bigg(\Bigg[\sqrt{\log_\tau\left(1+\frac{\left(2^{\gamma^{l\,2}}-1\right)^3}{(2-1)^{3-1}}\right)},$$

$$\sqrt{\log_2\left(1+\frac{\left(2^{\gamma^{u\,2}}-1\right)^3}{(2-1)^{3-1}}\right)}\Bigg],\Bigg[\sqrt{1-\log_2\left(1+\frac{\left(2^{1-\eta^{l\,2}}-1\right)^3}{(2-1)^{3-1}}\right)},\sqrt{1-\log_2\left(1+\frac{\left(2^{1-\eta^{u\,2}}-1\right)^3}{(2-1)^{3-1}}\right)}\Bigg]\Bigg)\Bigg\}$$

$$=\{([0.1926,0.3262],[0.6233,0.6876]),([0.4675,0.5420],[0.6876,0.7801])\}.$$

The arithmetic operations on IVPHFNs based on different commonly used At-N&t-CNs are presented above. Now, based on those operations, IVPHF Archimedean aggregation operators are derived in the next section.

5. IVPHF archimedean aggregation operators

In this section some IVPHF Archimedean averaging operators and geometric operators are proposed using the operational rules based on At-N&t-CN. Also, some special cases using specific generating functions are presented. Moreover, several important properties of those operators are also discussed in this section.

5.1. IVPHF archimedean averaging operators

To aggregate IVPHFNs, Archimedean operation-based IVPHF averaging aggregation operators are introduced through the following definition.

Definition 12. For any collection of IVPHFNs, $\tilde{k}_i$ $(i = 1, 2, \ldots, n)$ and weight vector, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, an A$t$-N&$t$-CN-based IVPHFWA (AIVPHFWA) operator is defined by a mapping, *AIVPHFWA*:$K^n \to K$ such that

$$\mathit{AIVPHFWA}\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right) = \oplus_{A}{}_{i=1}^{n}\left(\omega_i \tilde{k}_i\right),$$

where $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$.

Theorem 1. The aggregated value of any collections of IVPHFNs, $\tilde{k}_i$ $(i = 1, 2, \ldots, n)$, using AIVPHFWA operator is also an IVPHFN and is given by

$$\mathit{AIVPHFWA}\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right)$$

$$= \bigcup_{\left(\left[\gamma_i^l, \gamma_i^u\right], \left[\eta_i^l, \eta_i^u\right]\right) \in \tilde{k}_i, i=1,2,\ldots,n} \left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g\left(\gamma_i^{l\,2}\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g\left(\gamma_i^{u\,2}\right)\right)}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(\left(\eta_i^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(\left(\eta_i^u\right)^2\right)\right)}\right]\right)\right\}.$$

Proof: For $n = 2$,

$$\omega_1 \tilde{k}_1 = \bigcup_{\left(\left[\gamma_1^l, \gamma_1^u\right], \left[\eta_1^l, \eta_1^u\right]\right) \in \tilde{k}_1} \left\{\left(\left[\sqrt{g^{-1}\left(\omega_1 g\left(\left(\gamma_1^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\omega_1 g\left(\left(\gamma_1^u\right)^2\right)\right)}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{f^{-1}\left(\omega_1 f\left(\left(\eta_1^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\omega_1 f\left(\left(\eta_1^u\right)^2\right)\right)}\right]\right)\right\}.$$

$$\omega_2 \tilde{k}_2 = \bigcup_{\left(\left[\gamma_2^l, \gamma_2^u\right], \left[\eta_2^l, \eta_2^u\right]\right) \in \tilde{k}_2} \left\{\left(\left[\sqrt{g^{-1}\left(\omega_2 g\left(\left(\gamma_2^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\omega_2 g\left(\left(\gamma_2^u\right)^2\right)\right)}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{f^{-1}\left(\omega_2 f\left(\left(\eta_2^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\omega_2 f\left(\left(\eta_2^u\right)^2\right)\right)}\right]\right)\right\}.$$

Now,

$$\omega_1 \tilde{k}_1 \oplus_A \omega_2 \tilde{k}_2 =$$
$$\bigcup_{\left(\left[\gamma_i^l, \gamma_i^u\right], \left[\eta_i^l, \eta_i^u\right]\right) \in \tilde{k}_i, i=1,2} \left\{\left(\left[\sqrt{g^{-1}\left(g\left(g^{-1}\left(\omega_1 g\left(\left(\gamma_1^l\right)^2\right)\right)\right) + g\left(g^{-1}\left(\omega_2 g\left(\left(\gamma_2^l\right)^2\right)\right)\right)\right)},\right.\right.\right.$$
$$\left.\sqrt{g^{-1}\left(g\left(g^{-1}\left(\omega_1 g\left(\left(\gamma_1^u\right)^2\right)\right)\right) + g\left(g^{-1}\left(\omega_2 g\left(\left(\gamma_2^u\right)^2\right)\right)\right)\right)}\right],$$
$$\left[\sqrt{f^{-1}\left(f\left(f^{-1}\left(\omega_1 f\left(\left(\eta_1^l\right)^2\right)\right)\right) + f\left(f^{-1}\left(\omega_2 f\left(\left(\eta_2^l\right)^2\right)\right)\right)\right)},\right.$$
$$\left.\left.\left.\sqrt{f^{-1}\left(f\left(f^{-1}\left(\omega_1 f\left(\left(\eta_1^u\right)^2\right)\right)\right) + f\left(f^{-1}\left(\omega_2 f\left(\left(\eta_2^u\right)^2\right)\right)\right)\right)}\right]\right)\right\}$$

$$= \bigcup_{\substack{\left(\left[\gamma_i^l, \gamma_i^u\right], \left[\eta_i^l, \eta_i^u\right]\right) \\ \in \tilde{k}_i, i=1,2}} \left\{\left(\left[\sqrt{g^{-1}\left(\omega_1 g\left(\left(\gamma_1^l\right)^2\right) + \omega_2 g\left(\left(\gamma_2^l\right)^2\right)\right)}, \ \sqrt{y^{-1}\left(\omega_1 g\left(\left(\gamma_1^u\right)^2\right) + \omega_2 g\left(\left(\gamma_2^u\right)^2\right)\right)}\right],\right.\right.$$

$$\left[\sqrt{f^{-1}\left(\omega_1 f\left(\left(\eta_1^l\right)^2\right)+\omega_2 f\left(\left(\eta_2^l\right)^2\right)\right)},\ \sqrt{f^{-1}\left(\omega_1 f\left(\left(\eta_1^u\right)^2\right)+\omega_2 f\left(\left(\eta_2^u\right)^2\right)\right)}\right]\right)\right\}$$

$$=\bigcup_{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i,i=1,2}\left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^{2}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)},\sqrt{g^{-1}\left(\sum_{i=1}^{2}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right],\right.\right.$$

$$\left.\left.\left[\sqrt{f^{-1}\left(\sum_{i=1}^{2}\omega_i f\left(\left(\eta_i^l\right)^2\right)\right)},\sqrt{f^{-1}\left(\sum_{i=1}^{2}\omega_i f\left(\left(\eta_i^u\right)^2\right)\right)}\right]\right)\right\}$$

i.e., the theorem holds for $n=2$.

Suppose that theorem is true for $n=m$, i.e.,

$$\textit{AIVPHFWA}\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_m\right)$$

$$=\bigcup_{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i,i=1,2,\ldots,m}\left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^{m}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)},\sqrt{g^{-1}\left(\sum_{i=1}^{m}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right],\right.\right.$$

$$\left.\left.\left[\sqrt{f^{-1}\left(\int_{i=1}^{m}\omega_i f\left(\left(\eta_i^l\right)^2\right)\right)},\sqrt{f^{-1}\left(\sum_{i=1}^{m}\omega_i f\left(\left(\eta_i^u\right)^2\right)\right)}\right]\right)\right\}.$$

Now, when $n=m+1$,

$$\textit{AIVPHFWA}\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_m,\tilde{k}_{m+1}\right)=\textit{AIVPHFWA}\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_m\right)\oplus_A\omega_{m+1}\tilde{k}_{m+1}$$

$$=\bigcup_{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i,i=1,2,\ldots,m}\left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^{m}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)},\sqrt{g^{-1}\left(\sum_{i=1}^{m}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right],\right.\right.$$

$$\left.\left.\left[\sqrt{f^{-1}\left(\int_{i=1}^{m}\omega_i f\left(\left(\eta_i^l\right)^2\right)\right)},\sqrt{f^{-1}\left(\sum_{i=1}^{m}\omega_i f\left(\left(\eta_i^u\right)^2\right)\right)}\right]\right)\right\}\oplus_A$$

$$\bigcup_{\left(\left[\gamma_{m+1}^l,\gamma_{m+1}^u\right],\left[\eta_{m+1}^l,\eta_{m+1}^u\right]\right)\in\tilde{k}_{m+1}}\left\{\left(\left[\sqrt{g^{-1}\left(\omega_{m+1}g\left(\left(\gamma_{m+1}^l\right)^2\right)\right)},\sqrt{g^{-1}\left(\omega_{m+1}g\left(\left(\gamma_{m+1}^u\right)^2\right)\right)}\right],\right.\right.$$

$$\left.\left.\left[\sqrt{f^{-1}\left(\omega_{m+1}f\left(\left(\eta_{m+1}^l\right)^2\right)\right)},\sqrt{f^{-1}\left(\omega_{m+1}f\left(\left(\eta_{m+1}^u\right)^2\right)\right)}\right]\right)\right\}$$

$$=\bigcup_{\substack{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i\\ i=1,2,\ldots,m,m+1}}\left\{\left(\left[\sqrt{g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{m}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)\right)+g\left(g^{-1}\left(\omega_{k+1}g\left(\left(\gamma_{m+1}^l\right)^2\right)\right)\right)\right)},\right.\right.\right.$$

$$\left.\sqrt{g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{m}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)\right)+g\left(g^{-1}\left(\omega_{m+1}g\left(\left(\gamma_{m+1}^u\right)^2\right)\right)\right)\right)}\right],$$

$$\left[\sqrt{f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{m}\omega_i f\left(\left(\eta_i^l\right)^2\right)\right)\right)+f\left(f^{-1}\left(\omega_{m+1}f\left(\left(\eta_{m+1}^l\right)^2\right)\right)\right)\right)},\right.$$

$$\left.\left.\left.\sqrt{f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{m}\omega_i f\left(\left(\eta_i^u\right)^2\right)\right)\right)+f\left(f^{-1}\left(\omega_{m+1}f\left(\left(\eta_{m+1}^u\right)^2\right)\right)\right)\right)}\right]\right)\right\}$$

$$=\bigcup_{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i,i=1,2,\ldots,m,m+1}\left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^{m}\omega_i g\left(\left(\gamma_i^l\right)^2\right)+\omega_{m+1}g\left(\left(\gamma_{m+1}^l\right)^2\right)\right)},\right.\right.\right.$$

$$\sqrt{g^{-1}\left(\sum_{i=1}^{m}\omega_i g\left({\gamma_i^{u}}^2\right)+\omega_{m+1} g\left({\gamma_{m+1}^{u}}^2\right)\right)}\Bigg], \Bigg[\sqrt{f^{-1}\left(\sum_{i=1}^{m}\omega_i f\left({\eta_i^{l}}^2\right)+\omega_{m+1} f\left({\eta_{m+1}^{l}}^2\right)\right)},$$

$$\sqrt{f^{-1}\left(\sum_{i=1}^{m}\omega_i f\left(\left(\eta_i^{u}\right)^2\right)+\omega_{m+1} f\left(\left(\eta_{m+1}^{u}\right)^2\right)\right)}\Bigg]\Bigg)\Bigg\}$$

$$=\bigcup_{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i, i=1,2,\ldots,m,m+1}\left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^{m+1}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{m+1}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right],\right.\right.$$

$$\left.\left.\left[\sqrt{f^{-1}\left(\sum_{i=1}^{m+1}\omega_i f\left(\left(\eta_i^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{m+1}\omega_i f\left(\left(\eta_i^u\right)^2\right)\right)}\right]\right)\right\}.$$

Hence, the above is true for $n{=}m{+}1$, also. Thus, the theorem is true for all integers.
This completes the proof of the theorem.

Considering specific decreasing generating functions, different forms of AIVPHFWA operator can be generated, which are shown in the following manners.

- **Algebraic t-N and t-CN Operations on AIVPHFWA:** For $f(t) = -\log t$, the AIVPHFWA operator is reduced to the IVPHFWA operator and is described as:

$$\textit{IVPHFWA}\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_n\right)$$

$$=\bigcup_{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i, i=1,2,\ldots,n}\left\{\left(\left[\sqrt{1-\prod_{i=1}^{n}\left(1-{\gamma_i^l}^2\right)^{\omega_i}}, \sqrt{1-\prod_{i=1}^{n}\left(1-{\gamma_i^u}^2\right)^{\omega_i}}\right],\right.\right.$$

$$\left.\left.\left[\prod_{i=1}^{n}\left(\eta_i^l\right)^{\omega_i}, \prod_{i=1}^{n}\left(\eta_i^u\right)^{\omega_i}\right]\right)\right\}. \tag{6}$$

- **Einstein t-N and t-CN Operations on AIVPHFWA:** For $f(t) = \log\left(\frac{2-t}{t}\right)$, the AIVPHFWA operator is reduced into IVPHFEWA operator and is presented as:

$$\textit{IVPHFEWA}\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_n\right)$$

$$=\bigcup_{\substack{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i,\\ i=1,2,\ldots,n}}\left\{\left(\left[\frac{\sqrt{\prod_{i=1}^{n}\left(1+\left(\gamma_i^l\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\gamma_i^l\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+\left(\gamma_i^l\right)^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\left(\gamma_i^l\right)^2\right)^{\omega_i}}},\right.\right.\right.$$

$$\left.\frac{\sqrt{\prod_{i=1}^{n}\left(1+\left(\gamma_i^u\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\gamma_i^u\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+\left(\gamma_i^u\right)^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\left(\gamma_i^u\right)^2\right)^{\omega_i}}}\right],$$

$$\left.\left.\left[\frac{\sqrt{2}\prod_{i=1}^{n}\left(\eta_i^l\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(2-\left(\eta_i^l\right)^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(\left(\eta_i^l\right)^2\right)^{\omega_i}}}, \frac{\sqrt{2}\prod_{i=1}^{n}\left(\eta_i^u\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(2-\left(\eta_i^u\right)^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(\left(\eta_i^u\right)^2\right)^{\omega_i}}}\right]\right)\right\}. \tag{7}$$

- **Hamacher t-N and t-CN Operations on AIVPHFWA:** For $f(t) = \log\left(\frac{\psi+(1-\psi)t}{t}\right)$, $\psi > 0$, the AIVPHFWA operator is reduced into the IVPHFHWA operator which is described as:

$$\textit{IVPHFHWA}\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_n\right)$$

$$= \bigcup_{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i, i=1,2,\ldots,n} \left\{ \left(\left[\frac{\sqrt{\prod_{i=1}^n \left(1+(\psi-1)\left(\gamma_i^l\right)^2\right)^{\omega_i} - \prod_{i=1}^n \left(1-\left(\gamma_i^l\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^n \left(1+(\psi-1)\left(\gamma_i^l\right)^2\right)^{\omega_i} + (\psi-1)\prod_{i=1}^n \left(1-\left(\gamma_i^l\right)^2\right)^{\omega_i}}}, \right.\right.\right.$$

$$\left. \frac{\sqrt{\prod_{i=1}^n \left(1+(\psi-1)\left(\gamma_i^u\right)^2\right)^{\omega_i} - \prod_{i=1}^n \left(1-\left(\gamma_i^u\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^n \left(1+(\psi-1)\left(\gamma_i^u\right)^2\right)^{\omega_i} + (\psi-1)\prod_{i=1}^n \left(1-\left(\gamma_i^u\right)^2\right)^{\omega_i}}} \right],$$

$$\left[\frac{\sqrt{\psi}\prod_{i=1}^n \left(\eta_i^l\right)^{\omega_i}}{\sqrt{\prod_{i=1}^n \left(1+(\psi-1)\left(1-\left(\eta_i^l\right)^2\right)\right)^{\omega_i} + (\psi-1)\prod_{i=1}^n \left(\left(\eta_i^l\right)^2\right)^{\omega_i}}}, \right.$$

$$\left.\left.\left. \frac{\sqrt{\psi}\prod_{i=1}^n \left(\eta_i^u\right)^{\omega_i}}{\sqrt{\prod_{i=1}^n \left(1+(\psi-1)\left(1-\left(\eta_i^u\right)^2\right)\right)^{\omega_i} + (\psi-1)\prod_{i=1}^n \left(\left(\eta_i^u\right)^2\right)^{\omega_i}}} \right] \right) \right\}. \tag{8}$$

For $\psi = 1$ and $\psi = 2$, IVPHFHWA operator is converted into IVPHFWA and IVPHFEWA operators, respectively. So, IVPHFHWA operator is said to be a generalised version of IVPHFWA and IVPHFEWA operators.

- **Frank t-N and t-CN Operations on AIVPHFWA:** For $f(t) = \log\left(\frac{\tau-1}{\tau^t-1}\right)$, $\tau > 1$, the AIVPHFWA operator is reduced to IVPHFFWA operator in the form of

$$\mathit{IVPHFFWA}\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right)$$

$$= \bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i, \\ i=1,2,\ldots,n}} \left\{ \left(\left[\sqrt{1 - \frac{\log\left(1+\prod_{i=1}^n \left(\tau^{1-\left(\gamma_i^l\right)^2}-1\right)^{\omega_i}\right)}{\log\tau}}, \right.\right.\right.$$

$$\left. \sqrt{1 - \frac{\log\left(1+\prod_{i=1}^n \left(\tau^{1-\left(\gamma_i^u\right)^2}-1\right)^{\omega_i}\right)}{\log\tau}} \right],$$

$$\left.\left.\left[\sqrt{\frac{\log\left(1+\prod_{i=1}^n \left(\tau^{\left(\eta_i^l\right)^2}-1\right)^{\omega_i}\right)}{\log\tau}}, \sqrt{\frac{\log\left(1+\prod_{i=1}^n \left(\tau^{\left(\eta_i^u\right)^2}-1\right)^{\omega_i}\right)}{\log\tau}} \right] \right) \right\}.$$

For $\tau \to 1$, IVPHFFWA operator is reduced to IVPHFWA operator. So, IVPHFFWA operator may be considered as a generalisation of IVPHFWA operator.

Some important properties of the proposed AIVPHFWA aggregation operators are presented below.

Theorem 2. (Boundary) Let $\tilde{k}_i$ $(i = 1, 2, \ldots, n)$ be a collections of IVPHFNs, and for all $i = 1, 2, \ldots, n$, let

$$\gamma_{min}^l = min\left\{\gamma_{i_{min}}^l | \gamma_{i_{min}}^l = min_{[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i}\left\{\gamma_i^l\right\}\right\},$$

$$\gamma_{max}^u = max\left\{\gamma_{i_{max}}^u | \gamma_{i_{max}}^u = max_{[\gamma_i^l,\gamma_i^u]\in\tilde{h}_i}\left\{\gamma_i^u\right\}\right\},$$

$$\eta_{max}^l = max\left\{\eta_{i_{max}}^l | \eta_{i_{max}}^l = max_{[\eta_i^l,\eta_i^u]\in\tilde{h}_i}\left\{\eta_i^l\right\}\right\},$$

$$\eta^u_{min} = min\left\{\eta^u_{i_{min}} | \eta^u_{i_{min}} = min_{[\eta^l_i, \eta^u_i] \in \tilde{h}_i}\{\eta^u_i\}\right\},$$

and also γ^l_{max}, γ^u_{min}, η^l_{min} and η^u_{max} convey the alike meanings as above. Then

$$\tilde{k}_- \leqslant AIVPHFWA\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right) \leqslant \tilde{k}_+, \tag{9}$$

where $\tilde{k}_- = \left(\left[\gamma^l_{min}, \gamma^u_{min}\right], \left[\eta^l_{max}, \eta^u_{max}\right]\right)$ and $\tilde{k}_+ = \left(\left[\gamma^l_{max}, \gamma^u_{max}\right], \left[\eta^l_{min}, \eta^u_{min}\right]\right)$.

Proof: For any $i = 1, 2, \ldots, n$, $\gamma^l_{min} \leqslant \gamma^l_i \leqslant \gamma^l_{max}$ and $\gamma^u_{min} \leqslant \gamma^u_i \leqslant \gamma^u_{max}$.
i.e., $\left(\gamma^l_{min}\right)^2 \leqslant \left(\gamma^l_i\right)^2 \leqslant \left(\gamma^l_{max}\right)^2$ and $\left(\gamma^u_{min}\right)^2 \leqslant \left(\gamma^u_i\right)^2 \leqslant \left(\gamma^u_{max}\right)^2$.
Since $g(t)$ $(t \in [0, 1])$ is a monotonic increasing function,

$$g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma^l_{min}\right)^2\right)\right) \leqslant g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma^l_i\right)^2\right)\right) \leqslant g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma^l_{max}\right)^2\right)\right),$$

which implies that $\left(\gamma^l_{min}\right)^2 \leqslant g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma^l_i\right)^2\right)\right) \leqslant \left(\gamma^l_{max}\right)^2$.

$$\text{So, } \gamma^l_{min} \leqslant \sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma^l_i\right)^2\right)\right)} \leqslant \gamma^l_{max}. \tag{10}$$

$$\text{Similarly, } \gamma^u_{min} \leqslant \sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma^u_i\right)^2\right)\right)} \leqslant \gamma^u_{max}. \tag{11}$$

Now, for any $i = 1, 2, \ldots, n$, $\left(\eta^l_{min}\right)^2 \leqslant \left(\eta^l_i\right)^2 \leqslant \left(\eta^l_{max}\right)^2$.
Since $f(t)$ $(t \in [0, 1])$ is a decreasing function,

$$f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta^l_{min}\right)^2\right)\right) \leqslant f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta^l_i\right)^2\right)\right) \leqslant f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta^l_{max}\right)^2\right)\right),$$

which implies that $\left(\eta^l_{min}\right)^2 \leqslant f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta^l_i\right)^2\right)\right) \leqslant \left(\eta^l_{max}\right)^2$.

$$\text{So, } \eta^l_{min} \leqslant \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta^l_i\right)^2\right)\right)} \leqslant \eta^l_{max}. \tag{12}$$

$$\text{Similarly, } \eta^u_{min} \leqslant \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta^u_i\right)^2\right)\right)} \leqslant \eta^u_{max}. \tag{13}$$

From Eqs (10) and (12),

$$\gamma^l_{min} - \eta^l_{max} \leqslant \sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma^l_i\right)^2\right)\right)} - \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta^l_i\right)^2\right)\right)} \leqslant \gamma^l_{max} - \eta^l_{min}.$$

From Eqs (11) and (13),

$$\gamma^u_{min} - \eta^u_{max} \leqslant \sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma^u_i\right)^2\right)\right)} - \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta^u_i\right)^2\right)\right)} \leqslant \gamma^u_{max} - \eta^u_{min}.$$

Then, $S\left(\tilde{k}_-\right) \leqslant S\left(AIVPHFWA\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right)\right) \leqslant S\left(\tilde{k}_+\right)$.
Therefore, $\tilde{k}_- \leqslant AIVPHFWA\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right) \leqslant \tilde{k}_+$.

Theorem 3. Let $\tilde{k}_i$ $(i = 1, 2, \ldots, n)$ be a collection of IVPHFNs, $\omega_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ be their corresponding weight vectors, and $\sum_{i=1}^n \omega_i = 1$. If $\tilde{k}$ be an IVPHFN, then

$$AIVPHFWA\left(\tilde{k}_1 \oplus_A \tilde{k}, \tilde{k}_2 \oplus_A \tilde{k}, \ldots, \tilde{k}_n \oplus_A \tilde{k}\right) = AIVPHFWA\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right) \oplus_A \tilde{k}. \tag{14}$$

Proof.
It is clear that $\tilde{k}_i \oplus_A \tilde{k} =$

$$\bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i\\ i=1,2,\ldots,n,\\ ([\gamma^l,\, \gamma^u],[\eta^l,\, \eta^u])\in\tilde{k}}} \left\{\left(\left[\sqrt{g^{-1}\left(g\left(\left(\gamma_i^l\right)^2\right)+g\left(\left(\gamma^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(g\left(\left(\gamma_i^u\right)^2\right)+g\left(\left(\gamma^u\right)^2\right)\right)}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{f^{-1}\left(f\left(\left(\eta_i^l\right)^2\right)+f\left(\left(\eta^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(f\left(\left(\eta_i^u\right)^2\right)+f\left(\left(\eta^u\right)^2\right)\right)}\right]\right)\right\}.$$

Let *AIVPHFWA* $\left(\tilde{k}_1\oplus_A\tilde{k}, \tilde{k}_2\oplus_A\tilde{k}, \ldots, \tilde{k}_n\oplus_A\tilde{k}\right)$

$$=\bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i\\ i=1,2,\ldots,n\\ ([\gamma^l,\, \gamma^u],[\eta^l,\, \eta^u])\in\tilde{k}}} \left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(g^{-1}\left(g\left(\left(\gamma_i^l\right)^2\right)+g\left(\left(\gamma^l\right)^2\right)\right)\right)\right)},\right.\right.\right.$$
$$\left.\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(g^{-1}\left(g\left(\left(\gamma_i^u\right)^2\right)+g\left(\left(\gamma^u\right)^2\right)\right)\right)\right)}\right],$$
$$\left[\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(f^{-1}\left(f\left(\left(\eta_i^l\right)^2\right)+f\left(\left(\eta^l\right)^2\right)\right)\right)\right)},\right.$$
$$\left.\left.\left.\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(f^{-1}\left(f\left(\left(\eta_i^u\right)^2\right)+f\left(\left(\eta^u\right)^2\right)\right)\right)\right)}\right]\right)\right\}$$

$$=\bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i\\ ,\ i=1,2,\ldots,n,\\ ([\gamma^l,\, \gamma^u],[\eta^l,\, \eta^u])\in\tilde{k}}} \left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i \left(g\left(\left(\gamma_i^l\right)^2\right)+g\left(\left(\gamma^l\right)^2\right)\right)\right)},\right.\right.\right.$$
$$\left.\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i \left(g\left(\left(\gamma_i^u\right)^2\right)+g\left(\left(\gamma^u\right)^2\right)\right)\right)}\right],$$
$$\left.\left.\left[\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i \left(f\left(\left(\eta_i^l\right)^2\right)+f\left(\left(\eta^l\right)^2\right)\right)\right)},\ \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i \left(f\left(\left(\eta_i^u\right)^2\right)+f\left(\left(\eta^u\right)^2\right)\right)\right)}\right]\right)\right\}$$

$$=\bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i\\ i=1,2,\ldots,n\\ ([\gamma^l,\, \gamma^u],[\eta^l,\, \eta^u])\in\tilde{k}}} \left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_i^l\right)^2\right)+g\left(\left(\gamma^l\right)^2\right)\right)},\right.\right.\right.$$
$$\left.\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_i^u\right)^2\right)+g\left(\left(\gamma^u\right)^2\right)\right)}\right], \left[\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta_i^l\right)^2\right)+f\left(\left(\eta^l\right)^2\right)\right)},\right.$$
$$\left.\left.\left.\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta_i^u\right)^2\right)+f\left(\left(\eta^u\right)^2\right)\right)}\right]\right)\right\}.$$

Now, *AIVPHFWA* $\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right)\oplus_A\tilde{k}$

$$=\bigcup_{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i, i=1,2,\ldots,n} \left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^n \omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta_i^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^n \omega_i f\left(\left(\eta_i^u\right)^2\right)\right)}\right]\right)\right\}\oplus_A\left\{\left(\left[\gamma^l,\ \gamma^u\right],\left[\eta^l,\ \eta^u\right]\right)\right\}$$

$$= \bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i \\ i=1,2,\ldots,n, \\ ([\gamma^l,\,\gamma^u],[\eta^l,\,\eta^u])\in\tilde{k}}} \left\{ \left(\left[\sqrt{g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)\right)+g\left(\left(\gamma^l\right)^2\right)\right)}\,, \right.\right.\right.$$

$$\left.\sqrt{g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)\right)+g\left(\left(\gamma^u\right)^2\right)\right)}\right],$$

$$\left[\sqrt{f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\eta_i^l\right)^2\right)\right)\right)+f\left(\left(\eta^l\right)^2\right)\right)},\right.$$

$$\left.\left.\left.\sqrt{f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\eta_i^u\right)^2\right)\right)\right)+f\left(\left(\eta^u\right)^2\right)\right)}\right]\right)\right\}$$

$$= \bigcup_{\substack{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i \\ i=1,2,\ldots,n \\ ([\gamma^l,\,\gamma^u],[\eta^l,\,\eta^u])\in\tilde{k}}} \left\{ \left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\gamma_i^l\right)^2\right)+g\left(\left(\gamma^l\right)^2\right)\right)}, \right.\right.\right.$$

$$\left.\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\gamma_i^u\right)^2\right)+g\left(\left(\gamma^u\right)^2\right)\right)}\right],\left[\sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\eta_i^l\right)^2\right)+f\left(\left(\eta^l\right)^2\right)\right)},\right.$$

$$\left.\left.\left.\sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\eta_i^u\right)^2\right)+f\left(\left(\eta^u\right)^2\right)\right)}\right]\right)\right\}.$$

Hence the theorem.

Theorem 4. (Idempotency) If all $\tilde{k}_i$ $(i=1,2,\ldots,n)$ are equal, and let $\tilde{k}_i = \left\{\left(\left[\gamma^l,\,\gamma^u\right],\left[\eta^l,\,\eta^u\right]\right)\right\}$ for all $(i=1,2,\ldots,n)$, then

$$\mathit{AIVPHFWA}\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_n\right)=\tilde{k}=\left\{\left(\left[\gamma^l,\,\gamma^u\right],\left[\eta^l,\,\eta^u\right]\right)\right\}. \quad (15)$$

Proof: Here, *AIVPHFWA* $\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_n\right)$

$$= \bigcup_{([\gamma_i^l,\gamma_i^u],[\eta_i^l,\eta_i^u])\in\tilde{k}_i,i=1,2,\ldots,n} \left\{\left(\left[\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\gamma_i^l\right)^2\right)\right)},\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left(\left(\gamma_i^u\right)^2\right)\right)}\right],\right.\right.$$

$$\left.\left.\left[\sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\eta_i^l\right)^2\right)\right)},\sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left(\left(\eta_i^u\right)^2\right)\right)}\right]\right)\right\}.$$

Now, since $\tilde{k}_i=\tilde{k}=\left\{\left(\left[\gamma^l,\gamma^u\right],\left[\eta^l,\eta^u\right]\right)\right\}$ for all $(i=1,2,\ldots,n)$, then $\gamma_i^l=\gamma^l$, $\gamma_i^u=\gamma^u$, $\eta_i^l=\eta^l$ and $\eta_i^u=\eta^u$ for all $(i=1,2,\ldots,n)$. Therefore,

$$\mathit{AIVPHFWA}\left(\tilde{k},\tilde{k},\ldots,\tilde{k}\right)$$

$$\bigcup_{\substack{([\gamma^l,\,\gamma^u],[\eta^l,\,\eta^u])\in\tilde{k} \\ i=1,2,\ldots,n}} \left\{\left(\left[\sqrt{g^{-1}\left(g\left(\left(\gamma^l\right)^2\right)\sum_{i=1}^{n}\omega_i\right)},\sqrt{g^{-1}\left(g\left(\left(\gamma^u\right)^2\right)\sum_{i=1}^{n}\omega_i\right)}\right],\right.\right.$$

$$\left.\left.\left[\sqrt{f^{-1}\left(f\left(\left(\eta^l\right)^2\right)\sum_{i=1}^{n}\omega_i\right)},\sqrt{f^{-1}\left(f\left(\left(\eta^u\right)^2\right)\sum_{i=1}^{n}\omega_i\right)}\right]\right)\right\}$$

$$= \bigcup_{\substack{([\gamma^l,\,\gamma^u],[\eta^l,\,\eta^u])\in\tilde{k} \\ i=1,2,\ldots,n}} \left\{\left(\left[\gamma^l,\gamma^u\right],\left[\eta^l,\eta^u\right]\right)\right\}=\left\{\left(\left[\gamma^l,\gamma^u\right],\left[\eta^l,\eta^u\right]\right)\right\}.$$

Hence the theorem.

5.2. IVPHF Archimedean geometric operators

In the following, some IVPHF Archimedean geometric operators based on the Archimedean operations of IVPHFNs is proposed.

Definition 13. For any collection of IVPHFNs, $\tilde{k}_i$ $(i = 1, 2, \ldots, n)$ and weight vector, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, an A$t$-N&$t$-CN-based IVPHFWG (AIVPHFWG) operator is defined by a mapping, $AIVPHFWG: K^n \to K$ such that

$$AIVPHFWG\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right) = \otimes_{A}{}_{i=1}^{n}\left(\tilde{k}_i^{\omega_i}\right),$$

where $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$.

Theorem 5. The aggregated value of any collections of IVPHFNs, $\tilde{k}_i$ $(i = 1, 2, \ldots, n)$ using AIVPHFWG operator is also an IVPHFN, and is given by

$$AIVPHFWG\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right)$$
$$= \bigcup_{\left(\left[\gamma_i^l, \gamma_i^u\right], \left[\eta_i^l, \eta_i^u\right]\right) \in \tilde{k}_i, i=1,2,\ldots,n} \left\{\left(\left[\sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(\left(\gamma_i^l\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left(\left(\gamma_i^u\right)^2\right)\right)}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g\left(\left(\eta_i^l\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g\left(\left(\eta_i^u\right)^2\right)\right)}\right]\right)\right\}. \tag{16}$$

Proof. Proof is same as the proof of Theorem 1.

It is worthy to mention here that for different forms of the decreasing generator, f, several forms of familiar IVPHFWG operators can be obtained, which are described below.

Algebraic t-N and t-CN Operations on AIVPHFWG: If $f(t) = -\log t$, then the AIVPHFWG operator reduces to the IVPHF WG (IVPHFWG) operator, and is defined as:

$$IVPHFWG\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right)$$
$$= \bigcup_{\left(\left[\gamma_i^l, \gamma_i^u\right], \left[\eta_i^l, \eta_i^u\right]\right) \in \tilde{k}_i, i=1,2,\ldots,n} \left\{\left(\left[\prod_{i=1}^{n} \left(\gamma_i^l\right)^{\omega_i}, \prod_{i=1}^{n} \left(\gamma_i^u\right)^{\omega_i}\right],\right.\right.$$
$$\left.\left.\left[\sqrt{1 - \prod_{i=1}^{n}\left(1 - \left(\eta_i^l\right)^2\right)^{\omega_i}}, \sqrt{1 - \prod_{i=1}^{n}\left(1 - \left(\eta_i^u\right)^2\right)^{\omega_i}}\right]\right)\right\}. \tag{17}$$

Einstein t-N and t-CN Operations on AIVPHFWG: If $f(t) = \log\left(\frac{2-t}{t}\right)$ is considered, then AIVPHFWG operator reduces to IVPHF Einstein WG (IVPHFEWG) operator as:

$$IVPHFEWG\left(\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n\right) =$$
$$\bigcup_{\left(\left[\gamma_i^l, \gamma_i^u\right], \left[\eta_i^l, \eta_i^u\right]\right) \in \tilde{k}_i, i=1,2,\ldots,n} \left\{\left(\left[\frac{\sqrt{2}\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(2 - \left(\gamma_i^l\right)^2\right)^{\omega_i} + \prod_{i=1}^{n}\left(\left(\gamma_i^l\right)^2\right)^{\omega_i}}},\right.\right.\right.$$
$$\left.\frac{\sqrt{2}\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(2 - \left(\gamma_i^u\right)^2\right)^{\omega_i} + \prod_{i=1}^{n}\left(\left(\gamma_i^u\right)^2\right)^{\omega_i}}}\right],$$

$$\left[\frac{\sqrt{\prod_{i=1}^{n}\left(1+\left(\eta_i^l\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\eta_i^l\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+\left(\eta_i^l\right)^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\left(\eta_i^l\right)^2\right)^{\omega_i}}},\right.$$

$$\left.\frac{\sqrt{\prod_{i=1}^{n}\left(1+\left(\eta_i^u\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\eta_i^u\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+\left(\eta_i^u\right)^2\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\left(\eta_i^u\right)^2\right)^{\omega_i}}}\right]\Bigg)\Bigg\}. \tag{18}$$

- **Hamacher t-N and t-CN Operations on AIVPHFWG:** When $f(t)=\log\left(\frac{\psi+(1-\psi)t}{t}\right)$, $\psi>0$, AIVPHFWG operator reduces to IVPHF Hamacher WG (IVPHFHWG) operator in the form of

$$\mathit{IVPHFHWG}\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_n\right)$$

$$=\bigcup_{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i,i=1,2,\ldots,n}$$

$$\Bigg\{\Bigg(\Bigg(\left[\frac{\sqrt{\psi}\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1+(\psi-1)\left(1-\left(\gamma_i^l\right)^2\right)\right)^{\omega_i}+(\psi-1)\prod_{i=1}^{n}\left(\left(\gamma_i^l\right)^2\right)^{\omega_i}}},\right.$$

$$\left.\frac{\sqrt{\psi}\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1+(\psi-1)\left(1-\left(\gamma_i^u\right)^2\right)\right)^{\omega_i}+(\psi-1)\prod_{i=1}^{n}\left(\left(\gamma_i^u\right)^2\right)^{\omega_i}}}\right],$$

$$\left[\frac{\sqrt{\prod_{i=1}^{n}\left(1+(\psi-1)\left(\eta_i^l\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\eta_i^l\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+(\psi-1)\left(\eta_i^l\right)^2\right)^{\omega_i}+(\psi-1)\prod_{i=1}^{n}\left(1-\left(\eta_i^l\right)^2\right)^{\omega_i}}}\right.$$

$$\left.\frac{\sqrt{\prod_{i=1}^{n}\left(1+(\psi-1)\left(\eta_i^u\right)^2\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\left(\eta_i^u\right)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+(\psi-1)\left(\eta_i^u\right)^2\right)^{\omega_i}+(\psi-1)\sum_{i=1}^{n}\left(1-\left(\eta_i^u\right)^2\right)^{\omega_i}}}\right]\Bigg)\Bigg\}. \tag{19}$$

  For $\psi=1$ and $\psi=2$, IVPHFHWG operator is turned into IVPHFWG and IVPHFEWG operators, respectively. So, IVPHFHWG operator becomes a generalised version of IVPHFWG and IVPHFEWG operators.

- **Frank t-N and t-CN Operations on AIVPHFWG:** When $f(t)=\log\left(\frac{\tau-1}{\tau^t-1}\right)$, $\tau>1$, AIVPHFWG operator reduces to IVPHF Frank WG (IVPHFFWG) operator; thus, IVPHFFWG operator is defined as:

$$\mathit{IVPHFFWG}\left(\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_n\right)$$

$$=\bigcup_{\left(\left[\gamma_i^l,\gamma_i^u\right],\left[\eta_i^l,\eta_i^u\right]\right)\in\tilde{k}_i,i=1,2,\ldots,n}$$

$$\Bigg\{\Bigg(\Bigg(\left[\sqrt{\frac{\log\left(1+\prod_{i=1}^{n}\left(\tau^{\left(\gamma_i^l\right)^2}-1\right)^{\omega_i}\right)}{\log\tau}},\sqrt{\frac{\log\left(1+\prod_{i=1}^{n}\left(\tau^{\left(\gamma_i^u\right)^2}-1\right)^{\omega_i}\right)}{\log\tau}}\right],$$

$$\left[\sqrt{1-\frac{\log\left(1+\prod_{i=1}^{n}\left(\tau^{1-\left(\eta_i^l\right)^2}-1\right)^{\omega_i}\right)}{\log\tau}},\sqrt{1-\frac{\log\left(1+\prod_{i=1}^{n}\left(\tau^{1-\left(\eta_i^u\right)^2}-1\right)^{\omega_i}\right)}{\log\tau}}\right]\Bigg)\Bigg\}. \tag{20}$$

For $\tau \to 1$, IVPHFFWG operator is converted into IVPHFWG operator. So, IVPHFFWG operator is considered as a generalisation of IVPHFWG operator.

It is to be mentioned here that AIVPHFWG operator possesses same properties as like AIVPHFWA operator as discussed above. The proofs are also the same as that of AIVPHFWA operator. So, the proofs are skipped here.

6. MCDM method based on IVPHF information

In the present section, an approach to MCDM using IVPHF information is developed by utilising the proposed AIVPHFWA and AIVPHFWG operators.

Let $A = \{A_1, A_2, \ldots, A_m\}$ and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of alternatives and a collection of criteria, respectively, such that weight vector of the criteria is given by $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ with $\omega_j \in [0, 1]$ for $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} \omega_j = 1$. The IVPHF decision matrix $\tilde{D}=(\tilde{d}_{ij})_{m\times n}$ is constructed only when, all the performance scores of the alternatives are obtained. In solving MCDM problems, the cost criteria are required to transform into benefit criteria using the following method. Thus the decision matrix, $\tilde{D}=(\tilde{d}_{ij})_{m\times n}$ is converted into a normalised decision matrix, $\tilde{R}=(\tilde{r}_{ij})_{m\times n}$ by the following way:

$$\tilde{r}_{ij} = \begin{cases} \tilde{d}_{ij} \text{ for benefit criteria } C_j \\ \tilde{d}^c_{ij} \text{ for cost criteria } C_j \end{cases} \tag{21}$$

$i= 1, 2,\ldots,m$ and $j= 1, 2,\ldots,n$. Here $\tilde{d}^c_{ij}$ denotes the complement of $\tilde{d}_{ij}$.

Now, the newly defined AIVPHFWA (or AIVPHFWG) operator is utilized to develop an approach for solving MCDM problems under IVPHF environment. The whole process is described through the following steps:

Step 1: Transform the decision matrix, $\tilde{D}=\left(\tilde{d}_{ij}\right)_{m\times n}$ into the normalised form, $\tilde{R}=(\tilde{r}_{ij})_{m\times n}$ using Eq. (21).

Step 2: Aggregate the IVPHFNs, $\tilde{r}_{ij}$ for each alternative, A_i using AIVPHFWA (or AIVPHFWG) operator for a suitable weight vector, w and the parameters, ψ, τ as follows:

$$\begin{aligned}\tilde{r}_i^A &= \mathit{AIVPHFWA}\left(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}\right) \\ &= \bigcup_{\left([\gamma^l_{ij},\gamma^u_{ij}],[\eta^l_{ij},\eta^u_{ij}]\right)\in\tilde{r}_{ij}, j=1,2,\ldots,n} \\ &\left\{\left(\left[\sqrt{g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\left(\gamma^l_{ij}\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\left(\gamma^u_{ij}\right)^2\right)\right)}\right],\right.\right. \\ &\left.\left.\left[\sqrt{f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left(\left(\eta^l_{ij}\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_j f\left(\left(\eta^u_{ij}\right)^2\right)\right)}\right]\right)\right\}\end{aligned} \tag{22}$$

or

$$\begin{aligned}\tilde{r}_i^G &= \mathit{AIVPHFWG}\left(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in}\right) \\ &= \bigcup_{\left([\gamma^l_{ij},\gamma^u_{ij}],[\eta^l_{ij},\eta^u_{ij}]\right)\in\tilde{r}_{ij}, j=1,2,\ldots,n} \\ &\left\{\left(\left[\sqrt{f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left(\left(\gamma^l_{ij}\right)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left(\left(\gamma^u_{ij}\right)^2\right)\right)}\right],\right.\right. \\ &\left.\left.\left[\sqrt{g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\left(\eta^l_{ij}\right)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_j g\left(\left(\eta^u_{ij}\right)^2\right)\right)}\right]\right)\right\} \\ & i = 1,\ 2, \ldots, m; j = 1,\ 2, \ldots, n.\end{aligned} \tag{23}$$

Step 3: Utilizing the score function as mentioned in Definition 2, the rank of all alternatives is evaluated.

The above method is validated through the following illustrative examples.

Table 1
IVPHF decision matrix

	C_1	C_2
A_1	$\{([.1,.2],[.3,.6]),([.2,.3],[.4,.7])\}$	$\left\{\begin{array}{c}([.3,.4],[.6,.9]),([.3,.5],[.4,.8]),\\([.2,.7],[.2,.6])\end{array}\right\}$
A_2	$\{([.2,.4],[.3,.5]),([.5,.6],[.4,.7])\}$	$\{([.2,.5],[.6,.7]),([.3,.6],[.4,.8])\}$
A_3	$\{([.6,.9],[.1,.2]),([.7,.8],[.2,.3])\}$	$\{([.4,.6],[.2,.4]),([.5,.7],[.3,.4])\}$
A_4	$\{([.6,.7],[.3,.5]),([.7,.8],[.3,.4])\}$	$\{([.4,.5],[.2,.5]),([.5,.6],[.4,.5])\}$
A_5	$\{([.3,.7],[.3,.4]),([.6,.8],[.4,.6])\}$	$\{([.3,.5],[.3,.6]),([.3,.7],[.4,.7])\}$
	C_3	C_4
A_1	$\{([.2,.5],[.5,.8]),([.2,.6],[.3,.8])\}$	$\{([.2,.6],[.3,.8]),([.2,.7],[.4,.6])\}$
A_2	$\{([.2,.5],[.3,.7]),([.3,.5],[.6,.7])\}$	$\{([.2,.5],[.4,.7]),([.4,.7],[.5,.6])\}$
A_3	$\left\{\begin{array}{c}([.5,.6],[.1,.2]),([.7,.8],[.4,.6]),\\([.8,.9],[.1,.2])\end{array}\right\}$	$\{([.4,.7],[.3,.5]),([.6,.9],[.2,.4])\}$
A_4	$\{([.4,.5],[.2,.3]),([.5,.6],[.4,.5])\}$	$\left\{\begin{array}{c}([.4,.6],[.2,.6]),([.3,.6],[.3,.7]),\\([.5,.8],[.4,.6])\end{array}\right\}$
A_5	$\{([.2,.6],[.4,.6]),([.3,.7],[.4,.7])\}$	$\{([.3,.6],[.4,.6]),([.5,.6],[.3,.7])\}$

7. Illustrative examples

To establish application potentiality of the proposed approach, two illustrative examples are considered and solved in this section.

7.1. Example 1

At first, a revised problem (adapted from Wei et al. [25]) relating to green supply chain management (GSCM) is considered and solved under IVPHF context.

It is well known that the choice of suitable green supplier is a key factor of GSCM. Due to the fact that the entire supply chain depends upon the quality of the suppliers; and the ecological performance of industrial companies, organizations is directly or indirectly balanced by the suppliers' characteristics, so, in the view of sociological or environmental impact of the suppliers, appropriate green supplier assessment has now become an emerging research topic. In this section, the proposed methodology is applied to GSCM with IVPHF data for supplier evaluation and choosing most potential supplier.

The problem is discussed in the following manner:

In a GSCM five green suppliers are available as alternatives which are given by the set, $A = \{A_i | i = 1, 2, 3, 4, 5\}$.

The four criteria on which the suppliers are evaluated by the experts are given by

- C_1: The product quality factors.
- C_2: Environmental factors.
- C_3: Delivery factors.
- C_4: Price factors.

The criteria weight vector is considered as $w = (0.4, 0.1, 0.2, 0.3)^T$.

The DM provided judgement values on the alternatives considering the above-mentioned criteria by using IVPHFNs, and the resulting IVPHF decision matrix is presented in Table 1.

Now, the developed method is applied to find the best supplier in the decision making context. The steps of the methodology are presented below:

Step 1: All the criteria as shown in Table 1, being the benefit criteria, the decision matrix need not be normalised further.

Step 2: Utilising IVPHFHWA operator to aggregate IVPHFNs for each alternative, A_i, considering associated weight vector, $w = (0.4, 0.1, 0.2, 0.3)^T$, and the value of the parameter, $\psi = 2$, the aggregated IVPHFNs are calculated. For simplicity of presentation, only the aggregated IVPHFNs corresponding to the alternative, A_2 is presented below.

$$r_2^A = \textit{IVPHFHWA}\left(\tilde{r}_{21}, \tilde{r}_{22}, \tilde{r}_{23}, \tilde{r}_{24}\right)$$

$$= \{([0.2000, 0.4631], [0.3520, 0.6155]), ([0.2762,\ 0.5414], [0.3774, 0.5867]), ([0.2237, 0.4631],$$

Table 2
Score values for $\psi = 2$

Operator	Parameter	$S\left(\tilde{r}_1^A\right)$	$S\left(\tilde{r}_2^A\right)$	$S\left(\tilde{r}_3^A\right)$	$S\left(\tilde{r}_4^A\right)$	$S\left(\tilde{r}_5^A\right)$
IVPHFHWA	$\psi = 2$	0.4153	0.4545	0.7240	0.6049	0.5259

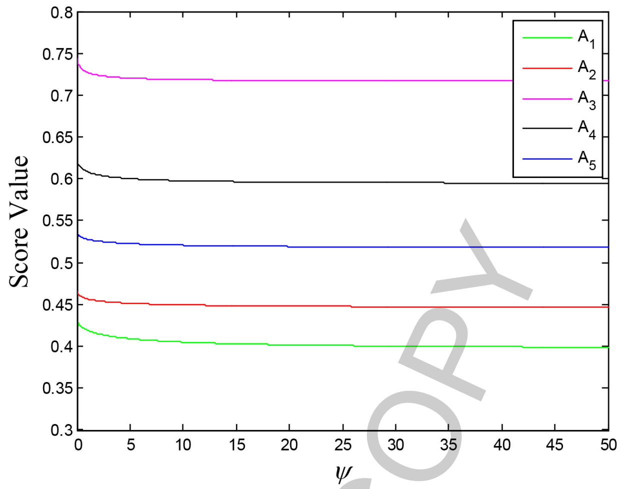

Fig. 1. Variation of the score values using IVPHFHWA operator.

$[0.4061, 0.6155]), ([0.2937, 0.5414], [0.4346, 0.5867]), ([0.2122, 0.4754], [0.3370, 0.6248]),$
$([0.2851, 0.5515], [0.3614, 0.5957]), ([0.2346, 0.4754], [0.3891, 0.6248]), ([0.3021, 0.5515],$
$[0.4167, 0.5957]), ([0.3543,\ 0.5431], [0.39450.7000,]), ([0.4015, 0.6081], [0.4224, 0.6693]),$
$([0.3679, 0.5431], [0.4538, 0.7000]), ([0.4134, 0.6081], [0.4849, 0.6693]), ([0.3612, 0.5532],$
$[0.3780, 0.7098]), ([0.4075, 0.6166], [0.4049, 0.6790]), ([0.3746, 0.5532], [0.4353, 0.7098]),$
$([0.4193, 0.6166], [0.4654, 0.6790])\}$

Similarly, other values of $\tilde{r}_i^A$ $(i = 1, 2, \ldots, 5)$ are calculated.

Step 3: The score values of each alternatives A_i $(i = 1, 2, \ldots, 5)$ are calculated and is presented in Table 2. The alternatives are ranked based on the score values.

In accordance with the score values, the alternatives are ranked, and the ordering of alternatives are obtained as $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$. Therefore, the best alternative is identified as A_3.

7.1.1. Sensitivity analysis

Now, based on the DM's preferences, the parameter, ψ can presume different values. To observe the variation of the ranking of the five alternatives based on the parameter, ψ, the values between 0 to 50 are assigned. The achieved score values of the five alternatives are shown in Figs 1 and 2.

From Fig. 1, it is observed that, using IVPHFHWA operator, the ordering of the alternatives does not change, but the score value of the alternatives decreases monotonically.

In a similar manner, IVPHFHWG operator is used on the given example, and the score value of alternatives are calculated by varying the parameter, ψ between 0 to 50. The achieved results are presented in Fig. 2. It is to be noted that the ordering of alternatives does not change as like using IVPHFHWA operator. But the score value of the alternatives increases monotonically.

Now, if IVPHFFWA and IVPHFFWG operators are used, individually, instead of using IVPHFHWA or IVPHFHWG operators for aggregating the attribute values of the alternatives, then the score values are presented in the Figs 3 and 4, respectively. As like above cases, similar observations are viewed through the figures corresponding to averaging and geometric operators.

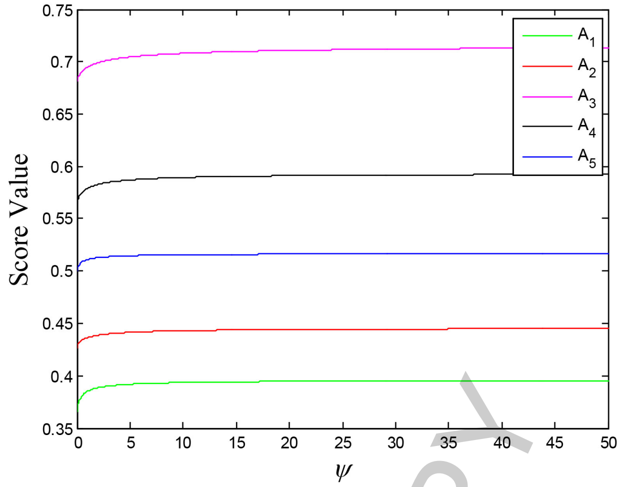

Fig. 2. Variation of the score values using IVPHFHWG operator.

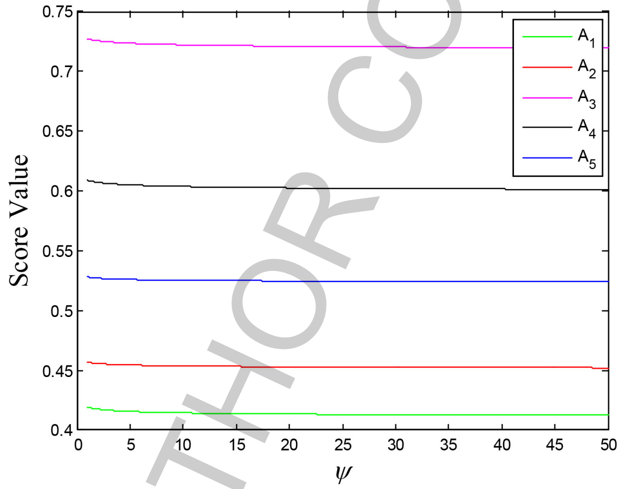

Fig. 3. Variation of the score values using IVPHFFWA operator.

It is worthy to mention here that no changes in the orderings of the alternatives are found while making the decision using different aggregation operators. Thus it stands that the proposed methodology possesses a strong consistency.

7.1.2. Comparison with existing method [25]

In the context of solving MCDM problems, Wei et al. [25] used PHFHWA and PHFHWG operators. It is to be noted here that the orderings of alternatives achieved by Wei et al. [25] are the same as like the proposed method. In the method developed by Wei et al. [25], PHF values are used. But in the proposed method, DM can evaluate the problem more adequately using IVPHFNs. So, the proposed method is advantageous for considering DM's flexibility for making proper assessment in real-life MCDM contexts. Further, the existing Hamacher aggregation operators [25] for PHFNs become a particular case of the At-N&t-CN based proposed operators. Moreover, the superiority of the proposed method is reflected by comparing the differences between two consecutive score values of the alternatives, which are ranked using the proposed method and the method described by Wei et al. [25]. The comparisons are presented through Figs 5 and 6.

The above figures show that the difference between any two score values of the consecutively ranked alternatives increases, significantly, in the proposed method. So the rank of the alternatives can be identified in a better way

Table 3
IVPHF decision matrix

	C_1	C_2	C_3
A_1	$\{([0.4, 0.5], [0.3, 0.4])\}$	$\left\{\begin{array}{c}([0.4, 0.6], [0.2, 0.4]),\\ [0.5, 0.7], [0.3, 0.5]\end{array}\right\}$	$\left\{\begin{array}{c}([0.1, 0.2], [0.8, 0.9]),\\ ([0.1, 0.3], [0.5, 0.6])\end{array}\right\}$
A_2	$\left\{\begin{array}{c}([0.6, 0.7], [0.2, 0.3]),\\ ([0.75, 0.9], [0.1, 0.3])\end{array}\right\}$	$\{([0.6, 0.7], [0.2, 0.3])\}$	$\left\{\begin{array}{c}([0.4, 0.7], [0.1, 0.2]),\\ ([0.7, 0.9], [0.2, 0.3])\end{array}\right\}$
A_3	$\{([0.3, 0.6], [0.3, 0.4])\}$	$\left\{\begin{array}{c}([0.5, 0.6], [0.3, 0.4]),\\ ([0.6, 0.7], [0.4, 0.5])\end{array}\right\}$	$\{([0.5, 0.6], [0.1, 0.3])\}$
A_4	$\left\{\begin{array}{c}([0.7, 0.8], [0.1, 0.2]),\\ ([0.6, 0.8], [0.2, 0.3])\end{array}\right\}$	$\{([0.6, 0.7], [0.1, 0.3])\}$	$\left\{\begin{array}{c}([0.3, 0.4], [0.1, 0.2]),\\ ([0.6, 0.7], [0.1, 0.25])\end{array}\right\}$

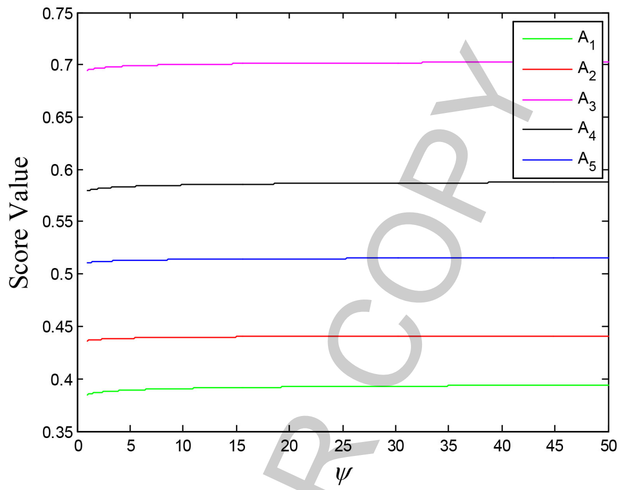

Fig. 4. Variation of the score values using IVPHFFWG operator.

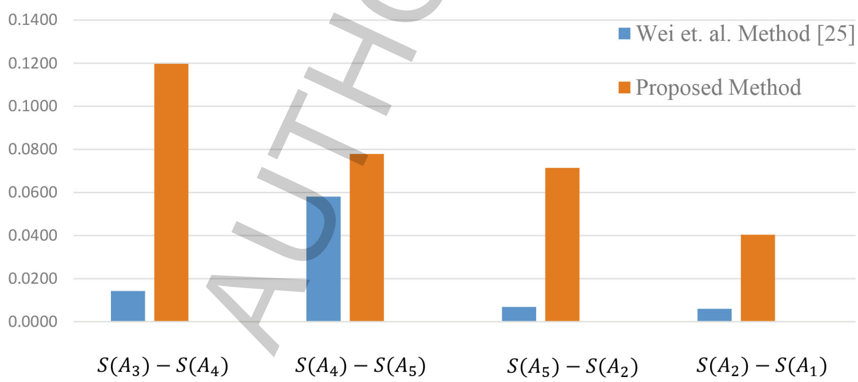

Fig. 5. Difference between two consecutive score values of the ranked alternatives using IVPHFHWA and PHFHWA operators.

than the existing method. Thus in comparison with the existing methods, the proposed methodology contains better efficiency in ranking the alternatives.

7.2. Example 2

Another problem is considered in this section to show the applicability and efficiency of the proposed methodology, more clearly. The problem related to investment of funds in an appropriate company. It is collected from a research article published by Garg [31] and revised under IVPHF environment. There are four investment companies,

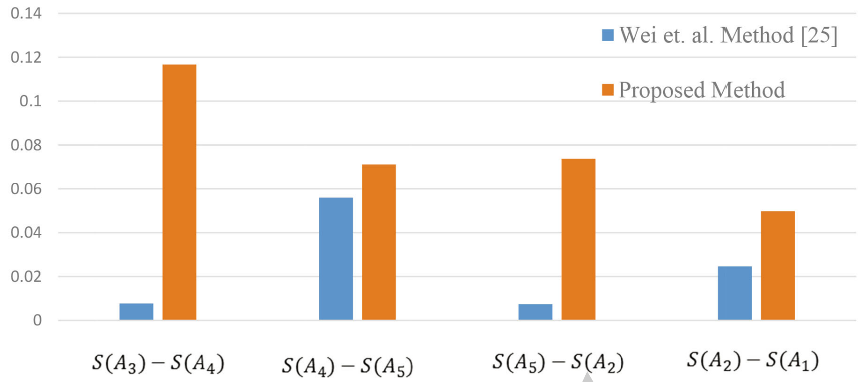

Fig. 6. Difference between two consecutive score values of the ranked alternatives using IVPHFHWG and PHFHWG operators.

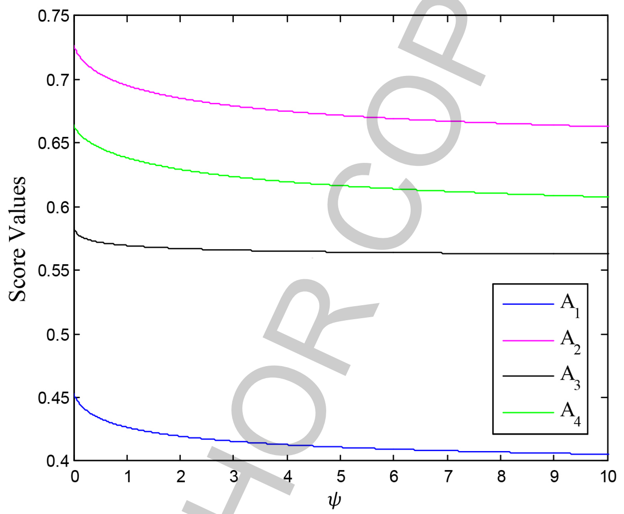

Fig. 7. Variation of the score values using IVPHFHWA operator.

$A_i\,(i = 1, 2, 3, 4)$ satisfying three criteria, $C_j\,(j = 1, 2, 3)$ with the weight vector, $w = (0.35, 0.25, 0.40)^T$. To choose the best investment company, the DM evaluates the alternatives based on the above criteria using IVPHFNs, and their evaluation values are listed in the following Table 3.

Using IVPHFHWA (Considering Hamacher parameter $\psi = 0.5$), the score values are obtained as $S(A_1) = 0.4338$, $S(A_2) = 0.7047$, $S(A_3) = 0.5721$, $S(A_4) = 0.6465$.

So, the rank of the alternatives is $A_2 \succ A_4 \succ A_3 \succ A_1$.

Further, using IVPHFHWG operator (Considering Hamacher parameter $= 10$), the score values are achieved as $S(A_1) = 0.3812$, $S(A_2) = 0.6226$, $S(A_3) = 0.5468$, $S(A_4) = 0.5638$.

So, the rank of the alternatives becomes $A_2 \succ A_4 \succ A_3 \succ A_1$.

7.2.1. Sensitivity analysis

Changing the Hamacher parameter, ψ in $(0, 10]$, and using IVPHFHWA and IVPHFHWG operators, the corresponding score values are presented through the Figs 7 and 8, respectively. From Fig. 7, it is clear that the ranking of the alternatives remains unchanged using IVPHFHWA operator for $\psi \in (0, 10]$. But, Fig. 8 discloses that the ranking of the alternatives differs for some values of ψ using IVPHFHWG operator. The ranking results are listed below.

Using IVPHFHWA operator, the ranking of alternative becomes $A_2 \succ A_3 \succ A_4 \succ A_1$ for $\psi \in (0, 10]$. Thus, the best alternative remains the same as A_2 throughout the given range.

Besides, using IVPHFHWG operator, the rankings of alternatives become

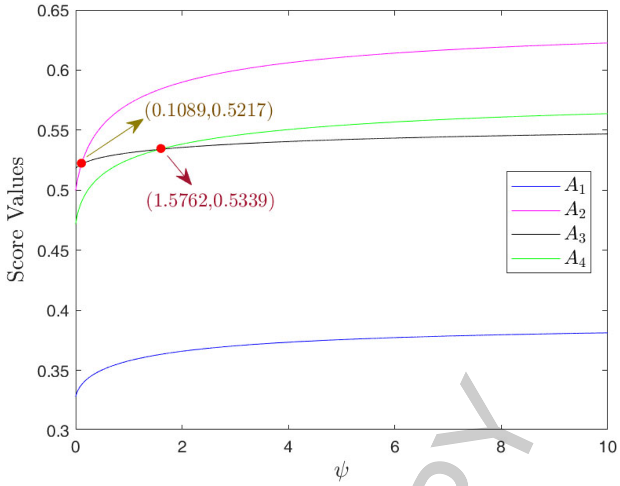

Fig. 8. Variation of the score values using IVPHFHWG operator.

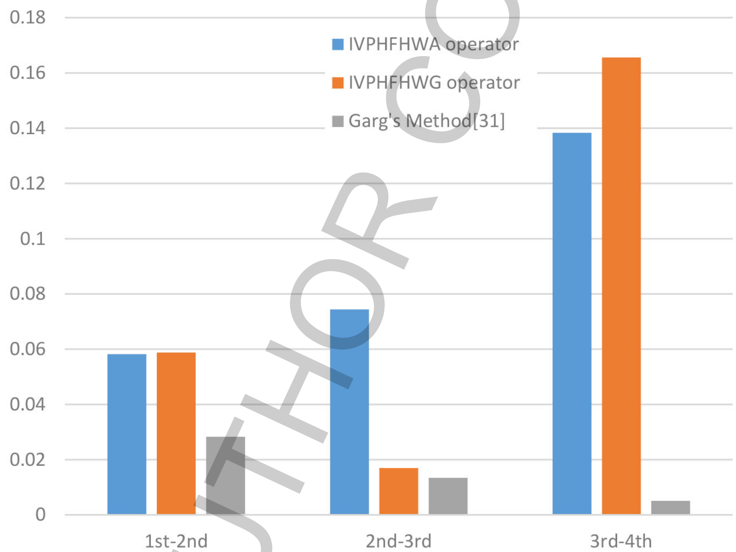

Fig. 9. The differences of overall values of the consecutively ranked alternatives.

$A_3 \succ A_2 \succ A_4 \succ A_1$ for $\psi \in (0, 0.1089)$;
$A_2 \succ A_3 \succ A_4 \succ A_1$ for $\psi \in (0.1089, 1.5762)$; and
$A_2 \succ A_4 \succ A_3 \succ A_1$ for $\psi \in (1.5762, 10]$.
Hence the best alternative is either A_2 or A_3 according to the choice of the parameter.

7.2.2. Comparison with other Method [31]

Using the Method developed by Garg [31], the ranking of the alternatives is found as $A_2 \succ A_4 \succ A_3 \succ A_1$; which is also acquired by the proposed operators by choosing particular value of the Hamacher parameter, ψ. Hence, in dealing real life MCDM problems, the proposed method is justified. Moreover, from the view point of the ranking results for different values of Hamacher parameter, it can be concluded that, by changing the values of the Hamacher parameter, ψ, according to the needs of the DMs, exact decisions can be taken. Further, in Fig. 9, the differences of overall values of the consecutively ranked alternatives are presented to compare with the existing method [31]. It is figured out from the figure that, using the proposed operators, the differences of overall values of the consecutively ranked alternatives are considerably increased than the existing method [31]. This shows that the proposed method is more effective than the existing method [31] in order to accomplish the ranking results.

8. Conclusions

In this paper, it has been shown that the proposed At-N&t-CN based aggregation operators for IVPHFNs are more flexible than the existing operators due to the presence of the parameters, ψ, τ. Depending on the preferences of the DMs, different values of the parameter can be chosen. As a consequence, several classes of aggregation operators can be generated. Thus the proposed method is capable enough to capture preferences of the DMs in MCDM problems in more flexible way. Through the illustrative examples, it has been established the fact that the proposed method not only capture the existing Hamacher operation based aggregation operators for PHFNs (Wei et al. [25]), but also extends the scope of using aggregation operators in IVPHF environments. The superiority of the proposed method is established by comparing the differences between any two score values of the consecutively ranked alternatives through the proposed method and the existing methods [25,31]. The comparison shows that all the differences have significantly increased. Hence ordering of the alternatives using the proposed methodology is more powerful. In future, the proposed operators may be extended to other domains, viz., q-rung orthopair fuzzy [49], bipolar fuzzy [50], cubic fuzzy [51,52], Pythagorean cubic fuzzy [53], cubic bipolar fuzzy [54], hesitant Pythagorean fuzzy [55] and other environments to capture uncertainties in more efficient ways. However, it is hoped that the proposed method would open up new direction for resolving uncertainties associated with real life MCDM problems.

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Interval-valued dual hesitant fuzzy prioritized aggregation operators based on Archimedean *t*-conorm and *t*-norm and their applications to multi-criteria decision making

Arun SARKAR and Animesh BISWAS

Multi-criteria decision making (MCDM) technique and approach have been a trending topic in decision making and systems engineering to choosing the probable optimal options. The primary purpose of this article is to develop prioritized operators to multi-criteria decision making (MCDM) based on Archimedean *t*-conorm and *t*-norms (A*t*-CN&*t*-Ns) under interval-valued dual hesitant fuzzy (IVDHF) environment. A new score function is defined for finding the rank of alternatives in MCDM problems with IVDHF information based on priority levels of criteria imposed by the decision maker. This paper introduces two aggregation operators: A*t*-CN&*t*-N-based IVDHF prioritized weighted averaging (AIVDHFPWA), and weighted geometric (AIVDHFPWG) aggregation operators. Some of their desirable properties are also investigated in details. A methodology for prioritization-based MCDM is derived under IVDHF information. An illustrative example concerning MCDM problem about a Chinese university for appointing outstanding oversea teachers to strengthen academic education is considered. The method is also applicable for solving other real-life MCDM problems having IVDHF information.

Key words: multi-criteria decision-making, interval-valued dual hesitant fuzzy elements, Archimedean *t*-conorm and *t*-norm, prioritized weighted averaging operator, prioritized weighted geometric operator

1. Introduction

The ambiguity of information is becoming an unalterable situation due to the rising complexity of our lifestyle rapidly. Multi-criteria decision making (MCDM) methods are a handy tool to grip this type of situation. Therefore,

A. Sarkar (corresponding author, e-mail: asarkarmth@gmail.com) is with Department of Mathematics, Heramba Chandra College, Kolkata – 700029, India.

A. Biswas (e-mail: abiswaskln@rediffmail.com) is with Department of Mathematics, University of Kalyani, Kalyani – 741235, India.

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MCDM has been an inexorable process to assess an object precisely. Besides the prior several decades, various methods have been proposed for solving different MCDM problems. Decision-maker (DM) can give their opinion by hesitant fuzzy (HF) set (HFS) [1,2] to defeat any hesitations. Generally, aggregation operators are essential tools for dealing with such MCDM problems. Xia and Xu [3] proposed a series of weighted averaging (WA) and weighted geometric (WG) aggregation operators based on HF environment viz., HF WA, HF ordered WA and their geometric operators. Based on Einstein operation, Zhou and Li [4] defined HF Einstein WG, and HF Einstein ordered WG operators and established the connections between the proposed operators. Zhang [5] proposed a method for deriving the weights of DMs and solved a multi-criteria group decision making (MCGDM) problem under HF information. Based on Hamacher t-conorm (t-CN) and t-norms (t-Ns), Son et al. [6] introduced some new HF power aggregation operators. Inspired by the concept of intuitionistic fuzzy (IF) set (IFS) and HFS, Zhu et al. [7] introduced dual HF (DHF) set (DHFS) by considering possible membership degrees and non-membership degrees with the condition that sum of maximum membership and non-membership degrees is less or equal to one. Under the DHF context, Wang et al. [8] defined some WA and WG aggregation operators: DHF WA, DHF WG, DHF ordered WA and DHF ordered WG operators. With Hamacher operations, Ju et al. [9] developed several aggregation operators viz., DHF Hamacher WA, DHF Hamacher WG, DHF Hamacher ordered WA, DHF Hamacher ordered WG operators, etc. Yu et al. [10] introduced the aggregation operators for aggregating DHF elements (DHFEs) and described these operators' properties. Zhao et al. [11] proposed some arithmetic operations of DHFEs based on Einstein t-CN and t-N, and some DHF aggregation operators are also introduced. Tang et al. [12] proposed the generalized rules of DHFS based on Frank t-CN and t-N, and used to construct the aggregation operators on DHF assessments in the context of MCDM.

However, in several real-life MCDM models, due to insufficiency in available information, DM are unable to exert their opinion with a crisp number but are comfortable to putting the decision values by interval numbers within $[0,1]$. To address this situation, Ju et al. [13] introduced the concept of interval-valued DHF (IVDHF) sets (IVDHFSs), which takes the hesitant membership and non-membership degrees in the form of interval-valued fuzzy numbers. It should be noted that when both the membership degree and non-membership degree of each element to a given set have single interval value, the IVDHFS reduces to the interval-valued IFS [14] and when the upper and lower limits of interval values are identical, IVDHFS becomes DHFS [7]. Thus, it is clear that IVDHFS is a more generalized form than other extensions of fuzzy sets. To aggregate the IVDHF elements (IVDHFEs), Ju et al. [13] developed IVDHF WA aggregation operator. Further, Zhang et al. [15] imposed Einstein t-CN and t-N on IVDHF environment to develop IVDHF Einstein WA and IVDHF Einstein WG operators.

During the aggregation process, the selection of appropriate operational laws is a crucial phase. The Archimedean t-CN and t-N (At-CN&t-N) provides a general rule of operational laws and more choices for DM. Different classes of t-CNs and t-Ns can be derived from At-CN&t-N [16, 17], such as t-CNs and t-Ns of the Algebraic, Einstein, Hamacher, Frank, and so on. Based on At-CN&t-N, Xia et al. [18] introduced At-CN&t-N-based IF WA and WG operators. Zhang and Wu [19] developed several At-CN&t-N-based interval-valued HF (IVHF) WA and WG aggregation operators. On DHF environment, Yu [20] proposed DHF WA and WG aggregation operators based on At-CN&t-N operations. Recently, Sarkar and Biswas [21] introduced At-CN&t-N operations on Pythagorean HF sets and defined a class of At-CN&t-N-based Pythagorean HF WA and WG operators. Again Sarkar and Biswas [22] applied At-CN&t-N on the IVDHF information and introduced a class of aggregation operators.

The above methods are all used under the premise that all criteria are in the same priority level. Most applications involve selecting or ordering of a group of alternatives based upon their satisfaction to a collection of criteria. To deal with this issue, Yager [23] developed prioritized average (PA) operators by modelling the criteria priority on the weights associated with criteria, which are dependent on the satisfaction of higher priority criteria. Yager [24] further focused on PA operators and proposed two methods for formulating this type of aggregation process. It is well known that the PA operator has many advantages over other operators. On HF environment, Yu [25] developed a family of aggregation operators based on Einstein t-CN and t-N, such as HF Einstein prioritized WA, WG and power WA operators. Wei [26] developed two prioritized aggregation operators for aggregating HFEs: HF prioritized WA (HFPWA), and HF prioritized WG (HFPWG) operators. Chen [27] introduced interval-valued IF prioritized aggregation operator and illustrated the proposed methodology by solving the watershed site selection problem. Liang et al. [28] derived generalized intuitionistic trapezoidal fuzzy prioritized WA and WG operators, also construct an approach for MCGDM under intuitionistic trapezoidal fuzzy environment. Under the IVHF context, Ye [29] proposed IVHF prioritized WA and WG operators and presented some properties of the proposed aggregation operators. Jin et al. [30] introduced Einstein operational laws on IVHF sets, and also developed two prioritized aggregation operators: IVHF Einstein prioritized WA (IVHFEPWA) and IVHF Einstein prioritized WG (IVHFEPWG) operators. Ren and Wei [31] proposed a prioritized multi-attribute decision-making method to solve decision problems under DHF environment. Recently, Biswas and Sarkar [32] introduced Einstein operations-based DHF prioritized WA (DHFPWA), and WG (DHFPWG) operators and constructed an approach for MCGDM. However, prioritized aggregation operators are applied in various contexts viz., IF, HF, IVHF, DHF for MCDM. But many prioritized-based MCDM problems can not be solved which are designed on IVDHF environment. And to overcome such situation, a methodology is proposed for IVDHF prioritized MCDM, which is the main motivation of this article. To do this at first define two prioritized aggregation operators based on At-CN&t-Ns under IVDHF information.

This article is organized as follows. Some preliminary concepts on DHFS, IVDHFS, At-CN&t-Ns and At-CN&t-Ns-based operations on IVDHFEs are studied in Section 2. A new score function of IVDHFE is defined in Section 3. In Section 4, At-CN&t-Ns-based IVDHF prioritized WA (AIVDHFPWA), and WG (AIVDHFPWG) aggregation operators are proposed to aggregate the IVDHFEs. After that classification of the proposed operators is made for different types of decreasing functions. Some desired properties and special cases of the proposed operators are also investigated. Section 5 gives an approach to MCDM under IVDHF environment. In Section 6, an illustrative example is solved using the proposed method, and sensitivity analysis is performed by varying the parameter. Finally, conclusion and scope for future studies have been described in Section 7.

2. Preliminaries

This section briefly reviews some basic concepts of DHFS, IVDHFS, At-CN&t-Ns and prioritized aggregation operators.

2.1. DHFS

Definition 1 [7] *The concept of DHFS was presented by Zhu et al. [7]. Let X be a fixed set. Then a DHFS is defined as*

$$P = \{\langle x, h_P(x), g_P(x)\rangle \mid x \in X\}, \tag{1}$$

where $\{\mu \mid \mu \in h_P(x)\}$ and $\{\nu \mid \nu \in g_P(x)\}$ denote the set of possible membership and non-membership degrees, respectively, of the element $x \in X$ to the set P, satisfying the conditions:

$0 \leqslant \mu, \nu \leqslant 1$, $0 \leqslant \max\{\mu\} + \max\{\nu\} \leqslant 0$*, for all $x \in X$. For convenience, $\langle h_P(x), g_P(x)\rangle$ is called the DHF element (DHFE) and denoted by $p = \langle h, g\rangle$.*

To compare among the DHFEs, Zhu et al. [7] derived the following comparison formula.

Definition 2 [7] *Let $p = \langle h, g\rangle$ be a DHFE. Then the score function $S(p)$ and accuracy function $A(p)$ of p is defined by*

$$S(p) = \hat{h} - \hat{g} \quad \text{and} \quad A(p) = \hat{h} + \hat{g},$$

where $\hat{h} = \frac{1}{\#h}\sum_{\mu\in h}\mu$ and $\hat{g} = \frac{1}{\#g}\sum_{\nu\in g}\nu$, and $\#h$ and $\#g$ denote the number of elements in h and g, respectively.

For any two DHFEs p_1 and p_2, if $S(p_1) > S(p_2)$ then $p_1 \succ p_2$.

To compute DMs' preference values by an interval number within [0, 1] instead of crisp numbers, Ju et al. [13] defined the concept of IVDHFSs.

Definition 3 [13] *Let X be a given set, then an IVDHFS $\widetilde{\mathrm{A}}$ on X is described as:*

$$\widetilde{\mathrm{A}} = \left\{ \left\langle x, \widetilde{h}_{\widetilde{\mathrm{A}}}(x), \widetilde{g}_{\widetilde{\mathrm{A}}}(x) \right\rangle \mid x \in X \right\}, \tag{2}$$

in which $\widetilde{h}_{\widetilde{\mathrm{A}}}(x) = \bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}(x)} \left\{ \left[\gamma^l, \gamma^u \right] \right\}$ and $\widetilde{g}_{\widetilde{\mathrm{A}}}(x) = \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}(x)} \left\{ \left[\eta^l, \eta^u \right] \right\}$ are two sets of interval values in [0, 1], representing the possible membership degree and non-membership degree of the element $x \in X$ to the set $\widetilde{\mathrm{A}}$, respectively, with $\left[\gamma^l, \gamma^u\right] \left[\eta^l, \eta^u\right] \subset [0, 1]$ and $0 \leqslant \max\{\gamma^u\} + \max\{\eta^u\} \leqslant 1$, for all $x \in X$. For convenience, Ju et al. [13] called the pair $\widetilde{\alpha}(x) = \left(\widetilde{h}(x), \widetilde{g}(x)\right)$ an IVDHF element (IVDHFE) and denoted by $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$.

To compare the IVDHFEs, Ju et al. [13] defined the score function and accuracy function in the following manner.

Definition 4 [13] *Score function of IVDHFE $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$ is defined as*

$$H\left(\widetilde{\alpha}\right) = \frac{1}{2}\left(\frac{1}{\Delta\widetilde{h}} \sum_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left(\gamma^l + \gamma^u\right) - \frac{1}{\Delta\widetilde{g}} \sum_{[\eta^l,\eta^u]\in\widetilde{g}} \left(\eta^l + \eta^u\right) \right), \tag{3}$$

and accuracy function of IVDHFE $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$ is defined as

$$A\left(\widetilde{\alpha}\right) = \frac{1}{2}\left(\frac{1}{\Delta\widetilde{h}} \sum_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left(\gamma^l + \gamma^u\right) + \frac{1}{\Delta\widetilde{g}} \sum_{[\eta^l,\eta^u]\in\widetilde{g}} \left(\eta^l + \eta^u\right) \right), \tag{4}$$

where $\Delta\widetilde{h}$ and $\Delta\widetilde{g}$ is the number of intervals in $\widetilde{h}$ and $\widetilde{g}$ respectively.

Definition 5 *Let $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ be any two IVDHFEs,*

(i) If $H\left(\widetilde{\alpha}_1\right) > H\left(\widetilde{\alpha}_2\right)$ then $\widetilde{\alpha}_1 > \widetilde{\alpha}_2$;

(ii) If $H\left(\widetilde{\alpha}_1\right) = H\left(\widetilde{\alpha}_2\right)$ then if $A\left(\widetilde{\alpha}_1\right) > A\left(\widetilde{\alpha}_2\right)$ then $\widetilde{\alpha}_1 > \widetilde{\alpha}_2$; if $A\left(\widetilde{\alpha}_1\right) = A\left(\widetilde{\alpha}_2\right)$ then $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$.

2.2. A t-CN&t-Ns

In this section, the definition of At-CN&t-Ns is displayed.

Definition 6 [16, 17] *A function $U : [0,1] \times [0,1] \to [0,1]$ is called a t-CN if it satisfies associativity, symmetricity, non-decreasing and $U(x,0) = x$ for all $x \in [0,1]$. If a binary operation $I : [0,1] \times [0,1] \to [0,1]$ satisfies associativity, symmetricity, non-decreasing and $I(x,1) = x$ for all $x \in [0,1]$ then I is known as a t-N.*

Archimedean t-CN (At-CN) and Archimedean t-N (At-N) operations are expressed as follows:

Definition 7 *[33] An At-CN U is formulated using increasing function g as*

$$U(x,y) = g^{(-1)}\left(g(x) + g(y)\right), \tag{5}$$

similarly, using decreasing function f, an At-N I is represented as

$$I(x,y) = f^{(-1)}\left(f(x) + f(y)\right) \quad \text{with } g(t) = f(1-t) \text{ for all } x, y, t \in [0,\ 1]. \tag{6}$$

Several t-CNs and t-Ns are derived by Klement and Mesiar [32] using different forms of increasing and decreasing functions; and using these functions Sarkar and Biswas [22] defined some operational rules for IVDHFEs based on algebraic, Einstein, Hamacher, and Frank classes of t-CN and t-Ns.

Definition 8 [22] *Let $\widetilde{\alpha}_i = \left(\widetilde{h}_i, \widetilde{g}_i\right)$ $(i = 1,2)$ and $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$ be any three IVDHFEs, $\lambda > 0$ be any scalar. At-CN&t-Ns-based operational laws for the IVDHFEs are presented bellow.*

(1) $\widetilde{\alpha}_1 \oplus_A \widetilde{\alpha}_2 =$

$$\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i\\ ,i=1,2}} \left\{\left[U\left(\gamma_1^l,\gamma_2^l\right), U\left(\gamma_1^u,\gamma_2^u\right)\right]\right\}, \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2}} \left\{\left[I\left(\eta_1^l,\eta_2^l\right), I\left(\eta_1^u,\eta_2^u\right)\right]\right\} \right)$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2}} \left\{\left[g^{-1}\left(g\left(\gamma_1^l\right)+g\left(\gamma_2^l\right)\right), g^{-1}\left(g\left(\gamma_1^u\right)+g\left(\gamma_2^u\right)\right)\right]\right\}, \right.$$

$$\left. \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2}} \left\{\left[f^{-1}\left(f\left(\eta_1^l\right)+f\left(\eta_2^l\right)\right), f^{-1}\left(f\left(\eta_1^u\right)+f\left(\eta_2^u\right)\right)\right]\right\} \right);$$

(2) $\widetilde{\alpha}_1 \otimes_A \widetilde{\alpha}_2 =$

$$\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2}} \left\{\left[I\left(\gamma_1^l,\gamma_2^l\right), I\left(\gamma_1^u,\gamma_2^u\right)\right]\right\}, \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2}} \left\{\left[U\left(\eta_1^l,\eta_2^l\right), U\left(,\eta_1^u,\eta_2^u\right)\right]\right\} \right)$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2}} \left\{\left[f^{-1}\left(f\left(\gamma_1^l\right)+f\left(\gamma_2^l\right)\right), f^{-1}\left(f\left(\gamma_1^u\right)+f\left(\gamma_2^u\right)\right)\right]\right\}, \right.$$

$$\left. \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2}} \left\{\left[g^{-1}\left(g\left(\eta_1^l\right)+g\left(\eta_2^l\right)\right), g^{-1}\left(g\left(\eta_1^u\right)+g\left(\eta_2^u\right)\right)\right]\right\} \right);$$

(3) $\lambda\widetilde{\alpha} =$

$$\left(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left\{\left[g^{-1}\left(\lambda g\left(\gamma^l\right)\right), g^{-1}\left(\lambda g\left(\gamma^u\right)\right)\right]\right\}, \right.$$

$$\left. \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \left\{\left[f^{-1}\left(\lambda f\left(\eta^l\right)\right), f^{-1}\left(\lambda f\left(\eta^u\right)\right)\right]\right\} \right);$$

(4) $\widetilde{\alpha}^\lambda =$

$$\left(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left\{\left[f^{-1}\left(\lambda f\left(\gamma^l\right)\right), f^{-1}\left(\lambda f\left(\gamma^u\right)\right)\right]\right\}, \right.$$

$$\left. \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \left\{\left[g^{-1}\left(\lambda g\left(\eta^l\right)\right), g^{-1}\left(\lambda g\left(\eta^u\right)\right)\right]\right\} \right).$$

2.3. PA Operator

PA operator for MCDM problems was introduced by Yager [23], which is defined in the following manner:

Definition 9 *[23] Let $\{C_i\}$ $(i = 1, 2, \ldots, n)$ be a collection of criteria, and their priority is expressed by the linear ordering $C_1 \succ C_2 \succ \ldots \succ C_n$. This ordering indicates criteria C_j has a higher priority than C_k if $j < k$. The value $C_j(z)$ is the performance of any alternative z under criteria C_j, and satisfies $C_j(z) \in [0, 1]$.*

$$\text{If } PA\left(C_j(z)\right) = \sum_{j=1}^{n} w_j C_j(z), \text{ where } w_j = \frac{T_j}{\sum_{j=1}^{n} T_j}, \; T_j = \prod_{k=1}^{j-1} C_k(z)$$

$(j = 2, \ldots, n)$, $T_1 = 1$. Then PA is called the PA operator.

In the following section, a new score function of IVDHFEs is introduced. The drawback of score function defined by Ju et al. [9] is that the score value may be negative.

3. Score value of IVDHFE

Definition 10 *Score function of IVDHFE $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$ is defined as*

$$S\left(\widetilde{\alpha}\right) = \left(\left(\frac{1}{2}\left(\frac{1}{\Delta\widetilde{h}}\left(\sum_{[\gamma^l, \gamma^u] \in h}\left(\gamma^l + \gamma^u\right)\right) - \frac{1}{\Delta\widetilde{g}}\left(\sum_{[\eta^l, \eta^u] \in g}\left(\eta^l + \eta^u\right)\right)\right)\right) + 1\right) \Big/ 2, \quad (7)$$

where $\Delta\widetilde{h}$ and $\Delta\widetilde{g}$ denote the number of intervals in $\widetilde{h}$ and $\widetilde{g}$, respectively.

To compare among the IVDHFEs, a comparative rule is presented as follows:

Definition 11 *Let $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ be any two IVDHFEs, then*
If $S\left(\widetilde{\alpha}_1\right) > S\left(\widetilde{\alpha}_2\right)$ then $\widetilde{\alpha}_1 > \widetilde{\alpha}_2$.

4. Development of At-CN&t-Ns-based IVDHF prioritized weighted aggregation operators

In this section, the IVDFEs are fused with PA operator based on At-CN&t-Ns and proposed the AIVDHFPWA and AIVDHFPWG operators.

Definition 12 *Let $\widetilde{\alpha}_i = \left(\widetilde{h}_i, \widetilde{g}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IVDHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the weight vectors of $\widetilde{\alpha}_i$ with $\omega_i \in [0, 1]$, where $w_i = \frac{T_i}{\sum_{i=1}^{n} T_i}$, $T_i = \prod_{k=1}^{i-1} S\left(\widetilde{\alpha}_k\right)$ $(i = 2, \ldots, n)$), $T_1 = 1$ and $S\left(\widetilde{\alpha}_i\right)$ is the score of $\widetilde{\alpha}_i$.*

Then, AIVDHFPWA operator is a mapping $\widetilde{\Omega}^n \to \widetilde{\Omega}$, where

$$AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) = \bigoplus_{i=1}^{n}{}_{A} \left(\frac{T_i}{\sum_{i=1}^{n} T_i} \widetilde{\alpha}_i\right).$$

$\bigoplus_A$ *conveys the meaning as described in Definition 8.*

Theorem 1 *Let $\widetilde{\alpha}_i = \left(\widetilde{h}_i, \widetilde{g}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IVDHFEs, then the aggregated value by using AIVDHFPWA operator is also an IVDHFE and*

$$\begin{aligned} AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) &= \bigoplus_{i=1}^{n}{}_{A} \left(\frac{T_i}{\sum_{i=1}^{n} T_i} \widetilde{\alpha}_i\right) \\ &= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2,\ldots,n}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^l\right) \right), g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^u\right) \right) \right] \right\}, \right. \\ &\qquad \left. \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, \\ i=1,2,\ldots,n}} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^l\right) \right), f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^u\right) \right) \right] \right\} \right). \end{aligned} \tag{8}$$

Proof. The theorem will be proved using the mathematical induction method.

The theorem is obvious for $n = 1$.

Assume that theorem is valid for $n = p$, it will prove that it is also valid for $n = p + 1$.

when $n = p$,

$$\begin{aligned} &AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p\right) = \\ &= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2,\ldots,p}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{p} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^l\right) \right), g^{-1}\left(\sum_{i=1}^{p} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^u\right) \right) \right] \right\}, \right. \end{aligned}$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,p}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right), f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right]\right\}\Bigg).$$

Now when $n = p + 1$,

$$AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_p,\widetilde{\alpha}_{p+1}\right) =$$

$$= AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_p\right)\bigoplus_A\left(\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}\widetilde{\alpha}_{p+1}\right),$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,p}} \left\{\left[g^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^l\right)\right), g^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^u\right)\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,p}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right), f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right]\right\}\right)\bigoplus_A$$

$$\left(\bigcup_{[\gamma_{p+1}^l,\gamma_{p+1}^u]\in\widetilde{h}_{p+1}} \left\{\left[g^{-1}\left(\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_{p+1}^l\right)\right), g^{-1}\left(\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_{p+1}^u\right)\right)\right]\right\},\right.$$

$$\left.\bigcup_{[\eta_{p+1}^l,\eta_{p+1}^u]\in\widetilde{g}_{p+1}} \left\{\left[f^{-1}\left(\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_{p+1}^l\right)\right), f^{-1}\left(\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_{p+1}^u\right)\right)\right]\right\}\right),$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,p,p+1}} \left\{\left[g^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^l\right)+fracT_{p+1}\sum_{i=1}^{n}T_i g\left(\gamma_{p+1}^l\right)\right),\right.\right.\right.$$

$$\left.\left. g^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^u\right)+\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_{p+1}^u\right)\right)\right]\right\}$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,p,p+1}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)+\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_{p+1}^l\right)\right),\right.\right.$$

$$\left.\left.f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)+\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_{p+1}^l\right)\right)\right]\right\}\Bigg),$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,p,p+1}} \left\{\left[g^{-1}\left(\sum_{i=1}^{p+1}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^l\right)\right),g^{-1}\left(\sum_{i=1}^{p+1}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^u\right)\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,p,p+1}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p+1}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right),f^{-1}\left(\sum_{i=1}^{p+1}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right]\right\}\right),$$

$$=\bigoplus_{i=1}^{p+1}{}_A\left(\frac{T_i}{\sum\limits_{i=1}^{n}T_i}\widetilde{\alpha}_i\right)=AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_p,\widetilde{\alpha}_{p+1}\right).$$

Hence the theorem is proved for $p+1$ and thus true for all n.

Hence $AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)$ is an IVDHFE.

This completes the proof.

Theorem 2 *(Boundary) Let* $\widetilde{\alpha}_i=\left(\widetilde{h}_i,\widetilde{g}_i\right)$ *(*$i=1,2,\ldots,n$*) be a collection of IVDHFEs, and let for all* $i=1,2,\ldots,n$.

$$\gamma_{\min}^l=\min\left\{\min_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i}\left\{\gamma_i^l\right\}\right\},\qquad \gamma_{\min}^u=\min\left\{\min_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i}\left\{\gamma_i^u\right\}\right\},$$

$$\gamma_{\max}^l=\max\left\{\max_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i}\left\{\gamma_i^l\right\}\right\},\qquad \gamma_{\max}^u=\max\left\{\max_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i}\left\{\gamma_i^u\right\}\right\},$$

$$\eta_{\min}^l=\min\left\{\min_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i}\left\{\eta_i^l\right\}\right\},\qquad \eta_{\min}^u=\min\left\{\min_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i}\left\{\eta_i^u\right\}\right\},$$

$$\eta_{\max}^l=\max\left\{\max_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i}\left\{\eta_i^l\right\}\right\},\qquad \eta_{\max}^u=\max\left\{\max_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i}\left\{\eta_i^u\right\}\right\}.$$

Then if $\widetilde{\alpha}_{-} = \left(\left[\gamma^l_{\min}, \gamma^u_{\min}\right], \left[\eta^l_{\max}, \eta^u_{\max}\right]\right)$ *and* $\widetilde{\alpha}_{+} = \left(\left[\gamma^l_{\max}, \gamma^u_{\max}\right], \left[\eta^l_{\min}, \eta^u_{\min}\right]\right)$,

$$\widetilde{\alpha}_{-} \leqslant AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) \leqslant \widetilde{\alpha}_{+}\,. \tag{9}$$

Proof. . For any $i = 1, 2, \ldots, n$, it is clear that $\gamma^l_{\min} \leqslant \gamma^l_i \leqslant \gamma^l_{\max}$ and $\gamma^u_{\min} \leqslant \gamma^u_i \leqslant \gamma^u_{\max}$. Since $g(t)$ $(t \in [0, 1])$ is a monotonic increasing function,

$$g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^l_{\min}\right)\right) \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^l_i\right)\right) \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^l_{\max}\right)\right),$$

which implies that

$$\gamma^l_{\min} \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^l_i\right)\right) \leqslant \gamma^l_{\max}\,. \tag{10}$$

Similarly, find that

$$\gamma^u_{\min} \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^u_i\right)\right) \leqslant \gamma^u_{\max}\,, \tag{11}$$

for any $i = 1, 2, \ldots, n$, $\eta^l_{\min} \leqslant \eta^l_i \leqslant \eta^l_{\max}$.

Since $f(t)$ $(t \in [0, 1])$ is a decreasing function,

$$f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^l_{\max}\right)\right) \leqslant f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^l_i\right)\right) \leqslant f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^l_{\min}\right)\right),$$

which implies that

$$\eta^l_{\max} \leqslant f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^l_i\right)\right) \leqslant \eta^l_{\min}\,. \tag{12}$$

Similarly,

$$\eta^u_{\max} \leqslant f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^u_i\right)\right) \leqslant \eta^u_{\min}\,. \tag{13}$$

From (10) and (12), it is obtained that

$$\gamma^l_{\min} - \eta^l_{\min} \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^l\right)\right) - f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^l\right)\right) \leqslant \gamma^l_{\max} - \eta^l_{\max}\,.$$

Also, from (11) and (13), it is found that

$$\gamma^u_{\min} - \eta^u_{\min} \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^u\right)\right) - f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^u\right)\right) \leqslant \gamma^u_{\max} - \eta^u_{\max}\,,$$

i.e., $S\left(\widetilde{\alpha}_-\right) \leqslant S\left(AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right)\right) \leqslant S\left(\widetilde{\alpha}_+\right)$.

Therefore, $\widetilde{\alpha}_- \leqslant AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) \leqslant \widetilde{\alpha}_+$.

Theorem 3 *Let $\widetilde{\alpha}_i$ ($i = 1, 2, \ldots, n$ be a collection of IVDHFEs, $\omega_i = \dfrac{T_i}{\sum_{i=1}^{n} T_i}$ ($i = 1, 2, \ldots, n$) be their corresponding weight vectors, if $\widetilde{\alpha}$ be an IVDHFE, then*

$$AIVDHFPWA\left(\widetilde{\alpha}_1 \oplus_A \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_A \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_A \widetilde{\alpha}\right) = \\ = AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) \oplus_A \widetilde{\alpha}.$$

Proof.

$$\widetilde{\alpha}_i \oplus_A \widetilde{\alpha} = \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ [\gamma^l, \gamma^u] \in \widetilde{h} \\ (i=1,2,\ldots,n)}} \left\{\left[g^{-1}\left(g\left(\gamma_i^l\right) + g\left(\gamma^l\right)\right), g^{-1}\left(g\left(\gamma_i^u\right) + g\left(\gamma^u\right)\right)\right]\right\}, \right.$$

$$\left. \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, \\ [\eta^l, \eta^u] \in \widetilde{g} \\ (i=1,2,\ldots,n)}} \left\{\left[f^{-1}\left(f\left(\eta_i^l\right) + f\left(\eta^l\right)\right), f^{-1}\left(f\left(\eta_i^u\right) + f\left(\eta^u\right)\right)\right]\right\} \right).$$

So,

$$AIVDHFPWA\left(\widetilde{\alpha}_1 \oplus_A \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_A \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_A \widetilde{\alpha}\right)$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ [\gamma^l,\gamma^u]\in\widetilde{h}\\ (i=1,2,\ldots,n)}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(g^{-1}\left(g\left(\gamma_i^l\right) + g\left(\gamma^l\right)\right)\right)\right), \right.\right.\right.$$

$$\left.\left. g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(g^{-1}\left(g\left(\gamma_i^u\right) + g\left(\gamma^u\right)\right)\right)\right)\right\}$$

$$\left. \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ [\eta^l,\eta^u]\in\widetilde{g}\\ (i=1,2,\ldots,n)}} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(f^{-1}\left(f\left(\eta_i^l\right) + f\left(\eta^l\right)\right)\right)\right) \right.\right.\right)$$

$$\left.\left.\left. f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(f^{-1}\left(f\left(\eta_i^l\right) + f\left(\eta^l\right)\right)\right)\right)\right]\right\}\right),$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ [\gamma^l,\gamma^u]\in\widetilde{h}\\ (i=1,2,\ldots,n)}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^l\right) + g\left(\gamma^l\right)\right), \right.\right.\right.$$

$$\left.\left. g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^u\right) + g\left(\gamma^u\right)\right)\right]\right\} \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ [\eta^l,\eta^u]\in\widetilde{g}\\ (i=1,2,\ldots,n)}} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^l\right) + f\left(\eta^l\right)\right) \right.\right.$$

$$\left.\left.\left. f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^u\right) + f\left(\eta^u\right)\right)\right]\right\}\right).$$

Now,

$$AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)\oplus_A\widetilde{\alpha}=$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i\\(i=1,2,\ldots,n)}}\left\{\left[g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^l\right)\right),g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^u\right)\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i\\(i=1,2,\ldots,n)}}\left\{\left[f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right),f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right]\right\}\right)\oplus_A\left(\left\{\left[\gamma^l,\gamma^u\right]\right\},\left\{\left[\eta^l,\eta^u\right]\right\}\right)$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ [\gamma^l,\gamma^u]\in\widetilde{h}\\(i=1,2,\ldots,n)}}\left\{\left[g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^l\right)\right)\right)+g\left(\gamma^l\right)\right),\right.\right.\right.$$

$$\left.\left.g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^u\right)\right)\right)+g\left(\gamma^u\right)\right)\right]\right\},$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ [\eta^l,\eta^u]\in\widetilde{g}\\(i=1,2,\ldots,n)}}\left\{\left[f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right)\right)+f\left(\eta^l\right)\right),\right.\right.$$

$$\left.\left.\left.f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right)+f\left(\eta^u\right)\right)\right]\right\}\right),$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ [\gamma^l,\gamma^u]\in\widetilde{h}\\(i=1,2,\ldots,n)}}\left\{\left[g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^l\right)+g\left(\gamma^l\right)\right),\ g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^u\right)+g\left(\gamma^u\right)\right)\right]\right\}\right.$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ [\eta^l,\eta^u]\in\widetilde{g}\\ (i=1,2,\ldots,n)}} \left\{\left[f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\eta_i^l\right)+f\left(\eta^l\right)\right),\ f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\eta_i^u\right)+f\left(\eta^u\right)\right)\right]\right\}\Bigg).$$

Therefore,

$$AIVDHFPWA\left(\widetilde{\alpha}_1\oplus_A\widetilde{\alpha},\widetilde{\alpha}_2\oplus_A\widetilde{\alpha},\ldots,\widetilde{\alpha}_n\oplus_A\widetilde{\alpha}\right)= \\ =AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)\oplus_A\widetilde{\alpha}.$$

Hence the theorem is proved.

Theorem 4 *(Idempotency) If all $\widetilde{\alpha}_i$ $(i=1,2,\ldots,n)$ are equal and let $\widetilde{\alpha}_i=\left(\left\{\left[\gamma^l,\gamma^u\right]\right\},\left\{\left[\eta^l,\eta^u\right]\right\}\right)$ for all $(i=1,2,\ldots,n)$, then*

$$AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=\left(\left\{\left[\gamma^l,\gamma^u\right]\right\},\left\{\left[\eta^l,\eta^u\right]\right\}\right).$$

Proof.

$$AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)= \\ =\left(\bigcup_{\substack{[\gamma^l,\gamma^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\gamma_i^l\right)\right),g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\gamma_i^u\right)\right)\right]\right\},\right. \\ \left.\bigcup_{\substack{[\eta^l,\eta^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right),f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right]\right\}\right).$$

Now, since $\widetilde{\alpha}_i=\left(\left\{\left[\gamma^l,\gamma^u\right]\right\},\left\{\left[\eta^l,\eta^u\right]\right\}\right)$ for all $(i=1,2,\ldots,n)$, $\gamma_i^l=\gamma^l$, $\gamma_i^u=\gamma^u$, $\eta_i^l=\eta^l$ and $\eta_i^u=\eta^u$ for all $(i=1,2,\ldots,n)$.

Therefore,

$$AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)= \\ =\left(\bigcup_{\substack{[\gamma^l,\gamma^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[g^{-1}\left(g\left(\gamma^l\right)\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}\right),g^{-1}\left(g\left(\gamma^u\right)\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}\right)\right]\right\},\right.$$

$$\bigcup_{\substack{[\eta^l,\eta^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}} \left\{\left[f^{-1}\left(f\left(\eta^l\right)\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}\right), f^{-1}\left(f\left(\eta^u\right)\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}\right)\right]\right\}\Bigg)$$

$$=\left(\bigcup_{\substack{[\gamma^l,\gamma^u]\in\widetilde{h}_i\\ i=1,2,\ldots,n}}\left\{\left[\gamma^l,\gamma^u\right]\right\}, \bigcup_{\substack{[\eta^l,\eta^u]\in\widetilde{g}_i\\ i=1,2,\ldots,n}}\left\{\left[\eta^l,\eta^u\right]\right\}\right),$$

$$=\left(\left\{\left[\gamma^l,\ \gamma^u\right]\right\},\left\{\left[\eta^l,\ \eta^u\right]\right\}\right).$$

Hence the theorem is proved.

At-CN&t-N-based IVDHF prioritized WG (AIVDHFPWG) operator is defined as follows.

Definition 13 *Let* $\widetilde{\alpha}_i=\left(\widetilde{h}_i,\widetilde{g}_i\right)$ *($i=1,2,\ldots,n$) be a collection of IVDHFEs and* $\frac{T_i}{\sum_{i=1}^{n}T_i}$ *indicates preference degree of* $\widetilde{\alpha}_i$*, where* $T_i=\prod_{k=1}^{i-1}S\left(\widetilde{\alpha}_k\right)$ *($i=2,\ldots,n$),* $T_1=1$ *and* $S\left(\widetilde{\alpha}_i\right)$ *is the score value of* $\widetilde{\alpha}_i$*. If*

$$AIVDHFPWG\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=\bigotimes_{i=1}^{n}{}_A\left(\frac{T_i}{\sum_{i=1}^{n}T_i}\widetilde{\alpha}_i\right),$$

then $AIVDHFPWG$ *is called the IVDHF prioritized WG (AIVDHFPWG) operator.*

$\otimes_A$ conveys the meaning as described in Definition 8.

Theorem 5 *Let* $\widetilde{\alpha}_i=\left(\widetilde{h}_i,\widetilde{g}_i\right)$ *($i=1,2,\ldots,n$) be a collection of IVDHFEs, then the aggregated value using AIVDHFPWG operator is also an IVDHFE and*

$$AIVDHFPWG\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=$$

$$\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\gamma_i^l\right)\right), f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\gamma_i^u\right)\right)\right]\right\},\right.$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\eta_i^l\right)\right),g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\eta_i^u\right)\right)\right]\right\}\Bigg). \tag{14}$$

Proof. The proof is similar to Theorem 1.

The proposed AIVDHFPWA and AIVDHFPWG operators provide a general expression with the generators $f(x)$ and $g(x)$. Some particular cases of the proposed PA operators are presented as follows:

Case 1 If $f(x) = -\log x$ is considered, then the AIVDHFPWA and AIVDHFPWG operators reduced to the IVDHF prioritized WA (IVDHFPWA) and WG (IVDHFPWG) operators, respectively, which are shown as follows:

$$IVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right) =$$
$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[1-\prod_{i=1}^{n}\left(1-\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}},1-\prod_{i=1}^{n}\left(1-\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\right]\right\},\right.$$
$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}},\prod_{i=1}^{n}\left(\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\right]\right\}\right), \tag{15}$$

and

$$IVDHFPWG\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right) =$$
$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}},\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\right]\right\},\right.$$
$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[1-\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}},1-\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\right]\right\}\right). \tag{16}$$

Case 2 For adopting the Einstein operations, the AIVDHFPWA operator turned to the IVDHF prioritized Einstein WA (IVDHFPEWA), and IVDHF prioritized Einstein WG (IVDHFPEWG) operators, sequentially, defined as:

$$IVDHFPEWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=$$
$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[\frac{\prod_{i=1}^{n}\left(1+\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(1-\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\frac{\prod_{i=1}^{n}\left(1+\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(1-\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\},\right.$$
$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[\frac{2\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(2-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\frac{2\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(2-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\}\right),\tag{17}$$

and

$$IVDHFEPWG\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=$$
$$\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[\frac{2\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(2-\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\frac{2\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(2-\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\},\right.$$
$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[\frac{\prod_{i=1}^{n}\left(1+\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\frac{\prod_{i=1}^{n}\left(1+\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\}\right).\tag{18}$$

Case 3 When putting $f(x) = \log\left(\frac{\sigma + (1-\sigma)x}{x}\right)$, $\sigma > 0$, i.e., for consideration of Hamacher operations, the AIVDHFPWA and AIVDHFPWA operators converted, respectively, to the IVDHF prioritized Hamacher WA (IVDHFPHWA), and IVDHF prioritized Hamacher WG (IVDHFPHWG) operators, which are described as:

$$IVDHFPHWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) =$$

$$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2,\ldots,n}} \left\{ \left[\frac{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} \left(1 - \gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1)\prod_{i=1}^{n} \left(1 - \gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.\right.\right.$$
$$\left.\left. \frac{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} \left(1 - \gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1)\prod_{i=1}^{n} \left(1 - \gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right]\right\}$$
$$\bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, \\ i=1,2,\ldots,n}} \left\{ \left[\frac{\sigma \prod_{i=1}^{n} \left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\left(1 - \eta_i^l\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1)\prod_{i=1}^{n} \left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.\right.$$
$$\left.\left.\left. \frac{\sigma \prod_{i=1}^{n} \left(\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\left(1 - \eta_i^u\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1)\prod_{i=1}^{n} \left(\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right]\right\}\right), \tag{19}$$

and,

$$IVDHFHPWG\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) =$$

$$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2,\ldots,n}} \left\{ \left[\frac{\sigma \prod_{i=1}^{n} \left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\left(1 - \gamma_i^l\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1)\prod_{i=1}^{n} \left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.\right.\right.$$

$$\left.\left.\frac{\sigma \prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n}\left(1+(\sigma-1)\left(1-\gamma_i^u\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}+(\sigma-1)\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right]\right\}\right)$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\dots,n}}\left\{\left[\frac{\prod_{i=1}^{n}\left(1+(\sigma-1)\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}-\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n}\left(1+(\sigma-1)\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}+(\sigma-1)\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}},\right.\right.$$

$$\left.\left.\left.\frac{\prod_{i=1}^{n}\left(1+(\sigma-1)\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}-\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n}\left(1+(\sigma-1)\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}+(\sigma-1)\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right]\right\}\right). \tag{20}$$

Case 4 The AIVDHFPWA and AIVDHFPWG operators switched to the IVDHF prioritized Frank WA (IVDHFPFWA), and IVDHF prioritized Frank WG (IVDHFPFWG) operators for calculating with the Frank t-CN and t-N, $f(x)=\log\left(\frac{\tau-1}{\tau^x-1}\right)$, $\tau>1$, respectively, which are expressed as:

$$IVDHFPFWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\dots,\widetilde{\alpha}_n\right)=$$

$$\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,\dots,n}}\left\{\left[1-\log_\tau\left(1+\frac{\prod_{i=1}^{n}\left(\tau^{1-\gamma_i^l}-1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau-1}\right),1-\log_\tau\left(1+\frac{\prod_{i=1}^{n}\left(\tau^{1-\gamma_i^u}-1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau-1}\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,\dots,n}}\left\{\left[\log_\tau\left(1+\frac{\prod_{i=1}^{n}\left(\tau^{\eta_i^l}-1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau-1}\right),\log_\tau\left(1+\frac{\prod_{i=1}^{n}\left(\tau^{\eta_i^u}-1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau-1}\right)\right]\right\}\right), \tag{21}$$

and

$$IVDHFFPWG\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) =$$
$$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2,\ldots,n}} \left\{ \left[\log_\tau \left(1 + \frac{\prod_{i=1}^{n} \left(\tau^{\gamma_i^l} - 1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau - 1} \right), \log_\tau \left(1 + \frac{\prod_{i=1}^{n} \left(\tau^{\gamma_i^u} - 1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau - 1} \right) \right] \right\}, \right.$$
$$\bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, \\ i=1,2,\ldots,n}} \left\{ \left[1 - \log_\tau \left(1 + \frac{\prod_{i=1}^{n} \left(\tau^{1-\eta_i^l} - 1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau - 1} \right), \right. \right.$$
$$\left. \left. \left. 1 - \log_\tau \left(1 + \frac{\prod_{i=1}^{n} \left(\tau^{1-\eta_i^u} - 1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau - 1} \right) \right] \right\} \right). \tag{22}$$

AIVDHFPWG operator also obeys the above properties as like AIVDHFPWA operator.

In the following sections, the methodological development of the MCDM method is incorporated and are described subsequently.

5. An approach to MCDM with the prioritization under IVDHF environment

In this section, the proposed AIVDHFPWA and AIVDHFPWG operators are applied on MCDM with IVDHFEs, in which the criteria are in different priority level. Let $\{z_1, z_2, \ldots, z_m\}$ be a set of alternatives, $\{C_1, C_2, \ldots, C_n\}$ be a set of criteria, and there prioritization relationship is $C_1 \succ C_2 \succ \ldots \succ C_n$. Suppose that $\widetilde{D} = \left[\widetilde{\alpha}_{ij}\right]_{m \times n}$ be an IVDHF decision matrix (IVDHFDM), where $\widetilde{\alpha}_{ij} = \left(\widetilde{h}_{ij}, \widetilde{g}_{ij}\right)$ is provided by the DM for the alternative z_i satisfying the criteria c_j. Then, the proposed AIVDHFPWA (or AIVDHFPWG) operators are used to develop an approach for solving MCDM problems in IVDHF environment. The proposed methodology is described through the following steps:

Step 1. In general, criteria are categorized into two types: one is benefit criteria, and the other one is cost criteria. If the IVDHFDM possesses cost type criteria, the matrix $\widetilde{D} = \left[\widetilde{\alpha}_{ij}\right]_{m \times n}$ can be converted into the normalized

IVDHFDM form as $\widetilde{R} = \left(\widetilde{r}_{ij}\right)_{m\times n}$ in the following way,

$$\widetilde{r}_{ij} = \begin{cases} \widetilde{\alpha}_{ij} & \text{for benefit criteria } C_j, \\ \widetilde{\alpha}_{ij}^c & \text{for cost criteria } C_j, \end{cases} \tag{23}$$

$i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Where $\widetilde{\alpha}_{ij}^c$ is the complement $\widetilde{\alpha}_{ij}$.

Step 2. Calculate the values of T_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ based on the following equations,

$$T_{ij} = \prod_{k=1}^{j-1} S\left(\widetilde{r}_{ik}\right) \qquad (i = 1, 2, \ldots, m;\ j = 2, \ldots, n); \tag{24}$$

$$T_{i1} = 1, \qquad i = 1, 2, \ldots, m. \tag{25}$$

Step 3. Aggregate the IVDHFEs $\widetilde{r}_{ij}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ for each alternative z_i using the IVDHFPHWA (or IVDHFPHWG) or IVDHFPFWA (or IVDHFPFWG) operator as follows:

$$\widetilde{r}_i = IVDHFPHWA\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots\ldots, \widetilde{r}_{in}\right)$$

$$= \left(\bigcup_{[\gamma_{ij}^l, \gamma_{ij}^u] \in \widetilde{h}_{ij}} \left\{ \left[\frac{\prod_{j=1}^{n} \left(1 + (\sigma - 1)\gamma_{ij}^l\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} - \prod_{j=1}^{n} \left(1 - \gamma_{ij}^l\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n} \left(1 + (\sigma - 1)\gamma_{ij}^l\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + (\sigma - 1)\prod_{j=1}^{n} \left(1 - \gamma_{ij}^l\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}, \right. \right. \right.$$

$$\left. \left. \left. \frac{\prod_{j=1}^{n} \left(1 + (\sigma - 1)\gamma_{ij}^u\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} - \prod_{i=1}^{n} \left(1 - \gamma_{ij}^u\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n} \left(1 + (\sigma - 1)\gamma_{ij}^u\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + (\sigma - 1)\prod_{j=1}^{n} \left(1 - \gamma_{ij}^u\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}} \right] \right\} \right)$$

$$\bigcup_{[\eta_{ij}^l, \eta_{ij}^u] \in \widetilde{g}_{ij}} \left\{ \left[\frac{\sigma \prod_{j=1}^{n} \left(\eta_{ij}^l\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n} \left(1 + (\sigma - 1)\left(1 - \eta_{ij}^l\right)\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + (\sigma - 1)\prod_{j=1}^{n} \left(\eta_{ij}^l\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}, \right. \right.$$

$$\left.\left.\left.\frac{\sigma\prod_{j=1}^{n}\left(\eta_{ij}^{u}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\prod_{j=1}^{n}\left(1+(\sigma-1)\left(1-\eta_{ij}^{u}\right)\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}+(\sigma-1)\prod_{j=1}^{n}\left(\eta_{ij}^{u}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}\right]\right\}\right), \tag{26}$$

or utilizing the proposed operator IVDHFPHWG, which is presented above by Eq. (16), aggregate the IVDHFEs elements as:

$$\widetilde{r}_i = IVDHFPHWG\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}\right)$$

or

$$\widetilde{r}_i = IVDHFPFWA\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}\right)$$
$$= \left(\bigcup_{[\gamma_{ij}^{l},\gamma_{ij}^{u}]\in\widetilde{h}_{ij}}\left\{\left[1-\log_{\tau}\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{1-\gamma_{i}^{l}j}-1\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\tau-1}\right),\right.\right.\right.$$
$$\left.\left.1-\log_{\tau}\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{1-\gamma_{ij}^{u}}-1\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\tau-1}\right)\right]\right\},$$
$$\left.\bigcup_{[\eta_{ij}^{l},\eta_{ij}^{u}]\in\widetilde{g}_{ij}}\left\{\left[\log_{\tau}\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{\eta_{ij}^{l}}-1\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\tau-1}\right),\log_{\tau}\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{\eta_{ij}^{u}}-1\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\tau-1}\right)\right]\right\}\right),$$

or aggregating the IVDHFEs using IVDHFPFWG operator already shown by Eq. (17).

Step 4. Using the proposed score function as in Definition 11, the rank of all alternatives are evaluated.

6. Illustrative example

In this section, an academic field related problem, adapted from an example previously studied by Jin et al. [30], is considered to illustrate the application of the proposed method and demonstrate its feasibility and effectiveness in a realistic scenario. For strengthening the academic environment of a Chinese university, the best alternative is to select among five alternatives, $\{A_1, A_2, A_3, A_4, A_5\}$, by considering four criteria: C_1: morality; C_2: research capability; C_3: teaching skill; and C_4: education background. The prioritization relationship for the criteria is $C_1 \succ C_2 \succ C_3 \succ C_4$. The alternatives are evaluated by the expert on the basis of the criteria under IVDHF environment, and the IVDHFDM is constructed as given in Table 1.

Table 1: IVDHFDM

	C_1	C_2
A_1	$(\{[0.3, 0.4], [0.5, 0.8]\}, \{[0.17, 0.2]\})$	$(\{[0.3, 0.4], [0.4, 0.7]\}, \{[0.2, 0.3]\})$
A_2	$(\{[0.3, 0.5]\}, \{[0.3, 0.4]\})$	$(\{[0.2, 0.3], [0.4, 0.5]\}, \{[0.3, 0.4], [0.4, 0.5]\})$
A_3	$(\{[0.3, 0.4], [0.5, 0.7]\}, \{[0.05, 0.1], [0.1, 0.2]\})$	$(\{[0.3, 0.5]\}, \{[0.1, 0.2]\})$
A_4	$(\{[0.3, 0.4], [0.4, 0.5], [0.5, 0.6]\}, \{[0.01, 0.1]\})$	$(\{[0.5, 0.7]\}, \{[0.05, 0.1], [0.1, 0.15]\})$
A_5	$[\{[0.3, 0.6], [0.7, 0.9]\}, \{[0.05, 0.1]\}]$	$(\{[0.4, 0.6]\}, \{[0.01, 0.1], [0.1, 0.15]\})$
	C_3	C_4
A_1	$(\{[0.6, 0.8]\}, \{[0.05, 0.1]\})$	$(\{[0.3, 0.4], [0.5, 0.6]\}, \{[0.2, 0.3], [0.3, 0.4]\})$
A_2	$\{[0.5, 0.6], [0.7, 0.8]\}, \{[0.15, 0.2]\}$	$(\{[0.4, 0.5]\}, \{[0.1, 0.2], [0.2, 0.3], [0.3, 0.4]\})$
A_3	$(\{[0.7, 0.8], [0.8, 0.9]\}, \{[0.02, 0.1]\})$	$(\{[0.6, 0.7]\}, \{[0.05, 0.1], [0.1, 0.2]\})$
A_4	$(\{[0.3, 0.5], [0.6, 0.8]\}, \{[0.06, 0.1], [0.1, 0.2]\})$	$(\{[0.8, 0.9]\}, \{[0.04, 0.1]\})$
A_5	$(\{[0.5, 0.7], [0.8, 0.9]\}, \{[0.01, 0.06]\})$	$(\{[0.7, 0.8]\}, \{[0.01, 0.1], [0.1, 0.2]\})$

To obtain the ranking results among the alternative(s), the developed AIVDHFPWA and AIVDHFPWG operators are used, and step by step execution of the proposed method is described below. In this context, it is to be noted here that three types of At-CN&t-N, viz., Hamacher, Dombi and Frank Classes are considered. Algebraic and Einstein classes can be derived as particular cases of Hamacher class of t-CN&t-Ns.

Step 1. Since all the criteria C_j $(j = 1, 2, 3, 4)$ are of the benefit type, then the criteria values do not need normalization and take $\left[\widetilde{r}_{ij}\right]_{m\times n} = \left[\widetilde{\alpha}_{ij}\right]_{m\times n}$.

Step 2. Calculating the values of T_{ij} $(i = 1, 2, \ldots, 5;\ j = 1, 2, 3, 4))$ based on the Eqs. (24) and (25) as follows:

$$T_{ij} = \begin{bmatrix} 1 & 0.6575 & 0.3945 & 0.3205 \\ 1 & 0.5250 & 0.2494 & 0.1839 \\ 1 & 0.6821 & 0.4263 & 0.3709 \\ 1 & 0.6975 & 0.5231 & 0.3753 \\ 1 & 0.7750 & 0.5464 & 0.4617 \end{bmatrix}.$$

Step 3. Utilizing the IVDHPFWA, IVDHPFEWA, IVDHFPHWA, and IVDHFPFWA operators, to aggregate all the preference values $\widetilde{r}_{ij}$, and get the overall preference values $\widetilde{r}_i$, which are shown in Tables 2–5.

Table 2: Overall preference values of $\widetilde{r}_i$ utilizing IVDHFPWA operator

$\widetilde{r}_1$ = ({[0.3622,0.5002], [.3905,.5268], [.3889,.5875], [.4160,.6095], [0.4465,0.6854], [0.4711,0.7022], [0.4697,0.7404], [0.4932,0.7542]}, {[0.1483,0.2107], [0.1567,0.2190]})
$\widetilde{r}_2$ = ({[0.3149,0.4681], [.3581,.5131], [.3658,.5140], [.4057,.5551]}, {[.2477,.3431], [0.2644,0.3564], [0.2747,0.3662], [0.2676,0.3643], [0.2856,0.3784], [0.2967,0.3888]})
$\widetilde{r}_3$ = ({[0.4435,0.5741], [0.4810,0.6220], [0.51410.6780], [0.5468,0.7142]}, {[0.0684,0.1600], [0.0758,0.1775], [0.0904,0.1885], [0.1003,0.2091]})
$\widetilde{r}_4$ = ({[0.4664,0.6295], [.5233,.6919], [.4972,.6546], [.5508,.7128], [0.5313,0.6831], [0.5813,0.7365]}, {[0.0656,0.1306], [0.0727,0.1502], [0.0790,0.1573], [0.0876,0.1809]})
$\widetilde{r}_5$ = ({[0.4546,0.6630], [0.5444,0.7284], [0.5977,0.7952], [0.6640,0.8350]}, {[0.0178,0.0905], [0.0261,0.1015], [0.0339,0.1013], [0.0496,0.1136]})

Table 3: Overall preference values of $\widetilde{r}_i$ utilizing IVDHFPEWA operator

$\widetilde{r}_1$ = ({[0.3569,0.4899], [.3848,.517], [.3842,.5776], [.4114,0.6013], [0.4417,0.6753], [0.4674,0.6946], [0.4668,0.7367], [0.4918,0.753]}, {[0.1494,0.2123], [0.1583,0.2214]})
$\widetilde{r}_2$ = ({[0.3114,0.4647], [.3475,.5042], [.3639,.5136], [.3985,.5506]}, {[.2497,.3457], [0.2654,0.3583], [0.2756,0.3680], [0.2708,0.3683], [0.2877,0.3815], [0.2986,0.3917]})
$\widetilde{r}_3$ = ({[0.4324,0.5653], [0.4642,0.6075], [0.5076,0.6748], [0.5366,0.7083]}, [0.0687,0.1607], [0.0762,0.1780], [0.0918,0.1905], [0.1017,0.2107]})
$\widetilde{r}_4$ = ({[0.4527,0.6168], [.5119,.6807], [.4869,.6459], [.5436,0.7059], [0.5230,0.677], [0.5768,0.7326]}, {[0.0657,0.1312], [0.0729,0.1509], [0.0792,0.1580], [0.0878,0.1814]})
$\widetilde{r}_5$ = ({[0.4470,0.6610], [0.5290,0.7227], [0.5924,0.7912], [0.6579,0.8316]}, {[0.0179,0.0905], [0.0263,0.1019], [0.0341,0.1015], [0.0500,0.1142]})

Table 4: Overall preference values of $\widetilde{r}_i$ utilizing IVDHFPHWA operator

$\widetilde{r}_1$ = ({[0.3541,0.4853], [.3818,.5126], [.3817,0.5729], [0.409,0.5975], [.4391,.6704], [0.4654,0.691], [0.4653,.735], [0.491,0.7524]}, {[0.1498,0.2129], [0.1589,0.2223]})
$\widetilde{r}_2$ = ({[0.3094,0.4629], [.3422,.5], [.3629,.5134], [.395,.5487]}, {[.2505,.3467], [0.2659,0.359], [0.276,0.3687], [0.2721,0.37], [0.2885,0.3828], [0.2993,0.3929]})
$\widetilde{r}_3$ = ({[0.4267,0.5613], [0.4560,0.6011], [0.5043,0.6734], [0.5316,0.7059]}, {[0.0689,0.1610], [0.0764,0.1782], [0.0923,0.1913], [0.1022,0.2113]})
$\widetilde{r}_4$ = ({[0.446,.6111], [.5063,0.6756], [.4821,.6422], [.5404,.7029], [0.519,0.6745], [0.5749,0.731]}, {[0.0658,0.1314], [0.073,0.1511], [0.0793,0.1583], [0.0878,0.1815]})
$\widetilde{r}_5$ = ({[0.4432,0.6602], [0.5215,0.7205], [0.5898,0.7895], [0.655,0.8302]}, {[0.0179,0.0906], [0.0264,0.102], [0.0342,0.1016], [0.0501,0.1144]})

Table 5: Overall preference values of $\widetilde{r}_i$ utilizing IVDHFPFWA operator

$\widetilde{r}_1$ = ({[0.3584,0.4919], [.3865,.5190], [.3855,0.5795], [0.4128,0.6029], [.4431,.677], [0.4685,0.6958], [0.4676,.7373], [0.4921,0.7531]}, {[0.1494,0.2122], [0.1583,0.2212]})
$\widetilde{r}_2$ = ({[0.3127,0.4657], [.3506,.5062], [.3645,.5137], [.4004,.5514]}, {[.2496,.3453], [0.2654,0.3580], [0.2755,0.3677], [0.2706,0.3677], [0.2875,0.3810], [0.2984,0.3912]})
$\widetilde{r}_3$ = ({[0.4353,0.5670], [0.4681,0.6097], [0.5093,0.6753], [0.5388,0.7090]}, {[0.0687,0.1607], [0.0763,0.1780], [0.0918,0.1905], [0.1017,0.2106]})
$\widetilde{r}_4$ = ({[0.4559,.6187], [.5145,0.6824], [.4891,.6470], [.5450,.7067], [0.5247,0.6777], [0.5776,0.7329]}, {[0.0657,0.1312], [0.0729,0.1509], [0.0792,0.158], [0.0878,0.1814]})
$\widetilde{r}_5$ = ({[0.4489,0.6613], [0.5324,0.7232], [0.5935,0.7914], [0.6590,0.8318]}, {[0.0179,0.0905], [0.0263,0.1019], [0.0341,0.1015], [0.0500,0.1142]})

Step 4. Calculating the score functions $S(\widetilde{r}_i)$ of the overall IVDHFEs.

Step 5. Rank all the candidates A_i $(i = 1, 2, \ldots, 5)$ in accordance with the score values $S(\widetilde{r}_i)$ of the overall IVDHFEs. From the Fig. 1–4, it is clear that when IVDHFPHWA, IVDHFPHWG, IVDHFPFWA and IVDHFPFWG operators are utilized, the same ordering of the candidates is obtained, and the most desirable candidate is A_5.

The overall IVDHF values $\widetilde{r}_i$ $(i = 1, 2, \ldots, 5$ of the candidates A_i are derived by aggregating IVDHFEs $\widetilde{r}_{ij}$ $(j = 1, 2, \ldots, 5))$ for all i with prioritized aggregation operator IVDHFPWA, and is presented in Table 2, whereas Table 3 represents the aggregating values of each candidate A_i using Einstein-based aggregation operator IVDHFPEWA instead of IVDHFPWA.

Subsequently, Hamacher $(\sigma = 3)$ and Frank $(\tau = 3)$ based aggregation operators IVDHFPHWA and IVDHFPFWA are utilized to aggregate the performance values of the alternatives A_i and is demonstrated in Tables 4 and 5, respectively.

The score values and the ranking results by varying parameters, σ and τ, in the IVDHFPHWA, IVDHFPHWG, IVDHFPFWA and IVDHFPFWG operators, are shown in Tables 6–9, respectively.

Table 6: Ranking results for different parameters of the IVDHFPHWA operator

Parameter	$S(z_1)$	$S(z_2)$	$S(z_3)$	$S(z_4)$	$S(z_5)$	Ordering
$\sigma = 1$	0.6752	0.5587	0.7190	0.7447	0.7968	$A_5 > A_4 > A_3 > A_1 > A_1$
$\sigma = 2$	0.6714	0.5550	0.7136	0.7401	0.7935	$A_5 > A_4 > A_3 > A_1 > A_1$
$\sigma = 3$	0.6696	0.5533	0.7112	0.7381	0.7920	$A_5 > A_4 > A_3 > A_1 > A_1$

Table 7: Ranking results for different parameters of the IVDHFPHWA operator

Parameter	$S(z_1)$	$S(z_2)$	$S(z_3)$	$S(z_4)$	$S(z_5)$	Ordering
$\sigma = 1$	0.6462	0.5370	0.6812	0.7039	0.7687	$A_5 > A_4 > A_3 > A_1 > A_1$
$\sigma = 2$	0.6502	0.5398	0.6858	0.7090	0.7724	$A_5 > A_4 > A_3 > A_1 > A_1$
$\sigma = 3$	0.6526	0.5414	0.6886	0.7121	0.7747	$A_5 > A_4 > A_3 > A_1 > A_1$

Table 8: Ranking results for different parameters of the IVDHFPFWA operator

Parameter	$S(z_1)$	$S(z_2)$	$S(z_3)$	$S(z_4)$	$S(z_5)$	Ordering
$\tau = 2$	0.6732	0.5568	0.7162	0.7422	0.7950	$A_5 > A_4 > A_3 > A_1 > A_1$
$\tau = 3$	0.6721	0.5558	0.7146	0.7409	0.7941	$A_5 > A_4 > A_3 > A_1 > A_1$
$\tau = 4$	0.6714	0.5552	0.7136	0.7400	0.7934	$A_5 > A_4 > A_3 > A_1 > A_1$

Table 9: RRanking results for different parameters of the IVDHFPFWG operator

Parameter	$S(z_1)$	$S(z_2)$	$S(z_3)$	$S(z_4)$	$S(z_5)$	Ordering
$\tau = 2$	0.6479	0.5383	0.6833	0.7151	0.7703	$A_5 > A_4 > A_3 > A_1 > A_1$
$\tau = 3$	0.6489	0.5390	0.6844	0.7160	0.7712	$A_5 > A_4 > A_3 > A_1 > A_1$
$\tau = 4$	0.6495	0.5394	0.6851	0.7167	0.7717	$A_5 > A_4 > A_3 > A_1 > A_1$

Now, based on the DMs' preferences, the parameter can take different values. Based on the Hamacher (or Frank) parameter σ (or τ) between 0 to 20 (or 1 to 20), the score values and ranking of the five alternatives are shown in Fig. 1–4.

From Fig. 1, when the given problem is solved with IVDHFPHWA operator, it is perceived that the ordering of the alternatives does not change. Still, with varying the Hamacher parameter σ the score value of the alternatives decreases monotonically.

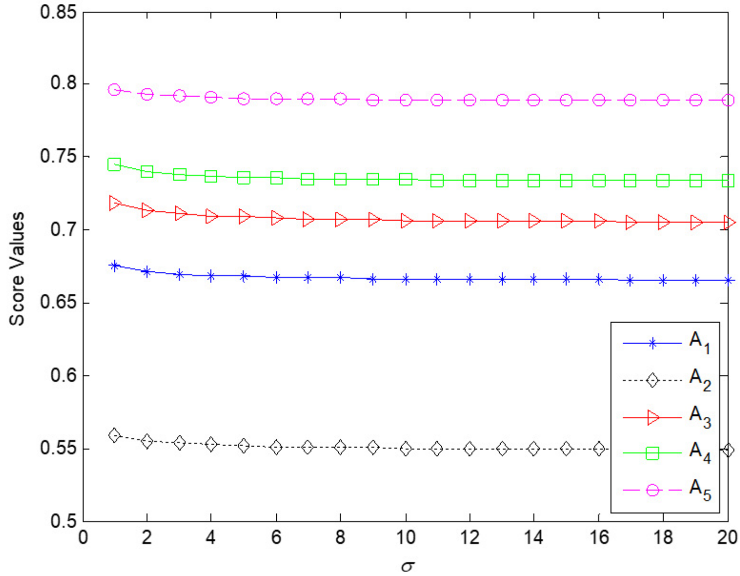

Figure 1: Changes of the score values applying

Similarly, if IVPHFHWG operator is used, the score value of alternatives are computed by varying the Hamacher parameter $\sigma \in [0, 20]$, the obtained results are advertised in the following Fig. 2. It is to be noted here that the ranking of

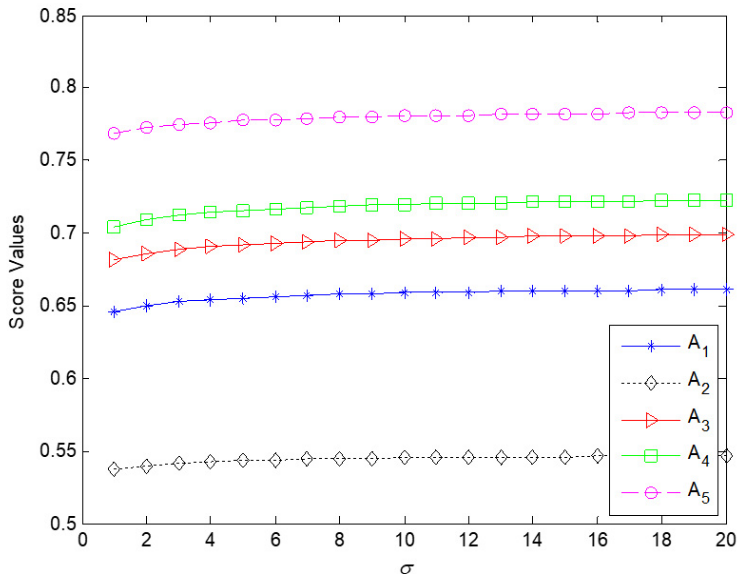

Figure 2: Changes of the score values applying IVDHFPHWG

alternatives does not modify as like using IVDHFPHWA operator. But the score value of the alternatives increases monotonically.

If IVDHFPFWA and IVDHFPFWG operators are used for the Frank parameter τ between 1 to 20, individually, the score values are presented in Fig. 3 and Fig. 4, respectively. As like the above cases, the equivalent observations are seen corresponding to averaging and geometric operators.

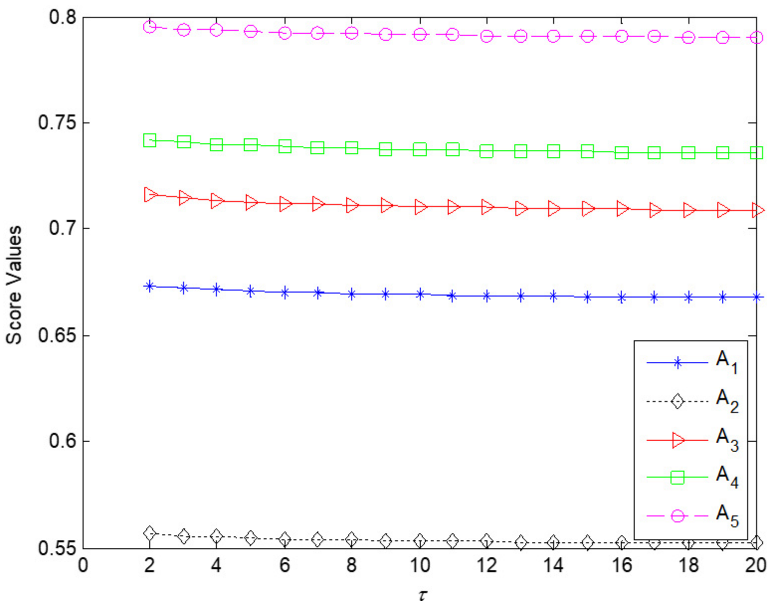

Figure 3: Changes of the score values applying IVDHFPFWA

It is decent to mention here that no changes in the ranking of the alternatives A_i $(i = 1, 2, \ldots, 5)$ are found while making the decision using different PA operators. Thus it persists that the suggested methodology has a durable consistency.

It is worthy to mention here that, same ranking results of alternatives are found using the proposed method and which also covers the result of Jin et al. [30]. The technique developed by Jin et al. [30] is based on Einstein operation under DHF environment, whereas the proposed approach is based on At-CN&t-Ns under IVDHF information. Because IFS, and DHFS are the particular cases of IVDHFS and also At-CN&t-Ns contains an adjustable parameter. So it is claimed that the approach of Jin et al. [30] is a special case of the proposed method. Thus, the proposed methodology is more consistent than the technique developed by Jin et al. [30].

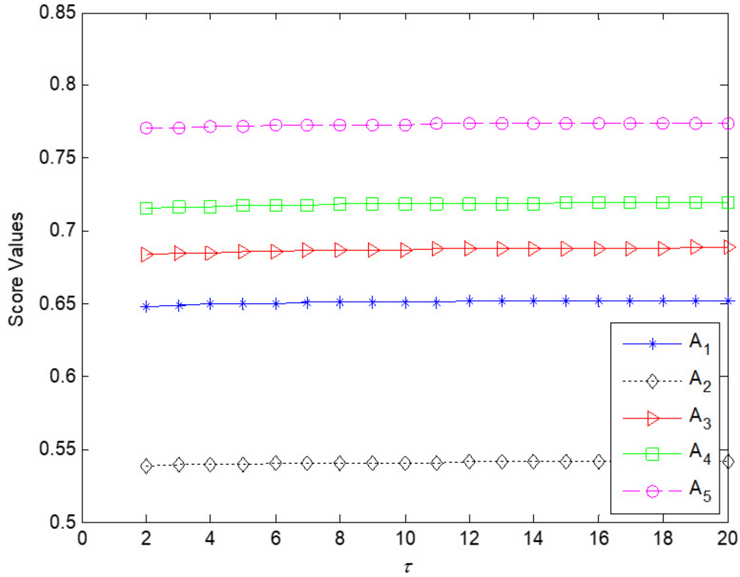

Figure 4: Changes of the score values applying IVDHFPFWG

7. Conclusion

The main contributions of this article is to define a score function of IVDHFE and to propose two prioritized aggregation operators AIVDHFPWA and AIVDHFPWG based on At-CN&t-Ns under the IVDHF context. Most of the prioritized-based aggregation operators can be constructed from AIVDHFPWA and AIVDHFPWG operators. Some desirable properties, such as idempotency, monotonicity, and boundedness of the proposed operators, are investigated. An approach for solving MCDM problem is presented in which the criteria are in different preference level. Through the illustrative example, it has been established the fact that the proposed method not only captures the existing Einstein operation based aggregation operators for IVHFEs [30] but also extends the scope of using aggregation operators in IVDHF environment. In future, the proposed operators may be extended to other domains, viz., q-rung orthopair fuzzy [34], Neutrosophic set [35,36], cubic bipolar fuzzy [37] and Pythagorean fuzzy [38–42] environments. Several types of AOs based on Schweizer-Sklar [43], Yager [44] and many other classes of t-CN&t-Ns can also be developed in IVDHF contexts.

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RESEARCH ARTICLE

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Dual hesitant *q*-rung orthopair fuzzy Dombi *t*-conorm and *t*-norm based Bonferroni mean operators for solving multicriteria group decision making problems

Arun Sarkar[1] | **Animesh Biswas**[2]

[1]Department of Mathematics, Heramba Chandra College, Kolkata, India

[2]Department of Mathematics, University of Kalyani, Kalyani, India

Correspondence
Animesh Biswas, Department of Mathematics, University of Kalyani, Kalyani 741235, India.
Email: abiswaskln@rediffmail.com

Abstract

In this paper, Bonferroni mean (BM) and Dombi t-conorms and t-norms (Dt-CN&t-Ns) are combined under dual hesitant q-rung orthopair fuzzy (DHq-ROF) environment to produce DHq-ROF-Dombi BM, weighted Dombi BM, Dombi geometric BM, and Dombi weighted geometric BM aggregation operators (AOs). Using these operators, the decision making processes would become more flexible and also would possess the capabilities of capturing interrelationships among input arguments under imprecise decision making environments. Apart from those, a large number of AOs either already developed or not yet developed may also be derived from the proposed AOs. In the process of developing the AOs, some operational laws of DHq-ROF numbers based on Dt-CN&t-Ns are defined first. Several important properties of the developed operators are discussed. The proposed AOs are used to frame a new methodology to solve multicriteria group decision making problems under DHq-ROF contexts. Several illustrative examples are solved to demonstrate effectiveness and benefits of the developed method. Sensitivity analysis is performed to show the variations of ranking values with the change of different parameters in the decision making contexts. Finally, the introduced method is compared with several existing techniques to establish superiority and effectiveness of the proposed method.

KEYWORDS

Bonferroni mean, Dombi operations, dual hesitant q-rung orthopair fuzzy set, multicriteria group decision making

1 | INTRODUCTION

Multicriteria group decision making (MCGDM) is a decision-making process to select the best one from a set of possible alternatives which are assessed with respect to a set of criteria by a group of decision-makers (DMs). Since decision-making problems are becoming more complex, most of the data associated with MCGDM are becoming enormously complex and possibilistically uncertain. To handle such uncertain phenomena, Zadeh[1] introduced the concept of fuzzy sets (FSs). Since then a large number of research works have been performed by pioneer researchers[2–5] using FSs and its extensions. Afterwards, Atanassov[6] presented intuitionistic fuzzy (IF) sets (IFSs) by adding the nonmembership grade along with membership grade by satisfying the condition that the sum of membership and nonmembership degrees is less than or equals to 1. Sometimes, violation of the condition for IFSs is observed in capturing uncertainties associated with several decision making problems. In those cases DMs wish to provide membership and nonmembership degrees in such a manner that the sum of those membership and nonmembership degrees may become greater than 1. To deal such situations, Yager[7,8] proposed the concept of Pythagorean fuzzy (PF) set (PFS) by extending IF space. For PFS, the square sum of membership and nonmembership degrees is bounded by 1. After the inception of PFS, a large number of research works have already been studied.[9–16]

Although, PFS has established its potentiality to solve MCGDM problems, but in some decision making situations, it is observed that the sum of the square of membership and nonmembership degrees go beyond the value 1. In such contexts, DMs cannot provide their decision values using PFSs. To tackle such situations, in 2017, Yager[17] further defined a novel concept of q-rung orthopair fuzzy (q-ROF) set (q-ROFS). The q-ROFS consists of a pair of membership and nonmembership degrees satisfying the condition that the sum of the qth power of membership and nonmembership degrees lies within a unit closed interval. So, DMs can describe the space of uncertain information with broader horizon. Due to the flexibility of q-ROFSs, DMs can adjust their judgment values by the parameter q to evaluate related information more accurately in the process of decision-making. So q-ROFSs are more efficient to deal MCDM problems. For better understanding of the satisfying region of intuitionistic fuzzy numbers, PFNs and q-ROFNs, the Figure 1 has been incorporated.

In tackling extremely complex real-life decision-making problems, DMs, sometimes, become confused to put single judgment value corresponding to some alternatives. Under this situation the concept of dual hesitant fuzzy sets[18] (DHFSs) came into account. Combining the concept of DHFSs[18] with q-ROFSs,[17] Xu et al.,[19] proposed dual hesitant q-ROF (DHq-ROF) sets (DHq-ROFSs) as a more generalized form of q-ROFSs. The DHq-ROFS is formulated by two sets of membership degrees and nonmembership degrees with the condition that the sum of qth power of the greatest membership and nonmembership values is less than or equals to 1. This provides more freedom to the DMs for expressing their evaluation values. It can easily be

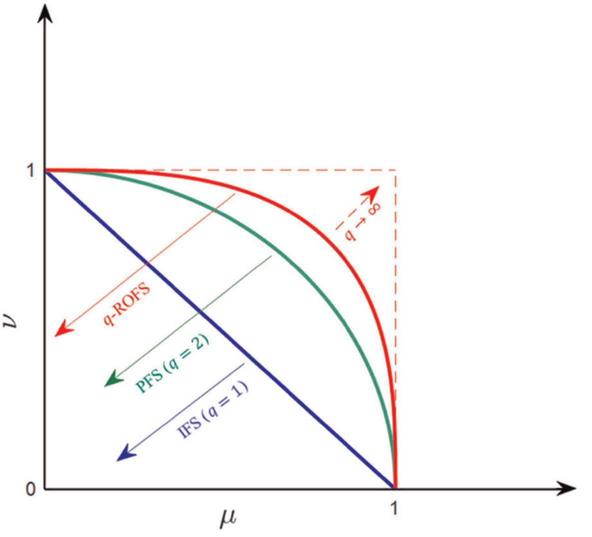

FIGURE 1 Satisfying spaces of IFNs, PFNs, and q-ROFNs. IFN, intuitionistic fuzzy number; PFN, Pythagorean fuzzy number; q-ROFN, q-rung orthopair fuzzy number [Color figure can be viewed at wileyonlinelibrary.com]

realized that DHq-ROFS possesses efficient capability to resolve hesitancy involved with the decision making processes than other variants of FSs, like q-ROFS, PFS, DHFS, and so on. In intelligent systems the information is generally aggregated using fuzzy reasoning and fuzzy rule base. Those systems are usually used in expert systems, computer vision, robotics, fuzzy control, decision support systems, and so on. With the advancement of the variants of fuzzy sets, different aggregation operators (AOs) under those imprecise environments have been developed which had been subsequently used to enrich the domain of intelligent systems.

In PF environments, Prof. R. R. Yager[7,8] introduced weighted averaging (WA) and weighted geometric (WG) AOs, namely, PFWA, PFWG, weighted PF power average and weighted PF power geometric AOs by extending the concept of AOs under fuzzy and IF domains.[20–24] Peng and Yuan[25] introduced generalized PFWA and generalized PF point AOs. In most of the above mentioned cases biasness of using membership and nonmembership degrees are observed under PF contexts. For removing that biasness, Ma and Xu[26] introduced symmetric PFWG, and symmetric PFWA operators to aggregate PFNs. Based on Einstein operations, Rahman et al.[27] developed PF Einstein WG operator to aggregate PFNs. Further, Garg[28,29] proposed generalized PFWA and PFWG AOs. Apart from those, various AOs are also presented by the researchers[30–32] to solve MCDM problems under PF decision making contexts.

Following the concept of PFSs, several AOs for q-ROFSs are developed.[33–35] Liu and Wang[36] presented q-ROF WA and WG operators. Peng et al.[37] proposed exponential operational laws on q-ROF numbers (q-ROFNs), and using those laws, derived q-ROF weighted exponential AO. Darko and Liang[38] introduced some q-ROF Hamacher AOs.

It is the fact that uncertainty handling capabilities to intelligent systems became more powerful with the development of DHq-ROFSs. Wang et al.[39] developed several WA and WG AOs based on Hamacher t-conorm and t-norms (t-CN&t-Ns) for aggregating DHq-ROF numbers (DHq-ROFNs). Wang et al.[40] further presented some Muirhead mean (MM), and dual MM (DMM) operators under DHq-ROF contexts.

In the existing literature, a major part of the research works concentrates on the isolated aggregation arguments. But, in some practical applications, the arguments may have heterogeneous connections. As a consequence, more conjunctional correlations among the various attributes are required. Hence, some decision making problems which are engaged with these types of input arguments cannot be solved by usual independent argument-based AOs. The BM[41] operator can successfully perform this task by producing a flexible function to aggregate the correlation values with the consideration of the interrelationship between two arguments. Following the concept of BM and geometric BM (GBM) various research works have been performed in various domains of fuzzy sets.[42–45]

Further, Dombi t-CN&t-Ns[46] (Dt-CN&t-Ns) have the properties of general t-CN&t-Ns, which can make information aggregation process more flexible by varying Dombi parameter. As a powerful tool, Dombi operations are applied to develop AOs in various environments. Jana et al.[47] introduced PF Dombi AOs, namely, PF Dombi-WA, OWA, hybrid WA, and corresponding geometric AOs. Based on Dombi operations, Jana et al.[48] further proposed several AOs to aggregate q-ROFNs, namely, q-ROF Dombi-WA, OWA, hybrid WA, along with their geometric variants. Considering priority level of each criterion in real-life MCDM problems, Aydemir and Gündüz[49] developed q-ROF Dombi prioritized WA and WG operators.

Thus, by taking the advantages of BM for considering interrelationship among input arguments, and Dt-CN&t-Ns for providing flexibility in decision making processes, it would be convenient to develop an efficient tool to manage higher level of imprecision in MCGDM problems under DHq-ROF environment.

From that view point, in this paper DHq-ROF Dombi BM (DHq-ROFDBM) and DHq-ROF Dombi GBM (DHq-ROFDGBM) operators along with their weighted variants, namely, DHq-ROF weighted Dombi BM (DHq-ROFWDBM) and DHq-ROF weighted Dombi GBM (DHq-ROFWDGBM) AOs are proposed. Those operators not only consider the interrelationship between input arguments through DHq-ROF BM, but also focus on hesitant circumstances. The proposed operators would be advantageous in the sense that those contain four parameters, namely, rung parameter, Dombi parameter, and two BM parameters. Based on preferences of the DMs, as well as different decision making situations, the proposed model would become very much flexible by changing values of those associated parameters. The proposed operators would also possess the inherent capability to capture various existing operators as their special cases. A flexible MCGDM method based on those operators under DHq-ROF environment is presented for solving real-life decision making problems.

The remaining part of this paper is designed in a manner that in Section 2, a brief review on basic concepts of DHq-ROFSs, BM operator, and Dt-CN&t-Ns are discussed. Section 3 describes some operations of DHq-ROFNs based on Dt-CN&t-Ns. In Section 4, DHq-ROFDBM, DHq-ROFWDBM, DHq-ROFDGBM, and DHq-ROFWDGBM AOs are introduced, and their properties are investigated. Also, some special cases of those operators by varying associated parametric values are discussed. Section 5 presents MCGDM method based on the proposed DHq-ROFWDBM and DHq-ROFWDGBM operators. In Section 6, four examples to illustrate and validate effectiveness of the proposed MCGDM method are solved and compared with existing methods. Section 7 presents novelty of the proposed method. Finally, the conclusions and scope for future studies are discussed in Section 8.

2 | PRELIMINARIES

In this section, some basic concepts which are required to develop the proposed methodology are briefly reviewed.

2.1 | q-ROFS

Definition 1 (Yager[17]). Let X be a universe of discourse. A q-ROFS, P on X is given by $P = \{(x, \mu_P(x), \nu_P(x)) \,|\, x \in X\}$, where $\mu_P : X \to [0,1]$ and $\nu_P : X \to [0,1]$ indicate the membership and nonmembership functions, respectively, to represent the respective degree of belongingness and nonbelongingness of the element $x \in X$ to the set P, satisfying the condition that

$$0 \le (\mu_P(x))^q + (\nu_P(x))^q \le 1, q \ge 1.$$

The degree of indeterminacy is formulated by

$$\pi_P(x) = [(\mu_P(x))^q + (\nu_P(x))^q - (\mu_P(x))^q(\nu_P(x))^q]^{\frac{1}{q}}.$$

For convenience, Yager[17] named $(\mu_P(x), \nu_P(x))$ as a q-ROFN and is denoted by $p = (\mu, \nu)$.

For q-ROFNs, Liu and Wang[36] introduced score and accuracy functions in the following manners.

Definition 2 (Liu and Wang[36]). For any q-ROFN, $p = (\mu, \nu)$, the score function of p is defined by

$$S(p) = \frac{1}{2}(1 + \mu^q - \nu^q), \;\text{ where }\; S(p) \in [0, 1],$$

and the accuracy function of p is defined by

$$A(p) = \mu^q + \nu^q.$$

Ranking method of q-ROFNs:

Liu and Wang[36] proposed a ranking method of q-ROFNs as follows.

Let p_1 and p_2 be any two q-ROFNs, then the ordering of those q-ROFNs is done by the following principles:

- $p_1 \succ p_2$ when $S(p_1) > S(p_2)$;
- For $S(p_1) = S(p_2)$:
  (1) $p_1 \succ p_2$ when $A(p_1) > A(p_2)$;
  (2) and $p_1 \approx p_2$ for $A(p_1) = A(p_2)$.

Four fundamental operations[32] on q-ROFNs are presented as follows:

Definition 3 (Liu and Wang[36]). Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, and $p_2 = (\mu_2, \nu_2)$ be three q-ROFNs, and $\lambda > 0$ be any scalar. Then four basic operations are defined as follows:

(1) $p_1 \oplus p_2 = (\sqrt[q]{\mu_1^q + \mu_2^q - \mu_1^q\mu_2^q}, \nu_1\nu_2)$;

(2) $p_1 \otimes p_2 = (\mu_1\mu_2, \sqrt[q]{\nu_1^q + \nu_2^q - \nu_1^q\nu_2^q})$;

(3) $\lambda p = (\sqrt[q]{1 - (1 - \mu^q)^\lambda}, \nu^\lambda)$;

(4) $p^\lambda = (\mu^\lambda, \sqrt[q]{1 - (1 - \nu^q)^\lambda})$.

2.2 | DHq-ROFS

Based on q-ROFSs[17] and DHFSs,[18,50] Xu et al.[19] proposed the concept of DHq-ROFSs and defined basic operations on them.

Definition 4 (Xu et al.[19]). Let $\mathcal{U}$ be a fixed set. A DHq-ROFS $\tilde{\mathcal{K}}$ on $\mathcal{U}$ is described as:

$$\tilde{\mathcal{K}} = (\langle x, \tilde{h}_{\tilde{\mathcal{K}}}(x), \tilde{g}_{\tilde{\mathcal{K}}}(x)\rangle | x \in \mathcal{U}),$$

where $\tilde{h}_{\tilde{\mathcal{K}}}(x)$ and $\tilde{g}_{\tilde{\mathcal{K}}}(x)$ are two sets of real numbers in [0,1], representing the possible membership degrees and nonmembership degrees, respectively, of the element $x \in \mathcal{U}$ to the set $\tilde{\mathcal{K}}$ satisfying the conditions:

$$0 \leq \gamma, \eta \leq 1 \text{ and } 0 \leq \left(max_{\gamma \in \tilde{h}_{\tilde{\mathcal{K}}}(x)}\{\gamma\}\right)^q + \left(max_{\eta \in \tilde{g}_{\tilde{\mathcal{K}}}(x)}\{\eta\}\right)^q \leq 1,$$

where $\gamma \in \tilde{h}_{\tilde{\mathcal{K}}}(x)$, $\eta \in \tilde{g}_{\tilde{\mathcal{K}}}(x)$ for all $x \in \mathcal{U}$.

For convenience, Xu et al.[19] called the pair $\tilde{\mathcal{K}} = (\tilde{h}_{\tilde{\mathcal{K}}}(x), \tilde{g}_{\tilde{\mathcal{K}}}(x))$ as a DHq-ROFN, and denoted it by $\tilde{\kappa} = (\tilde{h}, \tilde{g})$.

Definition 5 (Xu et al.[19]). Let $\tilde{\kappa} = (\tilde{h}, \tilde{g})$ be a DHq-ROFN. The score function of $\tilde{\kappa}$, denoted by $S(\tilde{\kappa})$, is given by

$$S(\tilde{\kappa}) = \frac{1}{2}\left(1 + \frac{1}{l_{\tilde{h}}}\sum\nolimits_{\gamma \in \tilde{h}} \gamma^q - \frac{1}{l_{\tilde{g}}}\sum\nolimits_{\eta \in \tilde{g}} \eta^q\right). \tag{1}$$

Also, the accuracy function of $\tilde{\kappa}$, denoted by $A(\tilde{\kappa})$, and is given by

$$A(\tilde{\kappa}) = \left(\frac{1}{l_{\tilde{h}}}\sum_{\gamma \in \tilde{h}} \gamma^q + \frac{1}{l_{\tilde{g}}}\sum_{\eta \in \tilde{g}} \eta^q\right), \tag{2}$$

where $l_{\tilde{h}}$ and $l_{\tilde{g}}$ are the number of elements in $\tilde{h}$ and $\tilde{g}$, respectively.

The ordering of DHq-ROFNs is done as follows:

Let $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)(i = 1,2)$ be any two DHq-ROFNs,

- If $S(\tilde{\kappa}_1) > S(\tilde{\kappa}_2)$, then $\tilde{\kappa}_1$ is superior to $\tilde{\kappa}_2$, denoted by $\tilde{\kappa}_1 \succ \tilde{\kappa}_2$;
- If $S(\tilde{\kappa}_1) = S(\tilde{\kappa}_2)$, then
  (1) If $A(\tilde{\kappa}_1) > A(\tilde{\kappa}_2)$, then $\tilde{\kappa}_1 \succ \tilde{\kappa}_2$;
  (2) If $A(\tilde{\kappa}_1) = A(\tilde{\kappa}_2)$, then $\tilde{\kappa}_1$ is equivalent to $\tilde{\kappa}_2$, denoted by $\tilde{\kappa}_1 \approx \tilde{\kappa}_2$.

2.3 | The BM operator

A mean type AO, namely, BM operator, initially introduced by Bonferroni,[41] is defined as follows.

Definition 6 (Bonferroni[41]). Let $\theta, \phi \geq 0$ be any two numbers, and $d_i(i = 1,2, ..., n)$ be a collection of nonnegative real numbers. If

$$BM^{\theta,\phi}(d_1, d_2, ..., d_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} d_i^{\theta} d_j^{\phi} \right)^{\frac{1}{\theta+\phi}}, \tag{3}$$

then $BM^{\theta,\phi}$ is called the BM operator.

Considering both BM operator and geometric mean, Zhu et al.[51] explored GBM operator as follows.

Definition 7 (Zhu et al.[51]). Let $\theta, \phi \geq 0$, and $d_i(i = 1,2, ..., n)$ be a collection of nonnegative real numbers. A GBM operator is defined as follows:

$$GBM^{\theta,\phi}(d_1, d_2, ..., d_n) = \frac{1}{\theta+\phi} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (\theta d_i + \phi d_j)^{\frac{1}{n(n-1)}}. \tag{4}$$

2.4 | D*t*-CN&*t*-Ns

Definition 8 (Dombi[46]). For any two real numbers, α and β, Dombi t-conorm, U_D, and Dombi t-norm, I_D, are defined, respectively, as follows:

$$U_D(\alpha, \beta) = 1 - \frac{1}{1 + \left(\left(\frac{\alpha}{1-\alpha} \right)^{\tau} + \left(\frac{\beta}{1-\beta} \right)^{\tau} \right)^{1/\tau}}, \tag{5}$$

$$I_D(\alpha, \beta) = \frac{1}{1 + \left(\left(\frac{1-\alpha}{\alpha} \right)^{\tau} + \left(\frac{1-\beta}{\beta} \right)^{\tau} \right)^{1/\tau}}, \tag{6}$$

where $\tau > 0$ and $(\alpha, \beta) \in [0,1] \times [0,1]$.

3 | OPERATIONS ON DH*q*-ROFNs BASED ON D*t*-CN&*t*-Ns

In this section, the concept of Dt-CN&t-Ns would be imposed on DHq-ROFNs to present operational laws on DHq-ROFNs. Suppose $\tilde{\kappa}_1$, $\tilde{\kappa}_2$, and $\tilde{\kappa}$ are any three DHq-ROFNs. The operational laws, namely, addition, multiplication, scalar multiplication, and exponent of DHq-ROFNs based on Dt-CN&t-Ns are presented (for $\lambda, \tau > 0$) as follows:

(1) Addition: $$\tilde{\kappa}_1 \oplus_D \tilde{\kappa}_2 = \left(\cup_{\substack{\gamma_i \in \tilde{h}_i \\ i=1,2}} \left\{ \sqrt[q]{1 - \frac{1}{1 + \left(\left(\frac{\gamma_1{}^q}{1-\gamma_1{}^q} \right)^{\tau} + \left(\frac{\gamma_2{}^q}{1-\gamma_2{}^q} \right)^{\tau} \right)^{\frac{1}{\tau}}}} \right\}, \cup_{\substack{\eta_i \in \tilde{g}_i \\ i=1,2}} \left\{ \frac{1}{\sqrt[q]{1 + \left(\left(\frac{1-\eta_1{}^q}{\eta_1{}^q} \right)^{\tau} + \left(\frac{1-\eta_2{}^q}{\eta_2{}^q} \right)^{\tau} \right)^{\frac{1}{\tau}}}} \right\} \right);$$

(2) Multiplication: $$\tilde{\kappa}_1 \otimes_D \tilde{\kappa}_2 = \left(\cup_{\substack{\gamma_i \in \tilde{h}_i \\ i=1,2}} \left\{ \frac{1}{\sqrt[q]{1 + \left(\left(\frac{1-\gamma_1^q}{\gamma_1^q} \right)^{\tau} + \left(\frac{1-\gamma_2^q}{\gamma_2^q} \right)^{\tau} \right)^{\frac{1}{\tau}}}} \right\}, \cup_{\substack{\eta_i \in \tilde{g}_i \\ i=1,2}} \left\{ \sqrt[q]{1 - \frac{1}{1 + \left(\left(\frac{\eta_1^q}{1-\eta_1^q} \right)^{\tau} + \left(\frac{\eta_2^q}{1-\eta_2^q} \right)^{\tau} \right)^{\frac{1}{\tau}}}} \right\} \right);$$

(3) Scalar multiplication: $\lambda\tilde{\kappa} = \left(\cup_{\gamma\in\tilde{h}}\left\{\sqrt[q]{1-\left(1\Big/\left(1+\left(\lambda\left(\frac{\gamma^q}{1-\gamma^q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)\right)}\right\}, \cup_{\eta\in\tilde{g}}\left\{1\Big/\sqrt[q]{1+\left(\lambda\left(\frac{1-\eta^q}{\eta^q}\right)^{\tau}\right)^{\frac{1}{\tau}}}\right\}\right)$;

(4) Exponent: $\tilde{\kappa}^{\lambda} = \left(\cup_{\gamma\in\tilde{h}}\left\{1\Big/\sqrt[q]{1+\left(\lambda\left(\frac{1-\gamma^q}{\gamma^q}\right)^{\tau}\right)^{\frac{1}{\tau}}}\right\}, \cup_{\eta\in\tilde{g}}\left\{\sqrt[q]{1-\left(1\Big/\left(1+\left(\lambda\left(\frac{\eta^q}{1-\eta^q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)\right)}\right\}\right)$.

For better understanding of the above operations, the following example is provided.

Example 1. Let $\tilde{\kappa}_1 = \langle\{0.8, 0.9\}, \{0.4, 0.5\}\rangle$, $\tilde{\kappa}_2 = \langle\{0.6, 0.7, 0.8\}, \{0.2, 0.3\}\rangle$ be two DHq-ROFNs. Now, assuming $q = 3$, $\tau = 2$ and $\lambda = 4$, the above operations are performed on $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$, and the following results are obtained:

$$\text{(i)}\ \tilde{\kappa}_1 \oplus_D \tilde{\kappa}_2 = \left(\bigcup_{\substack{\gamma_1\in\{0.8,0.9\},\\ \gamma_2\in\{0.6,0.7,0.8\}}}\left\{\sqrt[3]{1-\frac{1}{1+\left(\left(\frac{\gamma_1{}^3}{1-\gamma_1{}^3}\right)^2+\left(\frac{\gamma_2{}^3}{1-\gamma_2{}^3}\right)^2\right)^{\frac{1}{2}}}}\right\},\right.$$

$$\left.\bigcup_{\substack{\eta_1\in\{0.4,0.5\}\\ \eta_2\in\{0.2,0.3\}}}\left\{\frac{1}{\sqrt[3]{1+\left(\left(\frac{1-\eta_1{}^3}{\eta_1{}^3}\right)^2+\left(\frac{1-\eta_2{}^3}{\eta_2{}^3}\right)^2\right)^{\frac{1}{2}}}}\right\}\right)$$

$$= (\{0.7213, 0.7346, 0.7729, 0.8544, 0.8559, 0.8618\}, \{0.1995, 0.2927, 0.1999, 0.2982\})$$

$$\text{(ii)}\ \tilde{\kappa}_1 \otimes_D \tilde{\kappa}_2 = \left(\bigcup_{\substack{\gamma_1\in\{0.8,0.9\},\\ \gamma_2\in\{0.6,0.7,0.8\}}}\left\{\frac{1}{\sqrt[3]{1+\left(\left(\frac{1-\gamma_1{}^3}{\gamma_1{}^3}\right)^2+\left(\frac{1-\gamma_2{}^3}{\gamma_2{}^3}\right)^2\right)^{\frac{1}{2}}}}\right\},\right.$$

$$\left.\bigcup_{\substack{\eta_1\in\{0.4,0.5\}\\ \eta_2\in\{0.2,0.3\}}}\left\{\sqrt[3]{1-\frac{1}{1+\left(\left(\frac{\eta_1{}^3}{1-\eta_1{}^3}\right)^2+\left(\frac{\eta_2{}^3}{1-\eta_2{}^3}\right)^2\right)^{\frac{1}{2}}}}\right\}\right)$$

$$= (\{0.5948, 0.6829, 0.7524, 0.5992, 0.6972, 0.7907\}, \{0.2538, 0.2621, 0.3538, 0.3564\})$$

$$\text{(iii)}\ 4\tilde{\kappa}_1 = \left(\bigcup_{\gamma_1\in\{0.8,0.9\}}\left\{\sqrt[3]{1-\left(1\Big/\left(1+\left(4\left(\frac{\gamma_1^3}{1-\gamma_1^3}\right)^2\right)^{\frac{1}{2}}\right)\right)}\right\},\right.$$

$$\left.\bigcup_{\eta_1\in\{0.4,0.5\}}\left\{1\Big/\sqrt[3]{1+\left(4\left(\frac{1-\eta_1^3}{\eta_1^3}\right)^2\right)^{\frac{1}{2}}}\right\}\right)$$

$$= (\{0.8782, 0.9448\}, \{0.0361, 0.2963\})$$

(iv) $$\tilde{\kappa}_1^{\,4} = \left(\bigcup_{\gamma_1 \in \{0.8,0.9\}} \left\{ 1 \Big/ \sqrt[3]{1 + \left(4\left(\frac{1-\gamma_1^{\,3}}{\gamma_1^{\,3}} \right)^2 \right)^{\frac{1}{2}}} \right\}, \bigcup_{\eta_1 \in \{0.4,0.5\}} \left\{ \sqrt[3]{1 - \left(1 \Big/ \left(1 + \left(4\left(\frac{\eta_1^{\,3}}{1-\eta_1^{\,3}} \right)^2 \right)^{\frac{1}{2}} \right) \right)} \right\} \right)$$

$$= (\{0.0407, 0.1887\}, \{0.4937, 0.6057\})$$

4 | DEVELOPMENT OF DOMBI OPERATION-BASED BM OPERATORS UNDER DH*q*-ROF ENVIRONMENT

The advantages of Dt-CN&t-Ns, BM and GBM operations in the context of MCDM have already been discussed. Considering those advantages, several AOs under DHq-ROF environment would now be developed based on Dt-CN&t-Ns and BM.

4.1 | Construction of DH*q*-ROFDBM and DH*q*-ROFWDBM operators

Based on the operations on DHq-ROFNs, as defined in the previous section, DHq-ROFDBM AO is now generated to combine a finite number of DHq-ROFNs using Dombi operations and BM. This operator is defined as follows:

Definition 9. Let $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)$, $(i = 1,2, \ldots, n)$ be a collection of DHq-ROFNs. Also, let $\theta, \phi > 0$ be any two numbers. If

$$DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \left(\frac{1}{n(n-1)} \oplus_{D\, \substack{i,j=1 \\ i \neq j}}^{n} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_j^{\phi} \right) \right)^{\frac{1}{\theta+\phi}},$$

then $DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$ is called DHq-ROFDBM operator.

The operator, as defined above, satisfies some properties which are presented through the following theorems.

Theorem 1. *Let $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)(i = 1,2, \ldots, n)$ be a collection of DHq-ROFNs, and $\theta, \phi > 0$. The number which is obtained by using DHq-ROFDBM operator is still a DHq-ROFN, that is,*

$$DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \left(\frac{1}{n(n-1)} \oplus_{D}^{n}{}_{\substack{i,j=1 \\ i \neq j}} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_j^{\phi} \right) \right)^{\frac{1}{\theta+\phi}}$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i, \\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta + \phi} \left(1 \Bigg/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 \Big/ \left(\theta \left(\frac{1-\gamma_i^q}{\gamma_i^q} \right)^{\tau} + \phi \left(\frac{1-\gamma_j^q}{\gamma_j^q} \right)^{\tau} \right) \right) \right) \right) \right)^{\frac{1}{\tau}} \right)} \right\},$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i, \\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta + \phi} \left(1 \Bigg/ \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 \Big/ \left(\theta \left(\frac{\eta_i^q}{1-\eta_i^q} \right)^{\tau} + \phi \left(\frac{\eta_j^q}{1-\eta_j^q} \right)^{\tau} \right) \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\} \right) \tag{7}$$

Proof. From operations of DHq-ROFNs based on Dt-CN&t-N, it follows that

$$\tilde{\kappa}_i^{\theta} = \left(\bigcup_{\gamma_i \in \tilde{h}_i} \left\{ 1 \Bigg/ \sqrt[q]{1 + \left(\theta \left(\frac{1-\gamma_i^q}{\gamma_i^q} \right)^{\tau} \right)^{1/\tau}} \right\}, \bigcup_{\eta_i \in \tilde{g}_i} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\theta \left(\frac{\eta_i^q}{1-\eta_i^q} \right)^{\tau} \right)^{1/\tau} \right) \right)} \right\} \right),$$

$$\tilde{\kappa}_j^{\phi} = \left(\bigcup_{\gamma_j \in \tilde{h}_j} \left\{ 1 \Bigg/ \sqrt[q]{1 + \left(\phi \left(\frac{1-\gamma_j^q}{\gamma_j^q} \right)^{\tau} \right)^{1/\tau}} \right\}, \bigcup_{\eta_j \in \tilde{g}_j} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\phi \left(\frac{\eta_j^q}{1-\eta_j^q} \right)^{\tau} \right)^{1/\tau} \right) \right)} \right\} \right).$$

Then $\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_j^{\phi} = \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i, \\ \gamma_j \in \tilde{h}_j}} \left\{ 1 \Big/ \sqrt[q]{1 + \left(\theta \left(\frac{1-\gamma_i{}^q}{\gamma_i{}^q} \right)^{\tau} + \phi \left(\frac{1-\gamma_j{}^q}{\gamma_j{}^q} \right)^{\tau} \right)^{\frac{1}{\tau}}} \right\}, \right.$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i, \\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 - 1 \Bigg/ \left(1 + \left(\theta \left(\frac{\eta_i^q}{1-\eta_i^q} \right)^{\tau} + \phi \left(\frac{\eta_j^q}{1-\eta_j^q} \right)^{\tau} \right)^{\frac{1}{\tau}} \right)} \right\} \right).$$

Now, using mathematical induction, it is necessary to prove that

$$\oplus_{D\substack{i,j=1 \\ i \neq j}}^{n} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_j^{\phi} \right) = \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i, \\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 \Big/ \left(\theta \left(\frac{1-\gamma_i^q}{\gamma_i^q} \right)^{\tau} + \phi \left(\frac{1-\gamma_j^q}{\gamma_j^q} \right)^{\tau} \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\}, \right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i, \\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 \Big/ \left(\theta \left(\frac{\eta_i{}^q}{1-\eta_i{}^q} \right)^{\tau} + \phi \left(\frac{\eta_j^q}{1-\eta_j^q} \right)^{\tau} \right) \right) \right)^{\frac{1}{\tau}} \right)} \right\} \right). \tag{8}$$

For $n = 2$,

$$\oplus_{D\substack{i,j=1 \\ i \neq j}}^{2} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_j^{\phi} \right) = \left(\tilde{\kappa}_1^{\theta} \otimes_D \tilde{\kappa}_2^{\phi} \right) \oplus_D \left(\tilde{\kappa}_2^{\theta} \otimes_D \tilde{\kappa}_1^{\phi} \right)$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i, \\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 - \left(1 \middle/ \left(1 + \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{2} \left(1 \middle/ \left(\theta \left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} + \phi \left(\frac{1-\gamma_j^q}{\gamma_j^q}\right)^{\tau}\right)\right)\right)^{\frac{1}{\tau}}\right)\right)} \right\}, \right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i, \\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 \middle/ \left(1 + \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{2} \left(1 \middle/ \left(\theta \left(\frac{\eta_i^q}{1-\eta_i^q}\right)^{\tau} + \phi \left(\frac{\eta_j^q}{1-\eta_j^q}\right)^{\tau}\right)\right)\right)^{\frac{1}{\tau}}\right)} \right\} \right).$$

Assume now that Equation (8) is valid for $n = m$, that is,

$$\oplus_{D\ \substack{i,j=1 \\ i \neq j}}^{m} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_j^{\phi}\right) = \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i, \\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 - \left(1 \middle/ \left(1 + \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{m} \left(1 \middle/ \left(\theta \left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} + \phi \left(\frac{1-\gamma_j^q}{\gamma_j^q}\right)^{\tau}\right)\right)\right)^{\frac{1}{\tau}}\right)\right)} \right\}, \right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i, \\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 \middle/ \left(1 + \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{m} \left(1 \middle/ \left(\theta \left(\frac{\eta_i^q}{1-\eta_i^q}\right)^{\tau} + \phi \left(\frac{\eta_j^q}{1-\eta_j^q}\right)^{\tau}\right)\right)\right)^{\frac{1}{\tau}}\right)} \right\} \right). \tag{9}$$

Then, if $n = m + 1$,

$$\oplus_{D\ \substack{i,j=1 \\ i \neq j}}^{m+1} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_j^{\phi}\right) = \oplus_{D\ \substack{i,j=1 \\ i \neq j}}^{m} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_j^{\phi}\right) \oplus_D \left(\oplus_{D\ i=1}^{m} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_{m+1}^{\phi}\right)\right)$$
$$\oplus_D \left(\oplus_{D\ j=1}^{m} \left(\tilde{\kappa}_{m+1}^{\theta} \otimes_D \tilde{\kappa}_j^{\phi}\right)\right). \tag{10}$$

To prove the above Equation (10), it is necessary to prove that

$$\oplus_{D\ i=1}^{m} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_{m+1}^{\phi}\right) = \left(\bigcup_{\gamma_i \in \tilde{h}_i} \left\{ \sqrt[q]{1 - 1 \middle/ \left(1 + \left(\sum_{i=1}^{m} \frac{1}{\theta \left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} + \phi \left(\frac{1-\gamma_{m+1}^q}{\gamma_{m+1}^q}\right)^{\tau}}\right)^{\frac{1}{\tau}}\right)} \right\}, \right.$$

$$\left. \bigcup_{\eta_i \in \tilde{g}_i} \left\{ \sqrt[q]{1 \middle/ \left(1 + \left(\sum_{i=1}^{m} 1 \middle/ \left(\theta \left(\frac{\eta_i^q}{1-\eta_i^q}\right)^{\tau} + \phi \left(\frac{\eta_{m+1}^q}{1-\eta_{m+1}^q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)} \right\} \right). \tag{11}$$

It can easily be shown that Equation (11) is true for $m = 1,2$.
Let Equation (11) be true for $m = v$, that is,

$$\oplus_{D\ i=1}^{v}\left(\tilde{\kappa}_i^{\phi}\otimes_D\tilde{\kappa}_{v+1}^{\phi}\right)=\left(\bigcup_{\gamma_i\in\tilde{h}_i}\left\{\sqrt[q]{1-1\Bigg/\left(1+\left(\sum_{i=1}^{v}\frac{1}{\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau}+\phi\left(\frac{1-\gamma_{v+1}^q}{\gamma_{v+1}^q}\right)^{\tau}}\right)^{\frac{1}{\tau}}\right)}\right\},\right.$$

$$\left.\bigcup_{\eta_i\in\tilde{g}_i}\left\{\sqrt[q]{1\Bigg/\left(1+\left(\sum_{i=1}^{v}1\Bigg/\left(\theta\left(\frac{\eta_i{}^q}{1-\eta_i{}^q}\right)^{\tau}+\phi\left(\frac{\eta_{v+1}{}^q}{1-\eta_{v+1}{}^q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right\}\right).$$

Then for $m=v+1$,

$$\left(\oplus_{D_{i=1}^{v+1}}\left(\tilde{\kappa}_i{}^{\theta}\otimes_D\tilde{\kappa}_{v+2}{}^{\phi}\right)\right)=\left(\oplus_{D_{i=1}^{v}}\left(\tilde{\kappa}_i{}^{\theta}\otimes_D\tilde{\kappa}_{v+2}{}^{\phi}\right)\right)\oplus_D\left(\tilde{\kappa}_{v+1}{}^{\theta}\otimes_D\tilde{\kappa}_{v+2}{}^{\phi}\right)$$

$$=\left(\bigcup_{\gamma_i\in\tilde{h}_i}\left\{\sqrt[q]{1-\left(1\Bigg/\left(1+\left(\sum_{i=1}^{v+1}1\Bigg/\left(\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau}+\phi\left(\frac{1-\gamma_{v+2}{}^q}{\gamma_{v+2}{}^q}\right)^{\tau}\right)\right)^{1/\tau}\right)\right)}\right\},\right.$$

$$\left.\bigcup_{\eta_i\in\tilde{g}_i}\left\{\sqrt[q]{1\Bigg/\left(1+\left(\sum_{i=1}^{v+1}1\Bigg/\left(\theta\left(\frac{\eta_i^q}{1-\eta_i^q}\right)^{\tau}+\phi\left(\frac{\eta_{v+2}{}^q}{1-\eta_{v+2}{}^q}\right)^{\tau}\right)\right)^{1/\tau}\right)}\right\}\right).$$

Thus, Equation (11) is true for $m=v+1$, and so Equation (11) holds for all m. Similarly, it can be proved that

$$\left(\oplus_{D_{j=1}^{m}}(\tilde{\kappa}_{m+1}^{\theta}\otimes_D\tilde{\kappa}_j^{\phi})\right)$$

$$=\left(\bigcup_{\gamma_j\in\tilde{h}_j}\left\{\sqrt[q]{1-\left(1\Bigg/\left(1+\left(\sum_{j=1}^{m}1\Bigg/\left(\theta\left(\frac{1-\gamma_{m+1}^q}{\gamma_{m+1}^q}\right)^{\tau}+\phi\left(\frac{1-\gamma_j^q}{\gamma_j^q}\right)^{\tau}\right)\right)^{1/\tau}\right)\right)}\right\},\right.$$

$$\left.\bigcup_{\eta_j\in\tilde{g}_j}\left\{\sqrt[q]{1\Bigg/\left(1+\left(\sum_{j=1}^{m}1\Bigg/\left(\theta\left(\frac{\eta_{m+1}^q}{1-\eta_{m+1}^q}\right)^{\tau}+\phi\left(\frac{\eta_j^q}{1-\eta_j^q}\right)^{\tau}\right)\right)^{1/\tau}\right)}\right\}\right).\tag{12}$$

Using Equations (9), (11) and (12), the Equation (10) takes the form as follows:

$$\oplus_{D_{\substack{i,j=1\\i\neq j}}^{m+1}}\left(\tilde{\kappa}_i^{\theta}\otimes_D\tilde{\kappa}_j^{\phi}\right)$$

$$=\left(\bigcup_{\substack{\gamma_i\in\tilde{h}_i,\\\gamma_j\in\tilde{h}_j}}\left\{\sqrt[q]{1-\left(1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{m+1}\left(1\Bigg/\left(\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau}+\phi\left(\frac{1-\gamma_j^q}{\gamma_j^q}\right)^{\tau}\right)\right)\right)^{1/\tau}\right)\right)}\right\},\right.$$

$$\bigcup_{\substack{\eta_i \in \tilde{g}_i,\\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\sum_{\substack{i,j=1\\ i \neq j}}^{m+1} \left(1 \Bigg/ \left(\theta \left(\frac{\eta_i{}^q}{1-\eta_i{}^q} \right)^\tau + \phi \left(\frac{\eta_j{}^q}{1-\eta_j{}^q} \right)^\tau \right) \right) \right)^{1/\tau} \right)} \right\} \tag{13}$$

that is, Equation (10) is valid for all n.

Now, $\frac{1}{n(n-1)} \oplus_{D}{}_{\substack{i,j=1\\ i\neq j}}^{n} \left(\tilde{\kappa}_i{}^\theta \otimes_D \tilde{\kappa}_j{}^\phi \right)$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i,\\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{1}{n(n-1)} \left(\sum_{\substack{i,j=1\\ i \neq j}}^{n} \left(1 \Bigg/ \left(\theta \left(\frac{1-\gamma_i^q}{\gamma_i^q} \right)^\tau + \phi \left(\frac{1-\gamma_j{}^q}{\gamma_j{}^q} \right)^\tau \right) \right) \right) \right)^{1/\tau} \right) \right)} \right\}, \right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i,\\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{1}{n(n-1)} \left(\sum_{\substack{i,j=1\\ i \neq j}}^{n} \left(1 \Bigg/ \left(\theta \left(\frac{\eta_i{}^q}{1-\eta_i{}^q} \right)^\tau + \phi \left(\frac{\eta_j{}^q}{1-\eta_j{}^q} \right)^\tau \right) \right) \right) \right)^{1/\tau} \right)} \right\} \right). \tag{14}$$

Finally, the aggregated value using DHq-ROFDBM is obtained as follows:

$$\left(\frac{1}{n(n-1)} \oplus_{D}{}_{\substack{i,j=1\\ i\neq j}}^{n} \left(\tilde{\kappa}_i^\theta \otimes_D \tilde{\kappa}_j^\phi \right) \right)^{\frac{1}{\theta+\phi}}$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i,\\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \left(\sum_{\substack{i,j=1\\ i \neq j}}^{n} \left(1 \Bigg/ \left(\theta \left(\frac{1-\gamma_i^q}{\gamma_i^q} \right)^\tau + \phi \left(\frac{1-\gamma_j^q}{\gamma_j^q} \right)^\tau \right) \right) \right) \right) \right)^{1/\tau} \right)} \right\}, \right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i,\\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \sum_{\substack{i,j=1\\ i \neq j}}^{n} \left(1 \Bigg/ \left(\theta \left(\frac{\eta_i{}^q}{1-\eta_i{}^q} \right)^\tau + \phi \left(\frac{\eta_j{}^q}{1-\eta_j{}^q} \right)^\tau \right) \right) \right) \right)^{1/\tau} \right) \right)} \right\} \right),$$

which is also a DHq-ROFN.

Hence the theorem is proved. □

To show the usefulness of the above theorem, the following example is considered.

Example 2. Let $P = \{\langle\{0.6, 0.7, 0.8\}, \{0.2, 0.3\}\rangle, \langle\{0.7, 0.9\}, \{0.3, 0.5\}\rangle, \langle\{0.6, 0.75\}, \{0.5, 0.6\}\rangle\}$ and $Q = \{\langle\{0.6, 0.8\}, \{0.4, 0.6\}\rangle, \langle\{0.7, 0.8\}, \{0.3, 0.4\}\rangle, \langle\{0.8, 0.9\}, \{0.5\}\rangle\}$ be two sets of DHq-ROFNSs, each of which contains three elements. Now utilize the proposed

DHq-ROFDBM operator to aggregate those three input arguments of P and Q as follows (suppose $q = 3$, $\tau = 2$, $\theta = \phi = 1$):

Let the aggregated value of the elements contained in P and Q is denoted by $\tilde{p}$ and $\tilde{q}$, respectively. Thus

$$\tilde{p} = DHq\text{-}ROFDBM^{1,1}(\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\kappa}_3) = \left(\frac{1}{3(3-1)} \oplus_{D\, \substack{i,j=1 \\ i \neq j}}^{3} (\tilde{\kappa}_i \otimes_D \tilde{\kappa}_j) \right)^{1/(1+1)}$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i, \\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[3]{1 \Bigg/ \left(1 + \left(3 \left(\frac{1}{\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{1}{\left(\left(\frac{1-\gamma_i^3}{\gamma_i^3} \right)^2 + \left(\frac{1-\gamma_j^3}{\gamma_j^3} \right)^2 \right)} \right) \right)} \right) \right)^{1/2} \right)} \right\}, \right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i, \\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[3]{1 - \left(1 \Bigg/ \left(1 + \left(3 \left(\frac{1}{\sum_{\substack{i,j=1 \\ i \neq j}}^{3} \left(\frac{1}{\left(\left(\frac{\eta_i^3}{1-\eta_i^3} \right)^2 + \left(\frac{\eta_j^3}{1-\eta_j^3} \right)^2 \right)} \right)} \right) \right)^{1/2} \right) \right)} \right\} \right).$$

So, $\tilde{p} = \langle \{0.5398, 0.5914, 0.5537, 0.6491, 0.5772, 0.6295, 0.6118, 0.6794, 0.6017, 0.6626, 0.6917, 0.7294\}, \{0.3819, 0.3852, 0.5440, 0.5844, 0.4163, 0.4226, 0.5461, 0.5863\} \rangle$

$\tilde{q} = \langle \{0.6017, 0.6118, 0.6556, 0.6917, 0.6833, 0.7121, 0.7226, 0.7519\}, \{0.4852, 0.5122, 0.5863, 0.5957\} \rangle$.

Based on the score function defined in Definition 5 the score value of $\tilde{p}$ and $\tilde{q}$ are obtained as $S(\tilde{p}) = 0.5642$, $S(\tilde{q}) = 0.5762$, so $\tilde{q} \succ \tilde{p}$.

It is important to note that several forms of AOs can also be derived from the proposed DHq-ROFDBM operator by varying θ, ϕ as follows:

(i) When $\phi \to 0$, DHq-ROFDBM operator reduces to generalized DHq-ROF Dombi averaging (GDHq-ROFDA) operator. Thus

$$DHq\text{-}ROFDBM^{\theta,0}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i,\\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta} \left(1 \Bigg/ \left(\sum_{i=1}^{n} \left(1 \Bigg/ \left(\theta \left(\frac{1-\gamma_i^q}{\gamma_i^q} \right)^{\tau} \right) \right) \right) \right) \right)^{\frac{1}{\tau}} \right)} \right\},\right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i,\\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta} \left(1 \Bigg/ \sum_{i=1}^{n} \left(1 \Bigg/ \left(\theta \left(\frac{\eta_i^q}{1-\eta_i^q} \right)^{\tau} \right) \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\} \right).$$

(ii) Now, if $\theta = 1$ and $\phi \to 0$, then

$$DHq\text{-}ROFDBM^{1,0}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i,\\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(n(n-1) \left(1 \Bigg/ \left(\sum_{i=1}^{n} \left(1 \Bigg/ \left(\frac{1-\gamma_i^q}{\gamma_i^q} \right)^{\tau} \right) \right) \right) \right)^{1/\tau} \right)} \right\},\right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i,\\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(n(n-1) \left(1 \Bigg/ \sum_{i=1}^{n} \left(1 \Bigg/ \left(\frac{\eta_i^q}{1-\eta_i^q} \right)^{\tau} \right) \right) \right)^{1/\tau} \right) \right)} \right\} \right),$$

which is the DHq-ROF Dombi averaging (DHq-ROFDA) operator.

(iii) Again, for $\theta = \phi = 1$, DHq-ROFDBM leads to the following operator:

$$DHq\text{-}ROFDBM^{1,1}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \left(\frac{1}{n(n-1)} \oplus_{D\, \substack{i,j=1\\ i \neq j}}^{n} (\tilde{\kappa}_i \otimes_D \tilde{\kappa}_j) \right)^{\frac{1}{2}}$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i,\\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{2} \left(1 \Bigg/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 \Bigg/ \left(\left(\frac{1-\gamma_i^q}{\gamma_i^q} \right)^{\tau} + \left(\frac{1-\gamma_j^q}{\gamma_j^q} \right)^{\tau} \right) \right) \right) \right) \right)^{\frac{1}{\tau}} \right)} \right\},\right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i,\\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{2} \left(1 \Bigg/ \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 \Bigg/ \left(\left(\frac{\eta_i^q}{1-\eta_i^q} \right)^{\tau} + \left(\frac{\eta_j^q}{1-\eta_j^q} \right)^{\tau} \right) \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\} \right).$$

Theorem 2 (Idempotency). *Let $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)(i = 1,2, \ldots, n)$ be a collection of DHq-ROFNs. If all $\tilde{\kappa}_i$ are equal to $\tilde{\kappa} = (\tilde{h}, \tilde{g})$, that is, $\tilde{\kappa}_i = \tilde{\kappa}$ for all $i = 1,2, \ldots, n$, then*

$$DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \tilde{\kappa}.$$

Proof. From Definition 9 it is found that

$$DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$$

$$= \left(\frac{1}{n(n-1)} \oplus_{D\,\substack{i,j=1\\ i\neq j}}^{n} \left(\tilde{\kappa}_i^{\theta} \otimes_D \tilde{\kappa}_j^{\phi}\right) \right)^{\frac{1}{\theta+\phi}}$$

$$= \left(\bigcup_{\substack{\gamma_i\in\tilde{h}_i,\\ \gamma_j\in\tilde{h}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 \Bigg/ \left(\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j^q}{\gamma_j^q}\right)^{\tau} \right) \right) \right) \right) \right)^{\frac{1}{\tau}} \right)} \right\}, \right.$$

$$\left. \bigcup_{\substack{\eta_i\in\tilde{g}_i,\\ \eta_j\in\tilde{g}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 \Bigg/ \left(\theta\left(\frac{\eta_i^q}{1-\eta_i^q}\right)^{\tau} + \phi\left(\frac{\eta_j^q}{1-\eta_j^q}\right)^{\tau} \right) \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\} \right).$$

Now, since $\tilde{\kappa}_i = \tilde{\kappa}(i = 1, 2, \ldots, n)$, $DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$

$$= \left(\bigcup_{\gamma\in\tilde{h}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 \Bigg/ \left(\theta\left(\frac{1-\gamma^q}{\gamma^q}\right)^{\tau} + \phi\left(\frac{1-\gamma^q}{\gamma^q}\right)^{\tau} \right) \right) \right) \right) \right)^{\frac{1}{\tau}} \right)} \right\}, \right.$$

$$\left. \bigcup_{\eta\in\tilde{g}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 \Bigg/ \left(\theta\left(\frac{\eta^q}{1-\eta^q}\right)^{\tau} + \phi\left(\frac{\eta^q}{1-\eta^q}\right)^{\tau} \right) \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\} \right),$$

$$= \left(\bigcup_{\gamma\in\tilde{h}} \left\{ \sqrt[q]{1 \Big/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Big/ \left(\frac{n(n-1)}{(\theta+\phi)} \frac{\gamma^{\tau q}}{(1-\gamma)^{\tau q}} \right) \right) \right)^{\frac{1}{\tau}} \right)} \right\}, \right.$$

$$\left. \bigcup_{\eta\in\tilde{g}} \left\{ \sqrt[q]{1 - \left(1 \Big/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Big/ \left(\frac{n(n-1)}{(\theta+\phi)} \frac{(1-\eta)^{\tau q}}{\eta^{\tau q}} \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\} \right),$$

$$= \left(\bigcup_{\gamma\in\tilde{h}} \left\{ \sqrt[q]{1 \Big/ \left(1 + \frac{1-\gamma^q}{\gamma^q} \right)} \right\}, \bigcup_{\eta\in\tilde{g}} \left\{ \sqrt[q]{1 - \left(1 \Big/ \left(1 + \frac{\eta^q}{1-\eta^q} \right) \right)} \right\} \right),$$

$$= \left(\bigcup_{\gamma\in\tilde{h}} \{\gamma\}, \bigcup_{\eta\in\tilde{g}} \{\eta\} \right) = \tilde{\kappa}.$$

□

Theorem 3 (Monotonicity). *Let $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)$ and $\tilde{\kappa}_i' = (\tilde{h}_i', \tilde{g}_i')(i = 1,2, \ldots, n)$ be two sets of DHq-ROFNs, if $\gamma_i \le \gamma_i'$ and $\eta_i' \le \eta_i$ for all $i = 1,2, \ldots, n$, then*

$$\text{DHq-ROFDBM}^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) \preccurlyeq \text{DHq-ROFDBM}^{\theta,\phi}(\tilde{\kappa}_1', \tilde{\kappa}_2', \ldots, \tilde{\kappa}_n')$$

Proof. From the definition, $\text{DHq-ROFDBM}^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i,\\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 \middle/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi}\left(1 \middle/ \left(\sum_{\substack{i,j=1\\ i\ne j}}^{n} \left(1 \middle/ \left(\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j{}^q}{\gamma_j{}^q}\right)^{\tau}\right)\right)\right)\right)\right)^{\frac{1}{\tau}}\right)} \right\},\right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i,\\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 - \left(1 \middle/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi}\left(1 \middle/ \sum_{\substack{i,j=1\\ i\ne j}}^{n} \left(1 \middle/ \left(\theta\left(\frac{\eta_i{}^q}{1-\eta_i{}^q}\right)^{\tau} + \phi\left(\frac{\eta_j{}^q}{1-\eta_j{}^q}\right)^{\tau}\right)\right)\right)\right)^{\frac{1}{\tau}}\right)\right)} \right\} \right).$$

Since, $\gamma_i \le \gamma_i'$ for all $i = 1,2, \ldots, n$,

it is found that $\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} \ge \theta\left(\frac{1-\gamma_i'^q}{\gamma_i'^q}\right)^{\tau}$ and $\phi\left(\frac{1-\gamma_j{}^q}{\gamma_j{}^q}\right)^{\tau} \ge \phi\left(\frac{1-\gamma_j'^q}{\gamma_j'^q}\right)^{\tau}$,

that is, $\left(\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j{}^q}{\gamma_j{}^q}\right)^{\tau}\right) \ge \left(\theta\left(\frac{1-\gamma_i'^q}{\gamma_i'^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j'^q}{\gamma_j'^q}\right)^{\tau}\right)$,

that is, $1 \middle/ \left(\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j{}^q}{\gamma_j{}^q}\right)^{\tau}\right) \le 1/\left(\theta\left(\frac{1-\gamma_i'^q}{\gamma_i'^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j'^q}{\gamma_j'^q}\right)^{\tau}\right)$,

that is, $\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(1 \middle/ \left(\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j{}^q}{\gamma_j{}^q}\right)^{\tau}\right)\right) \le \sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(1 \middle/ \left(\theta\left(\frac{1-\gamma_i'^q}{\gamma_i'^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j'^q}{\gamma_j'^q}\right)^{\tau}\right)\right)$,

$$\text{i.e.,}\ 1 \middle/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi}\left(\frac{1}{\left(\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(\frac{1}{\left(\theta\left(\frac{1-\gamma_i^q}{\gamma_i^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j{}^q}{\gamma_j{}^q}\right)^{\tau}\right)}\right)\right)}\right)\right)^{\frac{1}{\tau}}\right)$$

$$\le 1 \middle/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi}\left(\frac{1}{\left(\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(\frac{1}{\left(\theta\left(\frac{1-\gamma_i'^q}{\gamma_i'^q}\right)^{\tau} + \phi\left(\frac{1-\gamma_j'^q}{\gamma_j'^q}\right)^{\tau}\right)}\right)\right)}\right)\right)^{\frac{1}{\tau}}\right). \tag{15}$$

Again, since $\eta_i' \le \eta_i$ for all $i = 1, 2, \ldots, n$, in a similar way, the following inequality can be proved:

$$1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta + \phi} \left(\frac{1}{\sum_{\substack{i,j=1 \\ i \ne j}}^{n} \left(\frac{1}{\left(\theta \left(\frac{\eta_i'^{q}}{1 - \eta_i'^{q}} \right)^{\tau} + \phi \left(\frac{\eta_j'^{q}}{1 - \eta_j'^{q}} \right)^{\tau} \right)} \right)} \right) \right)^{\frac{1}{\tau}} \right) \right)$$

$$\le 1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta + \phi} \left(\frac{1}{\sum_{\substack{i,j=1 \\ i \ne j}}^{n} \left(\frac{1}{\left(\theta \left(\frac{\eta_i^{q}}{1 - \eta_i^{q}} \right)^{\tau} + \phi \left(\frac{\eta_j^{q}}{1 - \eta_j^{q}} \right)^{\tau} \right)} \right)} \right) \right)^{\frac{1}{\tau}} \right) \right). \tag{16}$$

From Definition 5, Equations (15) and (16), the following inequality is found with respect to the score function of the aggregated DHq-ROFNs as follows:

$$S(DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)) \le S(DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1', \tilde{\kappa}_2', \ldots, \tilde{\kappa}_n')).$$

Hence the theorem is proved. □

On the basis of boundary conditions of the associated membership and nonmembership values of a collection of DHq-ROFNs, the boundary condition of the resultant DHq-ROFN, achieved using DHq-ROFDBM, is shown through the following theorem.

Theorem 4 (Boundary). *Let $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)(i = 1, 2, \ldots, n)$ be a collection of DHq-ROFNs. Also, let for all $i = 1, 2, \ldots, n$,*

$\gamma^* = max\{\gamma_{imax}\}$, where $\gamma_{imax} = max_{\gamma_i \in \tilde{h}_i}\{\gamma_i\}$;
$\gamma_\# = min\{\gamma_{imin}\}$, where $\gamma_{imin} = min_{\gamma_i \in \tilde{h}_i}\{\gamma_i\}$;
$\eta^* = max\{\eta_{imax}\}$, where $\eta_{imax} = max_{\eta_i \in \tilde{g}_i}\{\eta_i\}$;
and $\eta_\# = min\{\eta_{imin}\}$, where $\eta_{imin} = min_{\eta_i \in \tilde{g}_i}\{\eta_i\}$.
Now, if *$\tilde{\kappa}_- = (\gamma_\#, \eta^*)$ and $\tilde{\kappa}_+ = (\gamma^*, \eta_\#)$,*
then $\tilde{\kappa}_- \le DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) \le \tilde{\kappa}_+$.

Proof. Since $\gamma_\# \le \gamma_i \le \gamma^*$ and $\eta_\# \le \eta_i \le \eta^*$ for all $i = 1, 2, \ldots, n$, then

$\tilde{\kappa}_- \le \tilde{\kappa}_i$ for all $i = 1, 2, \ldots, n$, and, therefore, from monotonicity,

$$DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_-, \tilde{\kappa}_-, \ldots, \tilde{\kappa}_-) \le DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n).$$

So, by idempotency,

$$\tilde{\kappa}_- \leq DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n). \quad (17)$$

Similarly, it can be shown that

$$DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n) \leq \tilde{\kappa}_+. \quad (18)$$

Now, combining Equation (17) and (18), it is obtained that

$$\tilde{\kappa}_- \leq DHq\text{-}ROFDBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n) \leq \tilde{\kappa}_+. \qquad \square$$

It is worthy to mention here that DHq-ROFDBM operator can consider the interrelationship between the input parameters. However, it does not take into account the self-importance of parameters. In MCGDM problems, the importance of parameters is equivalent to attribute weights or experts' weights. Obviously, it is a shortcoming of the developed DHq-ROFDBM operator, which can be overcome by defining weighted variant of the developed DHq-ROFDBM operator, namely, DHq-ROFWDBM operator as follows.

Definition 10. Let $\tilde{\kappa}_i (i = 1,2, ..., n)$ be a collection of DHq-ROFNs and let $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Let $\theta, \phi > 0$ be any two numbers. If

$$DHq\text{-}ROFWDBM_{\omega}^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n) = \left(\frac{1}{n(n-1)} \oplus_{D\,\substack{i,j=1 \\ i \neq j}}^{n} \left((\omega_i \tilde{\kappa}_i)^{\theta} \otimes_D (\omega_j \tilde{\kappa}_j)^{\phi} \right) \right)^{\frac{1}{\theta+\phi}},$$

then $DHq\text{-}ROFWDBM_{\omega}^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n)$ is called DHq-ROFWDBM.
As like DHq-ROFDBM operator, several properties of DHq-ROFWDBM are also hold.

Theorem 5. *Let $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)(i = 1,2, ..., n)$ be a collection of DHq-ROFNs, whose weight vector is $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$, and satisfies $\omega_i \in [0,1]$, $\sum_{i=1}^{n} \omega_i = 1$. Also, let $\theta, \phi > 0$ be any two numbers. The aggregated value by using DHq-ROFWDBM operator is also a DHq-ROFN, and is expressed as follows:*

$$DHq\text{-}ROFWDBM_{\omega}^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n) = \left(\frac{1}{n(n-1)} \oplus_{D\,\substack{i,j=1 \\ i \neq j}}^{n} \left((\omega_i \tilde{\kappa}_i)^{\theta} \otimes_D (\omega_j \tilde{\kappa}_j)^{\phi} \right) \right)^{\frac{1}{\theta+\phi}}$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i, \\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 \Bigg/ \left(\frac{\theta}{\omega_i} \left(\frac{1-\gamma_i^q}{\gamma_i^q} \right)^{\tau} + \frac{\phi}{\omega_j} \left(\frac{1-\gamma_j^q}{\gamma_j^q} \right)^{\tau} \right) \right) \right) \right) \right)^{1/\tau} \right)} \right\},\right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i, \\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 \Bigg/ \left(\frac{\theta}{\omega_i} \left(\frac{\eta_i^q}{1-\eta_i^q} \right)^{\tau} + \frac{\phi}{\omega_j} \left(\frac{\eta_j^q}{1-\eta_j^q} \right)^{\tau} \right) \right) \right) \right)^{1/\tau} \right) \right)} \right\} \right). \quad (19)$$

Proof. The proof is similar to the proof of Theorem 1. □

As like previous, the following example is considered for a better understanding of the above theorem.

Example 3. In Example 2, if the weights of three elements associated with P and Q are taken as 0.1, 0.25 and 0.65, respectively, and if DHq-ROFWDBM operator is operated on P and Q for the values $q = 3$, $\tau = 2$, $\theta = \phi = 1$, the following results are obtained:

$$DHq\text{-}ROFWDBM_{\omega}^{1,1}(x_1, x_2, x_3) = \left(\frac{1}{3(3-1)} \oplus_{D\,\substack{i,j=1\\ i\neq j}}^{3} (\omega_i \tilde{\kappa}_i \otimes_D \omega_j \tilde{\kappa}_j)\right)^{1/(1+1)}$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i,\\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[3]{1 \Bigg/ \left(1 + \left(3\left(\frac{1}{\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\frac{1}{\left(\left(\frac{1-\gamma_i^3}{\gamma_i^3}\right)^2 + \left(\frac{1-\gamma_j^3}{\gamma_j^3}\right)^2\right)}\right)\right)}\right)\right)^{1/2}\right)} \right\},\right.$$

$$\left.\bigcup_{\substack{\eta_i \in \tilde{g}_i,\\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[3]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{3(3-1)}{1+1}\left(\frac{1}{\sum_{\substack{i,j=1\\ i\neq j}}^{3}\left(\frac{1}{\left(\left(\frac{\eta_i^3}{1-\eta_i^3}\right)^2 + \left(\frac{\eta_j^3}{1-\eta_j^3}\right)^2\right)}\right)}\right)\right)^{1/2}\right)\right)} \right\}\right).$$

So, $P = \langle\{0.5234, 0.5635, 0.5530, 0.6775, 0.5572, 0.5911, 0.5837, 0.6864, 0.5935, 0.6371, 0.6385, 0.7119\}, \{0.3986, 0.4075, 0.5253, 0.5794, 0.4509, 0.4717, 0.5354, 0.5839\}\rangle$; and $Q = \langle\{0.5691, 0.5740, 0.6495, 0.6693, 0.6442, 0.6507, 0.6929, 0.7067\}, \{0.5127, 0.5315, 0.5635, 0.5898\}\rangle$

Based on the score function defined in Equation (5), the score values are obtained as $S(\tilde{p}) = 0.5526$, $S(\tilde{q}) = 0.5523$. So, $\tilde{p} \succ \tilde{q}$.

It is to be mentioned here that the ordering of P and Q is reversed for the consideration of weights of the associated elements of DHq-ROFSs.

4.2 | Development of DH*q*-ROFDGBM and DH*q*-ROFWDGBM operators

In this section, GBM is combined with Dombi operations to generate DHq-ROFDGBM and DHq-ROFWDGBM AOs.

Definition 11. Let $\tilde{\kappa}_i(i = 1,2, ..., n)$ be a collection of DHq-ROFNs, and $\theta, \phi > 0$ be any numbers. If

$$DHq\text{-}ROFDGBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n) = \frac{1}{\theta+\phi}\left(\otimes_{D\,\substack{i,j=1\\ i\neq j}}^{n}((\theta\tilde{\kappa}_i) \oplus_D (\phi\tilde{\kappa}_j))\right)^{\frac{1}{n(n-1)}},$$

then $DHq\text{-}ROFDGBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n)$ is called DHq-ROFDGBM operator.

Theorem 6. *Let $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1,2, ..., n)$ be a collection of DHq-ROFNs and $\theta, \phi > 0$ be any two numbers. The aggregated value using DHq-ROFDGBM operator is also a DHq-ROFN and is stated as follows:*

$$DHq\text{-}ROFDGBM^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n)$$

$$= \left(\bigcup_{\substack{\gamma_i\in\tilde{h}_i,\\ \gamma_j\in\tilde{h}_j}}\left\{\sqrt[q]{1-\left(1\middle/\left(1+\left(\frac{n(n-1)}{\theta+\phi}\left(1\middle/\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1\middle/\left(\theta\left(\frac{\gamma_i^q}{1-\gamma_i^q}\right)^{\tau}+\phi\left(\frac{\gamma_j^q}{1-\gamma_j^q}\right)^{\tau}\right)\right)\right)\right)^{\frac{1}{\tau}}\right)\right)}\right\},\right.$$

$$\left.\bigcup_{\substack{\eta_i\in\tilde{g}_i,\\ \eta_j\in\tilde{g}_j}}\left\{\sqrt[q]{1\middle/\left(1+\left(\frac{n(n-1)}{\theta+\phi}\left(1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1\middle/\left(\theta\left(\frac{1-\eta_i^q}{\eta_i^q}\right)^{\tau}+\phi\left(\frac{1-\eta_j^q}{\eta_j^q}\right)^{\tau}\right)\right)\right)\right)\right)^{\frac{1}{\tau}}\right)}\right\}\right). \quad (20)$$

Proof. The proof is similar to Theorem 1. □

Now, some special cases of the above-defined DHq-ROFDGBM operator are investigated by considering several values of θ, ϕ.

(i) When $\phi \to 0$, DHq-ROFDGBM operator reduces to generalized DHq-ROF Dombi geometric (GDHq-ROFDG) operator as follows:

$$DHq\text{-}ROFDGBM^{\theta,0}(\tilde{\kappa}_1, \tilde{\kappa}_2, ..., \tilde{\kappa}_n)$$

$$= \left(\bigcup_{\substack{\gamma_i\in\tilde{h}_i,\\ \gamma_j\in\tilde{h}_j}}\left\{\sqrt[q]{1-\left(1\middle/\left(1+\left(\frac{1}{\theta}\left(1\middle/\sum_{i=1}^{n}\left(1\middle/\left(\theta\left(\frac{\gamma_i^q}{1-\gamma_i^q}\right)^{\tau}\right)\right)\right)\right)^{\frac{1}{\tau}}\right)\right)}\right\},\right.$$

$$\left.\bigcup_{\substack{\eta_i\in\tilde{g}_i,\\ \eta_j\in\tilde{g}_j}}\left\{\sqrt[q]{1\middle/\left(1+\left(\frac{1}{\theta}\left(1\middle/\left(\sum_{i=1}^{n}\left(1\middle/\left(\theta\left(\frac{1-\eta_i^q}{\eta_i^q}\right)^{\tau}\right)\right)\right)\right)\right)^{\frac{1}{\tau}}\right)}\right\}\right).$$

(ii) Further, if $\theta = 1$, $\phi \to 0$ is considered for DHq-ROFDGBM operator, DHq-ROF Dombi geometric (DHq-ROFDG) operator is found as follows:

$$DHq\text{-}ROFDGBM^{1,0}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i, \\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 - \left(1 \Big/ \left(1 + \left(1 \Big/ \sum_{i=1}^{n} \left(1 \Big/ \left(\left(\frac{\gamma_i^q}{1-\gamma_i^q} \right)^{\tau} \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\}, \right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i, \\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 \Big/ \left(1 + \left(1 \Big/ \left(\sum_{i=1}^{n} \left(1 \Big/ \left(\left(\frac{1-\eta_i^q}{\eta_i^q} \right)^{\tau} \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\} \right).$$

(iii) Also, for $\theta = \phi = 1$ in DHq-ROFDGBM operator, the following operator is found.

$$DHq - ROFDGBM^{1,0}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$$

$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i, \\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{2} \left(1 \Bigg/ \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 \Big/ \left(\left(\frac{\gamma_i^q}{1-\gamma_i^q} \right)^{\tau} + \left(\frac{\gamma_j^q}{1-\gamma_j^q} \right)^{\tau} \right) \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\}, \right.$$

$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i, \\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{2} \left(1 \Bigg/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 \Big/ \left(\left(\frac{1-\eta_i^q}{\eta_i^q} \right)^{\tau} + \left(\frac{1-\eta_j^q}{\eta_j^q} \right)^{\tau} \right) \right) \right) \right) \right)^{\frac{1}{\tau}} \right)} \right\} \right).$$

Adopting the concept of weighted GBM, DHq-ROFWDGBM operator is defined as follows.

Definition 12. Let $\tilde{\kappa}_i (i = 1,2, \ldots, n)$ be a collection of DHq-ROFNs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector with $\omega_i \in [0,1]$, $\sum_{i=1}^{n} \omega_i = 1$. Also, let $\theta, \phi > 0$ be any two numbers. Now, if

$$DHq\text{-}ROFWDGBM_{\omega}^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \frac{1}{\theta + \phi} \left(\otimes_{D} {}_{\substack{i,j=1 \\ i \neq j}}^{n} \left((\theta \tilde{\kappa}_i)^{w_i} \oplus_D (\phi \tilde{\kappa}_j)^{w_j} \right) \right)^{\frac{1}{n(n-1)}},$$

then $DHq\text{-}ROFWDGBM_{\omega}^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$ is called DHq-ROFWDGBM operator.

Theorem 7. *Let $\tilde{\kappa}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1,2, \ldots, n)$ be a collection of DHq-ROFNs, whose weight vectors is $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Let $\theta, \phi > 0$ be any two numbers. Then the aggregated value by using DHq-ROFWDGBM operator is also a DHq-ROFN, and is given by*

$$DHq\text{-}ROFWDGBM_{\omega}^{\theta,\phi}(\tilde{\kappa}_1, \tilde{\kappa}_2, \dots, \tilde{\kappa}_n)$$
$$= \left(\bigcup_{\substack{\gamma_i \in \tilde{h}_i,\\ \gamma_j \in \tilde{h}_j}} \left\{ \sqrt[q]{1 - \left(1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 \Bigg/ \left(\frac{\theta}{\omega_i}\left(\frac{\gamma_i^q}{1-\gamma_i^q}\right)^{\tau} + \frac{\phi}{\omega_j}\left(\frac{\gamma_j^q}{1-\gamma_j^q}\right)^{\tau} \right) \right) \right) \right)^{\frac{1}{\tau}} \right) \right)} \right\}, \right.$$
$$\left. \bigcup_{\substack{\eta_i \in \tilde{g}_i,\\ \eta_j \in \tilde{g}_j}} \left\{ \sqrt[q]{1 \Bigg/ \left(1 + \left(\frac{n(n-1)}{\theta+\phi} \left(1 \Bigg/ \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 \Bigg/ \left(\frac{\theta}{\omega_i}\left(\frac{1-\eta_i{}^q}{\eta_i{}^q}\right)^{\tau} + \frac{\phi}{\omega_j}\left(\frac{1-\eta_j{}^q}{\eta_j{}^q}\right)^{\tau} \right) \right) \right) \right) \right)^{\frac{1}{\tau}} \right)} \right\} \right). \tag{21}$$

Proof. Analogous to the proof of Theorem 1. □

Note: By changing the values of parameter, q, some special cases of DHq-ROFWDBM and DHq-ROFWDGBM operators are given as follows:

Case 1. If $q = 1$ is considered, then DHq-ROFWDBM and DHq-ROFWDGBM operators reduce to DHF weighted Dombi BM (DHFWDBM) and DHF weighted Dombi GBM (DHFWDGBM) operators, respectively.

Case 2. Again, for $q = 2$, DHq-ROFWDBM and DHq-ROFWDGBM operators are converted into DHPF weighted Dombi BM (DHPFDBM) and DHPF weighted Dombi GBM (DHPFDGBM) operators, respectively.

5 | AN APPROACH TO MCGDM WITH DHq-ROF INFORMATION

In this section, methodologies using DHq-ROFWDBM and DHq-ROFWDGBM operators are developed to solve MCGDM problems. An MCGDM problem under DHq-ROF environment is described below.

Let $X = \{x_1, x_2, \dots, x_m\}$ be a set of alternatives, which are assessed on the basis of a set of criteria, $C = \{C_1, C_2, \dots, C_n\}$ by a group of DMs, $E = \{e_1, e_2, \dots, e_p\}$. Also let, $w = (w_1, w_2, \dots, w_p)$ be the weight vector of the DMs, and $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ be the weight vector of criteria. The individual DM put his/her decision values in the form of DHq-ROFNs and the DHq-ROF decision matrix (DHq-ROFDM) is presented as $\tilde{K}_{m\times n}^{(l)} = [\tilde{\kappa}_{ij}{}^{(l)}]_{m\times n} = [(\tilde{h}_{ij}{}^{(l)}, \tilde{g}_{ij}{}^{(l)})]_{m\times n}$ $(l = 1,2, \dots, p)$. Each $\tilde{\kappa}_{ij}{}^{(l)}$ designates the decision value of the alternative, $x_i \in X$, on the basis of criteria, $C_j \in C$, provided by the DM, $e_l \in E$. Also, $\tilde{h}_{ij}{}^{(l)}$ represents the membership degree of the alternative, x_i that satisfies the criterion, C_j expressed by the DM, e_l $(i = 1,2, \dots, m;\ j = 1,2, \dots, n;\ l = 1,2, \dots, p)$; and $\tilde{g}_{ij}{}^{(l)}$ indicates corresponding degree of nonmembership.

Now, the developed DHq-ROFWDBM and DHq-ROFWDGBM operators are used to describe an approach for solving MCGDM problems. The proposed methodology is presented through the following steps:

Step 1. If DHq-ROFDM possesses cost type criteria, matrices $\tilde{K}^{(l)}_{m\times n} = [\tilde{\kappa}_{ij}{}^{(l)}]_{m\times n}$ are required to be converted into the normalized matrix DHq-ROFDM, $\tilde{R}^{(l)}_{m\times n} = (\tilde{r}_{ij}{}^{(l)})_{m\times n}$ by the following way;

$$\tilde{r}_{ij}{}^{(l)} = \begin{cases} \tilde{\kappa}_{ij}{}^{(l)} & \text{for benefit attribute } C_j, \\ \tilde{\kappa}^{c}_{ij}{}^{(l)} & \text{for cost attribute } C_j, \end{cases} \tag{22}$$

$i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. Where $\tilde{\kappa}^{c}_{ij}$ is the complement of $\tilde{\kappa}_{ij}$.
Then goto the next step.
But, if all the criteria are of benefit type, skip this step and go to Step 2.

Step 2. To aggregate all the individual DHq-ROFDMs, $R^{(l)}_{m\times n} = (\tilde{r}_{ij}{}^{(l)})_{m\times n}(l = 1,2, ..., p)$ into the collective DHq-ROFDM, $\tilde{R} = [\tilde{r}_{ij}]_{m\times n}, i = 1,2, ..., m; j = 1,2, ...,n$, the following DH$q$-ROFWDBM or DH$q$-ROFWDGBM operator is used. The weight vector, $w = (w_1, w_2, ..., w_n)^T$ corresponding to the DMs is considered in this context.

Using DHq-ROFWDBM operator, the collective DHq-ROFN is found, using Definition 9, as

$$\tilde{r}_{ij} = DHq\text{-}ROFWDBM^{\theta,\phi}_{\omega}\left(\tilde{r}_{ij}{}^{(1)}, \tilde{r}_{ij}{}^{(2)}, ..., \tilde{r}_{ij}{}^{(p)}\right). \tag{23}$$

Similar cases are observed for DHq-ROFWDGBM operator, and the collective DHq-ROFN is found as

$$\tilde{r}_{ij}{}' = DHq\text{-}ROFWDGBM^{\theta,\phi}_{\omega}\left(\tilde{r}_{ij}{}^{(1)}, \tilde{r}_{ij}{}^{(2)}, ..., \tilde{r}_{ij}{}^{(p)}\right). \tag{24}$$

Step 3. The resulting decision information, as found in matrix $\tilde{R}$ in the form of DHq-ROFNs, the criteria value of each alternative is aggregated by considering the criteria weight vector, $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ using DHq-ROFWDBM (or DHq-ROFWDGBM) operator, to get aggregated criteria value of each alternative using Definition 9 (or Definition 11), as

$$\tilde{r}_i = DHq\text{-}ROFWDBM^{\theta,\phi}_{\omega}(\tilde{r}_{i1}, \tilde{r}_{i2}, ..., \tilde{r}_{in}) \tag{25}$$

or

$$\tilde{r}_i{}' = DHq\text{-}ROFWDGBM^{\theta,\phi}_{\omega}(\tilde{r}_{i1}, \tilde{r}_{i2}, ..., \tilde{r}_{in}) \tag{26}$$

$$i = 1, 2, ..., m; j = 1, 2, ..., n.$$

Step 4. Calculate the score values $S(\tilde{r}_i)$ (or $S(\tilde{r}_i{}'))$ $(i = 1,2, ..., m)$ of the aggregated DHq-ROFNs, for finding ranking values of the alternatives, x_i. Through score functions, the aggregated decision values corresponding to each alternative in the form of DHq-ROFN are converted into crisp values for ranking purpose.

Step 5. Rank the alternatives.

6 | ILLUSTRATIVE EXAMPLES

To illustrate the proposed method and to establish the application potentiality of it, the following four examples are solved and compared with the existing approaches.

6.1 | Illustrative Example 1

To show effectiveness of the developed method, an example about investment to a suitable company among five possible companies, A_1, A_2, A_3, A_4, A_5, adapted from two articles[24,42] is considered. Three DMs, namely, $D^{(l)}(l = 1,2,3)$ with weight vector, $w = (0.35, 0.40, 0.25)^T$, are involved with the assessment process to evaluate the alternatives based on four criteria, namely, risk analysis (ξ_1), growth analysis (ξ_2), social-political impact analysis (ξ_3) and environmental impact analysis (ξ_4), with their weight vector $\omega = (0.2, 0.1, 0.3, 0.4)^T$. The judgment values corresponding to the alternatives are put in the form of DHq-ROFNs and the following three decision matrices, $\tilde{K}^{(l)} = [\tilde{\kappa}_{ij}{}^{(l)}]_{5\times4}(l = 1,2,3)$, are constructed which are presented as Tables 1–3.

The developed methodology is applied on this modified problem to find the best company for investment. The problem is solved using the proposed DHq-ROFWDBM and DHq-ROFWDGBM operators, consecutively, considering rung parameter, $q = 3$; Dombi parameter, $\tau = 2$; and BM parameter, $\theta = \phi = 1$, for convenience. The solution process is presented in the following section.

TABLE 1 DHq-ROFDM $\tilde{K}^{(1)}$ provided by $D^{(1)}$

	ξ_1	ξ_2	ξ_3	ξ_4
A_1	({0.5}, {0.4})	({0.5}, {0.3})	({0.2,0.3}, {0.6})	({0.4}, {0.4})
A_2	({0.7}, {0.3})	({0.7}, {0.3})	({0.6}, {0.2})	({0.6,0.9}, {0.2,0.3})
A_3	({0.4,0.5}, {0.4,0.6})	({0.5,0.6}, {0.4,0.9})	({0.6}, {0.2})	({0.5}, {0.3})
A_4	({0.8}, {0.2})	({0.7}, {0.2})	({0.4,0.6}, {0.2})	({0.5}, {0.2})
A_5	({0.4,0.5,0.7}, {0.3,0.4})	({0.4}, {0.2})	({0.4}, {0.5})	({0.4,0.6}, {0.6,0.7})

TABLE 2 DHq-ROFDM $\tilde{K}^{(2)}$ given by $D^{(2)}$

	ξ_1	ξ_2	ξ_3	ξ_4
A_1	({0.4,0.6}, {0.5,0.6})	({0.6}, {0.2})	({0.5}, {0.4})	({0.5}, {0.3,0.5})
A_2	({0.5}, {0.4})	({0.6}, {0.2})	({0.6,0.7,0.8}, {0.3})	({0.7}, {0.3})
A_3	({0.4}, {0.5})	({0.3}, {0.5})	({0.4}, {0.4})	({0.2,0.3}, {0.6,0.8})
A_4	({0.5}, {0.4})	({0.7,0.8}, {0.2,0.3})	({0.4,0.6}, {0.4})	({0.6}, {0.2})
A_5	({0.6}, {0.3})	({0.7}, {0.2})	({0.4}, {0.2})	({0.7}, {0.2,0.3})

TABLE 3 DHq-ROFDM $\tilde{K}^{(3)}$ provided by $D^{(3)}$

	ξ_1	ξ_2	ξ_3	ξ_4
A_1	({0.4}, {0.2})	({0.5}, {0.2})	({0.5}, {0.3,0.6})	({0.5}, {0.2})
A_2	({0.5}, {0.3})	({0.5,0.6,0.8}, {0.2,0.3})	({0.6}, {0.2})	({0.7}, {0.2})
A_3	({0.4}, {0.4})	({0.3}, {0.4})	({0.4}, {0.3})	({0.3,0.4}, {0.3,0.5,0.7})
A_4	({0.5,0.6,0.7}), ({0.3,0.4})	({0.5}, {0.3})	({0.3}, {0.5})	({0.5}, {0.2})
A_5	({0.6}, {0.2})	({0.6}, {0.4})	({0.4,0.5}, {0.3,0.4})	({0.6}, {0.3})

6.1.1 | Solution process using DHq-ROFWDBM operator

The steps for solving the problem using DHq-ROFWDBM operator is presented below.

Step 1. As all the criteria, ξ_j $(j = 1,2,3,4)$, in this example, represent benefit criteria, the original decision matrices $\tilde{K}^{(l)}$ $(l = 1,2,3)$ need not be normalized. Thus $\tilde{K}^{(l)} = \tilde{R}^{(l)}$ $(l = 1,2,3)$ for this example. Hence $[\tilde{\kappa}_{ij}{}^{(l)}]_{5\times4} = [\tilde{r}_{ij}{}^{(l)}]_{5\times4}$.

Step 2. To aggregate all the individual normalized decision matrices, $\tilde{R}^{(1)}$, $\tilde{R}^{(2)}$, and $\tilde{R}^{(3)}$ into a collective DHq-ROFDM, $\tilde{R}$, with associated weight vector, $w = (0.35, 0.40, 0.25)^T$, utilize DHq-ROFWDBM operator, as presented in Equation (23). The resultant DHq-ROFDM, $\tilde{R}$ is presented below.

$$\tilde{R} = \begin{bmatrix} \begin{pmatrix}\{0.3535, 0.4076\},\\ \{0.4721, 0.4906\}\end{pmatrix} & \begin{pmatrix}\{0.4406\},\\ \{0.2768\}\end{pmatrix} & \begin{pmatrix}\{0.3532, 0.3610\},\\ \{0.4970, 0.6579\}\end{pmatrix} & \begin{pmatrix}\{0.3877\},\\ \{0.3628, 0.4721\}\end{pmatrix} \\ \begin{pmatrix}\{0.4519\},\\ \{0.3984\}\end{pmatrix} & \begin{pmatrix}\{0.5038, 0.5373, 0.6049\},\\ \{0.2768, 0.3385\}\end{pmatrix} & \begin{pmatrix}\{0.5138, 0.5343, 0.5427\},\\ \{0.2771\}\end{pmatrix} & \begin{pmatrix}\{0.5748, 0.6492\},\\ \{0.2771, 0.3287\}\end{pmatrix} \\ \begin{pmatrix}\{0.3346, 0.3535\},\\ \{0.5136, 0.5905\}\end{pmatrix} & \begin{pmatrix}\{0.2697, 0.2707\},\\ \{0.5136, 0.5905\}\end{pmatrix} & \begin{pmatrix}\{0.3599\},\\ \{0.3783\}\end{pmatrix} & \begin{pmatrix}\left\{\begin{matrix}0.2314, 0.2957,\\ 0.2697, 0.3144\end{matrix}\right\},\\ \left\{\begin{matrix}0.4304, 0.5959, 0.6825,\\ 0.4337, 0.6351, 0.7920\end{matrix}\right\}\end{pmatrix} \\ \begin{pmatrix}\{0.4548, 0.5028, 0.5656\},\\ \{0.3783, 0.4411\}\end{pmatrix} & \begin{pmatrix}\{0.5511, 0.5801\},\\ \{0.2729, 0.3352\}\end{pmatrix} & \begin{pmatrix}\left\{\begin{matrix}0.3027, 0.3256,\\ 0.3307, 0.4476\end{matrix}\right\},\\ \{0.4730\}\end{pmatrix} & \begin{pmatrix}\{0.4406\},\\ \{0.2408\}\end{pmatrix} \\ \begin{pmatrix}\{0.4484, 0.4783, 0.5373\},\\ \{0.3287, 0.3628\}\end{pmatrix} & \begin{pmatrix}\{0.4670\},\\ \{0.2800\}\end{pmatrix} & \begin{pmatrix}\{0.3346, 0.3582\},\\ \{0.3961, 0.4911\}\end{pmatrix} & \begin{pmatrix}\{0.4670, 0.5343\},\\ \left\{\begin{matrix}0.4006, 0.4275,\\ 0.4020, 0.4296\end{matrix}\right\}\end{pmatrix} \end{bmatrix}$$

Step 3. Considering weight vector, $\omega = (0.2, 0.1, 0.3, 0.4)^T$ corresponding to four attributes, ξ_j $(j = 1,2,3,4)$, utilize DHq-ROFWDBM operator, as shown in Equation (19), to derive the overall evaluation values, $\tilde{r}_i = (\tilde{h}_{ij}, \tilde{g}_{ij})$ of each alternative, A_i, considering the same parameter values as in Step 2. The calculated overall evaluation value of the alternatives is presented as follows:

$$A_1 = \left(\{0.3127, 0.3108, 0.3007, 0.2988\}, \left\{\begin{matrix}0.5826, 0.5165, 0.5504, 0.5001,\\ 0.5757, 0.5129, 0.5450, 0.4968\end{matrix}\right\}\right),$$

$$A_2 = \left(\left\{\begin{matrix}0.4405, 0.4320, 0.4377, 0.4298, 0.4310, 0.4241,\\ 0.4222, 0.4146, 0.4192, 0.4123, 0.4118, 0.4065,\\ 0.4150, 0.4071, 0.4117, 0.4046, 0.4039, 0.3986\end{matrix}\right\}, \{0.4282, 0.4059, 0.4042, 0.3856\}\right),$$

$$A_3 = \left(\left\{\begin{matrix}0.2550, 0.2423, 0.2498, 0.2321,\\ 0.2549, 0.2421, 0.2497, 0.2320,\\ 0.2503, 0.2377, 0.2452, 0.2272,\\ 0.2502, 0.2376, 0.2451, 0.2271\end{matrix}\right\}, \left\{\begin{matrix}0.7404, 0.6986, 0.5948, 0.7145, 0.6832, 0.5929,\\ 0.7101, 0.6752, 0.5853, 0.6889, 0.6617, 0.5835,\\ 0.6978, 0.6669, 0.5796, 0.6796, 0.6539, 0.5780,\\ 0.6747, 0.6476, 0.5708, 0.6590, 0.6361, 0.5693\end{matrix}\right\}\right),$$

$$A_4 = \left(\left\{\begin{matrix}0.4009, 0.3770, 0.3764, 0.3740, 0.3945, 0.3695,\\ 0.3688, 0.3662, 0.3895, 0.3642, 0.3635, 0.3607,\\ 0.3843, 0.3585, 0.3578, 0.3547, 0.3786, 0.3519,\\ 0.3511, 0.3478, 0.3740, 0.3472, 0.3463, 0.3429\end{matrix}\right\}, \{0.4981, 0.4490, 0.4753, 0.4351\}\right),$$

$$A_5 = \left(\left\{\begin{matrix}0.3692, 0.3562, 0.3653, 0.3517, 0.3516, 0.3446,\\ 0.3469, 0.3396, 0.3427, 0.3376, 0.3375, 0.3323\end{matrix}\right\}, \left\{\begin{matrix}0.4882, 0.4783, 0.4874, 0.4778,\\ 0.4624, 0.4541, 0.4618, 0.4537,\\ 0.4746, 0.4649, 0.4738, 0.4645,\\ 0.4510, 0.4429, 0.4504, 0.4425\end{matrix}\right\}\right),$$

Step 4. Use the score function, as displayed in Equation (1), to find the score value of $\tilde{r}_i$ $(i = 1,2, ..., 5)$. The score values are found as $S(\tilde{r}_1) = 0.4370$, $S(\tilde{r}_2) = 0.5030$, $S(\tilde{r}_3) = 0.3688$, $S(\tilde{r}_4) = 0.4743$, and $S(\tilde{r}_5) = 0.4709$.

Step 5. Rank the alternatives based on the above score values, $(\tilde{r}_i)$ $(i = 1,2, ..., 5)$, using Definition 5. The ranking of the alternatives are obtained as $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$. So, the best alternative, that is, the best company for investment is identified as A_2.

6.1.2 | Alternative solution process using DHq-ROFWDGBM operator

Again, this problem is solved by utilizing DHq-ROFWDGBM operator. The steps for solving this problem are analogous to the steps as described in Section 6.1.1. The steps are presented briefly as follows:

Step 1′. is similar to Step 1 as above.

Step 2′. Utilize DHq-ROFWDGBM operator to aggregate the decision matrices $\tilde{R}^{(l)}$ $(l = 1,2,3)$, with weight vector $w = (0.35, 0.40, 0.25)^T$, into collective DHq-ROFDM, $\tilde{R}'$ as.

$$\tilde{R}' = \begin{bmatrix} \begin{pmatrix} \{0.5147,\ 0.5923\}, \\ \{0.3093,\ 0.3158\} \end{pmatrix} & \begin{pmatrix} \{0.6224\}, \\ \{0.1785\} \end{pmatrix} & \begin{pmatrix} \{0.5468,\ 0.5488\}, \\ \{0.3307,\ 0.4474\} \end{pmatrix} & \begin{pmatrix} \{0.5587\}, \\ \{0.2395,\ 0.3093\} \end{pmatrix} \\ \begin{pmatrix} \{0.6543\}, \\ \{0.2636\} \end{pmatrix} & \begin{pmatrix} \{0.6949,\ 0.7245,\ 0.7962\}, \\ \{0.1785,\ 0.2168\} \end{pmatrix} & \begin{pmatrix} \{0.6883,\ 0.7233,\ 0.7562\}, \\ \{0.1767\} \end{pmatrix} & \begin{pmatrix} \{0.7602,\ 0.8442\}, \\ \{0.1767,\ 0.2202\} \end{pmatrix} \\ \begin{pmatrix} \{0.4751,\ 0.5147\}, \\ \{0.3511,\ 0.4113\} \end{pmatrix} & \begin{pmatrix} \{0.4204,\ 0.4275\}, \\ \{0.3511,\ 0.4113\} \end{pmatrix} & \begin{pmatrix} \{0.5425\}, \\ \{0.2283\} \end{pmatrix} & \begin{pmatrix} \begin{Bmatrix} 0.3961,\ 0.4911, \\ 0.4204,\ 0.4956 \end{Bmatrix}, \\ \begin{Bmatrix} 0.2675,\ 0.3780,\ 0.4748, \\ 0.2677,\ 0.3858,\ 0.5497 \end{Bmatrix} \end{pmatrix} \\ \begin{pmatrix} \{0.6695,\ 0.7358,\ 0.7971\}, \\ \{0.2283,\ 0.2805\} \end{pmatrix} & \begin{pmatrix} \{0.7417,\ 0.7815\}, \\ \{0.1821,\ 0.2170\} \end{pmatrix} & \begin{pmatrix} \{0.4418,\ 0.5026,\ 0.4970,\ 0.6380\}, \\ \{0.3100\} \end{pmatrix} & \begin{pmatrix} \{0.6224\}, \\ \{0.1660\} \end{pmatrix} \\ \begin{pmatrix} \{0.6528,\ 0.6634,\ 0.7245\}, \\ \{0.2202,\ 0.2395\} \end{pmatrix} & \begin{pmatrix} \{0.6985\}, \\ \{0.1845\} \end{pmatrix} & \begin{pmatrix} \{0.4751,\ 0.5135\}, \\ \{0.2314,\ 0.2957\} \end{pmatrix} & \begin{pmatrix} \{0.6985,\ 0.7233\}, \\ \begin{Bmatrix} 0.2320,\ 0.2707, \\ 0.2322,\ 0.2709 \end{Bmatrix} \end{pmatrix} \end{bmatrix}$$

Step 3′. Obtain the collective preference values of each alternative using DHq-ROFWDGBM operator as follows:

$$\tilde{r}_1' = \left(\{0.7005, 0.6999, 0.6764, 0.6759\}, \begin{Bmatrix} 0.2452, 0.2191, 0.2323, 0.2114, \\ 0.2432, 0.2164, 0.2305, 0.2096 \end{Bmatrix} \right),$$

$$\tilde{r}_2' = \left(\begin{Bmatrix} 0.8651, 0.8427, 0.8571, 0.8345, 0.8493, 0.8266, \\ 0.8512, 0.8300, 0.8437, 0.8224, 0.8365, 0.8152, \\ 0.8447, 0.8238, 0.8372, 0.8163, 0.8300, 0.8091 \end{Bmatrix}, \{0.1673, 0.1561, 0.1608, 0.1482\} \right),$$

$$\tilde{r}_3' = \left(\begin{Bmatrix} 0.6108, 0.5987, 0.6099, 0.5963, \\ 0.6087, 0.5960, 0.6077, 0.5934, \\ 0.5992, 0.5858, 0.5981, 0.5831, \\ 0.5970, 0.5832, 0.5959, 0.5803 \end{Bmatrix}, \begin{Bmatrix} 0.2950, 0.2809, 0.2493, 0.2917, 0.2794, 0.2493, \\ 0.2809, 0.2663, 0.2341, 0.2774, 0.2647, 0.2340, \\ 0.2731, 0.2652, 0.2394, 0.2715, 0.2643, 0.2394, \\ 0.2551, 0.2499, 0.2291, 0.2541, 0.2492, 0.2290 \end{Bmatrix} \right),$$

$$\tilde{r}_4' = \left(\begin{Bmatrix} 0.8145, 0.7773, 0.7782, 0.7704, 0.8092, 0.7738, \\ 0.7747, 0.7672, 0.8023, 0.7687, 0.7695, 0.7624, \\ 0.7973, 0.7652, 0.7660, 0.7592, 0.7843, 0.7536, \\ 0.7543, 0.7479, 0.7799, 0.7504, 0.7511, 0.7450 \end{Bmatrix}, \{0.1834, 0.1792, 0.1662, 0.1599\} \right),$$

$$\tilde{r}_5' = \left(\begin{Bmatrix} 0.8002, 0.7912, 0.7991, 0.7896, 0.7824, 0.7741, \\ 0.7809, 0.7722, 0.7789, 0.7707, 0.7772, 0.7687 \end{Bmatrix}, \begin{Bmatrix} 0.1951, 0.1839, 0.1951, 0.1839, \\ 0.1792, 0.1739, 0.1792, 0.1738, \\ 0.1904, 0.1786, 0.1904, 0.1785, \\ 0.1738, 0.1691, 0.1738, 0.1690 \end{Bmatrix} \right).$$

Step $4'$. Calculate the score value of $\tilde{r}_i'(i = 1,2, ..., 5)$ as $S(\tilde{r}_1') = 0.6573$, $S(\tilde{r}_2') = 0.7879$, $S(\tilde{r}_3') = 0.5973$, $S(\tilde{r}_4') = 0.7277$, $S(\tilde{r}_5') = 0.7364$.

Step $5'$. Rank the alternatives based on the score values, $\tilde{r}_i'(i = 1,2, ..., 5)$. The ordering of the alternatives are obtained as $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$. So, the best company for investment is identified as the same as A_2.

It is worthy to mention here that the ranking of alternatives, achieved by Liu et al.,[42] is $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$ using IF weighted Dombi-BM and GBM operators. But using the proposed approach, different ranking results of alternatives are found, which also covers the said result.[42] The technique developed by Liu et al.[42] is based on IFS, whereas the proposed approach is considered using DHq-ROF information. Since, it is already mentioned that IFS can be treated as a special case of IFS, it appears that the proposed methodology is more general and consistent than the technique developed by Liu et al.[42]

6.1.3 | The influence of BM parameters on decision-making results

Here, the influence of BM parameters, θ and ϕ on decision-making results based on DHq-ROFWDBM and DHq-ROFWDGBM operators are discussed. It is observed that the parameters, θ and ϕ play a vital role in the ranking method of alternatives. Different score values are obtained by assigning different values to θ and ϕ. The ranking results are shown in Tables 4 and 5 by changing the values of θ and ϕ, and by keeping the fixed values of the parameters $q(=3)$, and $\tau(=2)$. It is observed from the achieved result that if DHq-ROFWDBM operator is used, score values increase with variation of parameter θ or ϕ in between 0 and 20, which is displayed in Table 4. The ranking result is found as $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$. The best alternative is identified as A_2 and the worst choice is A_3.

Further, Table 5 shows that when θ or ϕ increases, the score values based on DHq-ROFWDGBM operator decreases. By using this AO, the ordering of the alternatives are slightly differ as $A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$, when the values $q = 3$, and $\tau = 2$ are considered. It is also observed that the position of two alternatives, A_4 and A_5 are just interchanging when DHq-ROFWDGBM operator is used instead of DHq-ROFWDBM operator. However, the best and the worst alternatives remain the same for both the operators. Geometrically, this fact is clearly visible in Figures 2 and 3, in which all the figures are symmetric with respect to the line segment joining two points $(0,0,0)$ and $(1,1,1)$, that is, when $\theta = \phi$, for both the operators, the score values and the ranking results remain the same.

6.1.4 | The influence of the Dombi parameter, τ on decision-making results

Now, the influence of the Dombi parameter, τ on decision-making results based on DHq-ROFWDBM and DHq-ROFWDGBM operators are discussed. In this case, considering $\theta = 1 = \phi$ and $q = 3$, the graphical interpretation of the score values of the alternatives for the different values of τ are shown in Figures 4 and 5, respectively. Utilizing DHq-ROFWDBM and

TABLE 4 Ranking results using DHq-ROFWDBM operator by varying θ and ϕ ($q = 3, \tau = 2$)

Varying θ and ϕ	A_1	A_2	A_3	A_4	A_5	Ranking order
$\theta = 0, \phi = 1$	0.4918	0.5680	0.4813	0.5331	0.5168	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 1, \phi = 1$	0.4370	0.5030	0.3688	0.4743	0.4709	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 1, \phi = 2$	0.4411	0.5056	0.3743	0.4782	0.4734	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 1, \phi = 5$	0.4549	0.5156	0.3952	0.4917	0.4819	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 1, \phi = 10$	0.4657	0.5258	0.4158	0.5030	0.4894	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 1, \phi = 15$	0.4709	0.5319	0.4276	0.5088	0.4935	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 5, \phi = 5$	0.4370	0.5030	0.3688	0.4743	0.4709	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 5, \phi = 15$	0.4466	0.5093	0.3820	0.4835	0.4767	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 5, \phi = 20$	0.4512	0.5127	0.3891	0.4880	0.4796	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 10, \phi = 15$	0.4384	0.5039	0.3707	0.4757	0.4718	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 15, \phi = 20$	0.4377	0.5035	0.3698	0.4750	0.4714	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 20, \phi = 0.5$	0.4800	0.5458	0.4513	0.5193	0.5018	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
$\theta = 20, \phi = 1$	0.4741	0.5362	0.4354	0.5124	0.4962	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$

TABLE 5 Ranking results using DHq-ROFWDGBM operator by varying θ and ϕ ($q = 3, \tau = 2$)

Varying θ and ϕ	A_1	A_2	A_3	A_4	A_5	Ranking order
$\theta = 0, \phi = 1$	0.5487	0.7423	0.5109	0.6052	0.6119	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 1, \phi = 1$	0.6573	0.7879	0.5973	0.7277	0.7364	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 1, \phi = 2$	0.6534	0.7867	0.5947	0.7221	0.7286	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 1, \phi = 5$	0.6390	0.7761	0.5844	0.7013	0.7008	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 1, \phi = 10$	0.6246	0.7663	0.5734	0.6809	0.6764	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 1, \phi = 15$	0.6157	0.7610	0.5665	0.6690	0.6639	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 5, \phi = 5$	0.6573	0.7879	0.5973	0.7277	0.7364	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 5, \phi = 15$	0.6480	0.7827	0.5910	0.7143	0.7180	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 5, \phi = 20$	0.6432	0.7791	0.5875	0.7073	0.7086	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 10, \phi = 15$	0.6559	0.7886	0.5964	0.7257	0.7337	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 15, \phi = 20$	0.6566	0.7891	0.5969	0.7267	0.7350	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 20, \phi = 0.5$	0.5947	0.7513	0.5506	0.6434	0.6423	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$
$\theta = 20, \phi = 1$	0.6094	0.7577	0.5616	0.6608	0.6563	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$

DHq-ROFWDGBM operators, Figures 4 and 5 indicate that the score values may be different for the different values of the parameter, τ. However, the best alternative is A_2. The DMs can choose the appropriate value of the parameter, τ by their preferences. Furthermore, from Figures 4 and 5, it can be easily seen that the score values according to the DHq-ROFWDBM operator becomes greater when the parameter τ increases. On the contrary, the DHq-

ROFWDGBM operator produces decreasing order of score values with the increase of the parameter, τ. Also, it is observed from those figures that, if the value of τ is taken away from the origin, differences of the score values of the consecutively ranked alternatives increase. Thus the ranking of alternatives become more prominent with the increasing value of the Dombi parameter, τ.

6.1.5 | The influence of the q-ROFNs parameter, q on decision-making result

In this section, the influence of the parameter, q, associated with q-ROFNs, on decision making results based on the DHq-ROFWDBM and DHq-ROFWDGBM operators is discussed by taking $\theta = \phi = 1$ and $\tau = 2$, for convenience. The ranking results using those two operators, with different values of q are shown, respectively, in Figures 6 and 7.

Figures 6 and 7 perceive that the achieved ranking results are different for the variations of q from 1 to 10 using both DHq-ROFWDBM and DHq-ROFWDGBM operators. For using DHq-ROFWDBM operator, ranking result is obtained as $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$. Whereas, while the DH$q$-ROFWDGBM operator is used, the ranking result becomes different for different values of q from 1 to 10. If q is taken in the open interval $(1, 2.1049)$, the ordering of alternatives is found as $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$, which is the same as using DHq-ROFWDBM operator. For $q \in (2.1049, 10)$, the ranking result slightly differs as $A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$. Whereas, for $q = 2.1049$, the ranking result becomes $A_2 \succ A_4 \approx A_5 \succ A_1 \succ A_3$. However, it is evident that the best alternative is identified as A_2 by applying both the AOs.

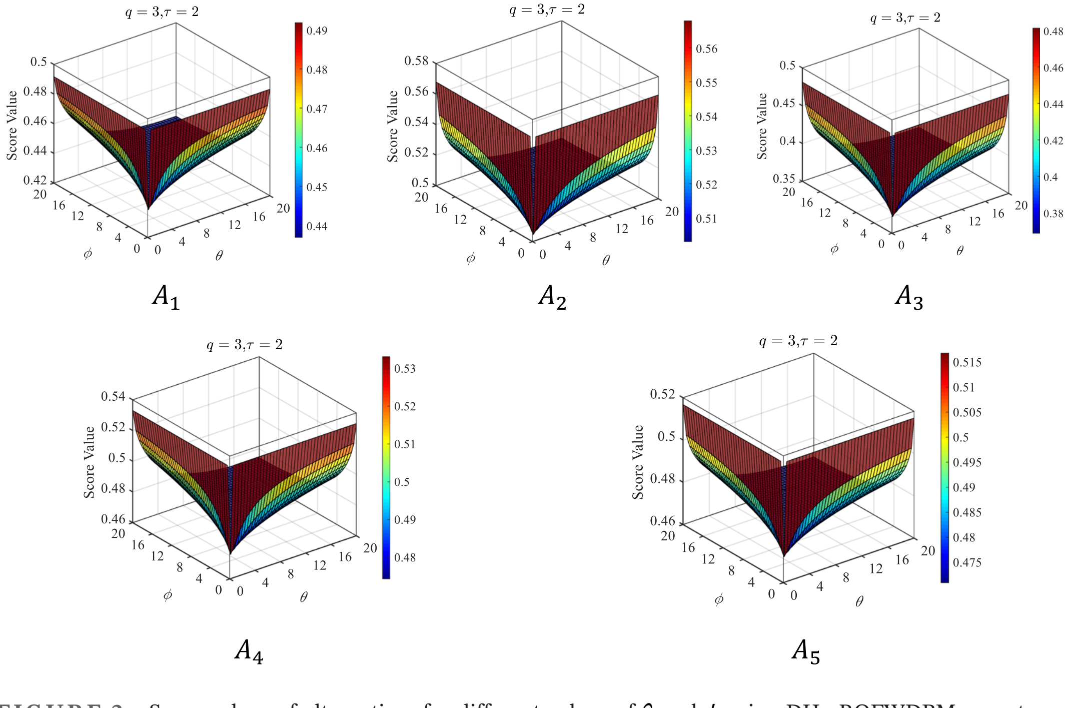

FIGURE 2 Score values of alternatives for different values of θ and ϕ using DHq-ROFWDBM operator [Color figure can be viewed at wileyonlinelibrary.com]

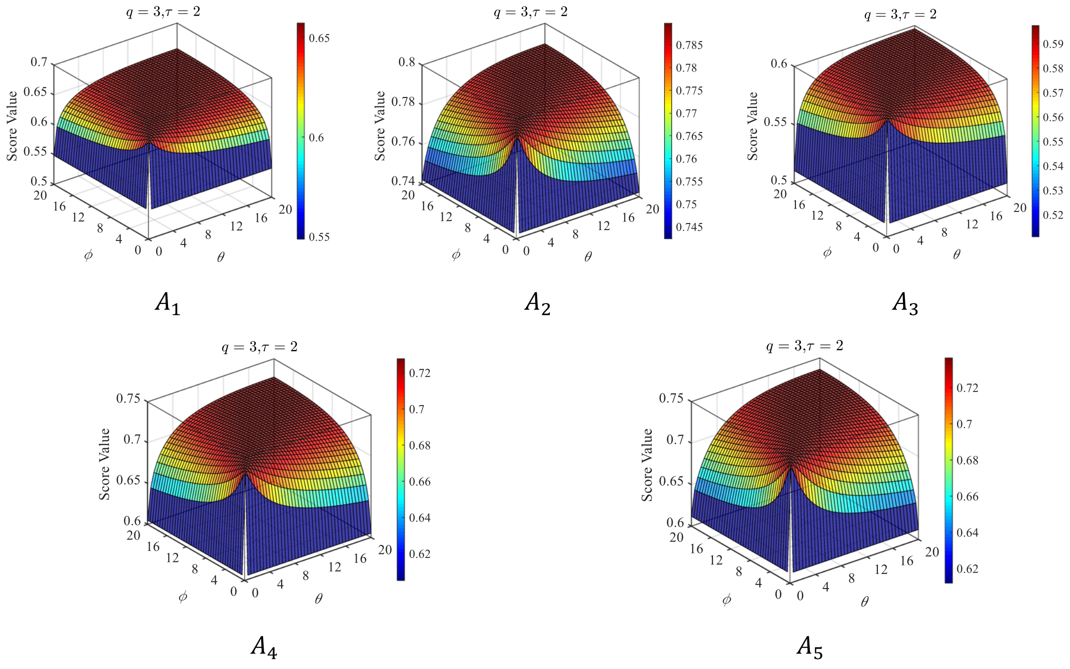

FIGURE 3 Score values of the alternatives for different values of θ and ϕ using DHq-ROFWDGBM operator [Color figure can be viewed at wileyonlinelibrary.com]

Again, from Figure 6, it is clear that the score values of the alternatives based on DHq-ROFWDBM operator become higher when the parameter q increases. Whereas, from Figure 7, it is observed that the score values based on DHq-ROFWDGBM operator become smaller, when the parameter q increases. Thus the DMs can choose the appropriate parameter value of q on the basis of their preferences.

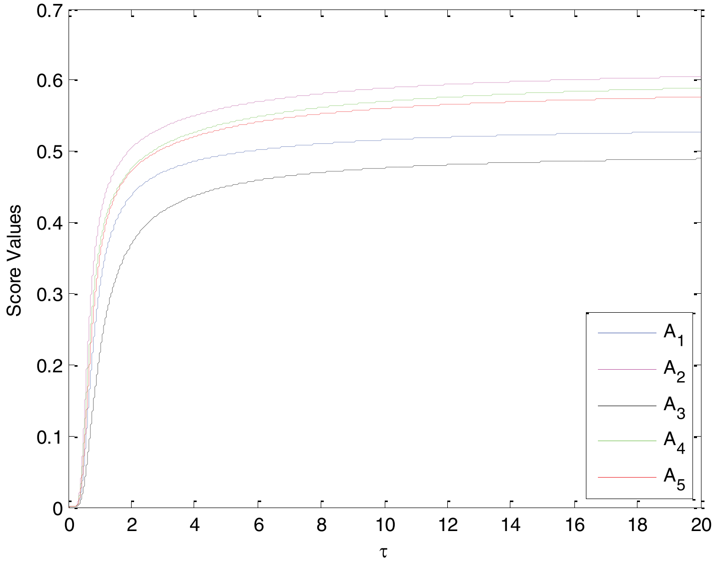

FIGURE 4 Variation of τ using DHq-ROFWDBM operator when $q = 3$, $\theta = \phi = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

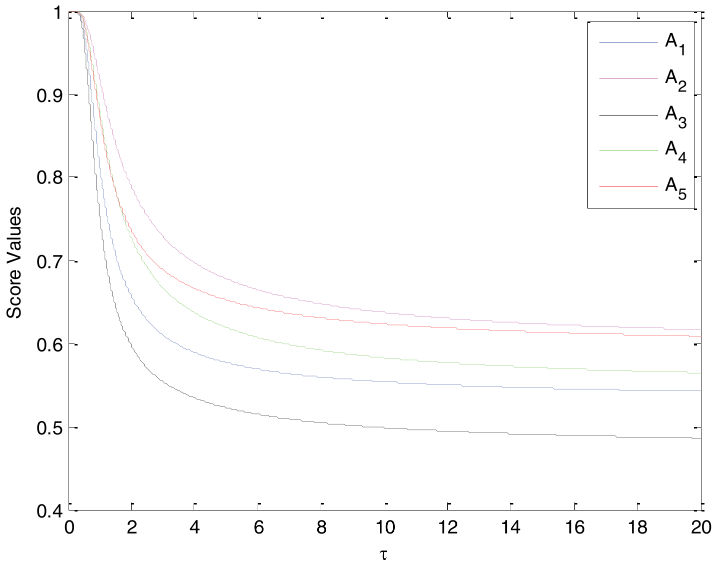

FIGURE 5 Variation of τ using DHq-ROFWDGBM operator when $q = 3$, $\theta = \phi = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

6.1.6 | Comparison with other approaches

To examine the effectiveness of the established method, the illustrative example as considered above, is solved by using several existing MCDM methods using different AOs, based on weighted hesitant PF MSM (WHPFMSM),[52] DHPF WA (DHPFWA[53]), DHPF weighted BM (DHPFWBM[45]), DHPF geometric weighted HM (DHPFGWHM),[54] DHPF weighted Hamy

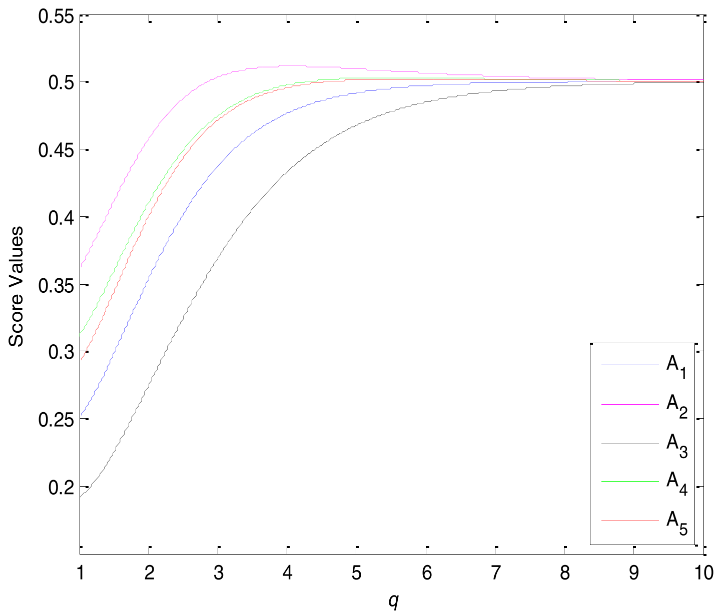

FIGURE 6 Variation of q using DHq-ROFWDBM operator when $\tau = 2$, $\theta = \phi = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

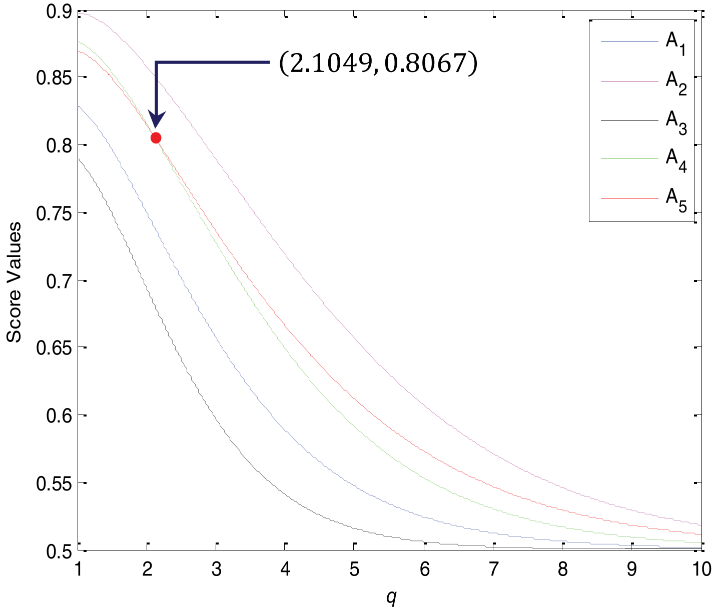

FIGURE 7 Variation of q using DHq-ROFWDGBM operator when $\tau = 2$, $\theta = \phi = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

mean,[55] DHq-ROF WA (DHq-ROFWA[39]), DHq-ROFWMM,[40] and q-rung DHF weighted HM (q-RDHFWHM[19]). The comparisons are performed in two different ways.

At first, the comparisons are performed on the basis of characteristic of the operators, and subsequently, on the basis of achieved results.

In comparing the method by the characteristic of the operator, it is worthy to mention here that all the above mentioned existing operators, including the developed operators, can capture HF information. The operators[45,52–55] which are developed in HPF environments can be treated as a special case of corresponding operators in DHq-ROF contexts by considering $q = 2$. Except the developed operators, none of the above-mentioned operators consider Dombi operations, which make the decision aggregation process more flexible. Combining BM with Dombi operations in DHq-ROF environment, the developed operators become more flexible and powerful than the existing operators, by also considering interrelationships among input arguments. The characteristics of the existing operators are summarized in Table 6. This table reflects the wide coverage of the existing methods by the proposed method.

Now, the existing methods,[19,39,40,45,52–55] under consideration, would be compared on the basis of achieved results. It is important to note that the ranking of alternatives corresponding to different methods is the same as the ranking through the developed method, for some specific values of the parameters associated with it. Table 7 displays the achieved score values and rankings obtained by applying all the existing methods, under consideration, together with the proposed method with the values $\theta = \phi = 1$, $q = 3$, and $\tau = 2$.

Thus, it can be claimed that existing methods now appear as some special cases of the developed process. However, different ranking results of the alternatives can be obtained by varying the Dombi parameter τ; BM parameters θ, ϕ; and the rung parameter q.

To show the robustness of the proposed method, three other problems,[33,35,56] considered previously, are further solved and compared with the existing methods.

TABLE 6 Characteristics of different methods under considerations

Methods	Consideration of interrelationships	Consideration of hesitancy	Flexibility due to Dombi operation	Capturing information by *q*-ROF
WHPFMSM[52]	Yes	Yes	No	No
DHPFWA[53]	No	Yes	No	No
DHPFWBM[45]	Yes	Yes	No	No
DHPFGWHM[54]	Yes	Yes	No	No
DHPF weighted Hamy Mean[55]	Yes	Yes	No	No
DH*q*-ROFWA[39]	No	Yes	No	Yes
DH*q*-ROFWMM[40]	Yes	Yes	No	Yes
q-RDHFWHM[19]	Yes	Yes	No	Yes
Proposed method	Yes	Yes	Yes	Yes

6.2 | Illustrative Example 2

To meet the growing demand for charging piles in Shanghai, Ju et al.[33] considered a problem relating to a company to select appropriate place to set up a large electric vehicle charging station from the four available areas, namely, A_1, A_2, A_3, and A_4 by considering four criteria, namely, ξ_1; environment impact factor; ξ_2: cost factor; ξ_3: society impact factor; and ξ_4: technology requirement, with weight vector $\omega = (0.2, 0.25, 0.3, 0.25)^T$. The company invited three experts, $D^{(l)}(l = 1,2,3)$, with weight vector, $w = (0.4, 0.3, 0.3)^T$, to evaluate four areas, by which appropriate area for the charging station would be chosen. The experts evaluated the areas, A_i $(i = 1,2,3,4)$ with regard to the criteria, ξ_j $(j = 1,2,3,4)$ and put judgment values in the form of q-ROFNs. To apply the proposed method, at first, the problem is adopted by considering input arguments of the decision matrices as DHq-ROFNs. The individual DHq-ROF evaluation matrices $\tilde{K}^{(l)} = [\tilde{\kappa}_{ij}^{l}]_{4\times4}(l = 1,2,3)$ provided by three experts are shown in Tables 8–10.

This problem is then solved using the developed DHq-ROFWDBM and DHq-ROFWDGBM operators. It is worthy to mention here that the ranking results achieved by Ju et al.[33] are $A_2 \succ A_3 \succ A_4 \succ A_1$ and $A_2 \succ A_4 \succ A_3 \succ A_1$ using q-ROF generalized power WA (q-ROFGPWA) and WG (q-ROFGPWG) operators, respectively. But using the proposed approach, different ranking results of alternatives are found by utilizing the proposed method, and which also cover the result of Ju et al.[33] The technique developed by Ju et al.[33] is based on q-ROFS, whereas, the proposed approach is based on DHq-ROF information. Thus, the proposed methodology is more general and consistent than the technique developed by Ju et al.[33]

As like the previous example, this example is also solved by considering different parameter values. Without loss of generality, the values of the BM parameters, θ and ϕ are primarily assumed to be 1, along with rung parameter, $q = 3$, and Dombi parameter, $\tau = 2$. The ordering of the alternatives using the proposed DHq-ROFWDBM operator is achieved as $A_2 \succ A_3 \succ A_4 \succ A_1$. However, the ranking changes when DHq-ROFWDGBM operator is used. The changes in ordering of the alternatives depend on the BM parameters, θ and ϕ, which is shown in Table 11. When the parameters, θ and ϕ are changed between 1 and 20, the ranking

TABLE 7 Comparison results with the existing methods in terms of score values and ranking of the alternatives

AOs used	Score values	Ranking
WHPFMSM[52]	$S(A_1) = 0.5003$, $S(A_2) = 0.6124$, $S(A_3) = 0.4374$, $S(A_4) = 0.5632$, $S(A_5) = 0.5439$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
DHPFWA[53]	$S(A_1) = 0.5442$, $S(A_2) = 0.6983$, $S(A_3) = 0.5087$, $S(A_4) = 0.6316$, $S(A_5) = 0.6077$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
DHPFWBM[45]	$S(A_1) = 0.5071$, $S(A_2) = 0.6358$, $S(A_3) = 0.4546$, $S(A_4) = 0.5737$, $S(A_5) = 0.5556$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
DHPFGWHM[54]	$S(A_1) = 0.5105$, $S(A_2) = 0.5263$, $S(A_3) = 0.5085$, $S(A_4) = 0.5167$, $S(A_5) = 0.5162$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
DHPF weighted Hamy Mean[55]	$S(A_1) = 0.9315$, $S(A_2) = 0.9629$, $S(A_3) = 0.9207$, $S(A_4) = 0.9475$, $S(A_5) = 0.9446$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
DH**q**-ROFWA[39]	$S(A_1) = 0.5296$, $S(A_2) = 0.6539$, $S(A_3) = 0.5083$, $S(A_4) = 0.5913$, $S(A_5) = 0.5772$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
DH**q**-ROFWMM[40]	$S(A_1) = 0.4881$, $S(A_2) = 0.5677$, $S(A_3) = 0.4319$, $S(A_4) = 0.5210$, $S(A_5) = 0.5068$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
q-RDHFWHM[19]	$S(A_1) = 0.5049$, $S(A_2) = 0.5172$, $S(A_3) = 0.5039$, $S(A_4) = 0.5096$, $S(A_5) = 0.5091$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$
Proposed operator (DH**q**-ROFWDBM)	$S(A_1) = 0.4370$, $S(A_2) = 0.5030$, $S(A_3) = 0.3688$, $S(A_4) = 0.4743$, $S(A_5) = 0.4709$	$A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_3$

TABLE 8 DHq-ROFDM $\tilde{K}^{(1)}$ for Example 6.2

	ξ_1	ξ_2	ξ_3	ξ_4
A_1	$\langle\{0.6,0.8\},\{0.2,0.4\}\rangle$	$\langle\{0.4,0.6\},\{0.6\}\rangle$	$\langle\{0.3\},\{0.5,0.7\}\rangle$	$\langle\{0.3,0.6\},\{0.5\}\rangle$
A_2	$\langle\{0.1\},\{0.8\}\rangle$	$\langle\{0.4,0.5,0.6\},\{0.3\}\rangle$	$\langle\{0.2,0.4\},\{0.6,0.9\}\rangle$	$\langle\{0.3,0.5\},\{0.4\}\rangle$
A_3	$\langle\{0.3,0.4\},\{0.6\}\rangle$	$\langle\{0.3,0.4\},\{0.5\}\rangle$	$\langle\{0.4\},\{0.5\}\rangle$	$\langle\{0.6\},\{0.2,0.3,0.5\}\rangle$
A_4	$\langle\{0.4,0.5,0.6\},\{0.5\}\rangle$	$\langle\{0.5\},\{0.4\}\rangle$	$\langle\{0.4,0.6\},\{0.5,0.7\}\rangle$	$\langle\{0.4,0.6\},\{0.6\}\rangle$

TABLE 9 DHq-ROFDM $\tilde{K}^{(2)}$ for Example 6.2

		ξ_2	ξ_3	ξ_4
A_1	$\langle\{0.7,0.9\},\{0.4,0.5\}\rangle$	$\langle\{0.4\},\{0.5\}\rangle$	$\langle\{0.4,0.7\},\{0.6\}\rangle$	$\langle\{0.4\},\{0.5\}\rangle$
A_2	$\langle\{0.3\},\{0.5\}\rangle$	$\langle\{0.6\},\{0.5,0.9\}\rangle$	$\langle\{0.2\},\{0.5,0.8\}\rangle$	$\langle\{0.3\},\{0.6,0.8\}\rangle$
A_3	$\langle\{0.2\},\{0.7,0.8\}\rangle$	$\langle\{0.4\},\{0.7\}\rangle$	$\langle\{0.3,0.4\},\{0.5,0.75\}\rangle$	$\langle\{0.3,0.5\},\{0.3\}\rangle$
A_4	$\langle\{0.4\},\{0.5\}\rangle$	$\langle\{0.5,0.6\},\{0.3,0.65\}\rangle$	$\langle\{0.4\},\{0.6\}\rangle$	$\langle\{0.4\},\{0.6,0.7\}\rangle$

TABLE 10 DHq-ROFDM $\tilde{K}^{(3)}$ for Example 6.2

	ξ_1	ξ_2	ξ_3	C_4
A_1	$\langle\{0.4\},\{0.6\}\rangle$	$\langle\{0.3\},\{0.3,0.5\}\rangle$	$\langle\{0.6\},\{0.4\}\rangle$	$\langle\{0.3\},\{0.3,0.6\}\rangle$
A_2	$\langle\{0.4,0.5,0.7\},\{0.2\}\rangle$	$\langle\{0.2\},\{0.6\}\rangle$	$\langle\{0.2\},\{0.8\}\rangle$	$\langle\{0.2\},\{0.5,0.75\}\rangle$
A_3	$\langle\{0.3\},\{0.5\}\rangle$	$\langle\{0.4\},\{0.6,0.7\}\rangle$	$\langle\{0.3\},\{0.7,0.85\}\rangle$	$\langle\{0.2,0.4\},\{0.6\}\rangle$
A_4	$\langle\{0.5\},\{0.4,0.6\}\rangle$	$\langle\{0.2,0.4\},\{0.7\}\rangle$	$\langle\{0.2\},\{0.6\}\rangle$	$\langle\{0.5\},\{0.4,0.5,0.6\}\rangle$

results remain the same as $A_2 \succ A_3 \succ A_4 \succ A_1$, when DH$q$-ROFWDBM operator is used; but different ordering of alternatives, namely, $A_4 \succ A_3 \succ A_2 \succ A_1$, $A_2 \succ A_3 \succ A_4 \succ A_1$, $A_2 \succ A_4 \succ A_3 \succ A_1$, and $A_4 \succ A_2 \succ A_3 \succ A_1$ are found when DHq-ROFWDGBM operator is applied. Some simple values of the BM parameters are taken into account from the computational point of views, such as $\theta = 0$, or $\phi = 0$, or $\theta = 1, \phi = 1$. More details can be followed in Figures 8 and 9.

Again, to show the influence of Dombi parameter τ, the score value is calculated with varying the parameter from 0 to 20 by utilizing the DHq-ROFWDBM and DHq-ROFWDGBM operators keeping the value of other parameters as $\theta = \phi = 1$, and $q = 3$. The effects of Dombi parameter τ on ordering of the alternatives are shown details in Figures 10 and 11. From Figure 10 it is observed that when DHq-ROFWDBM operator is used, the ordering remains the same as $A_2 \succ A_3 \succ A_4 \succ A_1$. On the other hand, different ordering, namely, $A_2 \succ A_3 \succ A_4 \succ A_1$ and $A_3 \succ A_2 \succ A_4 \succ A_1$ are found when τ is varying in DHq-ROFWDGBM operator.

As like Dombi parameter τ, rung parameter q has also various impacts on ordering results. To show that, this example is solved by the proposed method with varying parameter q between 1 and 10, and also by taking $\theta = \phi = 1$, and $\tau = 2$. Figures 12 and 13 depict the ordering of the alternatives based on the achieved score values by varying the rung parameter, q.

It is important to note that two different ranking results are found when the problem is solved using the method of Ju et al.[33] But if this problem is solved using the proposed method,

TABLE 11 Ranking order for different operational BM parameters θ and ϕ of Example 6.2 ($q = 3, \tau = 2$)

Varying θ and ϕ	Ranking order using DHq-ROFWDBM operator	Ranking Order using Hq-ROFWDGBM operator
$\theta = 0, \phi = 1$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$\theta = 1, \phi = 1$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$\theta = 1, \phi = 2$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$\theta = 1, \phi = 5$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$\theta = 1, \phi = 10$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$\theta = 1, \phi = 15$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\theta = 5, \phi = 5$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$\theta = 5, \phi = 15$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$\theta = 5, \phi = 20$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$\theta = 10, \phi = 15$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$\theta = 15, \phi = 20$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$\theta = 20, \phi = 0.5$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\theta = 20, \phi = 1$	$A_2 \succ A_3 \succ A_4 \succ A_1$	$A_4 \succ A_2 \succ A_3 \succ A_1$

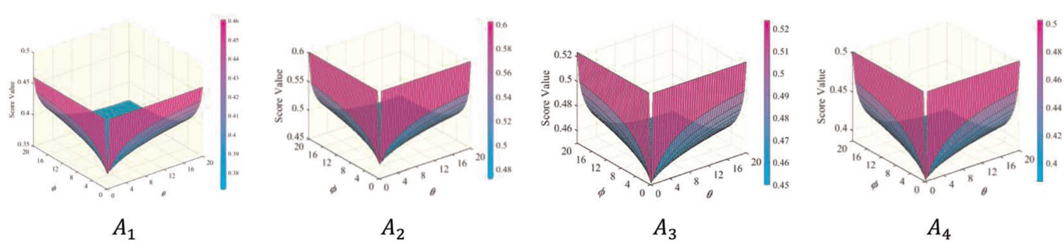

FIGURE 8 Scores of alternatives A_i based on DHq-ROFWDBM AO [Color figure can be viewed at wileyonlinelibrary.com]

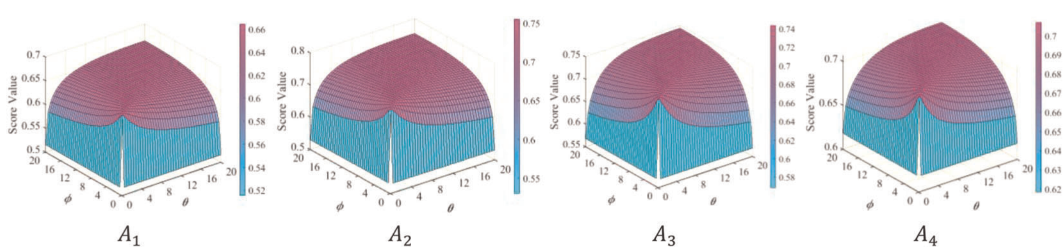

FIGURE 9 Scores of alternative A_i based on DHq-ROFWDGBM AO [Color figure can be viewed at wileyonlinelibrary.com]

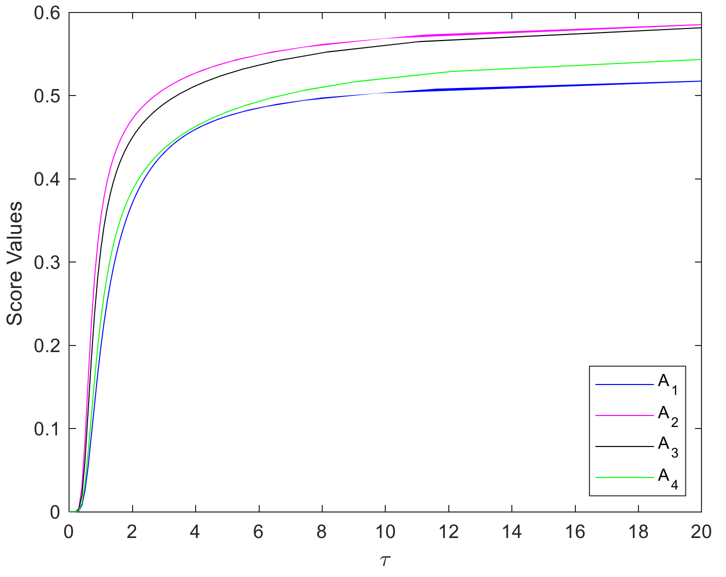

FIGURE 10 Impact of τ on ordering using DHq-ROFWDBM operator when $q = 3, \theta = \phi = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

three different ranking results are achieved, which also include Ju et al.'s[33] result. Further, Ju et al.[33] used simple power AOs in q-ROF environment. Those AOs do not consider the interrelationship of the attributes. On the contrary, the proposed method is based on BM operator under DHq-ROF context. So the developed operators successfully capture interrelationship between any two attributes, and can handle such situation where DMs are preferred to express

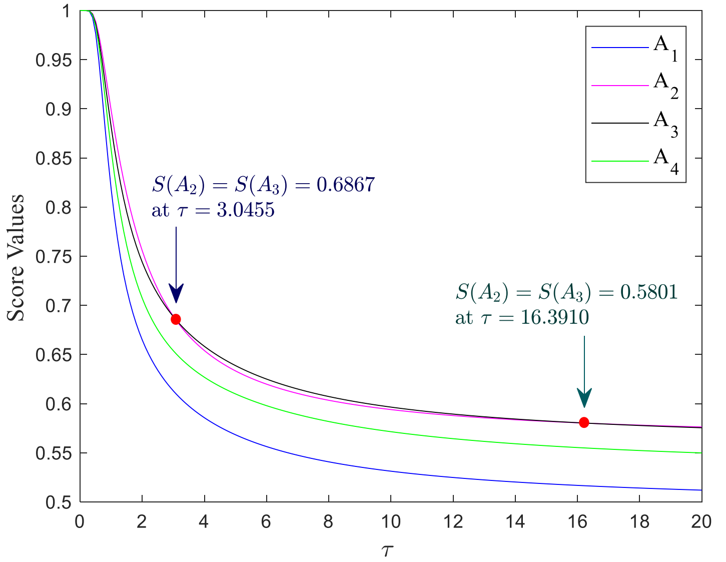

FIGURE 11 Impact of τ on ordering using DHq-ROFWDGBM operator when $q = 3, \theta = \phi = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

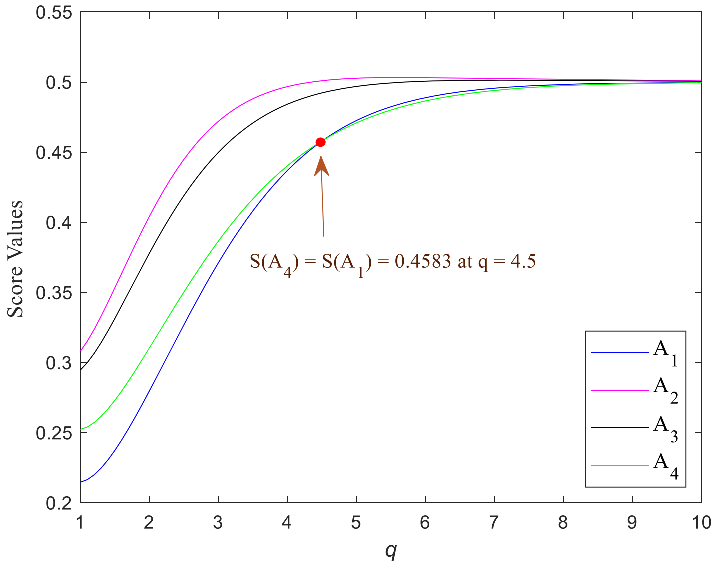

FIGURE 12 Impact of q on ordering using DHq-ROFWDBM operator when $\tau = 2, \theta = \phi = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

membership and nonmembership values using HF elements. Again, the existing method[33] is based on simple algebraic operations, whereas, the proposed methodology is based on DtCN&t-Ns. So, the proposed approach provides more flexibility to the DMs based on decision making situations by varying different parameters associated with it.

6.3 | Illustrative Example 3

Under HPF environment, Garg[56] solved an example relating to find a market for best investment, among five major available markets, A_1, A_2, A_3, A_4, A_5, following four major criteria, G_1, G_2, G_3, G_4. Solving the method using HPFWA operator,[56] the ranking results are appeared as $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$. It is evident that the best alternative is A_5. Now, that problem is solved using the developed DHq-ROFWDBM operator. By varying the Dombi parameter, τ, several ranking results are obtained. It is interesting to note here that among the rankings achieved for some value of τ through the proposed method, coincide with the result of earlier method.[56] Thus it is appeared that the proposed method includes the existing method.[56] Further, using the proposed DHq-ROFWDBM operator, and by varying the Dombi parameter, τ, between 0 and 20, the ordering of alternatives are found as follows:

(i) $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ for $\tau \in (0, 1.5362)$;
(ii) $A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$ for $\tau \in (1.5362, 3.3128)$; and
(iii) $A_4 \succ A_3 \succ A_5 \succ A_2 \succ A_1$ for $\tau \in (3.3128, 20)$.

The change of score values corresponding to the change of Dombi parameter is presented in Figure 14.

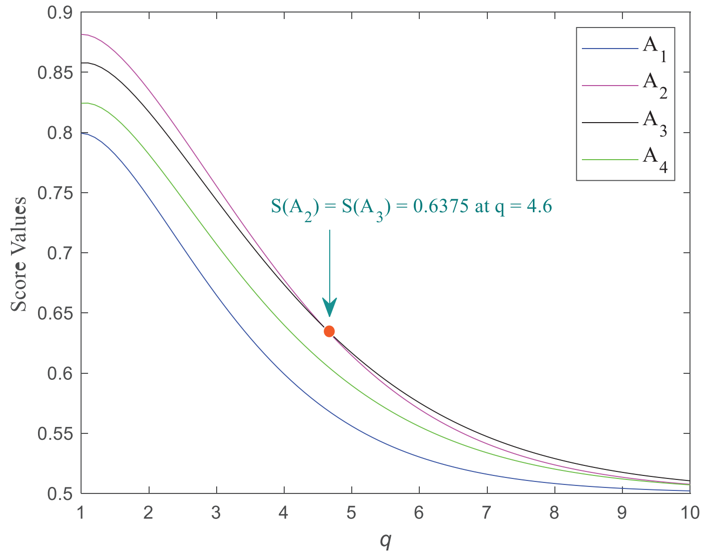

FIGURE 13 Impact of q on ordering using DHq-ROFWDGBM operator when $\tau = 2$, $\theta = \phi = 1$ [Color figure can be viewed at wileyonlinelibrary.com]

It is worth mentioning here that though different ranking results of alternatives are found using the proposed method but the best alternative remains the same.[56] The HPFWA operator[56] is based on algebraic operations under hesitant PF environment, whereas, the proposed approach is based on Dt-CN&t-Ns using DHq-ROF information. Also, the existing method[56] cannot handle the situation that involves interrelationship among arguments. So, it is claimed that the approach[56] is a particular case of the proposed method. Thus, the proposed methodology is more efficient than the technique developed by Garg.[56]

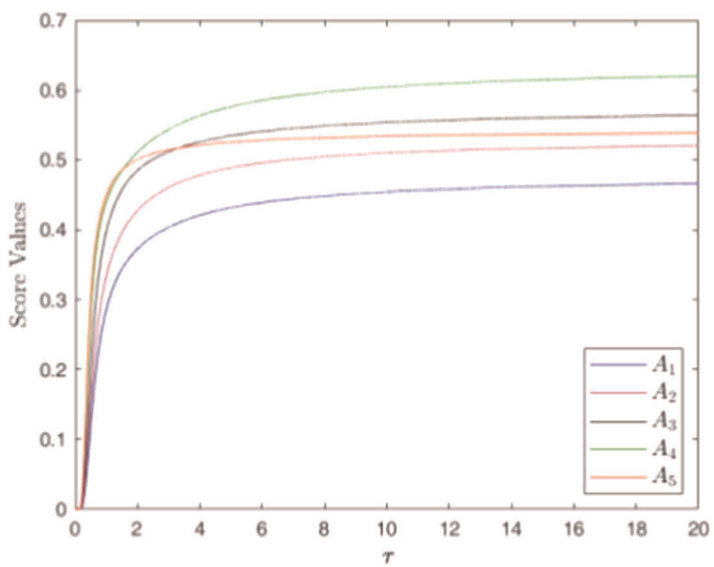

FIGURE 14 Impact of τ in DHq-ROFWDBM for Example 6.2 [Color figure can be viewed at wileyonlinelibrary.com]

6.4 | Illustrative Example 4

Garg and Chen[35] presented an MADM problem under q-ROF environment for assessing four alternatives A_1, A_2, A_3, A_4 based on four criteria $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4$, with weight vector $\omega = (0.15, 0.25, 0.35, 0.25)$. Utilizing q-ROFWNA and q-ROFOWNA AOs, the ranking results are found as $A_2 \succ A_4 \succ A_3 \succ A_1$ and $A_2 \succ A_4 \succ A_1 \succ A_3$, respectively. Now, if the proposed operators are used, same raking results are found as like the results achieved by Garg and Chen.[35] It is worth mentioning that the existing method[35] is developed in q-ROF environment, whereas, the proposed operators are developed under DHq-ROF environment. Thus the q-ROFWNA and q-ROFOWNA AOs[35] fail to capture hesitancy in q-ROFNs. But the proposed operator can capture that hesitancy. Also using DHq-ROFWDBM operator, the ordering of alternatives remains the same as $A_2 \succ A_4 \succ A_1 \succ A_3$ by varying τ. Again, when DHq-ROFWDGBM operator is used, two different ranking results are found by varying τ, namely, $A_2 \succ A_4 \succ A_1 \succ A_3$ for $\tau \in (0, 1.7072)$; and $A_2 \succ A_4 \succ A_3 \succ A_1$ for $\tau \in (1.7072, 20]$. The score values using the developed operator by varying τ are displayed through Figures 15 and 16.

From the above discussions, it is evidenced that the developed AOs possess higher capability, not only to cover the concepts of different existing operators, but also, a large number of AOs can be developed based on those operators.

It is also to be noted here that different ranking results of alternatives are found using the proposed method. The technique developed by Garg and Chen[35] is based on algebraic operations under q-ROF environment, whereas, the proposed approach is based on Dt-CN&t-Ns using DHq-ROF information. So, it is claimed that the approach of Garg and Chen[35] is a particular case of the proposed method. Again, it is worth mentioning that q-ROFWNA and q-ROFOWNA operators,[35] cannot consider the interrelationship between arguments. Thus, the proposed methodology is advantageous than the existing techniques.

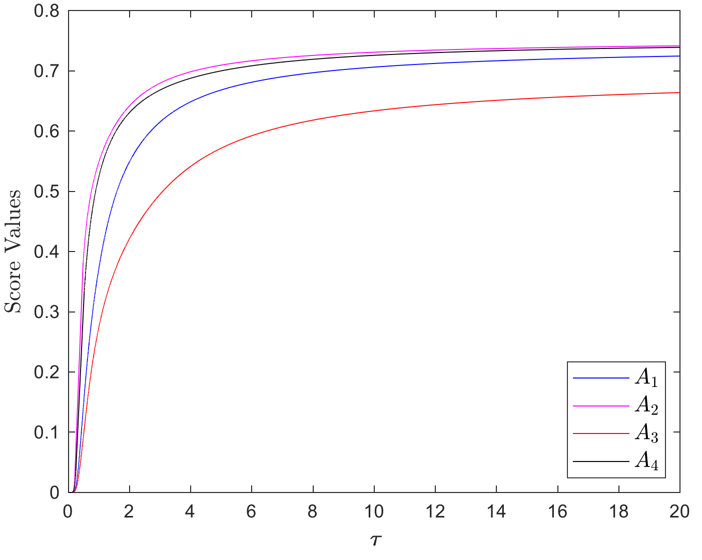

FIGURE 15 Impact of τ in DHq-ROFWDBM for Example 6.3 [Color figure can be viewed at wileyonlinelibrary.com]

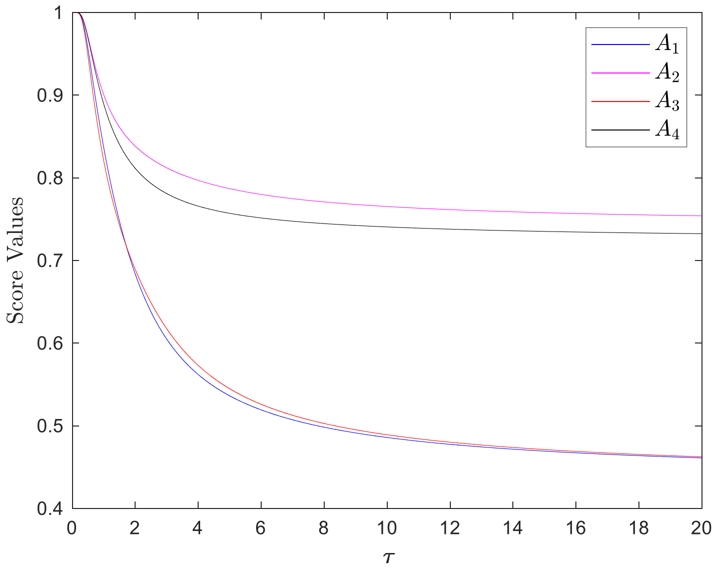

FIGURE 16 Impact of τ in DHq-ROFWDGBM for Example 6.3 [Color figure can be viewed at wileyonlinelibrary.com]

7 | NOVELTY OF THE PROPOSED METHOD

The proposed method meets the following requirements:

- By introducing Bonferroni mean in the aggregation operator, it can resolve the situations where the interrelationship among the decision making attributes are needed to consider. The proposed operator can successfully cope with the characteristics of interaction among the input arguments.
- It provides the decision makers more freedom to put judgment values due to the use of DHq-ROF information. The proposed method can capture the hesitancy of decision making processes in a more significant way than other variants of fuzzy sets.
- It makes the decision making process flexible using Dombi parameter. Decision makers can dynamically adjust the parameter according to their risk managements.
- It also can reflect the pessimistic or optimistic attitude of the decision makers towards the decision choices through selecting the Dombi parameters.

On the basis of the above discussions, it can be concluded that the development of DHq-ROF operators using Dombi Bonferroni mean parameters is a meaningful innovation for solving real life MCGDM problems.

8 | CONCLUSIONS

In this paper, an MCGDM method is developed to solve group decision making problems under DHq-ROF environment. From the perspective of interrelationship design among input arguments, the developed AOs possess the greater capability of making the decision process flexible by considering preferences of the DMs. Further, since, DHPFS, DHFS, PFS, IFS, HFS, and other variants of FSs are appeared as particular cases of DHq-ROFS, the developed operators are efficient enough not only to capture data during information processing phase by adapting rung parameter q, but also, the proposed method can adjust the degree of hesitancy during the investigation phases. Due to the presence of BM parameters, θ and ϕ, the proposed model is also capable of solving the decision making problems having the risk preference of experts. Also, the proposed method becomes more flexible by changing the Dombi parameter τ. Therefore, the introduced MCGDM method is more general and flexible to handle the problems containing correlated attributes. Several illustrative examples, adopted from different articles[33,35,42,56] are solved by the developed MCGDM method. Sensitivity analysis is performed to find the influence of the parameters q, τ, θ, and ϕ on the decision making process. From sensitivity analysis and comparison with other methods, it has already been established that the developed method can capture most of the existing methods by varying different parameters associated with it. It also absorbs the results of multiple number of decision making processes.

In future, the proposed method can be extended under DHq-ROF environment by integrating other averaging operations, namely, MM,[40] partitioned BM,[57] power BM,[58] HM,[54] MSM,[52] Hamy mean,[55] and so on, with DHqROFNs to generate different efficient AOs which may, subsequently, be used for solving MCDM problems. After introducing distance measure on DHq-ROF environment, several types of power AOs based on Frank, Hamacher, Schweizer-Sklar,[59] and many other classes of t-CN&t-Ns, can be developed. Inspired by the concept of complex q-ROFS,[60] DHq-ROFS can also be extended to develope complex DHq-ROFS. The presented method may also be extended to interval valued DHq-ROF[19] contexts. As like theoretical developments, the proposed method may also be applied to solve various real-life MCDM problems. In the future, more intelligent algorithms may be explored for development of a robust controller. However, it is hoped that the proposed method may add a new direction in the context of solving MCDM problems under imprecise decision making environments.

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ORCID

Arun Sarkar http://orcid.org/0000-0002-9263-4400

Animesh Biswas http://orcid.org/0000-0002-5813-1076

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RESEARCH ARTICLE

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Development of Archimedean t-norm and t-conorm-based interval-valued dual hesitant fuzzy aggregation operators with their application in multicriteria decision making

Arun Sarkar[1] | **Animesh Biswas**[2]

[1]Department of Mathematics, Heramba Chandra College, Kolkata, India

[2]Department of Mathematics, University of Kalyani, Kalyani, India

Correspondence
Animesh Biswas, Department of Mathematics, University of Kalyani, Kalyani, West Bengal 741235, India.
Email: abiswaskln@rediffmail.com

Funding information
University of Kalyani

Abstract

In this article, Archimedean t-norm and t-conorm (At-N&t-CN)-based aggregation operators are developed for aggregating the interval-valued dual hesitant fuzzy (IVDHF) elements (IVDHFEs), which can cover a wide variety of existing aggregation operators. After introducing the concept of IVDHF set, several related terms, viz., score function, accuracy function, and degree of hesitancy are defined. Using At-N&t-CN, different operations for IVDHFEs are presented. Conversion processes from the developed operators to other forms of operators in several variants of fuzzy environments are discussed. An approach to solve multicriteria decision making problem in IVDHF context is presented using the developed concepts. To demonstrate proficiency of the developed method, an illustrative example is presented. Furthermore, several numerical examples, studied previously, are also solved, and achieved solutions are compared with the existing ones to establish the robustness of the proposed operator.

KEYWORDS

Archimedean t-norm and t-conorm, hesitant fuzzy set, interval-valued dual hesitant fuzzy set, multicriteria decision making, weighted averaging and weighted geometric operators

1 | INTRODUCTION

Under real-world complex decision making situations, decision makers (DMs) often feel free to assign a set of possible evaluation values instead of putting exact decision values during the process of evaluating imprecise data. To conceptualize such concept, Torra and Narukawa[1] and Torra[2] introduced hesitant fuzzy (HF) set (HFS) as an extension of fuzzy set.[3] That empowers the membership degree of an element to consider a finite set of different possible values in [0, 1]. Afterward, generalizing the idea of HFSs, Chen et al[4] proposed interval-valued HF (IVHF) set (IVHFS) that assigns a set of possible subintervals of [0, 1] as membership degree.

In the process of decision making, the DMs sometimes prefer to assign membership value together with nonmembership value of an element to represent a decision situation. In such context, the concept of IFS[5,6] has been developed. Combining IFS with HFS, the concept of dual HF (DHF) sets (DHFSs)[7] comes into account. Thus DHFSs consist of a set of possible membership degrees and also a set of possible nonmembership degrees in [0, 1]. In recent time, DHFSs have received much attention to the researchers[8,9] in modelling various multicriteria decision making (MCDM) problems within a very short span after its initiation.

Extending the concept of DHFS, Ju et al[10] introduced interval-valued DHFS (IVDHFS) whose membership and non-membership values are represented through two possible sets of subintervals of [0, 1]. Therefore, using this perception, IVDHFS becomes more flexible to express the DM's hesitancy than IFS, HFS, IVHFS, and their extensions. After introduction of IVDHFS, it is interestingly noticed that the sets, viz., fuzzy set, IFS, IVIFS,[11] HFS, IVHFS, and DHFS can be treated as special cases of IVDHFS.

Research works on aggregation operators in imprecise decision making environments have now become an emerging area of research. On various imprecise circumstances, a large number of aggregation operators are developed using weighted averaging (WA) and weighted geometric (WG) operators. Using algebraic operations on HFSs, Xia and Xu[12] presented HF WA (HFWA), HF WG (HFWG) aggregation operators and applied them to solve MCDM problems. On the IVHF context, Chen et al[4] introduced IVHF WA (IVHFWA) and IVHF WG (IVHFWG) operators. Considering confidence level of the DMs, Zeng et al[13] very recently, presented a family of weighted aggregation operators, viz., weighted IVHF WA and the generalized weighted IVHF WA and corresponding to those WG operators. Wang et al[14] proposed DHF WA (DHFWA) and WG (DHFWG) aggregation operators. Using Einstein operations, Biswas and Sarkar[15] introduced DHF prioritized WA and WG operators in multicriteria group decision making context.

To aggregate the IVDHF elements (IVDHFEs), Ju et al[10] developed IVDHF weighted aggregation operator. Furthermore, Zhang et al[16] imposed Einstein WA and WG operators on IVDHF environment to develop IVDHF Einstein WA (IVDHFEWA) and IVDHF Einstein WG (IVDHFEWG) operators. Wei et al[17] presented IVDHF linguistic WG, IVDHF linguistic ordered WG, and IVDHF linguistic hybrid geometric operators for IVDHF linguistic information. Using the concept of IVDHFSs and linguistic term sets, Liu and Tang[18] proposed IVDHF uncertain linguistic sets. Wei[19] proposed some aggregation operators based on IVDHF uncertain linguistic information. Zang et al[20] defined distance measure for IVDHF elements and developed IVDHF Heronian mean (HM) and geometric HM aggregation operators[21] in the recent past. Recently, Sarkar and Biswas[22] applied IVDHF Bonferroni mean operator to solve MCDM problems.

In studying the above aggregation operators, it is the fact that those aggregation operators are mostly based on different kinds of t-norms and t-conorms (t-Ns&t-CNs) which are derived from Archimedean t-Ns&t-CNs (At-Ns&t-CNs).[23,24] In general, At-Ns&t-CNs produce different forms of t-Ns&t-CNs, which allow the information fusion process more flexible and robust and provide more freedom in decision making process to the DMs. It is well known[23] that Archimedean t-conorms are characterized by an increasing generator, g, and Archimedean t-norms characterized by a decreasing generator, f, maintaining the relationship $g(t) = f(1 - t)$. For different forms of generating function, At-N&t-CNs can produce many operators viz., Algebraic, Einstein, Hamacher, Frank, Dombi classes of operators, and others. Based on At-N&t-CN, Xia et al[25] constructed some operations on IFSs and developed At-N&t-CN-based intuitionistic fuzzy WA (AIFWA) and WG (AIFWG) aggregation operators. In DHF environment, Yu[26] proposed At-N&t-CN DHF WA (ADHFWA) and WG (ADHFWG) operators. Zhang et al[27] introduced some general aggregation operators based on At-N&t-CN for DHF linguistic (DHFL) elements (DHFLEs) viz., At-N&t-CN-based DHFL WA and WG, At-N&t-CN-based generalized DHFL WA and WG operators. Furthermore, Zhang et al[28] proposed power-geometric operators to aggregate DHFLEs based on At-N&t-CN. However, the previously proposed At-N&t-CN-based aggregation operators are not implemented to such situations in which the input arguments take the form of IVDHFEs. Therefore, the main contribution of this article is to develop several At-N&t-CN aggregation operators based on IVDHF information.

Thus the main objective of this article is to introduce At-N&t-CNs-based operations on IVDHFSs to aggregate IVDHFEs. Archimedean IVDHF WA (AIVDHFWA) and WG (AIVDHFWG) operators are proposed and their properties are discussed. Relationships between the newly introduced operators and the existing ones have been established. Based on the controlling parameter, the DMs now can adjust their preferences more accurately to express their judgement values. It is interesting to note that almost all the above-defined existing operators in different imprecise circumstances can be derived from the proposed operators. Conversion processes from the proposed aggregation operators in the IVDHF environment to various aggregation operators in other variants of fuzzy environments are also discussed. Finally, a methodology for solving MCDM problems using IVDHF information has been presented. Utilizing this proposed methodology, several existing problems on MCDM have been solved. It is to be noted here that all the solutions of these existing problems can be achieved for some particular value of the parameter in the proposed method.

The rest of this article is organized in such a manner that in Section 2 preliminary concepts of different existing variants of fuzzy sets with their respective score function, accuracy function, different operational laws on them have been presented. Different classes of t-Ns&t-CNs that can be derived from At-N&t-CNs are also presented in this section.

Section 3 contains At-N&t-CN-based aggregation operators for IVDHFEs using newly defined operations and their properties are studied. How different operators can be derived from the developed At-N&t-CN-based IVDHF aggregation operators is presented in Section 4. Section 5 discusses a methodology for solving MCDM problems in IVDHF environment. An illustrative example is solved using the proposed method and sensitivity analysis is performed by varying the parameter in Section 6. In Section 7, a comparative analysis of the proposed method with the existing methods by solving several examples, considered previously, in various variants of fuzzy environments is presented. Finally, conclusions and scope for future studies have been described.

2 | PRELIMINARIES

Some preliminary concepts, which are essential to develop the proposed methodology, are presented in this section.

Definition 1. [1, 2] Let X be any set. An HFS defined on X can be represented in the form of a function that maps each element of X with a subset consisting of a finite number of elements in [0, 1]. Symbolically, it is denoted as

$$E = \{\langle x, h_E(x)\rangle \mid x \in X\}, \tag{1}$$

where $h_E(x)$ is a collection of possible finite values lying within [0, 1], signifying membership degrees for $x \in X$ to the set E. Xia and Xu[12] simply denoted $h = h_E(x)$ as HF element (HFE).

The concept of IVHFS is presented by Chen et al[4] as follows.

Definition 2. [4] Let $I([0, 1])$ denote the set of all closed subintervals of [0, 1], that is,

$$I([0,1]) = \{[\gamma^l, \gamma^u] | \gamma^l \leq \gamma^u; \gamma^l, \gamma^u \in [0,1]\}. \tag{2}$$

The IVHFS is expressed by the symbol:

$$\widetilde{A} = \{\langle x, \widetilde{h}_{\widetilde{A}}(x)\rangle | x \in X\}. \tag{3}$$

where $\widetilde{h}_{\widetilde{A}}(x)$ denotes membership grade of element $x \in X$ to $\widetilde{A}$ as a set of possible intervals belongs to $I([0, 1])$. For simplicity, Chen et al[4] renamed $\widetilde{h} = \widetilde{h}_{\widetilde{A}}(x)$ as an IVHF element (IVHFE). The IVHFEs reduce to the HFEs, if $\gamma^l = \gamma^u$ is considered.

Definition 3. [7] Let X be a universe of discourse. Then a DHFS, D, on X is expressed as

$$D = \{\langle x, h(x), g(x)\rangle | x \in X\}, \tag{4}$$

where $h(x)$ and $g(x)$ denote the sets of possible degrees of membership and nonmembership of $x \in X$ in D, respectively, where.

$0 \leq \gamma, \eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1$, where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \max_{\gamma \in h(x)} \{\gamma\}$ and $\eta^+ = \max_{\eta \in h(x)} \{\eta\}$ for all $x \in X$.

DHFE is introduced by Zhu et al[7] and is expressed as $d = (h, g)$, where h and g are two HFEs. The score and accuracy functions of DHFE d are presented as

$$S(d) = \frac{1}{|h|}\sum_{\gamma \in h}\gamma - \frac{1}{|g|}\sum_{\eta \in g}\eta \text{ and } A(d) = \frac{1}{|h|}\sum_{\gamma \in h}\gamma + \frac{1}{|g|}\sum_{\eta \in g}\eta, \tag{5}$$

where $|h|$ and $|g|$ denote the scalar cardinality of γ and η, respectively.

For any two DHFEs d_1 and d_2, if $S(d_1) > S(d_2)$ then d_1 is larger than d_2.

Definition 4. [10] An IVDHFS $\widetilde{K}$ on a given set X is described as

$$\widetilde{K} = \{|x, \widetilde{h}(x), \widetilde{g}(x)| | x \in X\}, \tag{6}$$

in which $\widetilde{h}(x) = \bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}(x)} \{[\gamma^l,\gamma^u]\}$ and $\widetilde{g}(x) = \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}(x)} \{[\eta^l,\eta^u]\}$ denote two individual sets of subintervals in [0, 1], representing the respective possible degrees of membership and nonmembership of $x \in X$ to $\widetilde{K}$, with $[\gamma^l,\gamma^u], [\eta^l,\eta^u] \subset [0,1]$ and $0 \leq (\gamma^u)^+ + (\eta^u)^+ \leq 1$, where $(\gamma^u)^+ = \max\{\gamma^u\}$, and $(\eta^u)^+ = \max\{\eta^u\}$ for all $x \in X$. For computational simplicity Ju et al[10] named $\widetilde{k}(x) = (\widetilde{h}(x), \widetilde{g}(x))$ as an IVDHFE and used the notation as $\widetilde{k} = (\widetilde{h}, \widetilde{g})$.

To compare the IVDHFEs, Ju et al[10] presented score function and accuracy function in the following manner:

Definition 5. [10] Let $\widetilde{k} = (\widetilde{h}, \widetilde{g})$ be an IVDHFE. The score function of $\widetilde{k}$ is expressed as

$$S(\widetilde{k}) = \frac{1}{2}\left(\frac{1}{|\widetilde{h}|}\sum_{[\gamma^l,\gamma^u]\in\widetilde{h}}(\gamma^l+\gamma^u) - \frac{1}{|\widetilde{g}|}\sum_{[\eta^l,\eta^u]\in\widetilde{g}}(\eta^l+\eta^u)\right), \tag{7}$$

and accuracy function of IVDHFE $\widetilde{k} = (\widetilde{h}, \widetilde{g})$ is presented as

$$A(\widetilde{k}) = \frac{1}{2}\left(\frac{1}{|\widetilde{h}|}\sum_{[\gamma^l,\gamma^u]\in\widetilde{h}}(\gamma^l+\gamma^u) + \frac{1}{|\widetilde{g}|}\sum_{[\eta^l,\eta^u]\in\widetilde{g}}(\eta^l+\eta^u)\right), \tag{8}$$

where $|\widetilde{h}|$ and $|\widetilde{g}|$ is the number of intervals in $\widetilde{h}$ and $\widetilde{g}$, respectively.

Definition 6. [10] Let $\widetilde{k}_1$ and $\widetilde{k}_2$ be any two IVDHFEs,

- If $S(\widetilde{k}_1) > S(\widetilde{k}_2)$ then $\widetilde{k}_1 \succ \widetilde{k}_2$
- If $S(\widetilde{k}_1) = S(\widetilde{k}_2)$ then

if $A(\widetilde{k}_1) > A(\widetilde{k}_1)$ then $\widetilde{k}_1 \succ \widetilde{k}_2$; if $A(\widetilde{k}_1) = A(\widetilde{k}_2)$ then $\widetilde{k}_1 \approx \widetilde{k}_2$.

Definition 7. [10] If $\widetilde{k}_i = (\widetilde{h}_i, \widetilde{g}_i)$ $(i = 1, 2)$ and $\widetilde{k} = (\widetilde{h}, \widetilde{g})$ be three arbitrary IVDHFEs with $\lambda > 0$, then the following operational rules are defined as

1. $\widetilde{k}_1 \oplus \widetilde{k}_2 = \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i \\ i=1,2}}\{[\gamma_1^l+\gamma_2^l-\gamma_1^l\gamma_2^l, \gamma_1^u+\gamma_2^u-\gamma_1^u\gamma_2^u]\},\ \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i \\ i=1,2}}\{[\eta_1^l\eta_2^l, \eta_1^u\eta_2^u]\}\right)$,
2. $\widetilde{k}_1 \otimes \widetilde{k}_2 = \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i \\ i=1,2}}\{[\gamma_1^l\gamma_2^l, \gamma_1^u\gamma_2^u]\},\ \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i \\ i=1,2}}\{[\eta_1^l+\eta_2^l-\eta_1^l\eta_2^l, \eta_1^u+\eta_2^u-\eta_1^u\eta_2^u]\}\right)$,
3. $\lambda\widetilde{k} = (\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}}\{[1-(1-\gamma^l)^\lambda, 1-(1-\gamma^u)^\lambda]\}, \bigcup_{[\eta^l,\eta^l]\in\widetilde{g}}\{[(\eta^l)^\lambda, (\eta^u)^\lambda]\})$,
4. $\widetilde{k}^\lambda = (\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}}\{[(\gamma^l)^\lambda, (\gamma^u)^\lambda]\}, \bigcup_{[\eta^l,\eta^l]\in\widetilde{g}}\{[1-(1-\eta^l)^\lambda, 1-(1-\eta^u)^\lambda]\})$.

Definition 8. [23, 24] Archimedean t-norm is defined using a decreasing generator f such that

$$I(a,b) = f^{(-1)}(f(a)+f(b)), \text{for all } a, b \in [0,1]. \tag{9}$$

Similarly, Archimedean t-conorm is defined using an increasing generator g such that

$$U(a,b) = g^{(-1)}(g(a)+g(b)) \text{ with } g(t) = f(1-t) \text{ for all } a, b, t \in [0,1]. \tag{10}$$

Now, using the decreasing as well as increasing generators, different classes of t-Ns&t-CNs can be derived.

Let $f(t) = \log\left(\frac{\theta+(1-\theta)t}{t}\right)$, $\theta > 0$, $g(t) = \log\left(\frac{\theta+(1-\theta)(1-t)}{1-t}\right)$, $f^{-1}(t) = \frac{\theta}{e^t+\theta-1}$, $g^{-1}(t) = 1 - \frac{\theta}{e^t+\theta-1}$, then the At-N&t-CN is called Hamacher t-norm and t-conorm and are presented as $U_\theta^H(a,b) = \frac{a+b-ab-(1-\theta)ab}{1-(1-\theta)ab}$, $I_\theta^H(a,b) = \frac{ab}{\theta+(1-\theta)(a+b-ab)}$, $\theta > 0$. It is worthy to mention here that different types of operations can be obtained as some particular cases of Hamacher operators.

For example, if $\theta = 1$ is considered, then Hamacher operations represent algebraic operations. Also, if $\theta = 2$ is assumed, then Hamacher operations further reduce to Einstein operations. Several other forms can also be derived by varying the parameter θ.

Similarly, if the decreasing generator $f(t) = \log\left(\frac{\theta-1}{\theta^t-1}\right)$, $\theta > 1$ is considered then Frank class of t-Ns&t-CNs can be found.

A classification of t-Ns&t-CNs for particular decreasing generator f, is presented in Table 1 as follows:

Beliakov et al[29] proposed some arithmetic operations on IFNs based on At-N&t-CN which are presented below:

Definition 9. [29] Let $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ $(j = 1, 2)$ and $\alpha = (\mu_\alpha, \nu_\alpha)$ be three IFNs then

1. $\alpha_1 \oplus \alpha_2 = (U(\mu_{\alpha_1}, \mu_{\alpha_2}), I(\nu_{\alpha_1}, \nu_{\alpha_2})) = (g^{-1}(g(\mu_{\alpha_1}) + g(\mu_{\alpha_2})), f^{-1}(f(\nu_{\alpha_1}) + f(\nu_{\alpha_2})))$,
2. $\alpha_1 \otimes \alpha_2 = (I(\mu_{\alpha_1}, \mu_{\alpha_2}), U(\nu_{\alpha_1}, \nu_{\alpha_2})) = (f^{-1}(f(\mu_{\alpha_1}) + f(\mu_{\alpha_2})), g^{-1}(g(\nu_{\alpha_1}) + g(\nu_{\alpha_2})))$,
3. $\lambda\alpha = (g^{-1}(\lambda g(\mu_\alpha)), f^{-1}(\lambda f(\nu_\alpha)))$, $\lambda > 0$.
4. $\alpha^\lambda = (f^{-1}(\lambda f(\mu_\alpha)), g^{-1}(\lambda g(\nu_\alpha)))$, $\lambda > 0$.

Zhang and Wu[30] defined some At-N&t-CN-based operations for IVHFEs.

Definition 10. [30] Let $\widetilde{h}_i = (\{[\xi_i^l, \xi_i^u]\})$ $(i = 1, 2)$ and $\widetilde{h} = (\{[\xi^l, \xi^u]\})$ are three IVHFEs then

1. $\widetilde{h}_1 \oplus \widetilde{h}_2 = (\bigcup_{[\xi_i^l,\xi_i^u]\in\widetilde{h}_i, i=1,2}\{[g^{-1}(g(\xi_1^l) + g(\xi_2^l)), g^{-1}(g(\xi_1^u) + g(\xi_2^u))]\})$,
2. $\widetilde{h}_1 \otimes \widetilde{h}_2 = (\bigcup_{[\xi_i^l,\xi_i^u]\in\widetilde{h}_i, i=1,2}\{[f^{-1}(f(\xi_1^l) + f(\xi_2^l)), f^{-1}(f(\xi_1^u) + f(\xi_2^u))]\})$,
3. $\lambda\widetilde{h} = (\cup_{[\xi^l,\xi^u]\in\widetilde{h}}\{[g^{-1}(\lambda g(\xi^l)), g^{-1}(\lambda g(\xi^u))]\})$, $\lambda > 0$,
4. $\widetilde{h}^\lambda = (\cup_{([\xi^l\xi^u]\in\widetilde{h}}\{[f^{(-1)}(\lambda f(\xi^l)), f^{(-1)}(\lambda f(\xi^u))]\})$, $\lambda > 0$.

Yu[26] applied At-N&t-CN in DHF context and proposed the Archimedean operations there.

Definition 11. [26] Let $\widetilde{d}_i = (\widetilde{h}_i, \widetilde{g}_i)$ $(i = 1, 2)$ and $\widetilde{d} = (\widetilde{h}, \widetilde{g})$ are three DHFEs, then some operations based on At-N&t-CN are defined as follows.

1. $\widetilde{d}_1 \oplus \widetilde{d}_2 = (\bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2}\{g^{-1}(g(\gamma_1) + g(\gamma_2))\}, \bigcup_{\eta_1\in\widetilde{g}_1,\eta_2\in\widetilde{g}_2}\{f^{-1}(f(\eta_1) + f(\eta_2))\})$,
2. $\widetilde{d}_1 \otimes \widetilde{d}_2 = (\bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2}\{f^{-1}(f(\gamma_1) + f(\gamma_2))\}, \bigcup_{\eta_1\in\widetilde{g}_1,\eta_2\in\widetilde{g}_2}\{g^{-1}(g(\eta_1) + g(\eta_2))\})$,
3. $\lambda\widetilde{d} = (\cup_{\gamma\in\widetilde{h}}\{g^{-1}(\lambda g(\gamma))\}, \cup_{\eta\in\widetilde{g}}\{f^{-1}(\lambda f(\eta))\})$, $\lambda > 0$,
4. $\widetilde{d}^\lambda = (\cup_{\gamma\in\widetilde{h}}\{f^{-1}(\lambda f(\gamma))\}, \cup_{\eta\in\widetilde{g}}\{g^{-1}(\lambda g(\eta))\})$, $\lambda > 0$.

2.1 | Existing aggregation operators

As it is the fact that IVDHFS can cover a wide variety of fuzzy sets. So, if At-N&t-CN operators are introduced on IVDHFSs, it could cover existing aggregation operators, mostly. Focusing on this idea, a chart of various aggregation operators (viz., Algebraic, Einstein, Hamacher, Frank, etc.) and environments (viz., DHF, IVHF, HF, IVIF, IF, etc.) is shown in Figure 1.

TABLE 1 t-Conorms and t-norms generating from decreasing generator f

Classes of t-conorms and t-norms	Decreasing generator f	Archimedean t-conorm $U(a, b)$	Archimedean t-norm $I(a, b)$
Algebraic	$-\log t$	$a + b - ab$	ab
Einstein	$-\log\left(\frac{2-t}{t}\right)$	$\frac{a+b}{1+ab}$	$\frac{ab}{1+(1-a)(1-b)}$
Hamacher	$\log\left(\frac{\theta+(1-\theta)t}{t}\right)$, $\theta > 0$	$\frac{a+b-ab-(1-\theta)ab}{1-(1-\theta)ab}$	$\frac{ab}{\theta+(1-\theta)(a+b-ab)}$
Frank	$\log\left(\frac{\theta-1}{\theta^t-1}\right)$, $\theta > 1$	$1 - \log_\theta\left(1 + \frac{(\theta^{1-a}-1)(\theta^{1-b}-1)}{\theta-1}\right)$	$\log_\theta\left(1 + \frac{(\theta^a-1)(\theta^b-1)}{\theta-1}\right)$

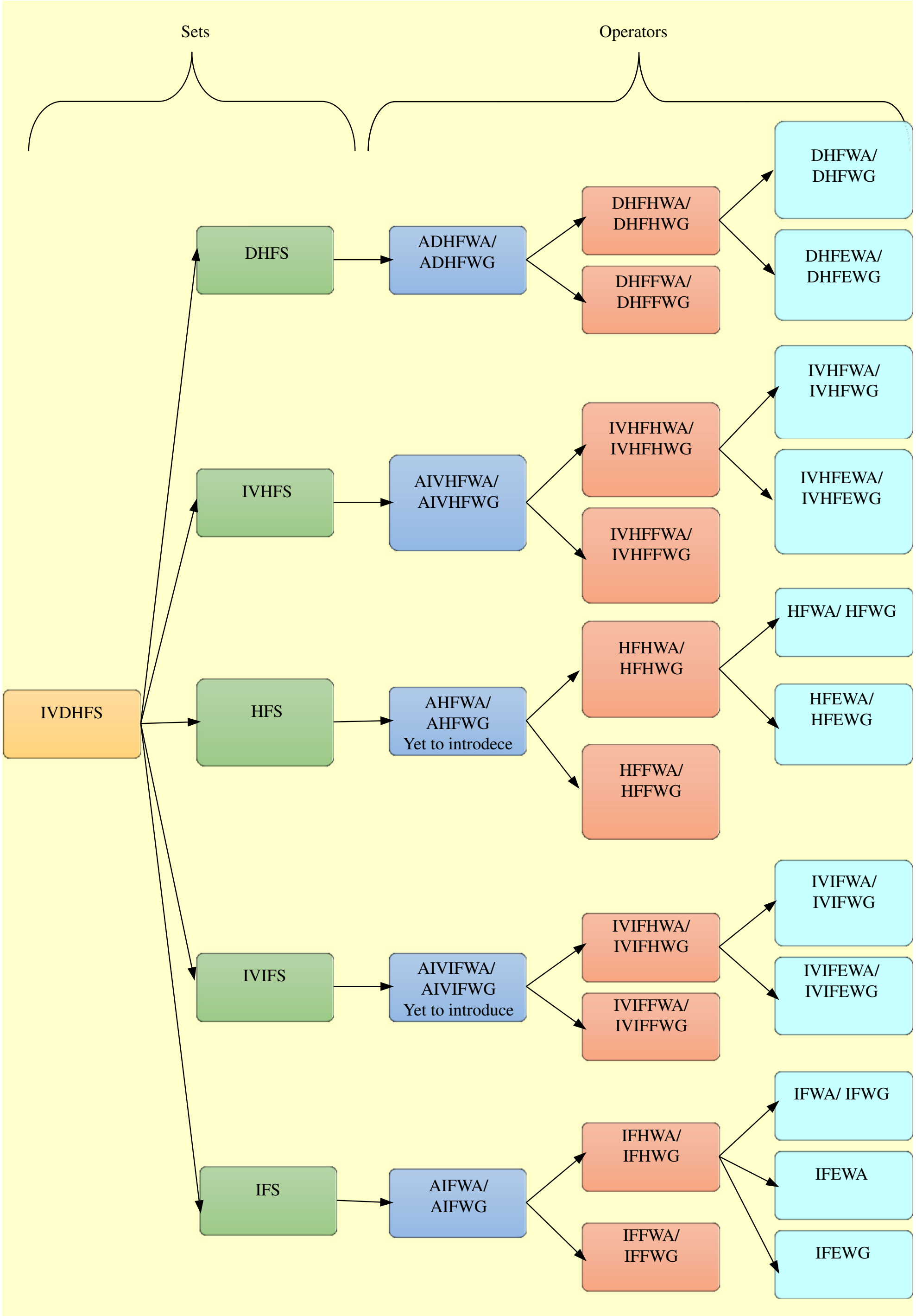


FIGURE 1 Classification of aggregation operators in various fuzzy environments

3 | DEVELOPMENT OF AT-N&T-CN-BASED AGGREGATION OPERATORS FOR IVDHFES

Based on the defined operations in Section 2.1, two aggregation operators, viz., AIVDHFWA and AIVDHFWG operators are presented. To develop those operators, At-N&t-CN-based operations on IVDHFEs are presented in the next subsection.

3.1 | At-N&t-CN-based operations on IVDHFEs

This section introduces At-N&t-CN-based operations for IVDHFEs.

Definition 12. Let $\widetilde{k}_i = (\widetilde{h}_i, \widetilde{g}_i)$ $(i = 1, 2)$ and $\widetilde{k} = (\widetilde{h}, \widetilde{g})$ be any three IVDHFEs. Now the following operations for the IVDHFEs based on At-N&t-CN are presented below.

- **Sum**. $$\widetilde{k}_1 \oplus \widetilde{k}_2 = \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i \\ i=1,2}} \{[U(\gamma_1^l, \gamma_2^l), U(\gamma_1^u, \gamma_2^u)]\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i \\ i=1,2}} \{[I(\eta_1^l, \eta_2^l), I(, \eta_1^u, \eta_2^u)]\} \right)$$ $$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i \\ i=1,2}} \{[g^{-1}(g(\gamma_1^l) + g(\gamma_2^l)), g^{-1}(g(\gamma_1^u) + g(\gamma_2^u))]\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i \\ i=1,2}} \{[f^{-1}(f(\eta_1^l) + f(\eta_2^l)), f^{-1}(f(\eta_1^u) + f(\eta_2^u))]\} \right)$$
- **Product**. $$\widetilde{k}_1 \otimes \widetilde{k}_2 = \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i \\ i=1,2}} \{[I(\gamma_1^l, \gamma_2^l), I(\gamma_1^u, \gamma_2^u)]\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i \\ i=1,2}} \{[U(\eta_1^l, \eta_2^l), U(, \eta_1^u, \eta_2^u)]\} \right)$$ $$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i \\ i=1,2}} \{[f^{-1}(f(\gamma_1^l) + f(\gamma_2^l)), f^{-1}(f(\gamma_1^u) + f(\gamma_2^u))]\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i \\ i=1,2}} \{[g^{-1}(g(\eta_1^l) + g(\eta_2^l)), g^{-1}(g(\eta_1^u) + g(\eta_2^u))]\} \right)$$
- **Scalar Multiplication**. $\lambda\widetilde{k} = \left(\bigcup_{[\gamma^l, \gamma^u] \in \widetilde{h}} \{[g^{-1}(\lambda g(\gamma^l)), g^{-1}(\lambda g(\gamma^u))]\}, \bigcup_{[\eta^l, \eta^u] \in \widetilde{g}} \{[f^{-1}(\lambda f(\eta^l)), f^{-1}(\lambda f(\eta^u))]\} \right), \quad \lambda > 0$
- **Exponentiation**. $\widetilde{k}^{\lambda} = \left(\bigcup_{[\gamma^l, \gamma^u] \in \widetilde{h}} \{[f^{-1}(\lambda f(\gamma^l)), f^{-1}(\lambda f(\gamma^u))]\}, \bigcup_{[\eta^l, \eta^u] \in \widetilde{g}} \{[g^{-1}(\lambda g(\eta^l)), g^{-1}(\lambda g(\eta^u))]\} \right), \quad \lambda > 0$

It is worthy to mention here that based on different forms of the decreasing generator, f, different types of operations can be derived as follows.

Case 1. **(Algebraic)** For $f(t) = -\log t$

1. $$\widetilde{k}_1 \oplus \widetilde{k}_2 = \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i \\ i=1,2}} \{[\gamma_1^l + \gamma_2^l - \gamma_1^l\gamma_2^l, \gamma_1^u + \gamma_1^u - \gamma_1^u\gamma_2^u]\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i \\ i=1,2}} \{[\eta_1^l\eta_2^l, \eta_1^u\eta_2^u]\} \right)$$
2. $$\widetilde{k}_1 \otimes \widetilde{k}_2 = \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i \\ i=1,2}} \{[\gamma_1^l\gamma_2^l, \gamma_1^u\gamma_2^u]\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i \\ i=1,2}} \{[\eta_1^l + \eta_2^l - \eta_1^l\eta_2^l, \eta_1^u + \eta_2^u - \eta_1^u\eta_2^u]\} \right)$$

3. $\lambda\widetilde{k} = \left(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \{[1-(1-\gamma^l)^\lambda, 1-(1-\gamma^u)^\lambda]\}, \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \{[(\eta^l)^\lambda, (\eta^u)^\lambda]\}\right),\ \lambda>0$

4. $\widetilde{k}^\lambda = \left(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \{[(\gamma^l)^\lambda, (\gamma^u)^\lambda]\}, \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \{[1-(1-\eta^l)^\lambda, 1-(1-\eta^u)^\lambda]\}\right),\ \lambda>0$

Case 2. **(Einstein Class)** For $f(t) = \log\left(\frac{2-t}{t}\right)$, $\lambda > 0$

1. $$\widetilde{k}_1 \oplus \widetilde{k}_2 = \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i\\ i=1,2}} \left\{\left[\frac{\gamma_1^l+\gamma_2^l}{1+\gamma_1^l\gamma_2^l}, \frac{\gamma_1^u+\gamma_2^u}{1+\gamma_1^u\gamma_2^u}\right]\right\}, \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i\\ i=1,2}} \left\{\left[\frac{\eta_1^l\eta_2^l}{1+(1-\eta_1^l)(1-\eta_2^l)}, \frac{\eta_1^u\eta_2^u}{1+(1-\eta_1^u)(1-\eta_2^u)}\right]\right\}\right)$$

2. $$\widetilde{k}_1 \otimes \widetilde{k}_2 = \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i\\ i=1,2}} \left\{\left[\frac{\gamma_1^l\gamma_2^l}{1+(1-\gamma_1^l)(1-\gamma_2^l)}, \frac{\gamma_1^u\eta_2^u}{1+(1-\gamma_1^u)(1-\gamma_2^u)}\right]\right\}, \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i\\ i=1,2}} \left\{\left[\frac{\eta_1^l+\eta_2^l}{1+\eta_1^l\eta_2^l}, \frac{\eta_1^u+\eta_2^u}{1+\eta_1^u\eta_2^u}\right]\right\}\right)$$

3. $$\lambda\widetilde{k} = \left(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left\{\left[\frac{(1+\gamma^l)^\lambda-(1-\gamma^l)^\lambda}{(1+\gamma^l)^\lambda+(1-\gamma^l)^\lambda}, \frac{(1+\gamma^u)^\lambda-(1-\gamma^u)^\lambda}{(1+\gamma^u)^\lambda+(1-\gamma^u)^\lambda}\right]\right\}, \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \left\{\left[\frac{2(\eta^l)^\lambda}{(2-\eta^l)^\lambda+(\eta^l)^\lambda}, \frac{2(\eta^u)^\lambda}{(2-\eta^u)^\lambda+(\eta^u)^\lambda}\right]\right\}\right)$$

4. $$\widetilde{k}^\lambda = \left(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left\{\left[\frac{2(\gamma^l)^\lambda}{(2-\gamma^l)^\lambda+(\eta^l)^\lambda}, \frac{2(\gamma^u)^\lambda}{(2-\gamma^u)^\lambda+(\gamma^u)^\lambda}\right]\right\}, \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \left\{\left[\frac{(1+\eta^l)^\lambda-(1-\eta^l)^\lambda}{(1+\eta^l)^\lambda+(1-\eta^l)^\lambda}, \frac{(1+\eta^u)^\lambda-(1-\eta^u)^\lambda}{(1+\eta^u)^\lambda+(1-\eta^u)^\lambda}\right]\right\}\right)$$

Case 3. **(Hamacher Class)** For $f(t) = \log\left(\frac{\theta+(1-\theta)t}{t}\right)$, $\theta > 0$

1. $$\widetilde{k}_1 \oplus \widetilde{k}_2 = \left(\bigcup_{[\gamma_1^l,\gamma_1^u]\in\widetilde{h}_1, [\gamma_2^l,\gamma_2^u]\in\widetilde{h}_2} \left\{\left[\frac{\gamma_1^l+\gamma_2^l-\gamma_1^l\gamma_2^l-(1-\theta)\gamma_1^l\gamma_2^l}{1-(1-\theta)\gamma_1^l\gamma_2^l}, \frac{\gamma_1^l+\gamma_2^l-\gamma_1^l\gamma_2^l-(1-\theta)\gamma_1^l\gamma_2^l}{1-(1-\theta)\gamma_1^l\gamma_2^l}\right]\right\},\right.$$
$$\left. \bigcup_{[\eta_1^l,\eta_1^u]\in\widetilde{g}_1, [\eta_2^l,\eta_2^u]\in\widetilde{g}_2} \left\{\left[\frac{\eta_1^l\eta_2^l}{\theta+(1-\theta)(\eta_1^l+\eta_2^l-\eta_1^l\eta_2^l)}, \frac{\eta_1^u\eta_2^u}{\theta+(1-\theta)(\eta_1^u+\eta_2^u-\eta_1^u\eta_2^u)}\right]\right\}\right)$$

2. $$\widetilde{k}_1 \otimes \widetilde{k}_2 = \left(\bigcup_{[\gamma_1^l,\gamma_1^u]\in\widetilde{h}_1, [\gamma_2^l,\gamma_2^u]\in\widetilde{h}_2} \left\{\left[\frac{\gamma_1^l\gamma_2^l}{\theta+(1-\theta)(\gamma_1^l+\gamma_2^l-\gamma_1^l\gamma_2^l)}, \frac{\gamma_1^u\gamma_2^u}{\theta+(1-\theta)(\gamma_1^u+\gamma_2^u-\gamma_1^u\gamma_2^u)}\right]\right\},\right.$$
$$\left. \bigcup_{[\eta_1^l,\eta_1^u]\in\widetilde{g}_1, [\eta_2^l,\eta_2^u]\in\widetilde{g}_2} \left\{\left[\frac{\eta_1^l+\eta_2^l-\eta_1^l\eta_2^l-(1-\theta)\eta_1^l\eta_2^l}{1-(1-\theta)\eta_1^l\eta_2^l}, \frac{\eta_1^u+\eta_2^u-\eta_1^u\eta_2^u-(1-\theta)\eta_1^u\eta_2^u}{1-(1-\theta)\eta_1^u\eta_2^u}\right]\right\}\right)$$

3. $$\lambda\widetilde{k} = \left(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left\{\left[\frac{(1+(\theta-1)\gamma^l)^\lambda-(1-\gamma^l)^\lambda}{(1+(\theta-1)\gamma^l)^\lambda+(\theta-1)(1-\gamma^l)^\lambda}, \frac{(1+(\theta-1)\gamma^u)^\lambda-(1-\gamma^u)^\lambda}{(1+(\theta-1)\gamma^u)^\lambda+(\theta-1)(1-\gamma^u)^\lambda}\right]\right\},\right.$$
$$\left. \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \left\{\left[\frac{\theta(\eta^l)^\lambda}{(1+(\theta-1)(1-\eta^l))^\lambda+(\theta-1)(\eta^l)^\lambda}, \frac{\theta(\eta^u)^\lambda}{(1+(\theta-1)(1-\eta^u))^\lambda+(\theta-1)(\eta^u)^\lambda}\right]\right\}\right), \quad \lambda>0$$

4. $$\widetilde{k}^\lambda = \left(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left\{\left[\frac{\theta(\gamma^l)^\lambda}{(1+(\theta-1)(1-\gamma^l))^\lambda+(\theta-1)(\gamma^l)^\lambda}, \frac{\theta(\gamma^u)^\lambda}{(1+(\theta-1)(1-\gamma^u))^\lambda+(\theta-1)(\gamma^u)^\lambda}\right]\right\},\right.$$
$$\left. \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \left\{\left[\frac{(1+(\theta-1)\eta^l)^\lambda-(1-\eta^l)^\lambda}{(1+(\theta-1)\eta^l)^\lambda+(\theta-1)(1-\eta^l)^\lambda}, \frac{(1+(\theta-1)\eta^u)^\lambda-(1-\eta^u)^\lambda}{(1+(\theta-1)\eta^u)^\lambda+(\theta-1)(1-\eta^u)^\lambda}\right]\right\}\right), \quad \lambda>0$$

It is to be noted here that considering $\theta = 1$ and 2, algebraic and Einstein classes can, respectively, be derived from Hamacher classes of operations.

Case 4. **(Frank Class)** For $f(t) = \log\left(\frac{\theta-1}{\theta^t-1}\right)$, $\theta > 1$,

1. $\widetilde{k}_1 \oplus \widetilde{k}_2 = \Big(\bigcup_{[\gamma_1^l,\gamma_1^u]\in\widetilde{h}_1,[\gamma_2^l,\gamma_2^u]\in\widetilde{h}_2} \Big\{ \Big[1-\log_\theta\Big(1+\frac{(\theta^{1-\gamma_1^l}-1)(\theta^{1-\gamma_2^l}-1)}{\theta-1}\Big), 1-\log_\theta\Big(1+\frac{(\theta^{1-\gamma_1^u}-1)(\theta^{1-\gamma_2^u}-1)}{\theta-1}\Big)\Big]\Big\},$
$\bigcup_{[\eta_1^l,\eta_1^u]\in\widetilde{g}_1,[\eta_2^l,\eta_2^u]\in\widetilde{g}_2} \Big\{ \Big[\log_\theta\Big(1+\frac{(\theta^{\eta_1^l}-1)(\theta^{\eta_2^l}-1)}{\theta-1}\Big), \log_\theta\Big(1+\frac{(\theta^{\eta_1^u}-1)(\theta^{\eta_2^u}-1)}{\theta-1}\Big)\Big]\Big\}\Big)$
2. $\widetilde{k}_1 \otimes \widetilde{k}_2 = \Big(\bigcup_{[\gamma_1^l,\gamma_1^u]\in\widetilde{h}_1,[\gamma_2^l,\gamma_2^u]\in\widetilde{h}_2} \Big\{ \Big[\log_\theta\Big(1+\frac{(\theta^{\gamma_1^l}-1)(\theta^{\gamma_2^l}-1)}{\theta-1}\Big), \log_\theta\Big(1+\frac{(\theta^{\gamma_1^u}-1)(\theta^{\gamma_2^u}-1)}{\theta-1}\Big)\Big]\Big\}, \bigcup_{[\eta_1^l,\eta_1^u]\in\widetilde{g}_1,[\eta_2^l,\eta_2^u]\in\widetilde{g}_2} \Big\{ \Big[1-\log_\theta\Big(1+\frac{(\theta^{1-\eta_1^l}-1)(\theta^{1-\eta_2^l}-1)}{\theta-1}\Big), 1-\log_\theta\Big(1+\frac{(\theta^{1-\eta_1^u}-1)(\theta^{1-\eta_2^u}-1)}{\theta-1}\Big)\Big]\Big\}\Big)$
3. $\lambda\widetilde{k} = \Big(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \Big\{ \Big[1-\log_\theta\Big(1+\frac{(\theta^{1-\gamma^l}-1)^\lambda}{(\theta-1)^{\lambda-1}}\Big), 1-\log_\theta\Big(1+\frac{(\theta^{1-\gamma^u}-1)^\lambda}{(\theta-1)^{\lambda-1}}\Big)\Big]\Big\},$
$\bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \Big\{ \Big[\log_\theta\Big(1+\frac{(\theta^{\eta^l}-1)^\lambda}{(\theta-1)^{\lambda-1}}\Big), \log_\theta\Big(1+\frac{(\theta^{\eta^u}-1)^\lambda}{(\theta-1)^{\lambda-1}}\Big)\Big]\Big\}\Big), \lambda>0.$
4. $\widetilde{k}^\lambda = \Big(\bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}} \Big\{ \Big[\log_\theta\Big(1+\frac{(\theta^{\gamma^l}-1)^\lambda}{(\theta-1)^{\lambda-1}}\Big), \log_\theta\Big(1+\frac{(\theta^{\gamma^u}-1)^\lambda}{(\theta-1)^{\lambda-1}}\Big)\Big]\Big\},$
$\bigcup_{[\eta^l,\eta^u]\in\widetilde{g}} \Big\{ \Big[1-\log_\theta\Big(1+\frac{(\theta^{1-\eta^l}-1)^\lambda}{(\theta-1)^{\lambda-1}}\Big), 1-\log_\theta\Big(1+\frac{(\theta^{1-\eta^u}-1)^\lambda}{(\theta-1)^{\lambda-1}}\Big)\Big]\Big\}\Big), \lambda>0.$

Thus it is clear from the above discussions that the proposed At-N&t-CN-based operations in IVDHF environment can cover a wide range of operations on the basis of various forms of generators.

3.2 | At-N&t-CNs-based IVDHFWA aggregation operator

Definition 13. Let $\widetilde{k}_i$ $(i = 1, 2, \dots, n)$ be a collection of IVDHFEs, and let $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. Then, an At-N&t-CNs-based AIVDHFWA operator is defined as a mapping: $\widetilde{K}^n \to \widetilde{K}$, such that AIVDHFWA$(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \oplus_{i=1}^n(\omega_i\widetilde{k}_i)$, where $\oplus$ has already been defined in Definition 12.

Theorem 1. Let $\widetilde{k}_i = (\widetilde{h}_i, \widetilde{g}_i)$ $(i = 1, 2, \dots, n)$ be a collection of IVDHFEs, then the aggregated value using AIVDHFWA operator is also an IVDHFE and is given by

$$\text{AIVDHFWA}(\widetilde{k}_1, \widetilde{k}_2, \widetilde{k}_3 \cdots, \widetilde{k}_\text{n}) = \Big(\cup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i, i=1,2,\dots,n} \Big\{ \Big[g^{-1}\Big(\sum_{i=1}^n \omega_i g(\gamma_i^l)\Big), g^{-1}\Big(\sum_{i=1}^n \omega_i g(\gamma_i^u)\Big)\Big]\Big\}, \cup_{([\eta_i^l,\eta_i^u]\in\widetilde{g}_i, i=1,2,\dots,n)} \Big\{ \Big[f^{(-1)}\Big(\sum_{(i=1)}^n \omega_i f(\eta_i^l)\Big), f^{(-1)}\Big(\sum_{(i=1)}^n \omega_i f(\eta_i^u)\Big)\Big]\Big\}\Big) \tag{11}$$

Proof. Induction method will be followed to prove this theorem.

For $n = 2$,

$$\omega_1\widetilde{k}_1 = \Big(\bigcup_{[\gamma_1^l,\gamma_1^u]\in\widetilde{h}_1} \{[g^{-1}(\omega_1 g(\gamma_1^l)), g^{-1}(\omega_1 g(\gamma_1^u))]\}, \bigcup_{[\eta_1^l,\eta_1^u]\in\widetilde{g}_1} \{[f^{-1}(\omega_1 f(\eta_1^l)), f^{-1}(\omega_1 f(\eta_1^u))]\} \Big)$$

$$\omega_2\widetilde{k}_2 = \Big(\bigcup_{[\gamma_2^l,\gamma_2^u]\in\widetilde{h}_2} \{[g^{-1}(\omega_1 g(\gamma_2^l)), g^{-1}(\lambda\omega_1(\gamma_2^u))]\}, \bigcup_{[\eta_2^l,\eta_2^u]\in\widetilde{g}_2} \{[f^{-1}(\omega_1 f(\eta_2^l)), f^{-1}(\omega_1 f(\eta_2^u))]\} \Big)$$

now, $\omega_1\widetilde{k}_1 \oplus \omega_2\widetilde{k}_2 = (\bigcup_{[\gamma_1^l,\gamma_1^u]\in\widetilde{h}_1,[\gamma_2^l,\gamma_2^u]\in\widetilde{h}_2}\{[g^{-1}(g(y^{-1}(\omega_1 g(\gamma_1^l))) + g(g^{-1}(\omega_2 g(\gamma_2^l)))),$

$$g^{-1}(g(g^{-1}(\omega_1 g(\gamma_1^u))) + g(g^{-1}(\omega_2 g(\gamma_2^u))))]\}, \bigcup_{\substack{[\eta_1^l,\eta_1^u]\in\widetilde{g}_1,\\ [\eta_2^l,\eta_2^u]\in\widetilde{g}_2}} \{[f^{-1}(f(f^{-1}(\omega_1 f(\eta_1^l))) + f(f^{-1}(\omega_2 f(\eta_2^l)))), f^{-1}(f(f^{-1}(\omega_1 f(\eta_1^u))) + f(f^{-1}(\omega_2 f(\eta_2^u))))]\})$$

$$= \left(\bigcup_{[\gamma_1^l,\gamma_1^u]\in\widetilde{h}_1,[\gamma_2^l,\gamma_2^u]\in\widetilde{h}_2} \left\{ \left[g^{-1}\left(\sum_{i=1}^{2} \omega_i g(\gamma_i^l) \right), g^{-1}\left(\sum_{i=1}^{2} \omega_i g(\gamma_i^u) \right) \right] \right\}, \right.$$
$$\left. \bigcup_{[\eta_1^l,\eta_1^u]\in\widetilde{g}_1,[\eta_2^l,\eta_2^u]\in\widetilde{g}_2} \left\{ \left[f^{-1}\left(\sum_{i=1}^{2} \omega_i f(\eta_i^l) \right), f^{-1}\left(\sum_{i=1}^{2} \omega_i f(\eta_i^u) \right) \right] \right\} \right)$$

that is, the theorem is true for $n = 2$.

Suppose now that the theorem is true for $n = p$, that is,

$$\mathrm{AIVDHFWA}(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_p) = \left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i, i=1,2,\dots,p} \left\{ \left[g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\gamma_i^l) \right), g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\gamma_i^u) \right) \right] \right\}, \right.$$
$$\left. \bigcup_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i, i=1,2,\dots,p} \left\{ \left[f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\eta_i^l) \right), f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\eta_i^u) \right) \right] \right\} \right)$$

For $n = p + 1$,

$$AIVDHFWA(k_1, \widetilde{k}_2, \dots, \widetilde{k}_p, \widetilde{k}_{p+1}) = AIVDHFWA(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_p) \oplus \omega_{p+1}\widetilde{k}_{p+1}$$
$$= \left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i, i=1,2,\dots,p} \left\{ \left[g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\gamma_i^l) \right), g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\gamma_i^u) \right) \right] \right\}, \right.$$
$$\left. \bigcup_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i, i=1,2,\dots,p} \left\{ \left[f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\eta_i^l) \right), f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\eta_i^u) \right) \right] \right\} \right) \oplus$$
$$\left(\bigcup_{[\gamma_{p+1}^l,\gamma_{p+1}^u]\in\widetilde{h}_{p+1}} \{[g^{-1}(\omega_{p+1} g(\gamma_{p+1}^l)), g^{-1}(\omega_{p+1} g(\gamma_{p+1}^u))]\}, \right.$$
$$\left. \bigcup_{[\eta_{p+1}^l,\eta_{p+1}^u]\in\widetilde{g}_{p+1}} \{[f^{-1}(\omega_{p+1} f(\eta_{p+1}^l)), f^{-1}(\omega_{p+1} f(\eta_{p+1}^u))]\} \right)$$
$$= \left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i, i=1,2,\dots,p,p+1} \left\{ \left[g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\gamma_i^l) \right) \right) \right. \right. \right. \right.$$
$$\left. \left. + g(g^{-1}(\omega_{p+1} g(\gamma_{p+1}^l)))\right),\ g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\gamma_i^u) \right) \right) + g(g^{-1}(\omega_{p+1} g(\gamma_{p+1}^u))) \right) \right] \right\},$$
$$\bigcup_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i, i=1,2,\dots,p,p+1} \left\{ \left[f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\eta_i^l) \right) \right) \right. \right. \right.$$
$$\left. \left. \left. + f(f^{-1}(\omega_{p+1} f(\eta_{p+1}^l)))\right), f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\eta_i^u) \right) \right) + f(f^{-1}(\omega_{p+1} f(\eta_{p+1}^u))) \right) \right] \right\} \right)$$
$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\dots,p,p+1}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\gamma_i^l) + \omega_{p+1} g(\gamma_{p+1}^l) \right), \right. \right. \right.$$
$$\left. \left. g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\gamma_i^u) + \omega_{p+1} g(\gamma_{p+1}^u) \right) \right] \right\},$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\dots,p,p+1}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p}\omega_i f(\eta_i^l)+\omega_{p+1}f(\eta_{p+1}^l)\right),\ f^{-1}\left(\sum_{i=1}^{p}\omega_i f(\eta_i^u)+\omega_{p+1}f(\eta_{p+1}^u)\right)\right]\right\}\Bigg)$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\dots,p,p+1}} \left\{\left[g^{-1}\left(\sum_{i=1}^{p+1}\omega_i g(\gamma_i^l)\right), g^{-1}\left(\sum_{i=1}^{p+1}\omega_i g(\gamma_i^u)\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\dots,p,p+1}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p+1}\omega_i f(\eta_i^l)\right), f^{-1}\left(\sum_{i=1}^{p+1}\omega_i f(\eta_i^u)\right)\right]\right\}\right)$$

Therefore, this is true for $n = p + 1$ also. Hence, it is true for all natural numbers. ■

Now, based on different forms of the decreasing generator f, some specific WA aggregation operators can be derived as follows:

Case 1. (**Algebraic**). When $f(t) = -\log t$, the AIVDHFWA operator converts to the IVDHF WA (IVDHFWA) operator introduced by Ju et al[10]:

$$IVDHFWA(\widetilde{k}_1,\widetilde{k}_2,\dots,\widetilde{k}_n)=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\dots,n}} \left\{\left[1-\prod_{i=1}^{n}(1-\gamma_i^l)^{\omega_i}, 1-\prod_{i=1}^{n}(1-\gamma_i^u)^{\omega_i}\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\dots,n}} \left\{\left[\prod_{i=1}^{n}(\eta_i^l)^{\omega_i}, \prod_{i=1}^{n}(\eta_i^u)^{\omega_i}\right]\right\}\right)$$

Case 2. (**Einstein Class**). When $f(t) = \log\left(\frac{2-t}{t}\right)$, the AIVDHFWA operator converts to the IVDHF Einstein WA (IVDHFEWA) operator which is described as:

$$IVDHFEWA(\widetilde{k}_1,\widetilde{k}_2,\dots,\widetilde{k}_n)=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\dots,n}} \left\{\left[\frac{\prod_{i=1}^{n}(1+\gamma_i^l)^{\omega_i}-\prod_{i=1}^{n}(1-\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n}(1+\gamma_i^l)^{\omega_i}+\prod_{i=1}^{n}(1-\gamma_i^l)^{\omega_i}}, \frac{\prod_{i=1}^{n}(1+\gamma_i^u)^{\omega_i}-\prod_{i=1}^{n}(1-\gamma_i^u)^{\omega_i}}{\prod_{i=1}^{n}(1+\gamma_i^u)^{\omega_i}+\prod_{i=1}^{n}(1-\gamma_i^u)^{\omega_i}}\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\dots,n}} \left\{\left[\frac{2\prod_{i=1}^{n}(\eta_i^l)^{\omega_i}}{\prod_{i=1}^{n}(2-\eta_i^l)^{\omega_i}+\prod_{i=1}^{n}(\eta_i^l)^{\omega_i}}, \frac{2\prod_{i=1}^{n}(\eta_i^l)^{\omega_i}}{\prod_{i=1}^{n}(2-\eta_i^l)^{\omega_i}+\prod_{i=1}^{n}(\eta_i^l)^{\omega_i}}\right]\right\}\right)$$

Case 3. (**Hamacher Class**). When $f(t) = \log\left(\frac{\theta+(1-\theta)t}{t}\right)$, $\theta > 0$, then the AIVDHFWA operator is converted into the IVDHF Hamacher WA (IVDHFHWA) operator which is expressed as:

$$IVDHFHWA(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i \\ i=1,2,\dots,n}} \left\{ \left[\frac{\prod_{i=1}^n (1+(\theta-1)\gamma_i^l)^{\omega_i} - \prod_{i=1}^n (1-\gamma_i^l)^{\omega_i}}{\prod_{i=1}^n (1+(\theta-1)\gamma_i^l)^{\omega_i} + (\theta-1)\prod_{i=1}^n (1-\gamma_i^l)^{\omega_i}}, \frac{\prod_{i=1}^n (1+(\theta-1)\gamma_i^u)^{\omega_i} - \prod_{i=1}^n (1-\gamma_i^u)^{\omega_i}}{\prod_{i=1}^n (1+(\theta-1)\gamma_i^u)^{\omega_i} + (\theta-1)\prod_{i=1}^n (1-\gamma_i^u)^{\omega_i}} \right] \right\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i \\ i=1,2,\dots,n}} \left\{ \left[\frac{\theta \prod_{i=1}^n (\eta_i^l)^{\omega_i}}{\prod_{i=1}^n (1+(\theta-1)(1-\eta_i^l))^{\omega_i} + (\theta-1)\prod_{i=1}^n (\eta_i^l)^{\omega_i}}, \frac{\theta \prod_{i=1}^n (\eta_i^u)^{\omega_i}}{\prod_{i=1}^n (1+(\theta-1)(1-\eta_i^u))^{\omega_i} + (\theta-1)\prod_{i=1}^n (\eta_i^u)^{\omega_i}} \right] \right\} \right)$$

Case 4. (**Frank Class**). When $f(t) = \log\left(\frac{\theta-1}{\theta^t-1}\right)$, $\theta > 1$, the AIVDHFWA operator is converted to IVDHF Frank WA (IVDHFFWA) operator which is presented as:

$$IVDHFFWA(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i \\ i=1,2,\dots,n}} \left\{ \left[1 - \frac{\log\left(1+\prod_{i=1}^n (\theta^{1-\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta}, 1 - \frac{\log\left(1+\prod_{i=1}^n (\theta^{1-\gamma_i^u}-1)^{\omega_i}\right)}{\log\theta} \right] \right\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i \\ i=1,2,\dots,n}} \left\{ \left[\frac{\log\left(1+\prod_{i=1}^n (\theta^{\eta_i^l}-1)^{\omega_i}\right)}{\log\theta}, \frac{\log\left(1+\prod_{i=1}^n (\theta^{\eta_i^u}-1)^{\omega_i}\right)}{\log\theta} \right] \right\} \right)$$

Now various properties of AIVDHFWA operator are proposed below.

Theorem 2. (**Boundary**). Let $\widetilde{k}_i = (\widetilde{h}_i, \widetilde{g}_i)$ $(i = 1, 2, \dots, n)$ be a collection of IVDHFEs, and let
$\gamma_{\min}^l = \min\{\gamma_{i_{\min}}^l\}$ where $\gamma_{i_{\min}}^l = \min_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i} \{\gamma_i^l\}$ for all $i = 1, 2, \dots, n$,
$\gamma_{\max}^u = \max\{\gamma_{i_{\max}}^u\}$ where $\gamma_{i_{\max}}^u = \max_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i} \{\gamma_i^u\}$ for all $i = 1, 2, \dots, n$.
Similarly, $\gamma_{\min}^u$ and $\gamma_{\max}^l$ can be defined.
Again, let $\eta_{\min}^u = \min\{\eta_{i_{\min}}^u\}$ where $\eta_{i_{\min}}^u = \min_{[\eta_i^l, \eta_i^u] \in \widetilde{h}_i} \{\eta_i^u\}$ for all $i = 1, 2, \dots, n$.
$\eta_{\max}^l = \max\{\eta_{i_{\max}}^l\}$ where $\eta_{i_{\max}}^l = \max_{[\eta_i^l, \eta_i^u] \in \widetilde{h}_i} \{\eta_i^l\}$ for all $i = 1, 2, \dots, n$.
Similarly, $\eta_{\min}^l$ and $\eta_{\max}^u$ can be defined.
Now, if $\widetilde{k}_- = ([\gamma_{\min}^l, \gamma_{\min}^u], [\eta_{\max}^l, \eta_{\max}^u])$ and $\widetilde{k}_+ = ([\gamma_{\max}^l, \gamma_{\max}^u], [\eta_{\min}^l, \eta_{\min}^u])$, then $\widetilde{k}_- \le \text{AIVDHFWA}(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) \le \widetilde{k}_+$.

Proof. It is given that $\gamma_{\min}^l \le \gamma_i^l \le \gamma_{\max}^l$ and $\gamma_{\min}^u \le \gamma_i^u \le \gamma_{\max}^u$ for all $i = 1, 2, \dots, n$.
Since $g(t)$ $(t \in [0, 1])$ is a monotonic increasing function,
$g^{-1}\left(\sum_{i=1}^n \omega_i g(\gamma_{\min}^l)\right) \le g^{-1}\left(\sum_{i=1}^n \omega_i g(\gamma_i^l)\right) \le g^{-1}\left(\sum_{i=1}^n \omega_i g(\gamma_{\max}^l)\right)$ for all i and so

$$\gamma_{\min}^l \le g^{(-1)}\left(\sum_{(i=1)}^{n} \omega_i g(\gamma_i^l)\right) \le \gamma_{\max}^l \text{ for all } i. \tag{12}$$

Similarly,

$$\gamma^u_{\min} \leq g^{(-1)}\left(\sum_{(i=1)}^{n} \omega_i g(\gamma^u_i)\right) \leq \gamma^u_{\max} \text{ for all } i. \tag{13}$$

Again $\eta^l_{\min} \leq \eta^l_i \leq \eta^l_{\max}$ for all i and since $f(t)$ is a decreasing function, $f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta^l_{\max})\right) \leq f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta^l_i)\right) \leq f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta^l_{\min})\right)$ for all i which implies that

$$\eta^l_{\max} \leq f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta^l_i)\right) \leq \eta^l_{\min} \text{ for all } i. \tag{14}$$

In a similar way it can be shown that

$$\eta^u_{\max} \leq f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta^u_i)\right) \leq \eta^u_{\min} \text{ for all } i. \tag{15}$$

Further from Equations (12) and (14) it is clear that

$$\gamma^l_{\min} - \eta^l_{\min} \leq g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\gamma^l_i)\right) - f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta^l_i)\right) \leq \gamma^l_{\max} - \eta^l_{\max} \text{ for all } i. \tag{16}$$

Also, from Equations (13) and (15) it is found that

$$\gamma^u_{\min} - \eta^u_{\min} \leq g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\gamma^u_i)\right) - f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta^u_i)\right) \leq \gamma^u_{\max} - \eta^u_{\max} \text{ for all } i. \tag{17}$$

Now using Equations (16), (17), and Definition 5, it can easily be shown that

$$S(\widetilde{k}_-) \leq S(AIVDHFWA(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n)) \leq S(\widetilde{k}_+)$$

and hence $\widetilde{k}_- \leq \text{AIVDHFWA}(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) \leq \widetilde{k}_+$. ∎

Theorem 3. Let $\widetilde{k}_i$ $(i = 1, 2, \dots, n)$ be a collection of IVDHFEs, and $\omega_i \in [0, 1]$ $(i = 1, 2, \dots, n)$ be their corresponding weights such that $\sum_{i=1}^{n} \omega_i = 1$. If $\widetilde{k}$ be an IVDHFE, then

$$AIVDHFWA(\widetilde{k}_1 \oplus \widetilde{k}, \widetilde{k}_2 \oplus \widetilde{k}, \dots, \widetilde{k}_n \oplus \widetilde{k}) = AIVDHFWA(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) \oplus \widetilde{k}$$

Proof. We have $\widetilde{k}_i \oplus \widetilde{k} = (\bigcup_{[\gamma^l_i, \gamma^u_i] \in \widetilde{h}_i, [\gamma^l, \gamma^u] \in \widetilde{h}} \{[g^{-1}(g(\gamma^l_i) + g(\gamma^l)), g^{-1}(g(\gamma^u_i) + g(\gamma^u))]\},$

$$\bigcup_{[\eta^l_i, \eta^u_i] \in \widetilde{g}_i, [\eta^l, \eta^u] \in \widetilde{g}} \{[f^{-1}(f(\eta^l_i) + f(\eta^l)), f^{-1}(f(\eta^u_i) + f(\eta^u))]\})$$

Then,

$$AIVDHFWA(\widetilde{k}_1 \oplus \widetilde{k}, \widetilde{k}_2 \oplus \widetilde{k}, \dots, \widetilde{k}_n \oplus \widetilde{k})$$
$$= \left(\bigcup_{\substack{[\gamma^l_i, \gamma^u_i] \in \widetilde{h}_i, [\gamma^l, \gamma^u] \in \widetilde{h} \\ i = 1, 2, \dots, n}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \omega_i g(g^{-1}(g(\gamma^l_i) + g(\gamma^l)))\right),\ g^{-1}\left(\sum_{i=1}^{n} \omega_i g(g^{-1}(g(\gamma^u_i) + g(\gamma^u)))\right) \right] \right\}, \right.$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,[\eta^l,\eta^u]\in\widetilde{g}\\ i=1,2,\dots,n}} \left\{\left[f^{-1}\left(\sum_{i=1}^{n}\omega_i f(f^{-1}(f(\eta_i^l)+f(\eta^l)))\right)\right),\ f^{-1}\left(\sum_{i=1}^{n}\omega_i f(f^{-1}(f(\eta_i^l)+f(\eta^l)))\right)\right]\right\}\Bigg)$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,[\gamma^l,\gamma^u]\in\widetilde{h}\\ i=1,2,\dots,n}}\left\{\left[g^{-1}\left(\sum_{i=1}^{n}\omega_i(g(\gamma_i^l)+g(\gamma^l))\right),\ g^{-1}\left(\sum_{i=1}^{n}\omega_i(g(\gamma_i^u)+g(\gamma^u))\right)\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,[\eta^l,\eta^u]\in\widetilde{g}\\ i=1,2,\dots,n}}\left\{\left[f^{-1}\left(\sum_{i=1}^{n}\omega_i(f(\eta_i^l)+f(\eta^l))\right),\ f^{-1}\left(\sum_{i=1}^{n}\omega_i(f(\eta_i^l)+f(\eta^l))\right)\right]\right\}\right)$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,[\gamma^l,\gamma^u]\in\widetilde{h}\\ i=1,2,\dots,n}}\left\{\left[g^{-1}\left(\sum_{i=1}^{n}\omega_i g(\gamma_i^l)+g(\gamma^l)\right),\ g^{-1}\left(\sum_{i=1}^{n}\omega_i g(\gamma_i^u)+g(\gamma^u)\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,[\eta^l,\eta^u]\in\widetilde{g}\\ i=1,2,\dots,n}}\left\{\left[f^{-1}\left(\sum_{i=1}^{n}\omega_i f(\eta_i^l)+f(\eta^l)\right),\ f^{-1}\left(\sum_{i=1}^{n}\omega_i f(\eta_i^u)+f(\eta^u)\right)\right]\right\}\right).$$

Now,

$$\text{AIVDHFWA}(\widetilde{k}_1,\widetilde{k}_2,\dots,\widetilde{k}_n)\oplus\widetilde{k}=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i\\ i=1,2,\dots,n}}\left\{\left[g^{-1}\left(\sum_{i=1}^{n}\omega_i g(\gamma_i^l)\right),g^{-1}\left(\sum_{i=1}^{n}\omega_i g(\gamma_i^u)\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i\\ i=1,2,\dots,n}}\left\{\left[f^{-1}\left(\sum_{i=1}^{n}\omega_i f(\eta_i^l)\right),f^{-1}\left(\sum_{i=1}^{n}\omega_i f(\eta_i^u)\right)\right]\right\}\right)\oplus(\{[\gamma^l,\gamma^u]\},\{[\eta^l,\eta^u]\})$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,[\gamma^l,\gamma^u]\in\widetilde{h}\\ i=1,2,\dots,n}}\left\{\left[g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g(\gamma_i^l)\right)\right)+g(\gamma^l)\right),\right.\right.\right.$$

$$\left.\left.g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g(\gamma_i^u)\right)\right)+g(\gamma^u)\right)\right]\right\},$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,[\eta^l,\eta^u]\in\widetilde{g}\\ i=1,2,\dots,n}}\left\{\left[f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f(\eta_i^l)\right)\right)+f(\eta^l)\right),\right.\right.$$

$$\left.\left.\left.f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f(\eta_i^u)\right)\right)+f(\eta^u)\right)\right]\right\}\right)$$

$$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, [\gamma^l, \gamma^u] \in \widetilde{h} \\ i = 1, 2, \ldots, n}} \left\{ \left[g^{-1} \left(\sum_{i=1}^{n} \omega_i g(\gamma_i^l) + g(\gamma^l) \right), g^{-1} \left(\sum_{i=1}^{n} \omega_i g(\gamma_i^u) + g(\gamma^u) \right) \right] \right\}, \right.$$

$$\left. \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, [\eta^l, \eta^u] \in \widetilde{g} \\ i = 1, 2, \ldots, n}} \left\{ \left[f^{-1} \left(\sum_{i=1}^{n} \omega_i f(\eta_i^l) + f(\eta^l) \right), f^{-1} \left(\sum_{i=1}^{n} \omega_i f(\eta_i^u) + f(\eta^u) \right) \right] \right\} \right)$$

Therefore, AIVDHFWA$(\widetilde{k}_1 \oplus \widetilde{k}, \widetilde{k}_2 \oplus \widetilde{k}, \ldots, \widetilde{k}_n \oplus \widetilde{k})$ = AIVDHFWA$(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) \oplus \widetilde{k}$.
Hence the theorem is proved. ■

Theorem 4. (**Idempotency**). If all $\widetilde{k}_i$ $(i = 1, 2, \ldots, n)$ are considered as equal and let $\widetilde{k}_i = (\{[\gamma^l, \gamma^u]\}, \{[\eta^l, \eta^u]\})$ for all $(i = 1, 2, \ldots, n)$, then

$$AIVDHFWA(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) = (\{[\gamma^l, \gamma^u]\}, \{[\eta^l, \eta^u]\}).$$

Proof. Here AIVDHFWA$(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i | i=1,2,\ldots,n} \left\{ \left[g^{-1} \left(\sum_{i=1}^{n} \omega_i g(\gamma_i^l) \right), g^{-1} \left(\sum_{i=1}^{n} \omega_i g(\gamma_i^u) \right) \right] \right\}, \right.$

$$\left. \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i | i=1,2,\ldots,n} \left\{ \left[f^{-1} \left(\sum_{i=1}^{n} \omega_i f(\eta_i^l) \right), f^{-1} \left(\sum_{i=1}^{n} \omega_i f(\eta_i^u) \right) \right] \right\} \right)$$

Now, since $\widetilde{k}_i = (\{[\gamma^l, \gamma^u]\}, \{[\eta^l, \eta^u]\})$ for all $(i = 1, 2, \ldots, n)$, then we have
$\gamma_i^l = \gamma^l$, $\gamma_i^u = \gamma^u$, $\eta_i^l = \eta^l$ and $\eta_i^u = \eta^u$ for all $(i = 1, 2, \ldots, n)$.
Therefore, AIVDHFWA$(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i | i=1,2,\ldots,n} \left\{ \left[g^{-1} \left(g(\gamma^l) \sum_{i=1}^{n} \omega_i \right), g^{-1} \left(g(\gamma^u) \sum_{i=1}^{n} \omega_i \right) \right] \right\}, \right.$

$$\left. \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i | i=1,2,\ldots,n} \left\{ \left[f^{-1} \left(f(\eta^l) \sum_{i=1}^{n} \omega_i \right), f^{-1} \left(f(\eta^u) \sum_{i=1}^{n} \omega_i \right) \right] \right\} \right)$$

$$= \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i | i=1,2,\ldots,n} \{[\gamma^l, \gamma^u]\}, \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i | i=1,2,\ldots,n} \{[\eta^l, \eta^u]\} \right)$$

$$= (\{[\gamma^l, \gamma^u]\}, \{[\eta^l, \eta^u]\})$$

This completes the proof of the theorem. ■

3.3 | At-N&t-CN-based IVDHFWG aggregation operator

Definition 14. Let $\widetilde{k}_i$ $(i = 1, 2, \ldots, n)$ be a collection of IVDHFEs, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Now, an At-N&t-CN-based IVDHFWG (AIVDHFWG) operator is a function: $\widetilde{K}^n \to \widetilde{K}$, given by AIVDHFWG$(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) = \otimes_{i=1}^{n} ((\widetilde{k}_i)^{\omega_i})$, where $\otimes$ has already been defined in Definition 12.

Theorem 5. Let $\widetilde{k}_i = (\widetilde{h}_i, \widetilde{g}_i)\,(i = 1, 2, \dots, n)$ be a family of IVDHFEs. Now, if the operator AIVDHFWG is used to aggregate $\widetilde{k}_i$, the aggregated value is also an IVDHFE and is given by

$$\mathrm{AIVDHFWG}(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i|i=1,2,\dots,n} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i^l)\right), f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i^u)\right) \right] \right\}, \bigcup_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i|i=1,2,\dots,n} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\eta_i^l)\right), g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\eta_i^u)\right) \right] \right\} \right) \tag{18}$$

Proof. According to Definition 12,

$$(\widetilde{k}_1)^{\omega_1} = \left(\bigcup_{[\gamma_1^l,\gamma_1^u]\in\widetilde{h}_1} \{[f^{-1}(\omega_1 f(\gamma_1^l)), f^{-1}(\omega_1 f(\gamma_1^u))]\}, \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}_1} \{[g^{-1}(\omega_1 g(\eta_1^l)), g^{-1}(\omega_1 g(\eta_1^u))]\} \right)$$

and

$$(\widetilde{k}_2)^{\omega_2} = \left(\bigcup_{[\gamma_2^l,\gamma_2^u]\in\widetilde{h}_2} \{[f^{-1}(\omega_2 f(\gamma_2^l)), f^{-1}(\omega_2 f(\gamma_2^u))]\}, \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}_2} \{[g^{-1}(\omega_2 g(\eta_2^l)), g^{-1}(\omega_2 g(\eta_2^u))]\} \right).$$

Now,

$$\begin{aligned}
\otimes_{i=1}^{2}((\widetilde{k}_i)^{\omega_i}) &= ((\widetilde{k}_1)^{\omega_1}) \otimes ((\widetilde{k}_2)^{\omega_2}) \\
&= \left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i, i=1,2} \{[f^{-1}(\omega_1 f(\gamma_1^l) + \omega_2 f(\gamma_2^l)), f^{-1}(\omega_1 f(\gamma_1^u) + \omega_2 f(\gamma_2^u))]\} \right. \\
&\quad \left. \bigcup_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i, i=1,2} \{[g^{-1}(\omega_1 g(\eta_1^l) + \omega_2 g(\eta_2^l)), g^{-1}(\omega_1 g(\eta_1^u) + \omega_2 g(\eta_2^u))]\} \right) \\
&= \left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i, i=1,2} \left\{ \left[f^{-1}\left(\sum_{i=1}^{2} \omega_i f(\gamma_i^l)\right), f^{-1}\left(\sum_{i=1}^{2} \omega_i f(\gamma_i^u)\right) \right] \right\}, \right. \\
&\quad \left. \bigcup_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i, i=1,2} \left\{ \left[g^{-1}\left(\sum_{i=1}^{2} \omega_i g(\eta_i^l)\right), g^{-1}\left(\sum_{i=1}^{2} \omega_i g(\eta_i^u)\right) \right] \right\} \right) \\
&\quad \left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i|i=1,2,\dots,n} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i^l)\right), f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i^u)\right) \right] \right\}, \right. \\
&\quad \left. \bigcup_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i|i=1,2,\dots,n} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\eta_i^l)\right), g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\eta_i^u)\right) \right] \right\} \right)
\end{aligned}$$

∎

that is, the theorem is valid for $n = 2$.

Suppose now that the theorem is true for $n = p$, that is,

$$AIVDHFWG(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \otimes_{i=1}^{p}((\widetilde{k}_i)^{\omega_i}) = \left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i|i=1,2,\dots,p} \left\{ \left[f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\gamma_i^l)\right), f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\gamma_i^u)\right) \right] \right\}, \bigcup_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i|i=1,2,\dots,p} \left\{ \left[g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\eta_i^l)\right), g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\eta_i^u)\right) \right] \right\} \right)$$

Now it is show that the theorem is true for $n = p + 1$, that is,

$$
\begin{aligned}
&AIVDHFWG(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_p, \widetilde{k}_{p+1}) \\
&= AIVDHFWA(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) \otimes (\widetilde{k}_{p+1})^{\omega_{p+1}} \\
&= \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i | i=1,2,\dots,p} \left\{ \left[f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\gamma_i^l) \right), f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\gamma_i^u) \right) \right] \right\}, \right. \\
&\qquad \left. \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i | i=1,2,\dots,p} \left\{ \left[g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\eta_i^l) \right), g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\eta_i^u) \right) \right] \right\} \right) \\
&\otimes \left(\bigcup_{[\gamma_{p+1}^l, \gamma_{p+1}^u] \in \widetilde{h}_{p+1}} \{ [f^{-1}(\omega_{p+1} f(\gamma_{p+1}^l)), f^{-1}(\omega_{p+1} f(\gamma_{p+1}^u))] \} \bigcup_{[\eta_{p+1}^l, \eta_{p+1}^u] \in \widetilde{g}_{p+1}} \{ [g^{-1}(\omega_{p+1} g(\eta_{p+1}^l)), g^{-1}(\omega_{p+1} g(\eta_{p+1}^u))] \} \right) \\
&= \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, i=1,2,\dots,p,p+1} \left\{ \left[f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\gamma_i^l) + \omega_{p+1} f(\gamma_{p+1}^l) \right), f^{-1}\left(\sum_{i=1}^{p} \omega_i f(\gamma_i^u) + \omega_{p+1} f(\gamma_{p+1}^u) \right) \right] \right\}, \right. \\
&\qquad \left. \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, i=1,2,\dots,p,p+1} \left\{ \left[g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\eta_i^l) + \omega_{p+1} g(\eta_{p+1}^l) \right), g^{-1}\left(\sum_{i=1}^{p} \omega_i g(\eta_i^u) + \omega_{p+1} g(\eta_{p+1}^u) \right) \right] \right\} \right) \\
&= \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, i=1,2,\dots,p,p+1} \left\{ \left[f^{-1}\left(\sum_{i=1}^{p+1} \omega_i f(\gamma_i^l) \right), f^{-1}\left(\sum_{i=1}^{p+1} \omega_i f(\gamma_i^u) \right) \right] \right\} \right. \\
&\qquad \left. \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, i=1,2,\dots,p,p+1} \left\{ \left[g^{-1}\left(\sum_{i=1}^{p+1} \omega_i g(\eta_i^l) \right), g^{-1}\left(\sum_{i=1}^{p+1} \omega_i g(\eta_i^u) \right) \right] \right\} \right)
\end{aligned}
$$

Therefore the theorem is true for all n.

Now, based on different forms of the decreasing generator, f, different WG aggregation operators are derived as follows:

Case. (**Algebric Class**). When $f(t) = -\log t$, the AIVDHFWG operator is converted into the IVDHFWG operator introduced by Ju et al[10]:

$$
\begin{aligned}
IVDHFWG(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = &\left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i | i=1,2,\dots,n} \left\{ \left[\prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}, \prod_{i=1}^{n} (\gamma_i^u)^{\omega_i} \right] \right\}, \right. \\
&\left. \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i | i=1,2,\dots,n} \left\{ \left[1 - \prod_{i=1}^{n} (1 - \eta_i^l)^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - \eta_i^u)^{\omega_i} \right] \right\} \right)
\end{aligned}
$$

Case. (**Einstein Class**). When $f(t) = \log\left(\frac{2-t}{t}\right)$, the AIVDHFWG operator converted into the IVDHF Einstein WG (IVDHFEWG) operator which is expressed as:

$$
\begin{aligned}
IVDHFEWG(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = &\left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i | i=1,2,\dots,n} \left\{ \left[\frac{2 \prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n} (2 - \gamma_i^l)^{\omega_i} + \prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}, \frac{2 \prod_{i=1}^{n} (\gamma_i^u)^{\omega_i}}{\prod_{i=1}^{n} (2 - \gamma_i^u)^{\omega_i} + \prod_{i=1}^{n} (\gamma_i^u)^{\omega_i}} \right] \right\}, \right. \\
&\left. \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i | i=1,2,\dots,n} \left\{ \left[\frac{\prod_{i=1}^{n} (1 + \eta_i^l)^{\omega_i} - \prod_{i=1}^{n} (1 - \eta_i^l)^{\omega_i}}{\prod_{i=1}^{n} (1 + \eta_i^l)^{\omega_i} + \prod_{i=1}^{n} (1 - \eta_i^l)^{\omega_i}}, \frac{\prod_{i=1}^{n} (1 + \eta_i^u)^{\omega_i} - \prod_{i=1}^{n} (1 - \eta_i^u)^{\omega_i}}{\prod_{i=1}^{n} (1 + \eta_i^u)^{\omega_i} + \prod_{i=1}^{n} (1 - \eta_i^u)^{\omega_i}} \right] \right\} \right)
\end{aligned}
$$

Case. (**Hamacher Class**). When $f(t) = \log\left(\frac{\theta+(1-\theta)t}{t}\right)$, $\theta > 0$, the AIVDHFWG operator is converted to the IVDHF Hamacher WG (IVDHFHWG) operator which is presented as:

$$IVDHFHWG(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) = \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i \\ i=1,2,\ldots,n.}} \left\{ \left[\frac{\theta \prod_{i=1}^n (\gamma_i^l)^{\omega_i}}{\prod_{i=1}^n (1+(\theta-1)(1-\gamma_i^l))^{\omega_i} + (\theta-1)\prod_{i=1}^n (\gamma_i^l)^{\omega_i}}, \frac{\theta \prod_{i=1}^n (\gamma_i^u)^{\omega_i}}{\prod_{i=1}^n (1+(\theta-1)(1-\gamma_i^u))^{\omega_i} + (\theta-1)\prod_{i=1}^n (\gamma_i^u)^{\omega_i}} \right] \right\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i \\ i=1,2,\ldots,n.}} \left\{ \left[\frac{\prod_{i=1}^n (1+(\theta-1)\eta_i^l)^{\omega_i} - \prod_{i=1}^n (1-\eta_i^l)^{\omega_i}}{\prod_{i=1}^n (1+(\theta-1)\eta_i^l)^{\omega_i} + (\theta-1)\prod_{i=1}^n (1-\eta_i^l)^{\omega_i}}, \frac{\prod_{i=1}^n (1+(\theta-1)\eta_i^u)^{\omega_i} - \prod_{i=1}^n (1-\eta_i^u)^{\omega_i}}{\prod_{i=1}^n (1+(\theta-1)\eta_i^u)^{\omega_i} + (\theta-1)\prod_{i=1}^n (1-\eta_i^u)^{\omega_i}} \right] \right\} \right)$$

Case. (**Frank Class**). When $f(t) = \log\left(\frac{\theta-1}{\theta^t-1}\right)$, $\theta > 1$, the AIVDHFWG operator is converted into the IVDHF Frank WG (IVDHFFWG) operator which is presented as:

$$IVDHFFWG(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i | i=1,2,\ldots,n.} \left\{ \left[\frac{\log\left(1+\prod_{i=1}^n (\theta^{\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta}, \frac{\log\left(1+\prod_{i=1}^n (\theta^{\gamma_i^u}-1)^{\omega_i}\right)}{\log\theta} \right] \right\}, \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i | i=1,2,\ldots,n.} \left\{ \left[1 - \frac{\log\left(1+\prod_{i=1}^n (\theta^{1-\eta_i^l}-1)^{\omega_i}\right)}{\log\theta}, 1 - \frac{\log\left(1+\prod_{i=1}^n (\theta^{1-\eta_i^u}-1)^{\omega_i}\right)}{\log\theta} \right] \right\} \right).$$

4 | DERIVATION OF OPERATORS IN VARIOUS CONTEXTS FROM THE DEVELOPED OPERATORS IN IVDHF ENVIRONMENT

In this section, the developed AIVDHFWA and AIVDHFWG operators in IVDHF environment are used to describe different aggregation functions for processing information containing other variants of fuzzy elements.

4.1 | Archimedean DHF WA (ADHFWA) and WG (ADHFWG) operators[26]

When $\gamma_i^l = \gamma_i^u = \gamma_i$ and $\eta_i^l = \eta_i^u = \eta_i$ are considered for all $i = 1, 2, \ldots, n.$, the developed AIVDHFWA and AIVDHFWG operators are converted into ADHFWA and ADHFWG, operators, respectively. Thus those two operators are given by

$$\text{ADHFWA}(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{ g^{-1}\left(\sum_{i=1}^n \omega_i g(\gamma_i) \right) \right\}, \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, i=1,2,\ldots,n} \left\{ f^{-1}\left(\sum_{i=1}^n \omega_i f(\eta_i) \right) \right\} \right)$$

and

$$ADHFWG(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, i=1,2,\dots,n} \left\{ f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i) \right) \right\}, \bigcup_{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, i=1,2,\dots,n} \left\{ g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\eta_i) \right) \right\} \right).$$

4.2 | Archimedean IVHF WA (AIVHFWA) and WG (AIVHFWG) operator[30]

If $\eta_i^l = \eta_i^u = 0$ and $\gamma_i^l \neq \gamma_i^u$ are assumed for all $i = 1, 2, \dots, n.$, then the developed AIVDHFWA and AIVDHFWG are reduced to AIVHFWA operator and AIVHFWG operator, respectively, which are defined below.

$$AIVHFWA(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, i=1,2,\dots,n} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\gamma_i^l) \right), g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\gamma_i^u) \right) \right] \right\} \right)$$

and

$$AIVHFWG(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, i=1,\dots,n} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i^l) \right), f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i^u) \right) \right] \right\} \right).$$

4.3 | Archimedean HF WA (AHFWA) and WG (AHFWG) operators

If $\gamma_i^l = \gamma_i^u = \gamma_i$ and $\eta_i^l = \eta_i^u = 0$ are considered for all $i = 1, 2, \dots, n$, the developed AIVDHFWA and AIVDHFWG operators are reduced to AHFWA and AHFWG operators, respectively. Thus

$$AHFWA(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, i=1,2,\dots,n} \left\{ g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\gamma_i) \right) \right\} \right)$$

and

$$AHFWG(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, i=1,2,\dots,n} \left\{ f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i) \right) \right\} \right)$$

4.4 | Archimedean interval-valued intuitionistic fuzzy (IF) WA (AIVIFWA) and WG (AIVIFWG) operator

Moreover, in IVDHF environment when each of the membership and nonmembership grades corresponding to an element is specified by only one interval value, AIVDHFWA and AIVDHFWA operators reduce to AIVIFWA and AIVIFWG operators, respectively, and are defined as

$\text{AIVIFWA}(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\left[g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\gamma_i^l) \right), g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\gamma_i^u) \right) \right], \left[f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta_i^l) \right), f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta_i^u) \right) \right] \right)$ and $\text{AIVIFWG}(\widetilde{k}_1, \widetilde{k}_2, \dots, \widetilde{k}_n) = \left(\left[f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i^l) \right), f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i^u) \right) \right], \left[g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\eta_i^l) \right), g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\eta_i^l) \right) \right] \right)$

4.5 | Archimedean IF WA (AIFWA) and WG (AIFWG) operator[25]

When membership and nonmembership values of each element in a set correspond only one interval value with the same upper and lower bounds, that is, $\gamma_i^l = \gamma_i^u = \gamma_i$ and $\eta_i^l = \eta_i^u = \eta_i$ for all $i = 1, 2, \dots, n.$, AIVDHFWA and AIVDHFWG operators reduce to AIFWA and AIFWG operators, respectively, and are defined as:

$$AIFWA(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) = \left(g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\gamma_i) \right),\ f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\eta_i) \right) \right)$$

and

$$AIFWG(\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_n) = \left(f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\gamma_i) \right),\ g^{-1}\left(\sum_{i=1}^{n} \omega_i g(\eta_i) \right) \right).$$

As like earlier discussions, it is to be noted here that each type of Archimedean averaging aggregating operators as defined above can derive several types of averaging aggregation functions with the help of different generators depending on the values of the parameter θ. Those are described below:

Case 1. (**Algebraic Class**). If $f(t) = -\log t$, ADHFWA, AIVHFWA, AHFWA, AIVIFWA, and AIFWA operators reduces to DHFWA, IVHFWA, HFWA, IVIFWA,[31] and IFWA[32] operators, respectively, which are presented in the following Table 2.

Case 2. (**Einstein Class**). If $f(t) = \log\left(\frac{2-t}{t}\right)$, then the ADHFWA, AIVHFWA, AHFWA, AIVIFWA, and AIFWA operators are reduced to DHFEWA, IVHF Einstein WA (IVHFEWA),[33] HF Einstein WA (HFEWA),[34] IVIF Einstein WA (IVIFEWA),[35] and IF Einstein WA (IFEWA)[36] operators, respectively, which are displayed in the following Table 3.

TABLE 2 Algebraic WA aggregation operators

Aggregation operators	WA aggregation operator
DHFWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n.} \left\{1 - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}\right\}, \bigcup_{\eta_i \in \widetilde{g}_i, i=1,2,\ldots,n.} \left\{\prod_{i=1}^{n} (\eta_i)^{\omega_i}\right\}\right)$
IVHFWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{\left[1 - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}, 1 - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}\right]\right\}\right)$
HFWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{1 - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}\right\}\right)$
IVIFWA	$\left(\left[1 - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}, 1 - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}\right], \left[\prod_{i=1}^{n} (\eta_i)^{\omega_i}, \prod_{i=1}^{n} (\eta_i)^{\omega_i}\right]\right)$
IFWA	$\left(1 - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i},\ \prod_{i=1}^{n} (\eta_i)^{\omega_i}\right)$

TABLE 3 Einstein class of WA aggregation operators

Aggregation operators	WA aggregation operator
DHFEWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,\ldots,n} \left\{\frac{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} + \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}\right\}, \bigcup_{\substack{\eta_i \in \widetilde{g}_i,\\ i=1,2,\ldots,n}} \left\{\frac{2\prod_{i=1}^{n}(\eta_i)^{\omega_i}}{\prod_{i=1}^{n}(2-\eta_i)^{\omega_i} + \prod_{i=1}^{n}(\eta_i)^{\omega_i}}\right\}\right)$
IVHFEWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{\left[\frac{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} + \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}, \frac{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} + \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}\right]\right\}\right)$
HFEWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{\frac{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} + \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}\right\}\right)$
IVIFEWA	$\left(\left[\frac{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} + \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}, \frac{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} + \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}\right], \left[\frac{2\prod_{i=1}^{n}(\eta_i)^{\omega_i}}{\prod_{i=1}^{n}(2-\eta_i)^{\omega_i} + \prod_{i=1}^{n}(\eta_i)^{\omega_i}}, \frac{2\prod_{i=1}^{n}(\eta_i)^{\omega_i}}{\prod_{i=1}^{n}(2-\eta_i)^{\omega_i} + \prod_{i=1}^{n}(\eta_i)^{\omega_i}}\right]\right)$
IFEWA	$\left(\frac{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} - \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n}(1+\gamma_i)^{\omega_i} + \prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}}, \frac{2\prod_{i=1}^{n}(\eta_i)^{\omega_i}}{\prod_{i=1}^{n}(2-\eta_i)^{\omega_i} + \prod_{i=1}^{n}(\eta_i)^{\omega_i}}\right)$

TABLE 4 Hamacher class of WA aggregation operators

Aggregation operators	WA aggregation operator
DHFHWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{ \frac{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}} \right\}, \bigcup_{\eta_i \in \widetilde{g}_i, i=1,2,\ldots,n} \left\{ \frac{\theta \prod_{i=1}^{n} (\eta_i)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\eta_i))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\eta_i)^{\omega_i}} \right\}\right)$
IVHFHWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{ \frac{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i^l)^{\omega_i} - \prod_{i=1}^{n} (1-\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i^l)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\gamma_i^l)^{\omega_i}} \right\}, \right) \bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{ \frac{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i^u)^{\omega_i} - \prod_{i=1}^{n} (1-\gamma_i^u)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i^u)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\gamma_i^u)^{\omega_i}} \right\}$
HFHWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{ \frac{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}} \right\}\right)$
IVIFHWA	$\left(\left[\frac{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}}, \frac{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}} \right],\right.$
	$\left.\left[\frac{\theta \prod_{i=1}^{n} (\eta_i)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\eta_i))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\eta_i)^{\omega_i}}, \frac{\theta \prod_{i=1}^{n} (\eta_i)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\eta_i))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\eta_i)^{\omega_i}} \right]\right)$
IFHWA	$\left(\frac{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} - \prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\gamma_i)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\gamma_i)^{\omega_i}}, \frac{\theta \prod_{i=1}^{n} (\eta_i)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\eta_i))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\eta_i)^{\omega_i}} \right)$

TABLE 5 Frank class of WA aggregation operators

Aggregation operators	WA aggregation operator
DHFFWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{ 1 - \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\gamma_i}-1)^{\omega_i}\right)}{\log\theta} \right\}, \bigcup_{\eta_i \in \widetilde{g}_i, i=1,2,\ldots,n} \left\{ \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\eta_i}-1)^{\omega_i}\right)}{\log\theta} \right\}\right)$
IVHFFWA	$\left(\bigcup_{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{\left[1 - \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta}, 1 - \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\gamma_i^u}-1)^{\omega_i}\right)}{\log\theta} \right]\right\}\right)$
HFFWA	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n} \left\{ 1 - \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\gamma_i}-1)^{\omega_i}\right)}{\log\theta} \right\}\right)$
IVIFFWA	$\left(\left[1 - \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta}, 1 - \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\gamma_i^u}-1)^{\omega_i}\right)}{\log\theta} \right],\right.$
	$\left.\left[\frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\eta_i^l}-1)^{\omega_i}\right)}{\log\theta}, \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\eta_i^u}-1)^{\omega_i}\right)}{\log\theta} \right]\right)$
IFFWA	$\left(1 - \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\gamma_i}-1)^{\omega_i}\right)}{\log\theta}, \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\eta_i}-1)^{\omega_i}\right)}{\log\theta} \right)$

TABLE 6 Algebraic WG aggregation operators

Aggregation operators	WG aggregation operator
DHFWG	$\left(\bigcup_{\gamma_i\in\widetilde{h}_i,i=1,2,\ldots,n.}\left\{\prod_{i=1}^{n}(\gamma_i)^{\omega_i}\right\},\bigcup_{\eta_i\in\widetilde{g}_i,i=1,2,\ldots,n.}\left\{1-\prod_{i=1}^{n}(1-\eta_i)^{\omega_i}\right\}\right)$
IVHFWG	$\left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,i=1,2,\ldots,n.}\left\{\left[\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i},\prod_{i=1}^{n}(\gamma_i^u)^{\omega_i}\right]\right\}\right)$
HFWG	$\left(\bigcup_{\gamma_i\in\widetilde{h}_i,i=1,2,\ldots,n.}\left\{\prod_{i=1}^{n}(\gamma_i)^{\omega_i}\right\}\right)$
IVIFWG	$\left(\left[\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i},\prod_{i=1}^{n}(\gamma_i^u)^{\omega_i}\right],\left[1-\prod_{i=1}^{n}(1-\eta_i^l)^{\omega_i},1-\prod_{i=1}^{n}(1-\eta_i^u)^{\omega_i}\right]\right)$
IFWG	$\left(\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i},\ 1-\prod_{i=1}^{n}(1-\eta_i)^{\omega_i}\right)$

TABLE 7 Einstein class of WG aggregation operators

Aggregation operators	WG aggregation operator
DHFEWG	$\left(\bigcup_{\gamma_i\in\widetilde{h}_i,\ i=1,2,\ldots,n.}\left\{\frac{2\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n}(2-\gamma_i^l)^{\omega_i}+\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}\right\},\bigcup_{\eta_i\in\widetilde{g}_i,\ i=1,2,\ldots,n.}\left\{\frac{\prod_{i=1}^{n}(1+\eta_i^u)^{\omega_i}-\prod_{i=1}^{n}(1-\eta_i^u)^{\omega_i}}{\prod_{i=1}^{n}(1+\eta_i^u)^{\omega_i}+\prod_{i=1}^{n}(1-\eta_i^u)^{\omega_i}}\right\}\right)$
IVHFEWG	$\left(\bigcup_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,i=1,2,\ldots,n.}\left\{\left[\frac{2\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n}(2-\gamma_i^l)^{\omega_i}+\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}},\frac{2\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n}(2-\gamma_i^l)^{\omega_i}+\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}\right]\right\}\right)$
HFEWG	$\left(\bigcup_{\gamma_i\in\widetilde{h}_i,i=1,2,\ldots,n.}\left\{\frac{2\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n}(2-\gamma_i^l)^{\omega_i}+\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}\right\}\right)$
IVIFEWG	$\left(\left[\frac{2\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n}(2-\gamma_i^l)^{\omega_i}+\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}},\frac{2\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n}(2-\gamma_i^l)^{\omega_i}+\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}\right],\left[\frac{\prod_{i=1}^{n}(1+\eta_i^u)^{\omega_i}-\prod_{i=1}^{n}(1-\eta_i^u)^{\omega_i}}{\prod_{i=1}^{n}(1+\eta_i^u)^{\omega_i}+\prod_{i=1}^{n}(1-\eta_i^u)^{\omega_i}},\frac{\prod_{i=1}^{n}(1+\eta_i^u)^{\omega_i}-\prod_{i=1}^{n}(1-\eta_i^u)^{\omega_i}}{\prod_{i=1}^{n}(1+\eta_i^u)^{\omega_i}+\prod_{i=1}^{n}(1-\eta_i^u)^{\omega_i}}\right]\right)$
IFEWG[41]	$\left(\frac{2\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n}(2-\gamma_i^l)^{\omega_i}+\prod_{i=1}^{n}(\gamma_i^l)^{\omega_i}},\frac{\prod_{i=1}^{n}(1+\eta_i^u)^{\omega_i}-\prod_{i=1}^{n}(1-\eta_i^u)^{\omega_i}}{\prod_{i=1}^{n}(1+\eta_i^u)^{\omega_i}+\prod_{i=1}^{n}(1-\eta_i^u)^{\omega_i}}\right)$

Case 3. (**Hamacher Class**). When $f(t)=\log\left(\frac{\theta+(1-\theta)t}{t}\right)$, $\theta>0$, then the ADHFWA, AIVHFWA, AHFWA, AIVIFWA, and AIFWA operators are converted into the DHFHWA, IVHFHWA, HFHWA,[37] IVIFHWA,[35] and IFHWA operators, respectively, which are depicted in the following Table 4.

Case 4. (**Frank Class**). When $f(t)=\log\left(\frac{\theta-1}{\theta^t-1}\right)$, $\theta>1$, the ADHFWA, AIVHFWA, AHFWA, AIVIFWA, and AIFWA operators are reduced to the DHFFWA,[38] IVHFFWA, HFFWA,[39] IVIFFWA,[40] and IFFWA operators, respectively, which are described in the following Table 5.

Similar cases arise for WG aggregation operators which are presented in Tables 6-9.

5 | A METHODOLOGY TO SOLVE MCDM PROBLEMS USING IVDHF

In this section, the developed AIVDHFWA and AIVDHFWG operators are applied to formulate a method for solving MCDM problems having IVDHF information. Let $A=\{A_1,A_2,\ldots,A_m\}$ be a collection of alternatives which are to be selected, and $C=\{C_1,C_2,\ldots,C_n\}$ be a set of criteria with their weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$, satisfying $\omega_j\in[0,1]$ for $j=1,2,\ldots,n$, and $\sum_{j=1}^{n}\omega_j=1$, where ω_j represents the preference grade of the criterion C_j.

After evaluating the alternatives based on criteria an IVDHF decision matrix (IVDHFDM), $\widetilde{K}=(\widetilde{k}_{ij})_{m\times n}$ is constructed by some DM. Two kinds of attributes may be associated with the problem, benefit attributes, and cost attributes.

TABLE 8 Hamacher class of WG aggregation operators

Aggregation operators	WG aggregation operator
DHFHWG	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i,\ i=1,2,\ldots,n.} \left\{ \frac{\theta \prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\gamma_i^l))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}} \right\}, \bigcup_{\eta_i \in \widetilde{g}_i,\ i=1,2,\ldots,n} \left\{ \frac{\prod_{i=1}^{n} (1+(\theta-1)\eta_i^u)^{\omega_i} - \prod_{i=1}^{n} (1-\eta_i^u)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\eta_i^u)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\eta_i^u)^{\omega_i}} \right\}\right)$
IVHFHWG	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i,\ i=1,2,\ldots,n.} \left\{ \left[\frac{\theta \prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\gamma_i^l))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}, \frac{\theta \prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\gamma_i^l))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}} \right] \right\}\right)$
HFHWG	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n.} \left\{ \frac{\theta \prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\gamma_i^l))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}} \right\}\right)$
IVIFHWG	$\left(\left[\frac{\theta \prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\gamma_i^l))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}, \frac{\theta \prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\gamma_i^l))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}} \right], \left[\frac{\prod_{i=1}^{n} (1+(\theta-1)\eta_i^u)^{\omega_i} - \prod_{i=1}^{n} (1-\eta_i^u)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\eta_i^u)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\eta_i^u)^{\omega_i}}, \frac{\prod_{i=1}^{n} (1+(\theta-1)\eta_i^u)^{\omega_i} - \prod_{i=1}^{n} (1-\eta_i^u)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\eta_i^u)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\eta_i^u)^{\omega_i}} \right]\right)$
IFHWG	$\left(\frac{\theta \prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)(1-\gamma_i^l))^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (\gamma_i^l)^{\omega_i}}, \frac{\prod_{i=1}^{n} (1+(\theta-1)\eta_i^u)^{\omega_i} - \prod_{i=1}^{n} (1-\eta_i^u)^{\omega_i}}{\prod_{i=1}^{n} (1+(\theta-1)\eta_i^u)^{\omega_i} + (\theta-1)\prod_{i=1}^{n} (1-\eta_i^u)^{\omega_i}} \right)$

TABLE 9 Frank class of WG aggregation operators

Aggregation operators	WG aggregation operator
DHFFWG	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i,\ i=1,2,\ldots,n} \left\{ \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta} \right\}, \bigcup_{\eta_i \in \widetilde{g}_i,\ i=1,2,\ldots,n} \left\{ 1-\frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\eta_i^l}-1)^{\omega_i}\right)}{\log\theta} \right\}\right)$
IVHFFWG	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i,\ i=1,2,\ldots,n} \left\{ \left[\frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta}, \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta} \right] \right\}\right)$
HFFWG	$\left(\bigcup_{\gamma_i \in \widetilde{h}_i, i=1,2,\ldots,n.} \left\{ \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta} \right\}\right)$
IVIFFWG	$\left(\left[\frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta}, \frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta} \right], \left[1-\frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\eta_i^l}-1)^{\omega_i}\right)}{\log\theta}, 1-\frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\eta_i^l}-1)^{\omega_i}\right)}{\log\theta} \right]\right)$
IFFWG	$\left(\frac{\log\left(1+\prod_{i=1}^{n} (\theta^{\gamma_i^l}-1)^{\omega_i}\right)}{\log\theta},\ 1-\frac{\log\left(1+\prod_{i=1}^{n} (\theta^{1-\eta_i^l}-1)^{\omega_i}\right)}{\log\theta} \right)$

Now, in MCDM if the model is associated with the later kind of attributes, then those cost attributes are required to be converted into former attributes, that is, convert the IVDHFDM $\widetilde{K} = (\widetilde{k}_{ij})_{m\times n}$ into a normalized IVDHFDM $\widetilde{sR} = (\widetilde{r}_{ij})_{m\times n}$ where

$$\widetilde{r}_{ij} = \begin{cases} \widetilde{k}_{ij} \text{ for benefit attribute } C_j \\ \widetilde{k}^c_{ij} \text{ for cost attribute } C_j \end{cases} \tag{19}$$

$i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Suppose now that $\widetilde{R} = (\widetilde{r}_{ij})_{m\times n}$ be an IVDHFDM.

Then for solving MCDM problems with IVDHFEs, the following step by step executions is presented:

Step 1. Transform the IVDHFDM $\widetilde{K} = (\widetilde{k}_{ij})_{m\times n}$ into the normalized IVDHFDM $\widetilde{R} = (\widetilde{r}_{ij})_{m\times n}$ on the basis of Equation (19).

Step 2. Aggregate the IVDHFEs $\widetilde{r}_{ij}$ for each alternative z_i using the AIVDHFWA (or AIVDHFWG) operator as follows:

$$\widetilde{r}^A_i = \text{AIVDHFWA}(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}) = \left(\bigcup_{[\gamma^l_{ij},\gamma^u_{ij}]\in\widetilde{h}_{ij}|j=1,2,\ldots,n.} \left\{ \left[g^{-1}\left(\sum_{j=1}^n \omega_j g(\gamma^l_{ij})\right), g^{-1}\left(\sum_{j=1}^n \omega_j g(\gamma^u_{ij})\right) \right] \right\}, \right.$$

$$\left. \bigcup_{[\eta^l_{ij},\eta^u_{ij}]\in\widetilde{g}_{ij}|j=1,2,\ldots,n.} \left\{ \left[f^{-1}\left(\sum_{j=1}^n \omega_j f(\eta^l_{ij})\right), f^{-1}\left(\sum_{i=1}^n \omega_j f(\eta^u_{ij})\right) \right] \right\} \right)$$

or

$$\widetilde{r}^G_i = \text{AIVDHFWG}(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}) = \left(\bigcup_{[\gamma^l_{ij},\gamma^u_{ij}]\in\widetilde{h}_{ij}|j=1,2,\ldots,n.} \left\{ \left[f^{-1}\left(\sum_{j=1}^n \omega_j f(\gamma^l_{ij})\right), f^{-1}\left(\sum_{j=1}^n \omega_j f(\gamma^u_{ij})\right) \right] \right\}, \right.$$

$$\left. \bigcup_{[\eta^l_{ij},\eta^u_{ij}]\in\widetilde{g}_{ij}|j=1,2,\ldots,n.} \left\{ \left[g^{-1}\left(\sum_{j=1}^n \omega_j g(\eta^l_{ij})\right), g^{-1}\left(\sum_{i=1}^n \omega_j g(\eta^u_{ij})\right) \right] \right\} \right)$$

$$i = 1, 2, \ldots, m; j = 1, 2, \ldots, n.$$

Step 3. Using Definition 5, find the score and accuracy values of the alternatives.

Step 4. Based on the achieved score values find the ranking of alternatives.

6 | ILLUSTRATIVE EXAMPLE

To establish the application potentiality of the developed methodology, a modified version of a practical problem adapted from an article presented by Wei et al[42] is considered.

A company wants to capitalize funds in the best company. There are five possible companies, viz., A_1, A_2, A_3, A_4, and A_5, in which the money can be invested. The investment company considered four criteria of the alternatives, viz., C_1, C_2, C_3, and C_4 with the weight vector $w = (0.3,0.1,0.2,0.4)^T$. For avoiding influence to each other, the DM evaluated the five possible alternatives A_i $(i = 1, 2, \ldots, 5)$ with the weight vector and the decision matrix $\widetilde{R} = (\widetilde{r}_{ij})_{m\times n}$ is presented in Table 10, where $\widetilde{r}_{ij}$ $(i = 1, 2, \ldots, 5; j = 1, 2, 3, 4)$ are in the form of IVDHFEs. Then the developed methodology is applied to find the most appropriate alternative(s).

Step 1. It is assumed here that all the criteria C_j $(j = 1, 2, 3, 4)$ are benefit criteria. So, the performance values of the alternatives A_i $(i = 1, 2, 3, 4, 5)$ are not required for normalization.

Step 2. Use the aggregation operators IVDHFHWA as described in Equation (3) for aggregation of preference values $\widetilde{r}_{ij}$ for every alternatives A_i and obtain $\widetilde{r}^A_i$ $(i = 1, 2, 3, 4, 5)$.

Step 3. The score value of $\widetilde{r}^A_i$ $(i = 1, 2, 3, 4, 5)$ for each candidate is calculated utilizing Definition 5.

Step 4. Since $S(\widetilde{r}^A_3) > S(\widetilde{r}^A_4) > S(\widetilde{r}^A_2) > S(\widetilde{r}^A_5) > S(\widetilde{r}^A_1)$, the ordering of the alternatives A_i $(i = 1, 2, \ldots, 5)$ is determined as $A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1$ for $\theta \in [0, 50]$. Thus, the best alternative is found as A_3.

Varying the parameter $\theta \in [0, 50]$, the variation of score values are found using the IVDHFHWA operator and are presented in Figure 2. Here it is evidenced that when the value of the parameter θ increases from 0 to 50, the score values decrease.

TABLE 10 Decision matrix with IVDHFEs

	C_1	C_2	C_3	C_4
A_1	({[0.2,0.3], [0.3,0.4]}, {[0.2,0.3], [0.4,0.5]})	({[0.2,0.5]}, {[0.3,0.4]})	({[0.7,0.8], [0.8,0.9]}, {[0.05,0.1]})	({[0.4,0.5]}, {[0.1,0.2], [0.2,0.3], [0.3,0.4]})
A_2	({[0.2,0.3],[0.4,0.5]}, {[0.3,0.4]})	({[0.3,0.4],[0.6,0.7]}, {[0.05,0.1],[0.1,0.2],[0.2,0.3]})	({[0.3,0.4]}, {[0, 2, 0.3], [0.4,0.5]})	({[0.5,0.6], [0.8,0.9]}, {[0.05,0.1]})
A_3	({[0.5,0.7]}, {[[0.1,0.2],[0.2,0.3]]})	({[0.2,0.3],[0.4,0.5]}, {[0.1,0.2]})	({[0.8,0.9], [0.9,0.98]}, {[0.01,0.05]})	({[0.3,0.5]}, {[0.3,0.4]})
A_4	({[0.3,0.4],[0.7,0.8]}, {[0.05,0.1]})	({[0.1,0.3]}, {[0.2,0.3],[0.4,0.5]})	({[0.6,0.7], [0.8,0.9]}, {[0.01,0.05]})	({[0.5,0.7]}, {[0.05,0.1], [0.2,0.3]})
A_5	({[0.2,0.3]}, {[0.4,0.6]})	({[0.4,0.6]}, {[0.3,0.4]})	({[0.2,0.3], [0.6,0.7]}, {[0.1,0.3]})	({[0.6,0.7]}, {[0.01,0.1], [0.1,0.2]})

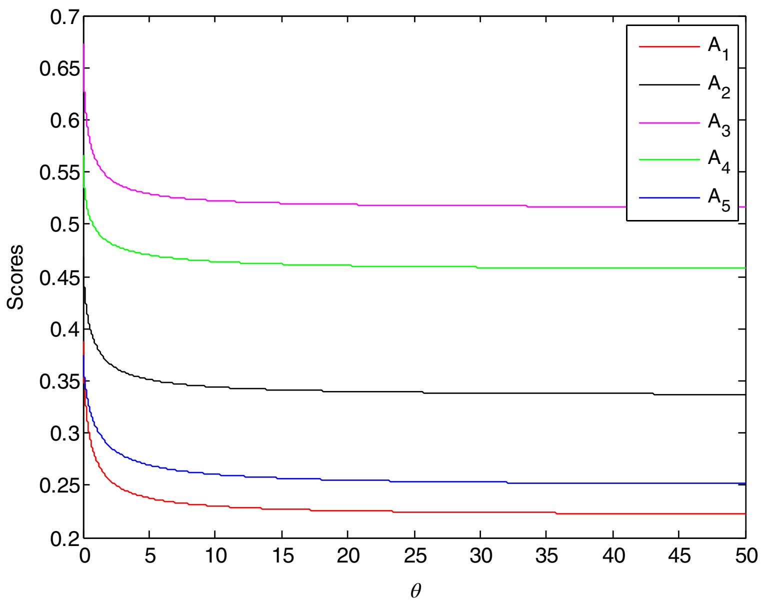


FIGURE 2 Geometric interpretation of score values for alternatives utilized by the operator IVDHFHWA

Now, when the IVDHFHWG operator is utilized, the obtained score values corresponding to alternatives are presented in Figure 3. From Figure 3, it is noticed that when the values of the parameter θ are increased from 0 to 50, the score values also increase.

Furthermore, the following observations are found:

1. when $\theta \in (0, 1.351)$, the ordering of the five alternatives is $A_3 \succ A_4 \succ A_2 \succ A_1 \succ A_5$, and the best choice is found as A_3.
2. when $\theta \in (1.351, 50]$, the ordering of the five alternatives is $A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1$, and the best choice is identified as A_3.
3. For $\theta = 1.351$ the ranking of the five alternatives is $A_3 \succ A_4 \succ A_2 \succ A_5 = A_1$, therefore A_5 and A_1 may be interchanged in the above ordering.

Now if IVDHFFWA operator is used then the ordering of alternatives become $A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1$ and varying the parameter $\theta \in [0, 50]$, score values of alternatives is presented in Figure 4. It is noticeable that when the values of the parameter θ are increased from 1 to 50, the score values decrease.

Again, if IVDHFFWG operator is used, the score value of the alternatives is presented in Figure 5. Here it is found that when the values of the parameter θ are increased from 1 to 50, the score values also increase.

Furthermore, the following variations are obtained in the ordering of alternatives.

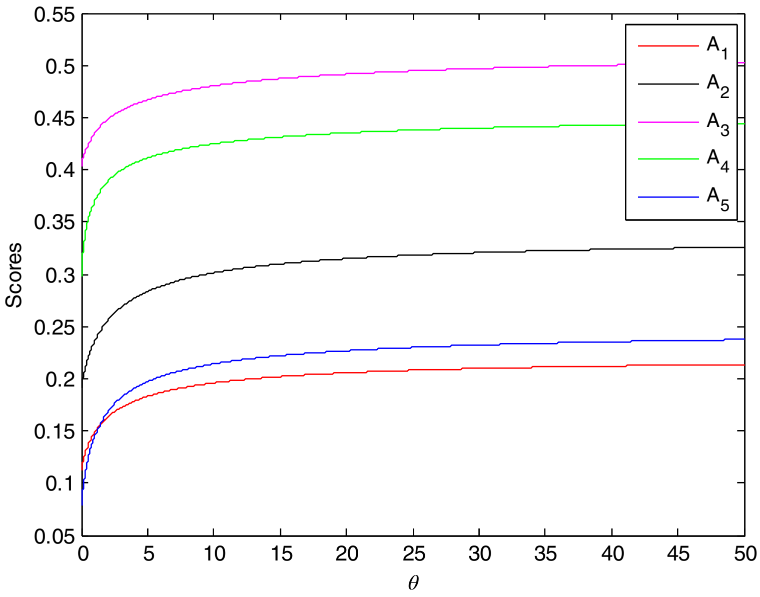


FIGURE 3 Geometric interpretation of score values for alternatives utilized by the operator IVDHFHWG

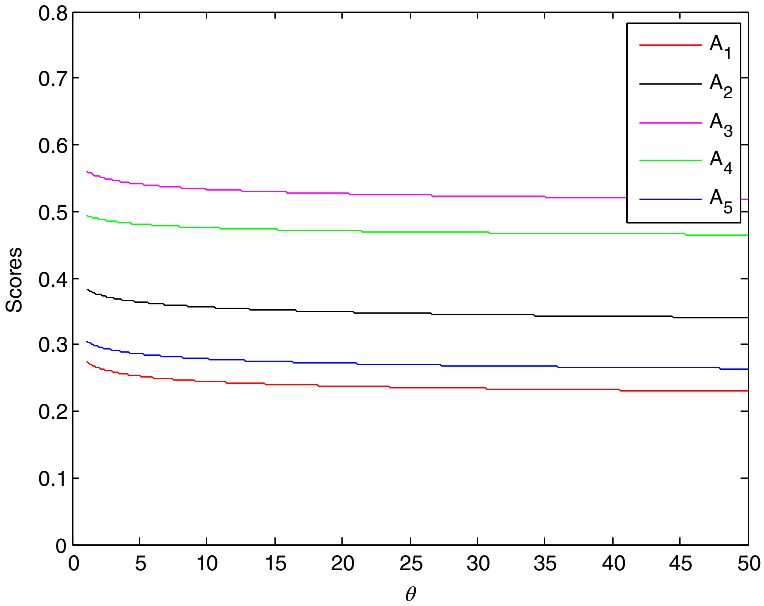


FIGURE 4 Geometric interpretation of score values for alternatives utilized by the operator IVDHFFWA

1. when $\theta \in (0, 1.905)$, the ordering of the five alternatives is $A_3 \succ A_4 \succ A_2 \succ A_1 \succ A_5$, and the best choice is A_3.
2. when $\theta \in (1.905, 50]$, the ordering of the five alternatives is $A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1$, and the best choice is A_3.
3. For $\theta = 1.905$ the ordering of the five alternatives is $A_3 \succ A_4 \succ A_2 \succ A_5 = A_1$, therefore A_5 and A_1 may be interchanged in the above ordering.

It is worthy to mention here that the ranking achieved by Wei et al[42] is $A_3 \succ A_4 \succ A_2 \succ A_1 \succ A_5$ and $A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$ using HIVFCOA and HIVFCOG operators, respectively. But using the proposed approach, the ranking of alternatives remains almost same using averaging as well as geometric operators. Thus, the proposed methodology is more consistent than the technique developed by Wei et al.[42]

From the above discussions, it is also clear that the score values are highly depending on the parameter θ, which reflects preference of the DMs in MCDM situations. Thus, the proposed method is flexible enough to establish DMs' preferences on the alternatives.

From the above comparisons, it is evidenced that the parameter θ is appeared as a reflection of the DMs' preferences. Based on the value of the parameter θ, the scores of the alternatives are different, and the rankings of the alternatives are

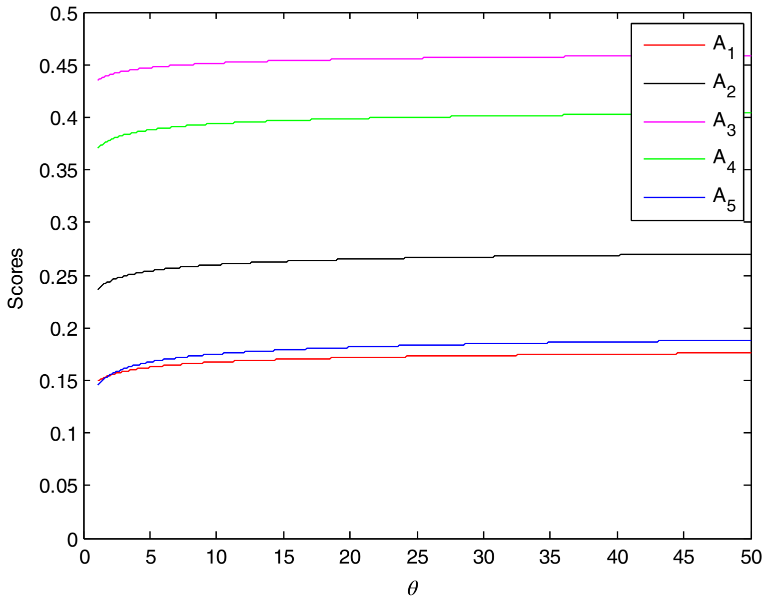


FIGURE 5 Geometric interpretation of score values for alternatives utilized by the operator IVDHFFWG

also varying. Therefore, the proposed aggregation operators can provide the DMs more choices and thus the proposed method is more flexible than the existing ones.

7 | COMPARATIVE STUDIES

It has already been mentioned that the proposed aggregation operators are capable of covering wide range of operators through At-N&t-CNs. Those operators are also applicable for aggregating many other variants of fuzzy sets. To establish these facts several examples are considered and solved using the proposed method. A list of methods, which are considered for this purpose, is presented in the following Table 11.

It is to be noted here that all the above problems are solved in the respective environments with the help of the proposed aggregation operators by considering different values of the parameter, θ. It is interesting to observe that the solutions of the existing methods are appeared as a special case for some particular value of θ in the respective environments.

The achieved solutions in different environments as described in Table 11 using the proposed averaging as well as geometric operators with the variation of the parameter $\theta \in [0, 50]$ is presented in Table 12.

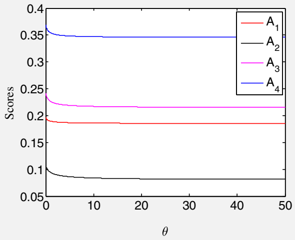
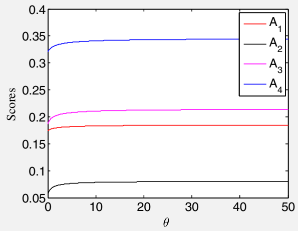
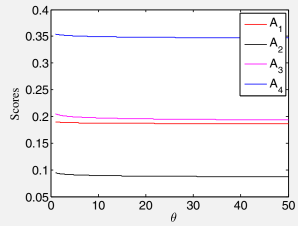
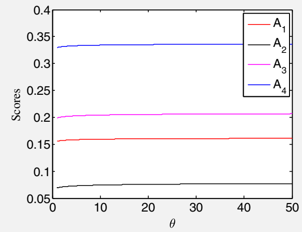
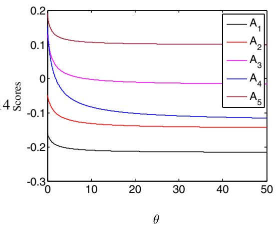
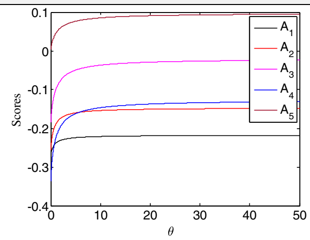
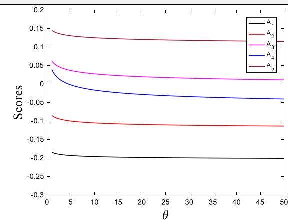
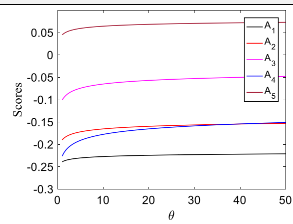
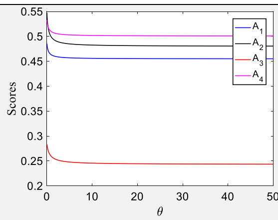
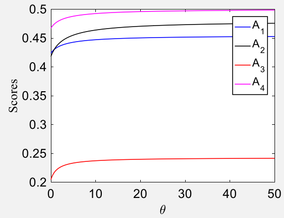
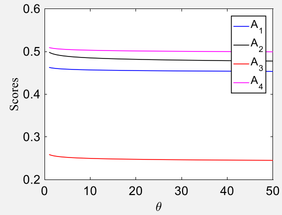
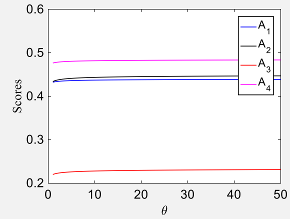
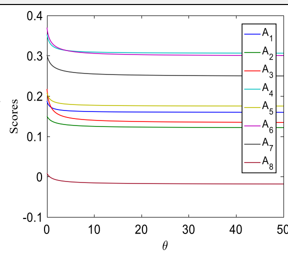
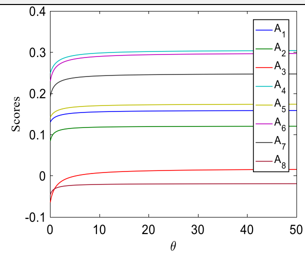
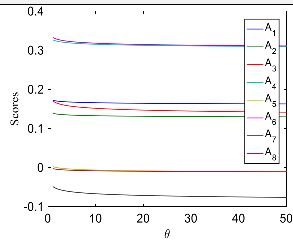
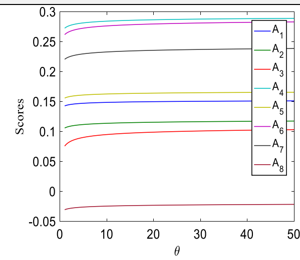
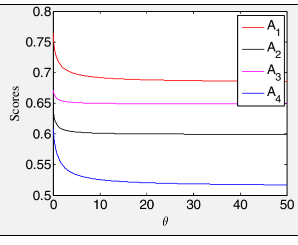
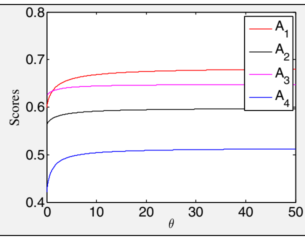
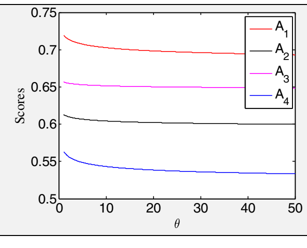
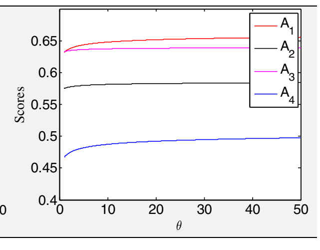
Note. Ju et al[10] solved a numerical example on IVDHF environment to find best alternative. Now, if that problem is solved by the proposed IVDHFHWA operator and the value of the parameter θ is considered as 1, the score value of the alternatives is found as same as the score values achieved by Ju et al. Therefore, the method introduced by Ju et al[10] is now appeared as a particular case of the proposed method. Apart from that the score value of A_is is checked for different values of the parameter θ by varying it in the range $[0, \infty)$. Moreover, this problem can be solved using the Frank class of aggregation operators. It is found that the ordering of the alternatives remains same as like the ordering achieved using Hamacher class of aggregation operators.

Now, the same problem[10] was solved by Zhang et al[16] using IVDHFEWA operator. Since Einstein operation can be constructed from Hamcher operation by considering the value of the parameter θ as 2, the result obtained by Zhang et al[16] appeared as a particular case of the proposed IVDHFHWA operator.

TABLE 11 Methods for solving MCDM in different variants of fuzzy contexts

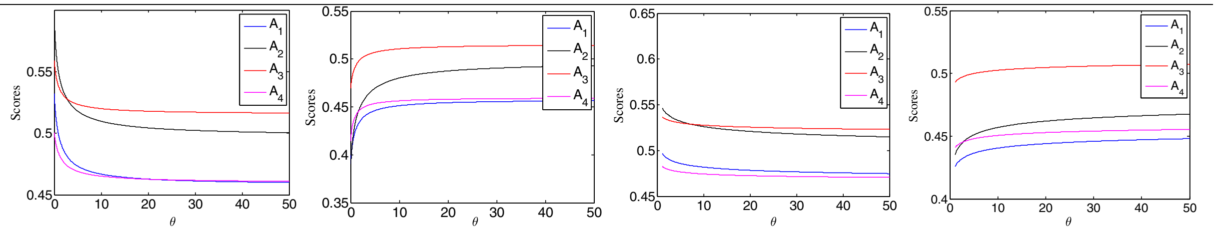
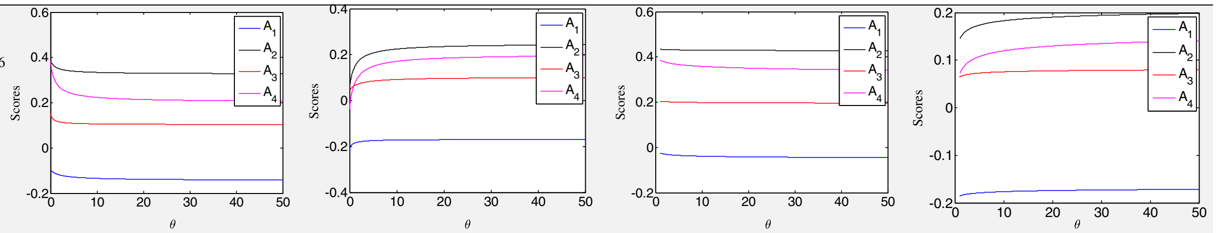
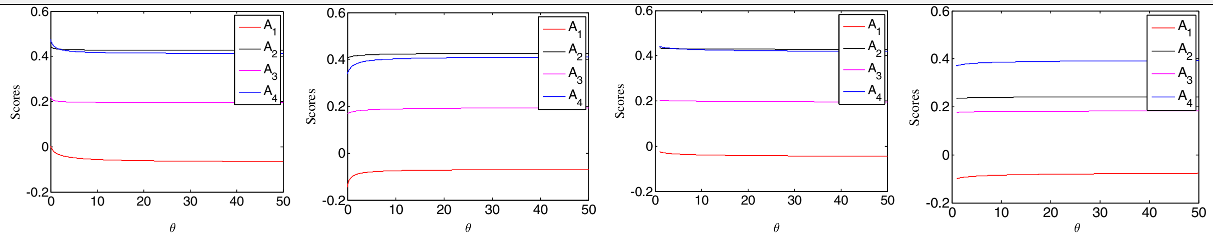
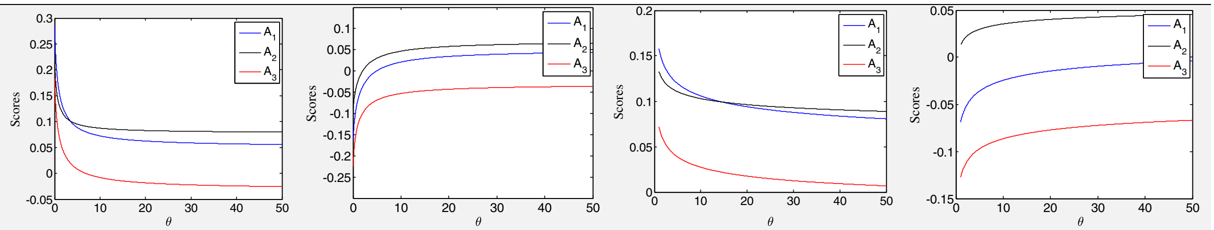
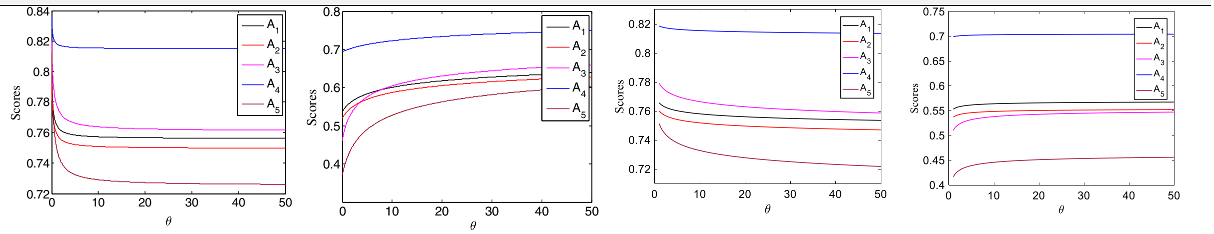
Method	Situations
Ju et al[10]; Zhang et al[16]	IVDHF
Wang et al[14]; Yu et al[43]; Zhao et al[44]	DHF
Yu[34]; Wang and Liu[45]	HF
Wang et al[46]; Nayagam et al[47]	IVIF
Wang and Liu[36]; Wang and Liu[48]	IF

TABLE 12 Comparison of results and achieved score values by varying θ with geometric interpretations

	Generalized Solution using Proposed Methodology			
Existing methods	**Hamacher Averaging Aggregation** Operator	**Hamacher Geometric Aggregation Operator**	**Frank Averaging Aggregation Operator**	**Frank Geometric Aggregation Operator**
Ju et al.[10] and Zhang et al.[16]				
Wang et al.[14]				
Yu et al.[43]				
Zhao et al.[44]				
Yu[34]				

(Continues)

TABLE 12 (Continued)

Wang and Liu[45]	
Wang et al.[46]	
Nayagam et al.[47]	
Wang and Liu[36]	
Wang and Liu[48]	

Furthermore, Wang et al[14] solved another MCDM problem for potential evaluation to choose best emerging technology enterprises utilizing DHFWA and DHFWG operators under DHF environment. As a special case of IVDHFS, Wang et al's results are easily achieved for the value of $\theta = 1$ of the DHFHWA and DHFHWG operators, respectively, and by transforming IVDHF to DHF by assuming the upper and lower limits as same.

Yu et al[43] considered a problem in DHF context for selecting foreign company for outsourcing human resource services using DHFWA and DHFWG operators. Considering $\theta = 1$ in the proposed DHFHWA and DHFHWG operators, the same ranking of alternatives as like Yu et al[43] are found as $A_1 \succ A_4 \succ A_2 \succ A_3$ and $A_4 \succ A_1 \succ A_3 \succ A_2$.

Zhao et al[44] in DHF environment solved a numerical example by considering eight alternatives using EDHFWA operator. The ordering of the alternatives is found as $A_4 \succ A_6 \succ A_7 \succ A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_8$ which corresponds to $\theta = 2$ in the proposed DHFHWA operator as like the previous arguments.

Yu[34] solved a numerical example using HF information based on Einstein operation and found best alternative with HFEWA and HFEWG operators as $A_1 \succ A_3 \succ A_2 \succ A_4$, which corresponds to $\theta = 2$ in the proposed HFHWA and HFHWG operators with similar arguments.

Wang and Liu[45] again solved another MCDM problem on HF environment using HFWG and HFEWG operators and found the ordering of the alternatives as $A_3 \succ A_4 \succ A_2 \succ A_1$ and $A_3 \succ A_2 \succ A_4 \succ A_1$, respectively. The similar results are obtained when $\theta = 1$ and 2 are considered in the proposed HFHWG operator. It is to be noted here that the changing of the ordering of A_4 and A_2 is clearly reflected in the graphical presentation of the proposed method.

Wang et al[46] solved another MCDM problem in a realistic scenario with four alternatives using IVIFWA operator and found the ordering as $A_2 \succ A_4 \succ A_3 \succ A_1$, which is the same as the ordering obtained by the proposed IVIFHWA operator considering $\theta = 1$.

Nayagam et al[47] presented an illustrative example of MCDM under IVIF situation. To aggregate the IVIFNs they utilized IVIFWA and IVIFWG operators and corresponding results are obtained as $A_2 \succ A_4 \succ A_3 \succ A_1$ and $A_2 \succ A_3 \succ A_4 \succ A_1$. These results correspond to $\theta = 1$ of the proposed IVIFHWA and IVIFHWG operators.

Wang and Liu[36] solved a numerical problem under intuitionistic fuzzy context. And using the aggregation operators IFWA and IFEWA, the ordering of three alternatives are found as $A_3 \succ A_2 \succ A_1$, which corresponds to $\theta = 1$ and 2, respectively, of the proposed method.

Furthermore, Wang and Liu[48] considered an MCDM problem on the same environment related to information technology improvement projects with five alternatives. Using IFWG and IFEWG operators, the ranking are presented as $A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_5$. The same result is achieved by the proposed IFHWG operator when $\theta = 1$ and 2, respectively, are assigned in the respective environment.

8 | SENSITIVITY ANALYSIS

It has already been mentioned that all the problems as presented in Table 11 are solved by varying the parameter $\theta \in [0, \infty)$. The ranking of alternatives based on the achieved score values are presented in Table 12 considering the parameter $\theta \in [0, 50]$. It is found that the ordering of alternatives is changed based on the value of θ. For clear presentation, Table 13 is presented by mentioning the ranges of θ in which the ranking of alternatives remain same for each of the method compared herein.

As for example, it is to be noted here that under the IVDHF environment, the ranking of alternatives achieved by Ju et al[10] and Zhang et al[16] is the same with the proposed Hamacher and Frank classes of t-Ns&t-CNs-based operators for all positive real values of the parameter θ.

When the problem of Wang et al[14] is solved by the introduced method, the change of the ordering of alternatives is found for the value of θ in several subinterval of $\mathbb{R}^+$. When the value of the parameter θ is considered in (0,0.3012), the ranking of the alternatives is $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$, and for $\theta \in (0.3012, \infty)$ the ordering is obtained as $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$, when DHFHWA operator is used. Furthermore, using the DHFHWG operator, ordering of the alternatives is changed in three intervals, but the best alternative would be the same as A_5. Beyond the value 5.7750 of θ, the ranking is fixed as $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$. Again, applying DHFFWA operator, the ranking is found as $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$ in $\theta \in (1, \infty)$ and for DHFFWG operator the ranking remains same after the value of $\theta = 39.1300$. Therefore, the best ranking result is $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$.

After solving the example of Yu et al[43] in DHF environment by DHFHWA operator, the first position of the alternatives differs between A_2 and A_4 about the point $\theta = 0.2453$. Whereas the ranking remains the same as $A_4 \succ A_2 \succ A_1 \succ A_3$ after the value of $\theta = 0.2453$. From that view point, the best alternative is A_4.

Furthermore, using DHFHWA operator to the example of Zhao et al,[44] the ordering of alternatives varies with the change of the parameter θ. For $\theta \in (0, 0.1445~)$ the ordering is $A_6 \succ A_4 \succ A_7 \succ A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_8$ and the position of A_3 and A_5 is interchanged when the value of the parameter is considered as $\theta \in (0.1445, 0.8915~)$. Again, it is noticed that the ranking becomes $A_6 \succ A_4 \succ A_7 \succ A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_8$ in the span of the interval $\theta \in (0.8915, 4.0550~)$. Though the ordering of the alternatives differs for $\theta \in [0, 4.0550]$, but the ranking is fixed for $\theta \in (4.0550, \infty)$ as $A_4 \succ A_6 \succ A_7 \succ A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_8$.

The HFHWA and HFHWG operators are used to solve Yu's[34] problem. When $\theta \in (0, \infty)$ in HFHWA, the ranking is found as $A_1 \succ A_3 \succ A_2 \succ A_4$, and by using HFHWG operator, the ordering between A_1 and A_3 slightly differs about the point $\theta = 1.135$.

After solving out an example, considered by Wang and Liu,[45] it is observed that the best alternative is A_2 for $0 < \theta < 2.6940$ and A_3 for $2.6940 < \theta < 21.110$ using HFHWA operator. For $\theta > 21.11$, the ordering of the alternatives is found as $A_3 \succ A_2 \succ A_4 \succ A_1$ and it is appeared that the best alternative as A_3. Again, utilizing HFHWG operator, the ordering is found as $A_3 \succ A_4 \succ A_2 \succ A_1$ in the range $\theta \in [0, 1.5970]$ and for $\theta \in (1.5970, \infty)$ the ranking is found as $A_3 \succ A_2 \succ A_4 \succ A_1$. The later result is achieved when Frank-based aggregation operators are used.

TABLE 13 Ranking of alternatives based on the value of θ in different variants of fuzzy contexts

Existing method				Proposed method		
Author	**Context**	**Method**	**Result**	**Operator**	**Parameter**	**Ranking**
Ju et al[10]	IVDHF	IVDHFWA	$A_4 \succ A_3 \succ A_1 \succ A_2$	IVDHFHWA	$\theta \in [0, 50]$	$A_4 \succ A_3 \succ A_1 \succ A_2$
				IVDHFHWA	$\theta \in [0, 50]$	$A_4 \succ A_3 \succ A_1 \succ A_2$
Zhang et al[16]	IVDHF	IVDHFEWA	$A_4 \succ A_3 \succ A_1 \succ A_2$	DHFHWA	$\theta \in (0, 0.3012)$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
				DHFHWA	$\theta \in (0.3012, \infty)$	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
Wang et al[14]	DHF	DHFWA	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$	DHFHWG	$\theta \in (0,0.6948)$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
				DHFHWG	$\theta \in (0.6948, 5.7750)$	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
				DHFHWG	$\theta \in (5.7750, \infty)$	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
Yu et al[43]	DHF	DHFWA	$A_1 \succ A_4 \succ A_2 \succ A_3$	DHFHWA	$\theta \in (0,0.2453)$	$A_2 \succ A_4 \succ A_1 \succ A_3$
		DHFWG	$A_4 \succ A_1 \succ A_3 \succ A_2$	DHFHWA	$\theta \in (0.2453, \infty)$	$A_4 \succ A_2 \succ A_1 \succ A_3$
				DHFHWG	$\theta \in (0, 0.7269)$	$A_4 \succ A_1 \succ A_2 \succ A_3$
				DHFHWG	$\theta \in (0.7269, \infty)$	$A_4 \succ A_2 \succ A_1 \succ A_3$
Zhao et al[44]	DHF	HFEWA	$A_4 \succ A_6 \succ A_7 \succ A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_8$	DHFHWA	$\theta \in (0,0.1445)$	$A_6 \succ A_4 \succ A_7 \succ A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_8$
				DHFHWA	$\theta \in (0.1445, 0.8915)$	$A_6 \succ A_4 \succ A_7 \succ A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_8$
				DHFHWA	$\theta \in (0.8915, 4.0550)$	$A_6 \succ A_4 \succ A_7 \succ A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_8$
				DHFHWA	$\theta \in (4.0550, \infty)$	$A_4 \succ A_6 \succ A_7 \succ A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_8$
				DHFHWG	$\theta \in (0, 0.8510)$	$A_4 \succ A_6 \succ A_7 \succ A_5 \succ A_1 \succ A_2 \succ A_8 \succ A_3$
				DHFHWG	$\theta \in (0.8510, \infty)$	$A_4 \succ A_6 \succ A_7 \succ A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_8$
Yu[34]	HFS	HFEWA	$A_1 \succ A_3 \succ A_2 \succ A_4$	HFHWA	$\theta \in (0, \infty)$	$A_1 \succ A_3 \succ A_2 \succ A_4$
		HFEWG	$A_1 \succ A_3 \succ A_2 \succ A_4$	HFHWG	$\theta \in (0, 1.135)$	$A_3 \succ A_1 \succ A_2 \succ A_4$
				HFHWG	$\theta \in (1.135, \infty)$	$A_1 \succ A_3 \succ A_2 \succ A_4$
Wang and Liu[45]	HFS	HFWG	$A_3 \succ A_4 \succ A_2 \succ A_1$	HFHWA	$\theta \in (0, 2.6940)$	$A_2 \succ A_3 \succ A_1 \succ A_4$
		HFEWG	$A_3 \succ A_2 \succ A_4 \succ A_1$	HFHWA	$\theta \in (2.6940, 21.110)$	$A_3 \succ A_2 \succ A_1 \succ A_4$
				HFHWA	$\theta \in (21.110, \infty)$	$A_3 \succ A_2 \succ A_4 \succ A_1$
				HFHWG	$\theta \in (0, 0.2477)$	$A_3 \succ A_4 \succ A_1 \succ A_2$
				HFHWG	$\theta \in (0.2477, 1.5970)$	$A_3 \succ A_4 \succ A_2 \succ A_1$
				HFHWG	$\theta \in (1.5970, \infty)$	$A_3 \succ A_2 \succ A_4 \succ A_1$
Wang et al[46]	IVIFS	IVIFWA	$A_2 \succ A_4 \succ A_3 \succ A_1$	IVIFHWA	$\theta \in (0, \infty)$	$A_2 \succ A_4 \succ A_3 \succ A_1$
				IVIFHWG	$\theta \in (0, 0.8479)$	$A_2 \succ A_3 \succ A_4 \succ A_1$
				IVIFHWG	$\theta \in (0.8479, \infty)$	$A_2 \succ A_4 \succ A_3 \succ A_1$
Nayagam et al[47]	IVIFS	IVIFWA	$A_2 \succ A_4 \succ A_3 \succ A_1$	IVIFHWA	$\theta \in (0, 2.0440)$	$A_4 \succ A_2 \succ A_3 \succ A_1$
		IVIFWG				
				IVIFHWA	$\theta \in (2.0440, \infty)$	$A_2 \succ A_4 \succ A_3 \succ A_1$
				IVIFHWG	$\theta \in (0, \infty)$	$A_2 \succ A_4 \succ A_3 \succ A_1$
Wang and Liu[36]	IFS	IFWA	$A_1 \succ A_2 \succ A_3$	IFHWA	$\theta \in (0, 3.4150)$	$A_1 \succ A_2 \succ A_3$
		IFEWA	$A_1 \succ A_2 \succ A_3$	IFHWA	$\theta \in (3.4150, \infty)$	$A_2 \succ A_1 \succ A_3$
Wang and Liu[48]	IFS	IFWG	$A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_5$	IFHWA	$\theta \in (0, 0.2435)$	$A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$
				IFHWA	$\theta \in (0.2435,0.4017)$	$A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$
				IFHWA	$\theta \in (0.4017, \infty)$	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
					$\theta \in (0,3.5950)$	$A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_5$
					$\theta \in (3.5950, 7.8870)$	$A_4 \succ A_1 \succ A_3 \succ A_2 \succ A_5$
					$\theta \in (7.8870, \infty)$	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
				IFFWA	$\theta \in (1, \infty)$	$A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$
				IFFWG	$\theta \in (1, \infty)$	$A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_5$

Under IVIF environment, the example of Wang et al[46] is solved using the proposed method. It is noticed that the ranking $A_2 \succ A_4 \succ A_3 \succ A_1$ is found for $\theta \in (0, \infty)$ when IVIFHWA operator is used and the best alternative is found as A_2. Moreover, when IVIFHWG operator is used, the ordering becomes $A_2 \succ A_3 \succ A_4 \succ A_1$ for $\theta \in (0,\ 0.8479)$ and $A_2 \succ A_4 \succ A_3 \succ A_1$ for $\theta \in (0.8479, \infty)$.

When a problem of Nayagam et al[47] is solved, the ranking is obtained as $A_4 \succ A_2 \succ A_3 \succ A_1$ for $\theta \in (0,\ 2.0440)$; $A_2 \succ A_4 \succ A_3 \succ A_1$ for $\theta \in (2.0440, \infty)$ by considering IVIFHWA operator. Again, using IVIFHWG operator, the ranking is found as $A_2 \succ A_4 \succ A_3 \succ A_1$ for all real values of the parameter θ. Utilizing the IVIFFWA operator, the ordering becomes $A_4 \succ A_2 \succ A_3 \succ A_1$ for $\theta \in (1,\ 4.4940)$; $A_2 \succ A_4 \succ A_3 \succ A_1$ for $\theta \in (4.4940, \infty)$. Whereas, if IVIFFWG operator is used, the result is stable for $\theta > 1$ as $A_2 \succ A_4 \succ A_3 \succ A_1$.

The ordering of the alternatives achieved by Wang and Liu[36] is $A_1 \succ A_2 \succ A_3$. When IFHWA operator is applied, this result coincides in the span of the parameter θ within (0, 3.4150), but the score value of A_1 and A_2 is same at the point $\theta = 3.4150$. Also it is observed that the ranking is found in $\theta \in (3.4150, \infty)$ as $A_2 \succ A_1 \succ A_3$. Again, for IFHWG operator, the ranking is $A_2 \succ A_1 \succ A_3$ for $\theta > 0$. It is also shown that, when that example is solved using the Frank-based averaging and geometric aggregation operators, the result is obtained as $A_2 \succ A_1 \succ A_3$.

While another numerical problem of Wang and Liu[48] is solved by the proposed Hamacher-based aggregation operator, IFHWA, the ordering of the alternatives fluctuates for different values of the parameter θ lies in several subinterval of $\mathbb{R}^+$ as: when $\theta \in (0, 0.2435)$, the ranking is $A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$; $A_4 \succ A_3 \succ A_1 \succ A_5 \succ A_2$ for $\theta \in (0.2435, 0.4017)$; and $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$ for $\theta \in (0.4017, \infty)$. Again, when it is solved by IFHWG operator, ordering of the alternatives are varied for θ, the ranking is found $A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_5$ for $\theta \in (0, 3.5950)$; $A_4 \succ A_1 \succ A_3 \succ A_2 \succ A_5$ for $\theta \in (3.5950, 7.8870)$; and $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$ for $\theta \in (7.8870, \infty)$. Furthermore, it is observed that when this problem is resolved using Frank-based aggregation operators IFFWA and IFFWG, the ranking is appeared as the same as $A_4 \succ A_1 \succ A_2 \succ A_3 \succ A_5$ for $\theta > 1$.

From the above comparisons, it has been established that the parameter θ is appeared as the preferences of the DMs. Varying the value of the parameter θ, it is found that the scores of the alternatives vary, and the orderings of the alternatives are also changing. Therefore, the DMs have more flexibility in preferences while the developed aggregation operators are used by them and so the developed methodology is flexible enough than the existing methods.

9 | CONCLUSIONS

In the proposed method, At-N&t-CNs are used to aggregate IVDHFEs to find the most appropriate alternative in MCDM circumstances through the development of AIVDHFWA and AIVDHFWG operators. The developed operators are efficiently capable of capturing the concepts of other types of aggregation operators in different variants of fuzzy environments as discussed in the previous section. Apart from that, many new types of aggregation operators can be generated from the proposed method. In solving numerical examples, it is observed that in some cases, studied previously, the ranking of the alternatives slightly differs when the authors have used their respective methods. These glitches can be clearly visible when those problems have been solved through the proposed method. Though the ranks differ in the initial stages due to the parameter, these differences have been reduced for a maximum span in the later stages. Due to this reason, it can be stated that the proposed method is the best one as it is represented in a more flexible and generalized format. The proposed method can be extended to solve GDM problems by considering ordered WA operators along with attribute weights to make reasonable decision. With the development of the new aggregation operators, the process of making decisions in complex MCDM would become less tedious by the DMs and to study the properties of those operators may open up new area of research in the current complex decision making arena. In future, the proposed operators may be extended to complex intuitionistic fuzzy,[49] Pythagorean fuzzy[50,51] environments.

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CONFLICT OF INTEREST

The authors are hereby declaring no conflict of interest for this article.

AUTHOR CONTRIBUTIONS

Arun Sarkar contributed to the data curation and investigation and supported the methodology and writing the original draft, review, and editing. Animesh Biswas contributed to the data curation, investigation, methodology, and writing the original draft, review, and editing.

ORCID

Animesh Biswas https://orcid.org/0000-0002-5813-1076

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RESEARCH ARTICLE

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Multicriteria decision-making using Archimedean aggregation operators in Pythagorean hesitant fuzzy environment

Arun Sarkar | **Animesh Biswas**

Department of Mathematics, University of Kalyani, Kalyani, India

Correspondence
Animesh Biswas, Department of Mathematics, University of Kalyani, Kalyani, West Bengal 741235, India.
Email: abiswaskln@rediffmail.com

Abstract

In multicriteria decision-making (MCDM), the existing aggregation operators are mostly based on algebraic *t*-conorm and *t*-norm. But, Archimedean *t*-conorms and *t*-norms are the generalized forms of *t*-conorms and *t*-norms which include algebraic, Einstein, Hamacher, Frank, and other types of *t*-conorms and *t*-norms. From that view point, in this paper the concepts of Archimedean *t*-conorm and *t*-norm are introduced to aggregate Pythagorean hesitant fuzzy information. Some new operational laws for Pythagorean hesitant fuzzy numbers based on Archimedean *t*-conorm and *t*-norm have been proposed. Using those operational laws, Archimedean *t*-conorm and *t*-norm-based Pythagorean hesitant fuzzy weighted averaging operator and weighted geometric operator are developed. Some of their desirable properties have also been investigated. Afterwards, these operators are applied to solve MCDM problems in Pythagorean hesitant fuzzy environment. The developed Archimedean aggregation operators are also applicable in Pythagorean fuzzy contexts also. To demonstrate the validity, practicality, and effectiveness of the proposed method, a practical problem is considered, solved, and compared with other existing method.

KEYWORDS

Archimedean *t*-conorm and *t*-norm, hesitant fuzzy set, multicriteria decision-making, Pythagorean fuzzy set, weighted averaging operator, weighted geometric operator

1 | INTRODUCTION

Atanassov[1–3] introduced the concept of intuitionistic fuzzy set (IFS), which is a generalization of the concept of fuzzy set.[4] In IFS, the sum of the degrees of membership and nonmembership is less than or equal to 1. In real-life, IFS has been successfully applied to solve multicriteria decision-making (MCDM) problems.[5–12] Recently, Yager[13,14] introduced Pythagorean fuzzy set (PFS) as a more generalized version of fuzzy sets. PFS has emerged as a highly efficient tool for depicting uncertainty in MCDM problems. As like IFS, PFS is also characterized by the membership degree and the nonmembership degree of its elements. But it extends in a manner that the square of the sum of membership and nonmembership values is less than or equal to 1. As a consequence, IFSs are appeared as a special type of PFSs. Within its short span, PFSs has been studied extensively and applied to many practical areas successfully after its inception. Zhang and Xu[15] provided the detailed mathematical expression for PFS and introduced the concept of Pythagorean fuzzy number (PFN). Yager[14] proposed Pythagorean fuzzy weighted average (PFWA) operator, Pythagorean fuzzy weighted geometric average operator, Pythagorean fuzzy weighted power average operator, and Pythagorean fuzzy weighted power geometric average operator. Garg[16] proposed Pythagorean fuzzy Einstein weighted averaging (PFEWA), Pythagorean fuzzy Einstein ordered weighted averaging (PFEOWA), generalized PFEWA (GPFEWA), and generalized PFEOWA (GPFEOWA) operators. Biswas and Sarkar[17] developed Pythagorean fuzzy-dependent operators, and also proposed[18] a new technique for order preference by similarity to ideal solution (TOPSIS)-based methodology on Pythagorean fuzzy environment. After that, Peng and Yang[19] introduced interval-valued Pythagorean fuzzy (IVPF) set (IVPFS) and developed two aggregation operators for aggregating the IVPF information, such as IVPHF weighted average operator and IVPF weighted geometric operator. Biswas and Sarkar[20] developed point operator-based similarity measures for IVPF sets. Wei and Lu[21] proposed some Pythagorean fuzzy Maclaurin symmetric mean operators in multiattribute decision-making (MADM). Zeng et al[22] developed a hybrid method for Pythagorean fuzzy MCDM. Wei[23] developed Pythagorean fuzzy interaction aggregation operators for MADM.

However, in many real-life MCDM problems decision-makers (DMs) are unable to determine the exact membership degree of an element to a set due to some sort of doubts among a few different values. To solve these problems, Torra and Narukawa[24] and Torra[25] proposed the concept of hesitant fuzzy set (HFS), which permits the membership degree of an element to be a set of several possible values between 0 and 1. Subsequently, Zhu et al[26] defined dual hesitant fuzzy (DHF) set (DHFS) by considering the membership degree and nonmembership degree corresponding to an element to a given fixed set by assigning two sets of crisp values belonging to [0, 1]. It is to be noted here that DHFS can be regarded as an extension of IFS. Wang et al[27] developed some aggregation operators for DHFSs, namely, DHF power average operators, DHF power geometric operators, Einstein DHF power average, and the Einstein DHF power geometric operators. Biswas and Sarkar[28] proposed DHF prioritized Einstein weighted averaging operator and DHF prioritized Einstein weighted geometric operators in the recent past. Inspired by the idea of HFS and PFS, Wei et al[29] proposed the concept of Pythagorean HFS (PHFS). Wei et al[29] also introduced Pythagorean hesitant fuzzy Hamacher aggregation operators to solve multiple attribute MADM problems.

It is customary to mention here that most of the above aggregation methods are based on widely used algebraic product and algebraic sum operations. However, Archimedean t-conorm and t-norm[30,31] are the generalizations of different types of fuzzy set theoretic operations. Changing the generators, a wide class of t-conorms and t-norms can be covered up, namely, algebraic, Einstein, Hamacher, Frank, and other classes of t-conorms and t-norms. Thus operators based on Archimedean t-conorm and t-norm possess important significance to research in Pythagorean fuzzy environment, especially in aggregation methods.

The motivation of this paper is to introduce Archimedean t-conorm and t-norm to aggregate PHF information. Some of the operational laws on PHFS based on Archimedean t-conorm and t-norm have been defined. Archimedean t-conorm and t-norm-based PHF weighted averaging (APHFWA) operator and Archimedean t-conorm and t-norm-based PHF weighted geometric (APHFWG) operator are developed and some of their desirable properties have also been described. The developed operators are then used to solve MCDM problems with PHF information. Finally, a practical example, previously studied by Wei et al,[29] is considered and solved to illustrate the proposed method.

2 | PRELIMINARIES

To develop the proposed methodologies, some basic definitions related to PFSs[13,14] and operations on them are briefly discussed in this section.

2.1 | Pythagorean fuzzy set

Definition 1 (Yager[13,14]). Let X be a universe of discourse. A PFS P in X is given by

$$P = \{\langle x, \mu_p(x), \nu_p(x)\rangle \mid x \in X\},$$

where μ_p: $X \to [0,1]$ denotes the degree of membership and ν_p: $X \to [0,1]$ denotes the degree of nonmembership of the element $x \in X$ to the set P with the condition that

$$0 \leq (\mu_p(x))^2 + (\nu_p(x))^2 \leq 1.$$

The degree of indeterminacy is given by

$$\pi_p(x) = \sqrt{1 - (\mu_p(x))^2 - (\nu_p(x))^2}.$$

For convenience, Zhang and Xu[15] called $(\mu_p(x), \nu_p(x))$ as a PFN and is denoted by $p = (\mu_p, \nu_p)$.

For PFNs, Yager and Abbasov[32] introduced score and accuracy functions in the following manners.

Definition 2 (Yager and Abbasov[32]). For any PFN $p = (\mu_p, \nu_p)$, the score function of p is defined as follows:

$$S(p) = (\mu_p)^2 - (\nu_p)^2,$$

where $S(p) \in [-1, 1]$.

For a PFN $p = (\mu_p, \nu_p)$, the accuracy function of p is defined as

$$A(p) = (\mu_p)^2 + (\nu_p)^2.$$

Yager and Abbasov[32] proposed a ranking method of PFNs as follows.

Definition 3 (Zhang and Xu[15]). Let p_1 and p_2 be any two PFNs, then the ordering of those PFNs are done by the following principles:

- if $S(p_1) > S(p_2)$, then $p_1 \succ p_1$;
- if $S(p_1) = S(p_2)$, then

(1) if $A(p_1) > A(p_2)$, then $p_1 \succ p_2$;
(2) if $A(p_1) = A(p_2)$, then $p_1 \approx p_2$.

Four basic operations on PFNs are presented by Yager[13] and Yager and Abbasov[32] as follows.

Definition 4 (Yager[13]; Yager and Abbasov[32]). Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$ and $p_2 = (\mu_2, \nu_2)$ be three PFNs, and $\lambda > 0$, then some basic operations are defined as follows:

(1) $p_1 \oplus p_2 = \left(\sqrt{{\mu_1}^2 + {\mu_2}^2 - {\mu_1}^2{\mu_2}^2}, \nu_1\nu_2\right)$,
(2) $p_1 \otimes p_2 = \left(\mu_1\mu_2, \sqrt{{\nu_1}^2 + {\nu_2}^2 - {\nu_1}^2{\nu_2}^2}\right)$,
(3) $\lambda p = \left(\sqrt{1 - (1 - \mu^2)^\lambda}, \nu^\lambda\right), \lambda > 0$,
(4) $p^\lambda = \left(\mu^\lambda, \sqrt{1 - \left(1 - \nu^2\right)^\lambda}\right), \lambda > 0$.

2.2 | Pythagorean hesitant fuzzy set

Following the concepts of HFSs and PFSs, Wei et al extended HFSs by introducing the Pythagorean concept in it and defined a new variety of fuzzy set, called PHFS which is described as follows:

Definition 5 (Wai et al[29]). Let X be a universe of discourse. A PHFS on X is defined as

$$P = \{\langle x, h_P(x)\rangle | x \in X\},$$

where $h_P(x)$ is a set of possible PFNs defined on X.

For convenience, Wei et al[29] called $p = h_P(x) = \bigcup_{(\gamma,\eta)\in h_P(x)}\{(\gamma, \eta)\}$ a Pythagorean hesitant fuzzy number (PHFN) where (γ, η) is a PFN. A PHFN is generally denoted by $p = h_P = (\mu, \upsilon)$.

To compare the PHFNs, Wei et al[29] introduced the following ranking method.

For any PHFN $p = (\mu, \upsilon)$, the score function of p be defined as follows:

$$S(p) = \frac{1}{2}\left(1 + \frac{1}{l_h}\sum_{(\gamma,\eta)\in(\mu,\upsilon)} (\gamma^2 - \eta^2)\right), \tag{1}$$

where $S(p) \in [-1,\ 1]$.

For a PHFN $p = (\mu,\ \upsilon)$, the accuracy function of p be defined as follows:

$$A(p) = \frac{1}{l_h}\sum_{(\gamma,\eta)\in(\mu,\upsilon)} (\gamma^2 - \eta^2), \tag{2}$$

where l_h are the number of elements in $(\mu,\ \upsilon)$.

Definition 6 (Wai et al[29]). Let p_1 and p_2 be any two PHFNs, then the ordering of those two PHFNs are given by

- if $S(p_1) > S(p_2)$, then p_1 is superior to p_2, denoted by $p_1 \succ p_2$;
- if $S(p_1) = S(p_2)$, then

(1) If $A(p_1) > A(p_2)$, then $p_1 \succ p_2$;

if $A(p_1) = A(p_2)$, then p_1 is equivalent to p_2, denoted by $p_1 \approx p_2$.

Definition 7 (Wai et al[29]). Let $p = (\mu,\ \nu)$, $p_1 = (\mu_1,\ \nu_1)$, and $p_2 = (\mu_2,\ \nu_2)$ be three PHFNs, and $\lambda > 0$, then four basic operations (algebraic) are defined as follows:

(1) $p_1 \oplus p_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{(\gamma_1)^2 + (\gamma_2)^2 - (\gamma_1)^2(\gamma_2)^2},\ \eta_1\eta_2\right)\right\}$,

(2) $p_1 \otimes p_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\gamma_1\gamma_2,\ \sqrt{{\eta_1}^2 + {\eta_2}^2 - {\eta_1}^2{\eta_2}^2}\right)\right\}$,

(3) $\lambda p = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\sqrt{1 - (1 - \gamma^2)^\lambda},\ \eta^\lambda\right)\right\}$, $\lambda > 0$,

(4) $p^\lambda = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\gamma^\lambda,\ \sqrt{1 - (1 - \eta^2)^\lambda}\right)\right\}$, $\lambda > 0$.

2.3 | Archimedean *t*-conorm and archimedean *t*-norm

Klir and Yuan[30] and Nguyen and Walker[31] introduced Archimedean *t*-conorms and Archimedean *t*-norms.

Definition 8 (Klir and Yuan[30]; Ngugen and Walker[31]). A fuzzy *t*-conorm is a function U: $[0,1] \times [0,1] \to [0,1]$ that satisfies the following axioms for all $a,\ b,\ d \in [0,1]$:

Axiom $U1$. $U(a,\ 0) = a$ for all a.

Axiom $U2$. If $b \le b^*$ and $d \le d^*$ then $U(b,\ d) \le U(b^*,\ d^*)$.

Axiom $U3$. $U(a,\ b) = U(b,\ a)$ for all a and for all b.

Axiom $U4$. $U(a, U(b,\ d)) = U(U(a,\ b),\ d)$ for all a, b, and d.

Definition 9 (Klir and Yuan[30]; Ngugen and Walker[31]). A fuzzy t-norm is a function I: $[0,1] \times [0,1] \to [0,1]$ that satisfies the following axioms for all a, b, $d \in [0,1]$:

Axiom $I1$. $I(a, 1) = a$ for all a.

Axiom $I2$. If $b \leq b^*$ and $d \leq d^*$ then $I(b, d) \leq I(b^*, d^*)$.

Axiom $I3$. $I(a, b) = I(b, a)$ for all a and for all b.

Axiom $I4$. $I(a, I(b, d)) = I(I(a, b), d)$ for all a, b, and d.

Definition 10 (Klir and Yuan[30]; Ngugen and Walker[31]). A t-conorm function $U(a, b)$ is called Archimedean t-conorm if it is continuous and $U(a, a) > a$ for all $a \in (0,1)$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for a, $b \in (0,1)$.

Definition 11 (Klir and Yuan[30]; Ngugen and Walker[31]). A t-norm function $I(a, b)$ is called Archimedean t-norm if it is continuous and $I(a, a) < a$ for all $a \in (0,1)$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for a, $b \in (0,1)$.

Definition 12. Let f be a continuous function from $[0,1]$ to $\mathbb{R}$ such that $f(1) = 0$ and f is strictly decreasing function, then f is called decreasing generators.

Definition 13. Let g be a continuous function from $[0,1]$ to $\mathbb{R}$ such that $y(0) = 0$ and y is strictly increasing function, then y is called increasing generators.

Definition 14 (Klement and Mesiar[33]). A strict Archimedean t-conorm is expressed by an increasing generator g such that

$$U(a, b) = g^{(-1)}(g(a) + g(b)) \text{ with } g(t) = f(1 - t) \text{ for all } a, b, t \in [0, 1], \tag{3}$$

and similarly t-norm is expressed by a decreasing generator f such that

$$I(a, b) = f^{(-1)}(f(a) + f(b)) \text{ for all } a, b \in [0, 1]. \tag{4}$$

Klement and Mesiar[33] proposed some t-conorms and t-norms for specific forms of the function f, as follows:

(1) Let $f(t) = -\log t$, then $g(t) = f(1 - t) = -\log(1 - t)$, $f^{-1}(t) = e^{-t}$, $g^{-1}(t) = 1 - e^{-t}$, and Algebric t-conorm and t-norm[34] are defined as $U^{A}(a, b) = a + b - ab$, $I^{A}(a, b) = ab$.

(2) Let $f(t) = -\log((2 - t)/t)$, then $g(t) = \log((2 - (1 - t))/(1 - t))$, $f^{-1}(t) = 2/(e^{t}+1)$, $g^{-1}(t) = 1 - 2/(e^{t} + 1)$, and Einstein t-conorm and t-norm[34] are defined as $U^{E}(a, b) = (a + b)/(1 + ab)$, $I^{E}(a, b) = ab/(1 + (1 - a)(1 - b))$.

(3) Let $f(t) = \log((\theta + (1 - \theta)t)/t)$, $\theta > 0$, then $g(t) = \log((\theta + (1 - \theta)(1 - t))/(1 - t))$, $f^{-1}(t) = \theta/(e^{t} + \theta - 1)$, $g^{-1}(t) = 1 - (\theta/(e^{t} + \theta - 1))$, and Hamacher t-conorm and t-norm[34] are defined as $U_{\theta}^{H}(a, b) = (a + b - ab - (1 - \theta)ab)/(1 - (1 - \theta)ab)$, $I_{\theta}^{H}(a, b) = ab/(\theta + (1 - \theta)(a + b - ab))$, $\theta > 0$. If $\theta = 1$, then Hamacher t-conorm and t-norm reduced to the algebraic t-conorm and t-norm, respectively. If $\theta = 2$ then

Hamacher t-conorm and t-norm reduced to the Einstein t-conorm and t-norm, respectively.

(4) Let $f(t) = \log((\theta - 1)/(\theta^t - 1))$, $\theta > 1$, then $g(t) = \log((\theta - 1)/(\theta^{1-t} - 1))$, $f^{-1}(t) = \log((\theta - 1 + e^t)/e^t)/\log\theta$, $g^{-1}(t) = 1 - (\log((\theta - 1 + e^t)/e^t)/\log\theta)$, and Frank t-conorm and t-norm[34] are defined as

$$U_\theta^{F}(a, b) = 1 - \log_\theta\left(1 + \frac{(\theta^{1-a} - 1)(\theta^{1-b} - 1)}{\theta - 1}\right),$$

$$I_\theta^{F}(a, b) = \log_\theta\left(1 + \frac{(\theta^{a} - 1)(\theta^{b} - 1)}{\theta - 1}\right),$$

$\theta > 1$. Especially, if $\theta \to 1$, then we have $\lim_{\theta\to 1} f(t) = -\log t$.

3 | ARCHIMEDEAN t-CONORM AND t-NORM ON PHFNs

Archimedean t-conorm and t-norm play an important role in aggregation of fuzzy numbers and is applied to solve different decision-making problems.[12,35] In this section, several operations based on Archimedean t-norm and t-conorm in PHF environment are introduced, which is defined by the following.

Definition 15. Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$, and $p_2 = (\mu_2, \nu_2)$ be three PHFNs, and $\lambda > 0$, now define some new operational laws for the PHFNs based on Archimedean t-conorm and Archimedean t-norm as follows:

(1) $$p_1 \oplus p_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{U((\gamma_1)^2, (\gamma_2)^2)}, \sqrt{I((\eta_1)^2, (\eta_2)^2)}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{g^{-1}(g((\gamma_1)^2) + g((\gamma_2)^2))}, \sqrt{f^{-1}(f((\eta_1)^2) + f((\eta_2)^2))}\right)\right\},$$

(2) $$p_1 \otimes p_1 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{I((\gamma_1)^2, (\gamma_2)^2)}, \sqrt{U((\eta_1)^2, (\eta_2)^2)}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{f^{-1}(f((\gamma_1)^2) + f((\gamma_2)^2))}, \sqrt{g^{-1}(g((\eta_1)^2) + g((\eta_2)^2))}\right)\right\},$$

(3) $$\lambda p = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\sqrt{g^{-1}(\lambda g((\gamma)^2))}, \sqrt{f^{-1}(\lambda f((\eta)^2))}\right)\right\},$$

(4) $p^{\lambda} = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\sqrt{f^{-1}(\lambda f((\gamma)^2))},\ \sqrt{g^{-1}(\lambda g((\eta)^2))}\right)\right\}.$

- **(Algebraic)** When $f(t) = -\log t$, the algebraic operations as described in Definition 7[29] are found.
- **(Einstein)** When $f(t) = \log((2 - t)/t)$, the following operations are defined as

(1) $$p_1 \oplus p_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\ (\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{\frac{(\gamma_1)^2+(\gamma_2)^2}{1+(\gamma_1)^2(\gamma_2)^2}},\ \frac{\eta_1\eta_2}{\sqrt{1+(1-(\eta_1)^2)(1-(\eta_2)^2)}}\right)\right\},$$

(2) $$p_1 \otimes p_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\ (\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\frac{\gamma_1\gamma_2}{\sqrt{1+(1-(\gamma_1)^2)(1-(\gamma_2)^2)}},\ \sqrt{\frac{(\eta_1)^2+(\eta_2)^2}{1+(\eta_1)^2(\eta_2)^2}}\right)\right\},$$

(3) $$\lambda p = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\sqrt{\frac{(1+(\gamma)^2)^{\lambda}-(1-(\gamma)^2)^{\lambda}}{(1+(\gamma)^2)^{\lambda}+(1-(\gamma)^2)^{\lambda}}},\ \frac{\sqrt{2}(\eta)^{\lambda}}{\sqrt{(2-(\eta)^2)^{\lambda}+((\eta)^2)^{\lambda}}}\right)\right\}, \lambda > 0,$$

(4) $$p^{\lambda} = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\frac{\sqrt{2}(\gamma)^{\lambda}}{\sqrt{(2-(\gamma)^2)^{\lambda}+((\gamma)^2)^{\lambda}}},\ \sqrt{\frac{(1+(\eta)^2)^{\lambda}-(1-(\eta)^2)^{\lambda}}{(1+(\eta)^2)^{\lambda}+(1-(\eta)^2)^{\lambda}}}\right)\right\}, \lambda > 0,$$

which are Einstein operations on PHFs.

- **(Hamacher)** When $f(t) = \log((\theta + (1 - \theta)t)/t)$, $\theta > 0$, $\lambda > 0$, the following operations are defined as

(1) $$p_1 \oplus p_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\ (\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{\frac{(\gamma_1)^2+(\gamma_2)^2-(\gamma_1)^2(\gamma_2)^2-(1-\theta)(\gamma_1)^2(\gamma_2)^2}{1-(1-\theta)(\gamma_1)^2(\gamma_2)^2}},\ \frac{\eta_1\eta_2}{\sqrt{\theta+(1-\theta)((\eta_1)^2+(\eta_2)^2-(\eta_1)^2(\eta_2)^2)}}\right)\right\},$$

(2) $$p_1 \otimes p_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\ (\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\frac{\gamma_1\gamma_2}{\sqrt{\theta+(1-\theta)((\gamma_1)^2+(\gamma_2)^2-(\gamma_1)^2(\gamma_2)^2)}},\ \sqrt{\frac{(\eta_1)^2+(\eta_2)^2-(\eta_1)^2(\eta_2)^2-(1-\theta)(\eta_1)^2(\eta_2)^2}{1-(1-\theta)(\eta_1)^2(\eta_2)^2}}\right)\right\},$$

(3) $$\lambda p = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\sqrt{\frac{(1+(\theta-1)(\gamma)^2)^{\lambda}-(1-(\gamma)^2)^{\lambda}}{(1+(\theta-1)(\gamma)^2)^{\lambda}+(\theta-1)(1-(\gamma)^2)^{\lambda}}},\ \frac{\sqrt{\theta}(\eta)^{\lambda}}{\sqrt{(1+(\theta-1)(1-(\eta)^2))^{\lambda}+(\theta-1)(\eta)^{2\lambda}}}\right)\right\},$$

(4) $$p^{\lambda} = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\frac{\sqrt{\theta}(\gamma)^{\lambda}}{\sqrt{(1+(\theta-1)(1-(\gamma)^2))^{\lambda}+(\theta-1)(\gamma)^{2\lambda}}},\ \sqrt{\frac{(1+(\theta-1)(\eta)^2)^{\lambda}-(1-(\eta)^2)^{\lambda}}{(1+(\theta-1)(\eta)^2)^{\lambda}+(\theta-1)(1-(\eta)^2)^{\lambda}}}\right)\right\},$$

which are Hamacher operations on PHFs.

- **(Frank)** When $f(t) = \log((\theta - 1)/(\theta^t - 1))$, $\theta > 1$, and $\lambda > 0$, then the following operations are defined as

(1) $$p_1 \oplus p_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{1-\log_\theta\left(1+\frac{(\theta^{1-(\gamma_1)^2}-1)(\theta^{1-(\gamma_2)^2}-1)}{\theta-1}\right)},\right.\right.$$
$$\left.\left.\sqrt{\log_\theta\left(1+\frac{(\theta^{(\eta_1)^2}-1)(\theta^{(\eta_2)^2}-1)}{\theta-1}\right)}\right)\right\},$$

(2) $$p_1 \otimes p_2 = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\(\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{\log_\theta\left(1+\frac{(\theta^{(\gamma_1)^2}-1)(\theta^{(\gamma_2)^2}-1)}{\theta-1}\right)},\right.\right.$$
$$\left.\left.\sqrt{1-\log_\theta\left(1+\frac{(\theta^{1-(\eta_1)^2}-1)(\theta^{1-(\eta_2)^2}-1)}{\theta-1}\right)}\right)\right\},$$

(3) $$\lambda p = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\sqrt{1-\log_\theta\left(1+\frac{(\theta^{1-(\gamma)^2}-1)^\lambda}{(\theta-1)^{\lambda-1}}\right)}, \sqrt{\log_\theta\left(1+\frac{(\theta^{(\eta)^2}-1)^\lambda}{(\theta-1)^{\lambda-1}}\right)}\right)\right\},$$

(4) $$p^\lambda = \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \left\{\left(\sqrt{\log_\theta\left(1+\frac{(\theta^{(\gamma)^2}-1)^\lambda}{(\theta-1)^{\lambda-1}}\right)}, \sqrt{1-\log_\theta\left(1+\frac{(\theta^{1-(\eta)^2}-1)^\lambda}{(\theta-1)^{\lambda-1}}\right)}\right)\right\},$$

which are Frank class of t-norms and t-conorms on PHFs.

4 | PYTHAGOREAN HESITANT FUZZY ARCHIMEDEAN AGGREGATION OPERATORS

In this section, two types of aggregation operators based on Archimedean t-norm and t-conorm, namely, averaging and geometric aggregation operators, as a general case, are developed. Subsequently, for different types of decreasing generators, several types of aggregation operators are derived.

4.1 | Pythagorean hesitant fuzzy archimedean averaging operators

Definition 16. Let p_i $(i = 1,2,\ldots,n) \in P$ be a collection of PHFNs, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then, an APHFWA operator is a mapping:$P^n \to P$, defined by APHFWA$(p_1, p_2, \ldots, p_n) = \oplus_{i=1}^{n}(\omega_i p_i)$, where $\oplus$ conveys the meaning as described in Definition 15.

Several properties of the above defined APHFWA operator are described as follows.

Theorem 1. *Let* $p_i = (\mu_i, \nu_i)$ $(i = 1,2,\ldots,n)$ *be a collection of* PHFNs*, and then the aggregated value by using* APHFWA *operator is also a* PHFN *and*

$$\text{APHFWA}(p_1, p_2, \ldots, p_n) = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \ldots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n)}} \left\{\left(\sqrt{g^{-1}\left(\sum\nolimits_{i=1}^{n} \omega_i g((\gamma_i)^2)\right)}, \sqrt{f^{-1}\left(\sum\nolimits_{i=1}^{n} \omega_i f((\eta_i)^2)\right)}\right)\right\}. \tag{5}$$

Proof. For $n = 2$,

$$\omega_1 p_1 = \bigcup_{(\gamma_1,\eta_1)\in(\mu_1,\nu_1)} \left\{\left(\sqrt{g^{-1}(\omega_1 g((\gamma_1)^2))}, \sqrt{f^{-1}(\omega_1 f((\eta_1)^2))}\right)\right\} \text{ and}$$

$$\omega_2 p_2 = \bigcup_{(\gamma_2,\eta_2)\in(\mu_2,\nu_2)} \left\{\left(\sqrt{g^{-1}(\omega_2 g((\gamma_2)^2))}, \sqrt{f^{-1}(\omega_2 f((\eta_2)^2))}\right)\right\}.$$

Now $\omega_1 p_1 \oplus \omega_2 p_2$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\ (\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{g^{-1}(g(g^{-1}(\omega_1 g((\gamma_1)^2))) + g(g^{-1}(\omega_2(g(\gamma_2)^2))))},\right.\right.$$

$$\left.\left.\sqrt{f^{-1}(f(f^{-1}(\omega_1 f((\eta_1)^2))) + f(f^{-1}(\omega_2 f((\eta_2)^2))))}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\ (\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{g^{-1}(\omega_1 g((\gamma_1)^2) + \omega_2 g((\gamma_2)^2))}, \sqrt{f^{-1}(\omega_1 f((\eta_1)^2) + \omega_2 f((\eta_2)^2))}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),\\ (\gamma_2,\eta_2)\in(\mu_2,\nu_2)}} \left\{\left(\sqrt{g^{-1}\left(\sum\nolimits_{i=1}^{2} \omega_i g((\gamma_i)^2)\right)}, \sqrt{f^{-1}\left(\sum\nolimits_{i=1}^{2} \omega_i f((\eta_i)^2)\right)}\right)\right\}.$$

Thus the theorem holds for $n = 2$. Suppose that theorem is true for $n = k$, ie,

$$\text{APHFWA}(p_1, p_2, \ldots, p_k)$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \ldots,(\gamma_k,\eta_k)\in(\mu_k,\nu_k)}} \left\{\left(\sqrt{g^{-1}\left(\sum\nolimits_{i=1}^{k} \omega_i g((\gamma_i)^2)\right)}, \sqrt{f^{-1}\left(\sum\nolimits_{i=1}^{k} \omega_i f((\eta_i)^2)\right)}\right)\right\}.$$

Now when $n = k + 1$,

$$\text{APHFWA}(p_1, p_2, \ldots, p_k, p_{k+1}) = \text{APHFWA}(p_1, p_2, \ldots, p_k) \oplus \omega_{k+1} p_{k+1}$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \ldots,(\gamma_k,\eta_k)\in(\mu_k,\nu_k)}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{k} \omega_i g((\gamma_i)^2)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{k} \omega_i f((\eta_i)^2)\right)}\right)\right\} \oplus$$

$$\bigcup_{(\gamma_{k+1},\eta_{k+1})\in(\mu_{k+1},\nu_{k+1})} \left\{\left(\sqrt{g^{-1}(\omega_{k+1} g((\gamma_{k+1})^2))}, \sqrt{f^{-1}(\omega_{k+1} f((\eta_{k+1})^2))}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i),\\ i=1,2,\ldots,k,\, k+1}} \left\{\left(\sqrt{g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{k} \omega_i g((\gamma_i)^2)\right)\right) + g(g^{-1}(\omega_{k+1} g((\gamma_{k+1})^2)))\right)},\right.\right.$$

$$\left.\left.\sqrt{f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{k} \omega_i f((\eta_i)^2)\right)\right) + f(f^{-1}(\omega_{k+1} f((\eta_{k+1})^2)))\right)}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i),\\ i=1,2,\ldots,k,\, k+1}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{k} \omega_i g((\gamma_i)^2) + \omega_{k+1} g((\gamma_{k+1})^2)\right)},\right.\right.$$

$$\left.\left.\sqrt{f^{-1}\left(\sum_{i=1}^{k} \omega_i f((\eta_i)^2) + \omega_{k+1} f((\eta_{k+1})^2)\right)}\right)\right\}$$

$$\bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i),\\ i=1,2,\ldots,k,\, k+1}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{k+1} \omega_i g((\gamma_i)^2)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{k+1} \omega_i f((\eta_i)^2)\right)}\right)\right\}.$$

Hence, the above is true for $n = k + 1$ also. Thus the theorem is true for all integers. This completes the proof of the theorem.

- **(Algebraic)** If $f(t) = -\log t$, then the APHFWA operator reduces to the Pythagorean hesitant fuzzy weighted averaging (PHFWA) operator defined as

$$\text{PHFWA}(p_1, p_2, \ldots, p_n) = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \ldots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n)}} \left\{\left(\sqrt{1 - \prod_{i=1}^{n} (1 - (\gamma_i)^2)^{\omega_i}}, \prod_{i=1}^{n} (\eta_i)^{\omega_i}\right)\right\}.$$

- **(Einstein)** If $f(t) = \log((2 - t)/t)$, then the APHFWA operator reduces to the Pythagorean hesitant fuzzy Einstein weighted averaging (PHFEWA) operator defined as

$$\mathrm{PHFEWA}(p_1, p_2, \dots, p_n) = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \dots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n)}} \left\{\left(\frac{\sqrt{\prod_{i=1}^{n}(1+(\gamma_i)^2)^{\omega_i}-\prod_{i=1}^{n}(1-\gamma_i^2)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}(1+(\gamma_i)^2)^{\omega_i}+\prod_{i=1}^{n}(1-(\gamma_i)^2)^{\omega_i}}}, \frac{\sqrt{2}\prod_{i=1}^{n}(\eta_i)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}(2-(\eta_i)^2)^{\omega_i}+\prod_{i=1}^{n}((\eta_i)^2)^{\omega_i}}}\right)\right\}.$$

- **(Hamacher)** When $f(t) = \log((\theta + (1-\theta)t)/t)$, $\theta > 0$, then the APHFWA operator reduces to the Pythagorean hesitant fuzzy Hamacher weighted averaging (PHFHWA) operator defined as

$$\mathrm{PHFHWA}(p_1, p_2, \dots, p_n) = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \dots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n)}} \left\{\left(\frac{\sqrt{\prod_{i=1}^{n}(1+(\theta-1)(\gamma_i)^2)^{\omega_i}-\prod_{i=1}^{n}(1-(\gamma_i)^2)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n}(1+(\theta-1)(\gamma_i)^2)^{\omega_i}+(\theta-1)\prod_{i=1}^{n}(1-(\gamma_i)^2)^{\omega_i}}}, \frac{\sqrt{\theta}\prod_{i=1}^{n}(\eta_i)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}(1+(\theta-1)(1-(\eta_i)^2))^{\omega_i}+(\theta-1)\prod_{i=1}^{n}((\eta_i)^2)^{\omega_i}}}\right)\right\}. \quad (6)$$

if the value of θ is considered as 1 and 2 in the above equation, then PHFHWA operator reduces to PHFWA and PHFEWA operators, respectively.

- **(Frank)** When $f(t) = \log((\theta-1)/(\theta^t-1))$, $\theta > 1$, then the APHFWA operator reduces to the Pythagorean hesitant fuzzy Frank weighted averaging (PHFFWA) operator defined as

$$\mathrm{PHFFWA}(p_1, p_2, \dots, p_n) = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \dots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n)}} \left\{\left(\sqrt{1-\frac{\log\left(1+\prod_{i=1}^{n}(\theta^{1-(\gamma_i)^2}-1)^{\omega_i}\right)}{\log\theta}}, \sqrt{\frac{\log\left(1+\prod_{i=1}^{n}(\theta^{(\eta_i)^2}-1)^{\omega_i}\right)}{\log\theta}}\right)\right\}. \quad (7)$$

Theorem 2 (Boundary). *Let $p_i = (\mu_i, \nu_i)$ $(i = 1, 2, \dots, n)$ be a collection of PHFNs, and let*
$\gamma_{\min} = \min\{\gamma_{i_{\min}}\}$, *where* $\gamma_{i_{\min}} = \min_{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)}\{\gamma_i\}$ *for all* $i = 1, 2, \dots, n$.
$\gamma_{\max} = \max\{\gamma_{i_{\max}}\}$, *where* $\gamma_{i_{\max}} = \max_{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)}\{\gamma_i\}$ *for all* $i = 1, 2, \dots, n$.
$\eta_{\min} = \min\{\eta_{i_{\min}}\}$, *where* $\eta_{i_{\min}} = \min_{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)}\{\eta_i\}$ *for all* $i = 1, 2, \dots, n$.
$\eta_{\max} = \max\{\eta_{i_{\max}}\}$, *where* $\eta_{i_{\max}} = \max_{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)}\{\eta_i\}$ *for all* $i = 1, 2, \dots, n$.
Let $p_- = (\gamma_{\min}, \eta_{\max})$ *and* $p_+ = (\gamma_{\max}, \eta_{\min})$.

$$p_- \le \mathrm{APHFWA}(p_1, p_2, \dots, p_n) \le p_+. \quad (8)$$

Let

$$\mathrm{APHFWA}(p_1, p_2, \dots, p_n) = p$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \dots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n)}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left((\gamma_i)^2\right)\right)},\ \sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left((\eta_i)^2\right)\right)}\right)\right\}.$$

Proof. For any $i = 1, 2, \dots, n$, we have $\gamma_{\min} \le \gamma_i \le \gamma_{\max}$,

$$\text{ie, } (\gamma_{\min})^2 \le (\gamma_i)^2 \le (\gamma_{\max})^2.$$

Since $g(t)$ $(t \in [0,1])$ is a monotonic increasing function, we get

$$g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left((\gamma_{\min})^2\right)\right) \le g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left((\gamma_i)^2\right)\right) \le g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left((\gamma_{\max})^2\right)\right),$$

which implies that

$$(\gamma_{\min})^2 \le g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left((\gamma_i)^2\right)\right) \le (\gamma_{\max})^2. \tag{9}$$

For any $i = 1, 2, \dots, n$, we have $(\eta_{\min})^2 \le (\eta_i)^2 \le (\eta_{\max})^2$ since $f(t)$ $(t \in [0,1])$ is a decreasing function, we get

$$f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left((\eta_{\min})^2\right)\right) \le f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left((\eta_i)^2\right)\right) \le f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left((\eta_{\max})^2\right)\right),$$

which implies that

$$(\eta_{\min})^2 \le f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left((\eta_i)^2\right)\right) \le (\eta_{\max})^2. \tag{10}$$

From (9) and (10),

$$(\gamma_{\min})^2 - (\eta_{\max})^2 \le g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left((\gamma_i)^2\right)\right) - f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left((\eta_i)^2\right)\right) \le (\gamma_{\max})^2 - (\eta_{\min})^2,$$

$$S(p_-) \le S(\text{APHFWA}(p_1, p_2, \dots, p_n)) \le S(p_+).$$

Therefore

$$p_- \le \text{APHFWA}(p_1, p_2, \dots, p_n) \le p_+.$$

Theorem 3. *Let p_i $(i = 1,2,\dots,n)$ be a collection of PHFNS, $\omega_i \in [0,1]$ $(i = 1,2,\dots,n)$ be their corresponding weight vectors, and $\sum_{i=1}^{n}\omega_i = 1$, if p be an PHFN, then*

$$\text{APHFWA}(p_1 \oplus p, p_2 \oplus p, \dots, p_n \oplus p) = \text{APHFWA}(p_1, p_2, \dots, p_n) \oplus p.$$

Proof.

$$p_i \oplus p = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i),\\ (\gamma,\eta)\in(\mu,\upsilon)}} \left\{\left(\sqrt{g^{-1}(g((\gamma_i)^2)+g((\gamma)^2))},\ \sqrt{f^{-1}(f((\eta_i)^2)+f((\eta)^2))}\right)\right\}.$$

Let

$$\mathrm{APHFWA}(p_1 \oplus p, p_2 \oplus p, \ldots, p_n \oplus p)$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \ldots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n),(\gamma,\eta)\in(\mu,\upsilon)}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g(g^{-1}(g((\gamma_i)^2)+g((\gamma)^2)))\right)},\right.\right.$$

$$\left.\left.\sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f(f^{-1}(f((\eta_i)^2)+f((\eta)^2)))\right)}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \ldots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n),(\gamma,\eta)\in(\mu,\upsilon)}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i (g((\gamma_i)^2)+g((\gamma)^2))\right)},\right.\right.$$

$$\left.\left.\sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i (f((\eta_i)^2)+f((\eta)^2))\right)}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \ldots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n),(\gamma,\eta)\in(\mu,\upsilon)}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g((\gamma_i)^2)+g((\gamma)^2)\right)},\right.\right.$$

$$\left.\left.\sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f((\eta_i)^2)+f((\eta)^2)\right)}\right)\right\}.$$

Now

$$\mathrm{APHFWA}(p_1, p_2, \ldots, p_n) \oplus p$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \ldots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n)}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g((\gamma_i)^2)\right)},\ \sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f((\eta_i)^2)\right)}\right)\right\}$$

$$\oplus \bigcup_{(\gamma,\eta)\in(\mu,\upsilon)} \{(\gamma, \eta)\}$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \dots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n),(\gamma,\eta)\in(\mu,\upsilon)}} \left\{\left(\sqrt{g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left((\gamma_i)^2\right)\right)\right)+g\left((\gamma)^2\right)\right)},\right.\right.$$

$$\left.\left.\sqrt{f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left((\eta_i)^2\right)\right)\right)+f\left((\eta)^2\right)\right)}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \dots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n),(\gamma,\eta)\in(\mu,\upsilon)}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left((\gamma_i)^2\right)+g\left((\gamma)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left((\eta_i)^2\right)+f\left((\eta)^2\right)\right)}\right)\right\}.$$

Hence the theorem.

Theorem 4 (Idempotency). *If all* $p_i = (\mu_i, \nu_i)$ $(i=1, 2, \dots, n)$ *are equal and let* $p_i = p = (\mu, \upsilon)$ *for all* $(i=1, 2, \dots, n)$*, then*

$$\mathrm{APHFWA}(p_1, p_2, \dots, p_n) = p.$$

Proof.

$$\mathrm{APHFWA}(p_1, p_2, \dots, p_n) = \bigcup_{\substack{(\gamma_1,\eta_1)\in(\mu_1,\nu_1),(\gamma_2,\eta_2)\in(\mu_2,\nu_2),\\ \dots,(\gamma_n,\eta_n)\in(\mu_n,\nu_n)}} \left\{\left(\sqrt{g^{-1}\left(\sum_{i=1}^{n}\omega_i g\left((\gamma_i)^2\right)\right)}, \sqrt{f^{-1}\left(\sum_{i=1}^{n}\omega_i f\left((\eta_i)^2\right)\right)}\right)\right\}.$$

Now since $p_i = (\mu, \upsilon)$, $(\gamma_i, \eta_i) = (\gamma, \eta)$ for all $(i = 1,2,\dots,n)$, then we have

$$\mathrm{APHFWA}(p_1, p_2, \dots, p_n) = \bigcup_{\substack{(\gamma,\eta)\in(\mu_i,\nu_i)\\ i=1,2,\dots,n}} \left\{\left(\sqrt{g^{-1}\left(g\left((\gamma)^2\right)\sum_{i=1}^{n}\omega_i\right)}, \sqrt{f^{-1}\left(f\left((\eta)^2\right)\sum_{i=1}^{n}\omega_i\right)}\right)\right\}$$

$$= \bigcup_{\substack{(\gamma,\eta)\in(\mu_i,\nu_i)\\ i=1,2,\dots,n}} \{(\gamma, \eta)\} = \{(\gamma, \eta)\} = p.$$

Hence the theorem.

4.2 | Pythagorean hesitant fuzzy archimedean geometric operators

In the following, we shall propose some Pythagorean hesitant fuzzy Archimedean geometric operator based on the Archimedean operations of PHFNs.

Definition 17. Let p_i $(i = 1,2,\dots,n)$ be a collection of PHFEs, and let $\omega=(\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then, an Archimedean t-conorm and t-norm-based Pythagorean hesitant fuzzy weighted geometric (APHFWG) operator is a

mapping $P^n \to P$, where $\mathrm{APHFWG}(p_1, p_2, \dots, p_n) = \otimes_{i=1}^{n}(p_i^{\omega_i})$

$\otimes$ conveys the meaning as described in the Definition 15.

Theorem 5. *Let* $p_i = (\mu_i, \nu_i)\,(i = 1,2,\ldots,n)$ *be a collections of* PHFN*s, then the aggregated value by using* APHFWG *operator is also a* PHFN *and*

$$\mathrm{APHFWG}(p_1, p_2, \ldots, p_n) = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}} \left\{\left(\sqrt{f^{-1}\left(\sum_{i=1}^{n} \omega_i f\left((\gamma_i)^2\right)\right)}, \sqrt{g^{-1}\left(\sum_{i=1}^{n} \omega_i g\left((\eta_i)^2\right)\right)}\right)\right\}. \tag{11}$$

Proof. Proof is same as the Theorem 1.

If the generator f is assigned different forms, then some specific interval-valued DHF weighted geometric operators can be obtained as follows:

- **(Algebraic)** If $f(t) = -\log t$, then the APHFWG operator reduces to the Pythagorean interval-valued hesitant fuzzy weighted geometric (PHFWG) operator defined as

$$\mathrm{PHFWG}(p_1, p_2, \ldots, p_n) = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}} \left\{\left(\prod_{i=1}^{n} (\gamma_i)^{\omega_i}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - (\eta_i)^2\right)^{\omega_i}}\right)\right\}.$$

- **(Einstein)** If $f(t) = \log((2 - t)/t)$, then the APHFWG operator reduces to the Pythagorean hesitant fuzzy Einstein weighted geometric (PHFEWG) operator defined as

$$\mathrm{PHFEWG}(p_1, p_2, \ldots, p_n)$$

$$= \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}} \left\{\left(\frac{\sqrt{2}\prod_{i=1}^{n} (\gamma_i)^{\omega_i}}{\sqrt{\prod_{i=1}^{n} \left(2 - (\gamma_i)^2\right)^{\omega_i} + \prod_{i=1}^{n} \left((\gamma_i)^2\right)^{\omega_i}}}, \frac{\sqrt{\prod_{i=1}^{n} \left(1 + (\eta_i)^2\right)^{\omega_i} - \prod_{i=1}^{n} \left(1 - (\eta_i)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n} \left(1 + (\eta_i)^2\right)^{\omega_i} + \prod_{i=1}^{n} \left(1 - (\eta_i)^2\right)^{\omega_i}}}\right)\right\}. \tag{12}$$

- **(Hamacher)** When $f(t) = \log((\theta + (1 - \theta)t)/t)$, $\theta > 0$, then the APHFWG operator reduces to the Pythagorean hesitant fuzzy Hamacher weighted geometric (PHFHWG) operator defined as

$$\mathrm{PHFHWG}(p_1, p_2, \ldots, p_n) = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}} \left\{\left(\frac{\sqrt{\theta}\prod_{i=1}^{n}(\gamma_i)^{\omega_i}}{\sqrt{\prod_{i=1}^{n}\left(1 + (\theta - 1)\left(1 - (\gamma_i)^2\right)\right)^{\omega_i} + (\theta - 1)\prod_{i=1}^{n}\left((\gamma_i)^2\right)^{\omega_i}}},\right.\right.$$

$$\left.\left.\frac{\sqrt{\prod_{i=1}^{n} \left(1 + (\theta - 1)(\eta_i)^2\right)^{\omega_i} - \prod_{i=1}^{n} \left(1 - (\eta_i)^2\right)^{\omega_i}}}{\sqrt{\prod_{i=1}^{n} \left(1 + (\theta-1)(\eta_i)^2\right)^{\omega_i} + (\theta - 1)\prod_{i=1}^{n} \left(1 - (\eta_i)^2\right)^{\omega_i}}}\right)\right\}, \tag{13}$$

if $\theta = 1$ and $\theta = 2$ is substituted in the above equation, then PHFHWG operator reduces to PHFWG and PHFEWG operator, respectively.

- **(Frank)** When $f(t) = \log((\theta - 1)/(\theta^t - 1))$, $\theta > 1$, then the APHFWG operator reduces to the Pythagorean hesitant fuzzy Frank weighted geometric (PHFFWG) operator defined as:

$$\text{PHFFWG}(p_1, p_2, \ldots, p_n) = \bigcup_{\substack{(\gamma_i,\eta_i)\in(\mu_i,\nu_i)\\ i=1,2,\ldots,n}} \left\{\left(\sqrt{\frac{\log\left(1+\prod_{i=1}^{n}\left(\theta^{(\gamma_i)^2}-1\right)^{\omega_i}\right)}{\log\theta}}, \sqrt{1-\frac{\log\left(1+\prod_{i=1}^{n}\left(\theta^{1-(\eta_i)^2}-1\right)^{\omega_i}\right)}{\log\theta}}\right)\right\}. \quad (14)$$

5 | AN APPROACH TO MCDM WITH PYTHAGOREAN HESITANT FUZZY INFORMATION

In this section, the proposed APHFWA and APHFWG operators are used to develop an approach to solve MCDM problems under PHF environment. For some MCDM, let $Z = \{z_1, z_2, \ldots, z_m\}$ be a set of alternatives to be selected, $C = \{C_1, C_2, \ldots, C_n\}$ be a collection of criterion such that their weight vector is given by $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_j \in [0,1]$ for $j = 1,2,\ldots,n$, and $\sum_{j=1}^{n} \omega_j = 1$. Suppose that $P = (p_{ij})_{m\times n}$ be a PHF decision matrix. Then, the developed APHFWA (and APHFWG) operators are used to develop an approach for solving MCDM problems in a PHF environment. The proposed methodology is described through the following steps:

Step 1. Aggregate the PHFNs, p_{ij} for each alternative z_i using the $APHFWA$ (or $APHFWG$) operator as follows:

$$p_i = \text{APHFWA}(p_{i1}, p_{i2}, \ldots, p_{in}) = \bigcup_{\substack{(\gamma_{ij},\eta_{ij})\in(\mu_{ij},\nu_{ij})\\ j=1,2,\ldots,n}} \left\{\left(\sqrt{g^{-1}\left(\sum_{j=1}^{n}\omega_j g((\gamma_{ij})^2)\right)}, \sqrt{f^{-1}\left(\sum_{j=1}^{n}\omega_j f((\eta_{ij})^2)\right)}\right)\right\}, \quad (15)$$

or

$$p_i = \text{APHFWG}(p_{i1}, p_{i2}, \ldots, p_{in}) = \bigcup_{\substack{(\gamma_{ij},\eta_{ij})\in(\mu_{ij},\nu_{ij})\\ j=1,2,\ldots,n}} \left\{\left(\sqrt{f^{-1}\left(\sum_{j=1}^{n}\omega_j f((\gamma_{ij})^2)\right)}, \sqrt{g^{-1}\left(\sum_{j=1}^{n}\omega_j g((\eta_{ij})^2)\right)}\right)\right\}, \quad (16)$$

$$i = 1, 2, \ldots, m.$$

Step 2. Use Equation (1) to calculate the score value of each alternative.

Step 3. According to the Definition 6, the rank of the alternatives is evaluated.

6 | ILLUSTRATIVE EXAMPLE

In the context of green supply chain management (GSCM) in PHF environment, one MCDM problem previously studied by Wei et al,[29] is considered and solved. With the fast growing industrialization, the environmental and ecological impacts of products have become a major issue in the society, since the civilization is facing various threats, like global warming, toxic environments, ozone layer depletion, natural resources depletion, etc. To overcome such situation GSCM is one of the emerging areas of research which is not just considers environmental impacts, but also looks into productivity and profit. The problem is described as follows.

The expert in a GSCM considers four criteria, namely (1) the product quality factor (C_1); (2) environmental factor (C_2); (3) delivery factor (C_3); and (4) price factor (C_4). The weight vector of the respective criterion is $w = (0.4, 0.1, 0.2, 0.3)^T$. There are five green suppliers available, and the set of all alternatives is denoted by $Z = \{z_i | i = 1, 2, 3, 4, 5\}$. The characteristics of the supplier $z_i (i = 1, 2, ..., 5)$ in terms of the criteria in are expressed by the following decision matrix represented through PHFNs (see Table 1):

To obtain the best alternative(s), the developed APHFWA and APHFWG operators are used and step-by-step execution of the proposed method is described below. In this context, it is to be noted here that two types of Archimedean t-norms and t-conorms, namely, Hamacher and Frank Classes are considered. Algebraic and Einstein classes can be derived as particular cases of Hamacher class of t-norms and t-conorms.

Step 1. Utilize the aggregation operators APHFWA and APHFWG as described in Equations (15) and (16), respectively, to aggregate all the preference values p_{ij} for each alternative z_i and get p_i (i = 1, 2, 3, 4, 5).

Step 2. The score value $S(z_i)$ for each alternative, $z_i (i = 1, 2, 3, 4, 5)$, is evaluated using Definition 5.

Step 3. Utilize the ranking method to determine the best alternatives among five alternatives.

TABLE 1 Pythagorean hesitant fuzzy decision matrix

	C_1	C_2
z_1	{(0.2,0.3), (0.3,0.4)}	{(0.4,0.6), (0.5,0.4), (0.7,0.2)}
z_2	{(0.4,0.3), (0.6,0.4)}	{(0.5,0.6), (0.6,0.4)}
z_3	{(0.6,0.2), (0.7,0.3)}	{(0.5,0.3), (0.5,0.4)}
z_4	{(0.6,0.5), (0.7,0.4)}	{(0.5,0.2), (0.6,0.5)}
z_5	{(0.3,0.3), (0.6,0.4)}	{(0.5,0.4), (0.7,0.4)}
	C_3	C_4
z_1	{(0.4,0.5), (0.6,0.3)}	{(0.6,0.3), (0.7,0.4)}
z_2	{(0.5,0.3), (0.5,0.6)}	{(0.5,0.4), (0.7,0.6)}
z_3	{(0.5,0.2), (0.8,0.6), (0.8,0.2)}	{(0.4,0.5), (0.6,0.4)}
z_4	{(0.4,0.3), (0.5,0.4)}	{(0.6,0.2), (0.6,0.3), (0.8,0.4)}
z_5	{(0.6,0.4), (0.7,0.4)}	{(0.4,0.6), (0.5,0.3)}

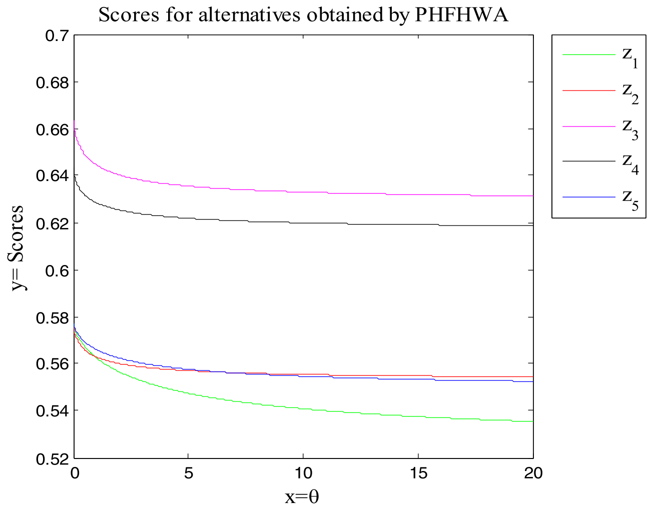

FIGURE 1 Scores for alternatives obtained by the PHFHWA operator. PHFHWA, Pythagorean hesitant fuzzy Hamacher weighted averaging [Color figure can be viewed at wileyonlinelibrary.com]

7 | RESULTS AND DISCUSSIONS

7.1 | Using PHFHWA operator

The proposed technique is applied to solve the above MCDM problem considering PHFHWA operator (Equation (6)). For different values of the parameter θ, ranging between 0 to 20, the change of score values obtained by the PHFHWA operator is depicted in Figure 1. It is to be observed that the score values decrease with the increase of θ.

The following changes are observed in ordering of alternatives:

(1) For $\theta \in [0, 0.8850]$, the decreasing order of the score values of the alternatives ranges as $S(z_1) \in [0.5635, 0.5739]$, $S(z_2) \in [0.5635, 0.5722]$, $S(z_3) \in [0.6454, 0.6575]$, $S(z_4) \in [0.6291, 0.6384]$, and $S(z_5) \in [0.5667, 0.5747]$.
Thus the ordering of the alternatives in $[0, 0.8850)$ appeared as $z_3 \succ z_4 \succ z_5 \succ z_1 \succ z_2$ and the best alternative is identified as z_3. At $\theta = 0.885$, the score value of the alternatives corresponding to z_1 and z_2 are equal with the values
$S(z_1) = 0.5635 = S(z_2)$. Thus the ranking of the five alternatives for $\theta = 0.885$ becomes $z_3 \succ z_4 \succ z_5 \succ z_2 = z_1$. Therefore, z_1 and z_2 are interchangeable in the above ordering.

(2) For $\theta \in (0.885, 6.631)$ the score values are obtained as $S(z_1) \in (0.5447, 0.5635)$, $S(z_2) \in (0.5563, 0.5635)$, $S(z_3) \in (0.6343, 0.6454)$, $S(z_4) \in (0.6209, 0.6291)$, and $S(z_5) \in (0.5563, 0.5667)$.
Thus in $(0.885, 6.631)$ the ordering of the alternatives becomes $z_3 \succ z_4 \succ z_5 \succ z_2 \succ z_1$. Thus the best alternative is identified as z_3 as like previous.
It is to be noted here that when $\theta = 6.631$, the score value of the alternatives corresponding to z_2 and z_5 are equal with $S(z_2) = 0.5563 = S(z_5)$.
Thus in the ranking of the five alternatives when $\theta = 6.631$, z_2 and z_5 are interchangeable.

TABLE 2 Result using PHFWA and PHFEWA operators

Operators	Score value	Ranking of the alternatives
PHFWA	$S(z_1) = 0.5626$, $S(z_2) = 0.5631$, $S(z_3) = 0.6446$, $S(z_4) = 0.6285$, $S(z_5) = 0.5661$.	$z_3 \succ z_4 \succ z_5 \succ z_2 \succ z_1$
PHFEWA	$S(z_1) = 0.5566$, $S(z_2) = 0.5601$, $S(z_3) = 0.6403$, $S(z_4) = 0.6254$, $S(z_5) = 0.5625$	$z_3 \succ z_4 \succ z_5 \succ z_2 \succ z_1$

Abbreviation: PHFWA, Pythagorean hesitant fuzzy weighted averaging; PHFEWA, Pythagorean hesitant fuzzy Einstein weighted averaging.

(3) Similarly, for $\theta \in (6.631,20]$, the score values of the alternatives are found as $S(z_1) = [0.5355, 0.5447)$, $S(z_2) = [0.5544, 0.5563)$, $S(z_3) = [0.6313, 0.6343)$, $S(z_4) = [0.\ 6186, 0.6209)$, and $S(z_5) = [0.\ 5525, 0.5563)$.

So the ordering of the five alternatives in this case is given by $z_3 \succ z_4 \succ z_2 \succ z_5 \succ z_1$ (Table 2).

Note 1 As described in (Equation (6)) Section 4.1, it is to be mentioned here that for if the value of the parameter θ, is considered as 1 and 2 then the PHFHWA operator reduced to the PHFWA and PHFEWA operator, respectively. Thus the score values and the ordering of alternatives are presented below by Table 2.

It is worthy to mention here that for all values of the parameter θ, the best alternative is identified as z_3.

Wei et al[29] considered the MCDM problem considering the value of the parameter $\theta = 3$, only, and the ranking of the alternatives found as the same as the proposed method using PHFHWA operator. Thus the proposed method appeared as a generalized method in which the technique of Wei et al[29] is a special case.

7.2 | Using PHFHWG operator

Now, if PHFHWG operator (Equation (13)) is used, the achieved score values of the alternatives are shown in Figure 2. From that it is clear that all score values obtained using PHFHWG operator increase, monotonically, as the parameter θ increases from 0 to 20.

For $\theta \in [0, 20]$, the score values of the alternatives using PHFHWG operator are ranges in increasing order as $S(z_1) = [0.4872, 0.5342]$, $S(z_2) = [0.5289, 0.5504]$, $S(z_3) = [0.5927, 0.6243]$, $S(z_4) = [0.5887, 0.6135]$, and $S(z_5) = [0.5342, 0.5477]$.

It is interesting to note here that for all values of the parameter, θ, the ordering of the alternatives is $z_3 \succ z_4 \succ z_2 \succ z_5 \succ z_1$ (Table 3 and Figure 3).

Note 2 As like previous discussions, it is to be pointed out that if $\theta = 1$, and $\theta = 2$ is considered in (Equation (13)) Section 4.2, then the score values and ordering of the alternatives corresponding to PHFWG and PHFEWG operators are presented below by Table 3.

Note 3 Considering $\theta = 3$, Wei et al[29] applied PHFHWG operator and found the ranking of the alternatives as $z_3 \succ z_4 \succ z_5 \succ z_2 \succ z_1$. The score value corresponding to z_5

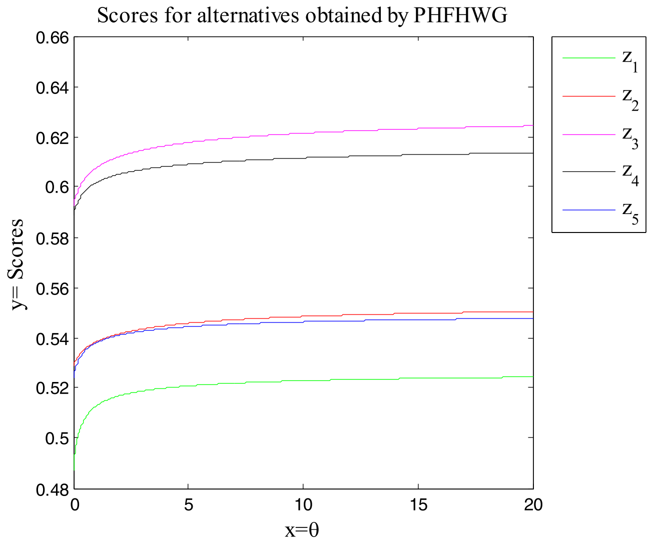

FIGURE 2 Scores for alternatives obtained by the PHFHWG operator. PHFHWG, Pythagorean hesitant fuzzy Hamacher weighted geometric [Color figure can be viewed at wileyonlinelibrary.com]

is presented by that method as $S(z_5) = 0.5508$, whereas, the score values corresponding to $\theta = 3$ for z_5 are found by the developed method as $S(z_5) = 0.5425$, keeping all the other score values same as like existing method (Wei et al[29]). Analyzing the characteristics of Figure 3 and calculating the score values, it is clear that for all values of the parameter θ, the ordering of the alternatives is $z_3 \succ z_4 \succ z_2 \succ z_5 \succ z_1$, and the curve corresponding to the score values z_2 and z_5 never intersects, though those two curves come very closer to each other in the neighborhood of the parameter $\theta = 1$. Under this context, the authors feel that this discrepancy arises due to some computational error of the existing method proposed by Wei et al.[29] The score of the alternatives z_2 and z_5 obtained by the PHFHWG operator is specifically presented in the following Figure 3.

Now, the deviation values between the scores obtained using PHFHWA and PHFHWG operators are presented in Figure 4. It is observed that the score values achieved through PHFHWG operator is smaller than those values obtained through PHFHWA operator. Also, the deviation among the score values decreases with the increase of the parameter θ.

TABLE 3 Result using PHFWG and PHFEWG operators

Operators	Score value	Ranking of the alternatives
PHFWG	$S(z_1) = 0.5128$, $S(z_2) = 0.5385$, $S(z_3) = 0.6075$, $S(z_4) = 0.6016$, $S(z_5) = 0.5379$	$z_3 \succ z_4 \succ z_2 \succ z_5 \succ z_1$
PHFEWG	$S(z_1) = 0.5168$, $S(z_2) = 0.5415$, $S(z_3) = 0.612$, $S(z_4) = 0.6050$, $S(z_5) = 0.5409$	$z_3 \succ z_4 \succ z_2 \succ z_5 \succ z_1$

Abbreviation: PHFWG, Pythagorean interval-valued hesitant fuzzy weighted geometric; PHFEWG, Pythagorean hesitant fuzzy Einstein weighted geometric.

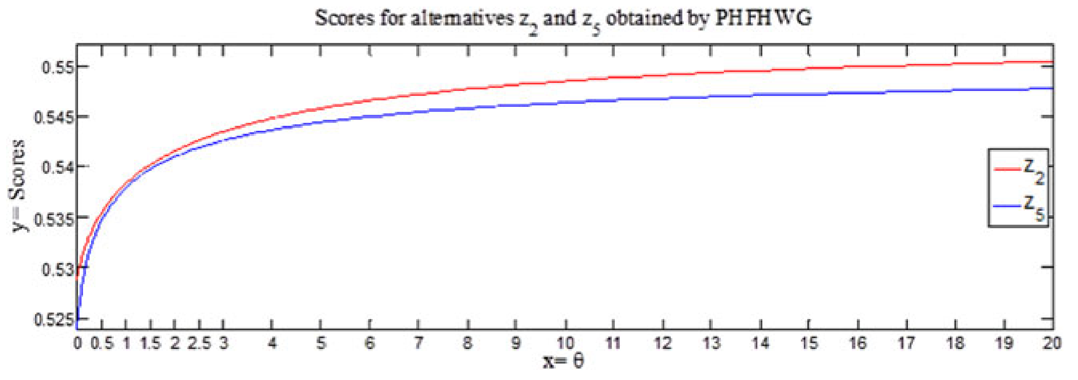

FIGURE 3 Scores for alternatives z_2 and z_5 obtained by the PHFHWG operator. PHFHWG, Pythagorean hesitant fuzzy Hamacher weighted geometric [Color figure can be viewed at wileyonlinelibrary.com]

7.3 | Using PHFFWA operator

The ordering of alternatives using PHFFWA operator as defined in (7) is found as $z_3 \succ z_4 \succ z_5 \succ z_2 \succ z_1$ and for different values of the parameter θ, score values of alternatives is shown in Figure 5. From that figure it is clear that scores of the alternatives decrease as the value of the parameter θ increases from 1 to 20.

7.4 | Using PHFFWG operator

Again if PHFFWG operator as defined in (14) is used then for different values of the parameter θ, score values of alternatives is shown in the Figure 6. From that figure, it is clear that scores of the alternatives increases as the value of the parameter θ increases from 1 to 20. It is to be observed here that using PHFFWG operator, the ordering

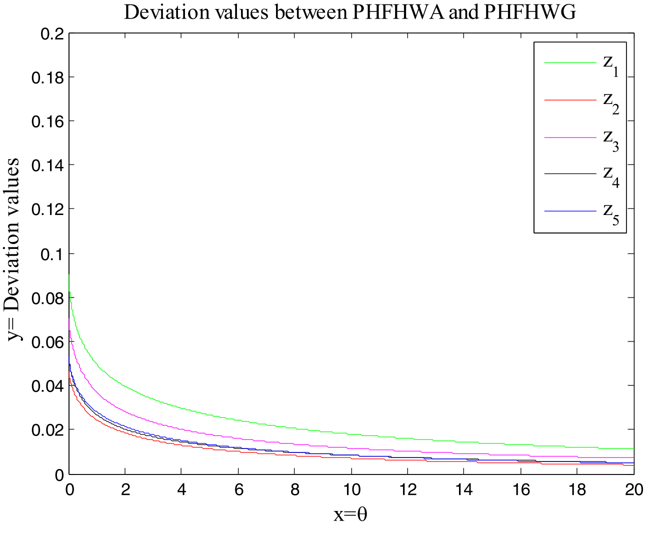

FIGURE 4 Deviation values for alternatives between the PHFHWA and PHFHWG operators. PHFHWA, Pythagorean hesitant fuzzy Hamacher weighted averaging; PHFHWG, Pythagorean hesitant fuzzy Hamacher weighted geometric [Color figure can be viewed at wileyonlinelibrary.com]

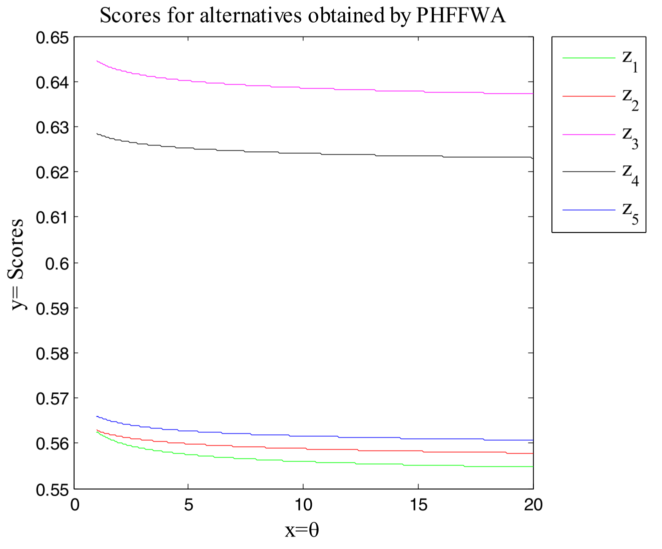

FIGURE 5 Scores for alternatives obtained by the PHFFWA operator. PHFFWA, Pythagorean hesitant fuzzy Frank weighted averaging [Color figure can be viewed at wileyonlinelibrary.com]

of alternatives changes at $\theta = 7.483$. The ordering of the alternatives are presented as follows:

(1) For $\theta \in [1,7.483)$, the ordering of the five alternatives is $z_3 \succ z_4 \succ z_2 \succ z_5 \succ z_1$.
(2) For $\theta \in (7.483,20]$, the ordering of the five alternatives is $z_3 \succ z_4 \succ z_5 \succ z_2 \succ z_1$.

Figure 7 illustrates the deviation values between the scores obtained by PHFFWA operator and by the PHFFWG operator. It is noted that the scores obtained by the PHFFWA

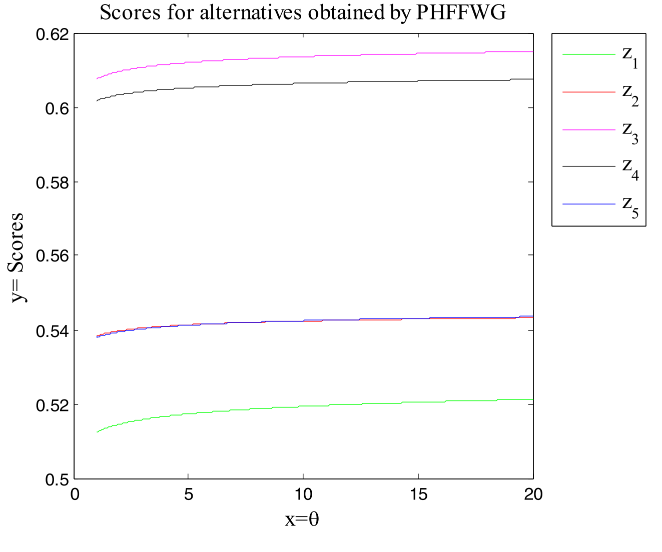

FIGURE 6 Scores for alternatives obtained by the PHFFWG operator. PHFFWG, Pythagorean hesitant fuzzy Frank weighted geometric [Color figure can be viewed at wileyonlinelibrary.com]

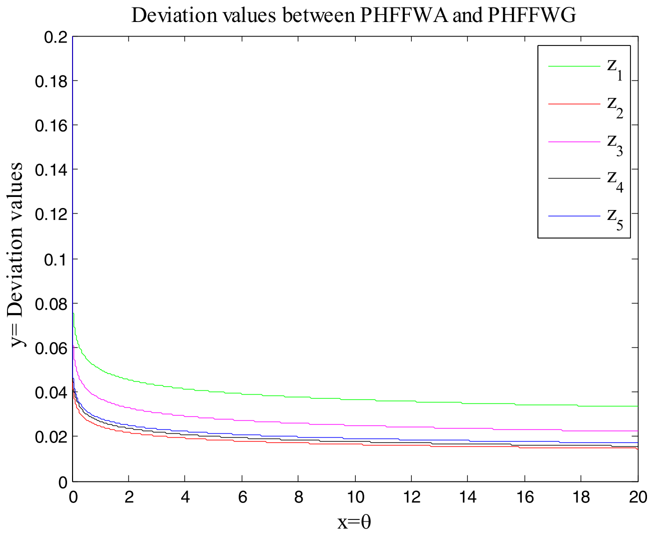

FIGURE 7 Deviation values for alternatives between the PHFFWA and PHFFWG operators. PHFFWA, Pythagorean hesitant fuzzy Frank weighted averaging; PHFFWG, Pythagorean hesitant fuzzy Frank weighted geometric [Color figure can be viewed at wileyonlinelibrary.com]

operator are larger than the PHFFWG operator, and as the value of θ increases, the deviation decreases.

8 | CONCLUSIONS

Archimedean t-conorm and t-norm are the generalizations of different classes of t-conorms and t-norms, namely, algebraic, Einstein, Hamacher, Frank, etc. In this paper, Archimedean t-conorm and t-norm are utilized to aggregate PHFNS to find best alternative in MCDM contexts through the development of two aggregation operators, namely, PHFHWA and PHFHWG operators PHFFWA and PHFFWG. Some operational laws on PHFNS together with their desirable properties are investigated. The changes of score values are observed with the change of the preferences of the alternatives by the DMs which are highly desirable in MCDM contexts. The proposed aggregation operators also satisfy all the properties that the existing ones have. The existing operators are appeared now as some special cases of Archimedean aggregation operators. Finally, a practical example[29] for green supplier selections in GSCM is given to verify the developed approach and to demonstrate its practicality and effectiveness.

In future, some other types of interval-valued Pythagorean hesitant fuzzy operators based on the developed concepts of Archimedean t-conorm and t-norm can easily be derived.

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Development of dual hesitant fuzzy prioritized operators based on Einstein operations with their application to multi-criteria group decision making

ANIMESH BISWAS and ARUN SARKAR

The purpose of this article is to develop a multicriteria group decision making (MCGDM) method in dual hesitant fuzzy (DHF) environment by evaluating the weights of the decision makers from the decision matrices using two newly defined prioritized aggregation operators based on score function to remove the inconsistencies in choosing the best alternative. Prioritized weighted averaging operator and prioritized weighted geometric operator based on Einstein operations are described first for aggregating DHF information. Some of their desirable properties are also investigated in details. A method for finding the rank of alternatives in MCGDM problems with DHF information based on priority levels of decision makers is developed. An illustrative example concerning MCGDM problem is considered to establish the application potentiality of the proposed approach. The method is efficient enough to solve different real life MCGDM problems having DHF information.

Key words: multi-criteria group decision-making, aggregation operator, dual hesitant fuzzy numbers, Einstein operations, prioritized weighted averaging operator, prioritized weighted geometric operator

1. Introduction

Theory of Fuzzy sets (FSs) [31] are widely and successfully applied in all areas of real life decision making problems to handle vagueness or possibilistic imprecisions. After introduction of FSs, several extensions are developed, such as type-2 FSs (T2FSs) [1–3, 8, 10], fuzzy multisets [10], interval-valued FSs [32], etc. As a generalization of FSs, Atanassov presented the concept of intuitionistic FS (IFS) [17] using two characteristic functions representing the degree of membership and the degree of non-membership of elements of the universal

A. Biswas, the corresponding author, is with Department of Mathematics, University of Kalyani, Kalyani – 741235, INDIA. E-mail: abiswaskln@rediffmail.com

A. Sarkar is with Department of Mathematics, Heramba Chandra College, Kolkata – 700029, INDIA. E-mail: asarkarmth@gmail.com

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set to the IFS. As like FSs, in IFSs several variants are also found in the form of intuitionistic linguistic FSs [7], interval-valued IFSs [3, 4, 12, 22], etc.

In real-life applications, when decision makers are confused with assigning exact preference information using FSs or IFSs, the concept of hesitant FSs (HFSs) came into the literature [15, 16] as a new generalization of FSs as well as IFSs. HFSs deal with the difficulties of establishing a common membership degree not because of a margin of error (IFSs), or some possibility distribution values (T2FS), but have a set of possible values. Torra [15] provided a definition corresponding to the envelope of HFSs. HFS has received a considerable attention to the researchers and is applied to various fields of decision-making [14, 23]. Xia and Xu [21] developed several series of aggregation operators for hesitant fuzzy information and discussed the relationships among them. In the context of multi-criteria decision-making (MCDM), Wei et al., [20] proposed hesitant fuzzy linguistic arithmetic aggregation operators. Based on the idea of prioritized aggregation operators [26] Wei [19] defined some prioritized aggregation operators for aggregating hesitant fuzzy information and then applied them to develop models for hesitant fuzzy multiple attribute decision making (MADM) problems in which the attributes are in different priority level.

Zhu et al. [34] proposed dual hesitant fuzzy (DHF) set (DHFS) by considering several possible values for the membership as well as non-membership degrees. Thus, DHFSs can take much more information than HFSs given by decision makers into account in MADM. Wei and Lu [18] developed Dual hesitant Pythagorean fuzzy Hamacher aggregation operators in MADM. Inspired by generalized ordered weighted average operator [25], Yu and Li [29] proposed some generalized aggregation operators for DHFEs. Different MADM theories and methods under DHF environments are developed using those aggregation operators. All the developed methods are under the assumption that the attributes are at the same priority level. However, in real and practical MADM situation, the attributes may have different priority levels. To overcome this drawback, in this paper, DHF prioritized weighted average (DHFPWA) operator and DHF prioritized weighted geometric (DHFPWG) operator are proposed and some of their properties have been discussed.

2. Some basic concepts and operations

In this section, some basic concepts, which are essential to develop the proposed methodology, are described.

Definition 1 [1] *Let a set X be fixed. An IFS α on X is represented in terms of two functions $\mu_\alpha\colon X \to [0,1]$ and $\nu_\alpha\colon X \to [0,1]$, and having the form $\alpha = \{\langle x, \mu_\alpha, \nu_\alpha(x)\rangle | x \in X\}$ with the condition $0 \leqslant \mu_\alpha(x) + \nu_\alpha(x) \leqslant 1$, for all $x \in X$,* *where $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of x in α.*

For convenience, Xu and Yager [24] called $\alpha = (\mu_\alpha, \nu_\alpha)$ an intuitionistic fuzzy number (IFN), where $\mu_\alpha \in [0,1]$, $\nu_\alpha \in [0,1]$ and $0 \leqslant \mu_\alpha + \nu_\alpha \leqslant 1$.

For any IFN $\alpha = (\mu_\alpha, \nu_\alpha)$, a score of α can be evaluated by a score function s [6] as

$$s(\alpha) = \mu_\alpha - \nu_\alpha \qquad \text{where} \quad s(\alpha) \in [-1,1]. \tag{1}$$

Definition 2 [15–16] *Let X be a universal set. A HFS A defined on X is represented by a function h_A that returns a subset of $[0,1]$ when it is applied to X.*

For convenience, Xia and Xu [21] represented the HFS A by using the mathematical symbol:

$$A = \{\langle x, h_A(x)\rangle \,|x \in X\}, \tag{2}$$

where $h_A(x)$ is a set of several different values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set A. Xia and Xu [21] called $h = h_A(x)$ a hesitant fuzzy element (HFE).

Definition 3 [34] *Let X be a fixed set, a DHFS D defined on X is represented as:*

$$D = \{\langle x, h(x), g(x)\rangle \,|x \in X\}, \tag{3}$$

where $h(x)$ and $g(x)$ are hesitant fuzzy elements, denoting respectively the membership and non-membership degree of the element x to D, with the conditions:

$$0 \leqslant \gamma, \;\; \tau \leqslant 1 \quad \textit{with} \quad 0 \leqslant \gamma^+ + \tau^+ \leqslant 1,$$

where $\gamma \in h(x) \subseteq [0,1]$, $\tau \in g(x) \subseteq [0,1]$ and $\gamma^+ = \max\{h(x)\}$, $\tau^+ = \max\{g(x)\}$.

For convenience $\langle h(x), g(x)\rangle$ is called the DHF element (DHFE) and is denoted as

$$\widetilde{\alpha} = (h, g).$$

2.1. Einstein operations

The set theoretical operators play an important role to aggregate different fuzzy information. Since the inception of fuzzy set theory, starting from Zadeh's operator, min and max, many other operators introduced in the literature. All types of the operators were included in the general concepts of the t-norms and t-conorms, which satisfy the requirement of the conjunction and disjunction operators, respectively.

There are various t-norm and t-conorm families available in the literature. Einstein operators include the Einstein product $\otimes_\varepsilon$ and Einstein sum $\oplus_\varepsilon$, which are examples of t-norm and t-conorm, respectively. The Einstein operators are defined as follows [11]:

$$a \oplus_\varepsilon b = \frac{a+b}{1+a\cdot b}, \quad a \otimes_\varepsilon b = \frac{a\cdot b}{1+(1-a)(1-b)} \quad \text{for all } (a,b) \in [0,1]^2. \tag{4}$$

Based on the concepts of Einstein operators Zhao et al. [33] introduced different operations on DHFEs as follows:

Definition 4 [33] *Let* $\widetilde{\alpha}_1 = (h_1, g_1)$, $\widetilde{\alpha}_2 = (h_2, g_2)$ *and* $\widetilde{\alpha} = (h, g)$ *be three DHFEs. Then*

$$\text{(i)} \quad \widetilde{\alpha}_1 \oplus_\varepsilon \widetilde{\alpha}_2 = \left(\bigcup_{\gamma_1\in h_1, \gamma_2\in h_2} \frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2}, \bigcup_{\tau_1\in g_1, \tau_2\in g_2} \frac{\tau_1\tau_2}{1+(1-\tau_1)(1-\tau_2)} \right);$$

$$\text{(ii)} \quad \widetilde{\alpha}_1 \otimes_\varepsilon \widetilde{\alpha}_2 = \left(\bigcup_{\gamma_1\in h_1, \gamma_2\in h_2} \frac{\gamma_1\gamma_2}{1+(1-\gamma_1)(1-\gamma_2)}, \bigcup_{\tau_1\in g_1, \tau_2\in g_2} \frac{\tau_1+\tau_2}{1+\tau_1\tau_2} \right);$$

$$\text{(iii)} \quad \lambda\widetilde{\alpha} = \left(\bigcup_{\gamma\in h} \frac{(1+\gamma)^\lambda-(1-\gamma)^\lambda}{(1+\gamma)^\lambda+(1-\gamma)^\lambda}, \bigcup_{\tau\in g} \frac{2\tau^\lambda}{(2-\tau)^\lambda+\tau^\lambda} \right), \quad \lambda > 0;$$

$$\text{(iv)} \quad \widetilde{\alpha}^\lambda = \left(\bigcup_{\gamma\in h} \frac{2\gamma^\lambda}{(2-\gamma)^\lambda+\gamma^\lambda}, \bigcup_{\tau\in g} \frac{(1+\tau)^\lambda-(1-\tau)^\lambda}{(1+\tau)^\lambda+(1-\tau)^\lambda} \right), \quad \lambda > 0.$$

2.2. Prioritized operators

The prioritized operators play also an important role in solving many MCDM problems. The prioritized averaging (PA) operator, introduced by Yager [26], is defined in the following manner:

Definition 5 [26] *Let* $C = \{C_1, C_2, \ldots, C_n\}$ *be a collection of criteria and that there is a prioritization between the criteria expressed by the linear ordering* $C_1 \succ C_2 \succ \ldots \succ C_n$, *indicate criteria* C_j *has a higher priority than* C_k *if* $j < k$. *The value* $C_j(x)$ *is the performance of any alternative* x *under criteria* C_j, *and satisfies* $C_j(x) \in [0,1]$. *If*

$$PA(C_j(x)) = \sum_{j=1}^{n} w_j C_j(x), \tag{5}$$

where $w_j = \dfrac{T_j}{\sum_{j=1}^n T_j}$, $T_j = \prod_{k=1}^{j-1} C_k(x)$ $(j = 2, \ldots, n)$, $T_1 = 1$. *Then PA is called the* PA *operator.*

In the following Sections, the methodological development of the paper is incorporated.

At first a new score function of DHFEs is introduced. In this context it is to be pointed out that a score function defined by Zhu et al. [34] is already exist in the literature. But, the drawback of that approach is that the score value becomes negative when average of membership degree is less than the average of non-membership degree.

Based on the concepts of score functions of DHFEs, Einstein operators and prioritized operators, dual hesitant fuzzy aggregation operators are defined. The defined operators are then used to solve a MCDM problem.

The methodological developments are described subsequently.

3. Score function of a dual hesitant fuzzy element (DHFE)

A new score function is defined in this section to find the ordering of DHFEs.

Definition 6 *Score function of DHFE is defined as*

$$s(\alpha) = \frac{1 + \sum_{\gamma \in h} \frac{\gamma}{l(h)} - \sum_{\tau \in g} \frac{\tau}{l(g)}}{2} \tag{6}$$

and the accuracy function of DHFE is described as follows

$$a(\alpha) = \sum_{\gamma \in h} \frac{\gamma}{l(h)} + \sum_{\tau \in g} \frac{\tau}{l(g)}. \tag{7}$$

where $l(h)$ and $l(g)$ represents the number of elements in h and g, respectively.

For comparison of DHFEs the following conditions are to be satisfied.

Let α_1 and α_2 be two DHFEs

1. If $s(\alpha_1) > s(\alpha_2)$ then $\alpha_1 > \alpha_2$;
2. If $s(\alpha_1) = s(\alpha_2)$ then
   if $a(\alpha_1) > a(\alpha_2)$ then $\alpha_1 > \alpha_2$; if $a(\alpha_1) = a(\alpha_2)$ then $\alpha_1 = \alpha_2$.

4. Development of Dual Hesitant fuzzy aggregation operator based on prioritized operators

Based on the score function, Einstein operations and prioritized operators as defined above, a dual hesitant fuzzy prioritized Einstein aggregation operator is defined as follows.

Definition 7 *Let* $\widetilde{\alpha}_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ *be a collections of* DHFEs *and* $w = (w_1, w_2, \ldots, w_n)$ *be the weight vectors of* $\widetilde{\alpha}_i$, *where*

$$w_i = \frac{T_i}{\sum_{i=1}^{n} T_i} \quad \text{and} \quad T_i = \prod_{k=1}^{i-1} s\left(\widetilde{\alpha}_k\right) \quad (i = 2, \ldots, n), \;\; T_1 = 1, \tag{8}$$

and $s(\widetilde{\alpha}_i)$ *is the score of DHFE* $\widetilde{\alpha}_i$. *If* $\text{DHFPEWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \oplus_{\varepsilon}{}_{i=1}^{n} w_i \alpha_i$ *then* DHFPEWA *is called a dual hesitant fuzzy prioritized Einstein weighted averaging operator.*

Theorem 1 *Let* $\widetilde{\alpha}_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ *be a collections of* DHFEs, *then the aggregated value by using DHFPEWA operator is also a DHFE and*

$$\begin{aligned}
&\text{DHFPEWA}\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) = \oplus_{\varepsilon}{}_{i=1}^{n} \left(\frac{T_i}{\sum_{i=1}^{n} T_i} \widetilde{\alpha}_i\right) \\
&= \left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \frac{\prod_{i=1}^{n} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{n} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right. \\
&\qquad \left. \bigcup_{\tau_1 \in g_1, \tau_2 \in g_2, \ldots, \tau_n \in g_n} \frac{2 \prod_{i=1}^{n} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} (2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{n} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right).
\end{aligned} \tag{9}$$

Proof. Using the mathematical induction method, the theorem will be proved.

The theorem is obvious for $n = 1$.

We assume that theorem is true for $n = p$, we shall prove that it is true for $n = p + 1$.

For $n = p$, we have

$$\begin{aligned}
&\text{DHFPEWA}\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p\right) = \oplus_{\varepsilon}{}_{i=1}^{p} \left(\frac{T_i}{\sum_{i=1}^{n} T_i} \widetilde{\alpha}_i\right) \\
&= \left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots\ldots, \gamma_p \in h_p} \frac{\prod_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right. \\
&\qquad \left. \bigcup_{\tau_1 \in g_1, \tau_2 \in g_2, \ldots, \tau_p \in g_p} \frac{2 \prod_{i=1}^{p} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{p} (2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{p} T_i}} + \prod_{i=1}^{p} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right).
\end{aligned}$$

Now when $n = p+1$,

$$\mathrm{DHFPEWA}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p, \widetilde{\alpha}_{p+1})$$

$$= \mathrm{DHFPEWA}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p) \oplus_\varepsilon \left(\frac{T_{p+1}}{\sum_{i=1}^{n} T_i} \widetilde{\alpha}_{p+1} \right)$$

$$= \mathrm{DHFPEWA}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p) \oplus_\varepsilon \left(\bigcup_{\gamma_{p+1} \in h_{p+1}} \frac{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} - (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} + (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}, \right.$$
$$\left. \bigcup_{\tau_{p+1} \in g_{p+1}} \frac{2\tau_{p+1}^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}{(2-\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} + \tau_{p+1}^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}} \right)$$

$$= \left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_p \in h_p} \frac{\prod_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} - \prod_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} + \prod_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}, \right.$$
$$\left. \bigcup_{\tau_1 \in g_1, \tau_2 \in g_2, \ldots, \tau_p \in g_p} \frac{2\prod_{i=1}^{p} (\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod_{i=1}^{p} (2-\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} + \prod_{i=1}^{p} (\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}} \right) \oplus_\varepsilon$$
$$\left(\bigcup_{\gamma_{p+1} \in h_{p+1}} \frac{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} - (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} + (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}, \right.$$
$$\left. \bigcup_{\tau_{p+1} \in g_{p+1}} \frac{2(\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}{(2-\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} + (\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}} \right)$$

$$= \left(\bigcup_{\substack{\gamma_1 \in h_1, \\ \gamma_2 \in h_2, \ldots, \\ \gamma_p \in h_p, \\ \gamma_{p+1} \in h_{p+1}}} \frac{\dfrac{\prod\limits_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} - \prod\limits_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod\limits_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} + \prod\limits_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}} + \dfrac{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} - (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} + (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}}{1 + \dfrac{\prod\limits_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} - \prod\limits_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod\limits_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} + \prod\limits_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}} \cdot \dfrac{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} - (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}} + (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^n T_i}}}}, \right.$$

$$\left.\bigcup_{\substack{\tau_1\in g_1,\\ \tau_2\in g_2,\\ \dots,\tau_p\in g_p,\\ \tau_{p+1}\in g_{p+1}}} \frac{\dfrac{2\prod_{i=1}^{p}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{p}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{p}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\cdot\dfrac{2\tau_{p+1}^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}}{(2-\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}+(\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}}}{1+\left(1-\dfrac{2\prod_{i=1}^{p}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{p}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{p}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)\left(1-\dfrac{2(\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}}{(2-\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}+(\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}}\right)}\right)$$

$$=\left(\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\dots,\\ \gamma_p\in h_p,\gamma_{p+1}\in h_{p+1}}}\frac{\prod_{i=1}^{p+1}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{p+1}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{p+1}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{p+1}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\right.$$

$$\left.\bigcup_{\substack{\tau_1\in g_1,\tau_2\in g_2,\dots,\\ \tau_p\in g_p,\tau_{p+1}\in g_{p+1}}}\frac{2\prod_{i=1}^{p+1}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{p+1}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{p+1}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)$$

$$=\oplus_{\varepsilon\, i=1}^{\;p+1}\left(\frac{T_i}{\sum_{i=1}^{n}T_i}\widetilde{\alpha}_i\right)=\mathrm{DHFPEWA}(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\dots,\widetilde{\alpha}_{p+1}).$$

Hence the theorem is proved for $p+1$ and thus true for all n.
Hence DHFPEWA $(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\dots,\widetilde{\alpha}_n)$ is a DHFE.
This completes the proof of the theorem.

Theorem 2 (Idempotency) *Let $\widetilde{\alpha}_i=(h_i,g_i)$ $(i=1,2,\dots,n)$ be a collections of* DHFE*s. If all $\widetilde{\alpha}_i$ $(i=1,2,\dots,n)$ are equal, i.e., $\widetilde{\alpha}_i=\widetilde{\alpha}$ for all i, where $\widetilde{\alpha}=(h,g)$ then*

$$DHFPEWA\,(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\dots,\widetilde{\alpha}_n)=\widetilde{\alpha}\,. \tag{10}$$

Proof. We have

$$
\begin{aligned}
&\mathrm{DHFPEWA}\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=\oplus_{\varepsilon}{}_{i=1}^{n}\left(\frac{T_i}{\sum_{i=1}^{n}T_i}\widetilde{\alpha}_i\right)\\
&=\left(\bigcup_{\gamma\in h}\frac{\prod_{i=1}^{n}(1+\gamma)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}(1-\gamma)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(1+\gamma)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(1-\gamma)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\ \bigcup_{\tau\in g}\frac{2\prod_{i=1}^{n}(\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(2-\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)\\
&=\left(\bigcup_{\gamma\in h}\frac{(1+\gamma)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}-(1-\gamma)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}}{(1+\gamma)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}+(1-\gamma)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}},\right.\\
&\qquad\left.\bigcup_{\tau\in g}\frac{2\tau^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}}{(2-\tau)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}+(\tau)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}}\right)\\
&=(h,g)=\widetilde{\alpha}\,.
\end{aligned}
$$

Hence the theorem is proved.

Theorem 3 (Boundary) *Let $\widetilde{\alpha}_i=(h_i,g_i)$ $(i=1,2,\ldots,n)$ be a collections of DHFEs, and let*

$$
\begin{aligned}
&\gamma_*=\min\{\gamma\in h_i\mid i=1,2,\ldots,n\}, \qquad \gamma^*=\max\{\gamma\in h_i\mid i=1,2,\ldots,n\},\\
&\tau_*=\min\{\tau\in g_i\mid i=1,2,\ldots,n\}, \qquad \tau^*=\max\{\tau\in g_i\mid i=1,2,\ldots,n\},\\
&\widetilde{\alpha}^-=(\gamma_*,\tau^*), \qquad \widetilde{\alpha}^+=(\gamma^*,\tau_*).
\end{aligned}
$$

then

$$
\widetilde{\alpha}^-\leqslant DHFPEWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)\leqslant\widetilde{\alpha}^+. \tag{11}
$$

Proof. We have

$$
\begin{aligned}
&DHFPEWA(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\oplus_{\varepsilon}{}_{i=1}^{n}\left(\frac{T_i}{\sum_{i=1}^{n}T_i}\widetilde{\alpha}_i\right)\\
&=\left(\bigcup_{\substack{\gamma_1\in h_1,\\ \gamma_2\in h_2,\\ \ldots,\\ \gamma_n\in h_n}}\frac{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\ \bigcup_{\substack{\tau_1\in g_1,\\ \tau_2\in g_2,\\ \ldots,\\ \tau_n\in g_n}}\frac{2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right).
\end{aligned}
$$

By the definition of γ_*, γ^*, τ_*, τ^*

$$\gamma_* \leqslant \gamma_i \leqslant \gamma^* \quad \text{for all } i, \text{ then}$$

Thus $\dfrac{1-\gamma^*}{1+\gamma^*} \leqslant \dfrac{1-\gamma_i}{1+\gamma_i} \leqslant \dfrac{1-\gamma_*}{1+\gamma_*}$ for all i

$$\text{i.e., } \prod_{i=1}^{n}\left(\frac{1-\gamma^*}{1+\gamma^*}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{1-\gamma_*}{1+\gamma_*}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} \quad \text{for all } i$$

$$\text{i.e., } \frac{1-\gamma^*}{1+\gamma^*} \leqslant \prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} \leqslant \frac{1-\gamma_*}{1+\gamma_*} \quad \text{for all } i$$

$$\text{i.e., } \frac{2}{1+\gamma^*} \leqslant 1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} \leqslant \frac{2}{1+\gamma_*} \quad \text{for all } i$$

$$\text{i.e., } \gamma_* \leqslant \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}}} - 1 \leqslant \gamma^* \quad \text{for all } i$$

$$\text{i.e., } \gamma_* \leqslant \frac{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} - \prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\Sigma_{i=1}^n T_i}}}{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} + \prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\Sigma_{i=1}^n T_i}}} \leqslant \gamma^* \quad \text{for all } i. \tag{12}$$

Similarly,
Since $\tau_* \leqslant \tau_i \leqslant \tau^*$ and $2-\tau^* \leqslant 2-\tau_i \leqslant 2-\tau_*$ then

$$\text{i.e., } \prod_{i=1}^{n}\left(\frac{2-\tau^*}{\tau^*}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{2-\tau_i}{\tau_i}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{2-\tau_*}{\tau_*}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} \quad \text{for all } i$$

$$\text{i.e., } \frac{2}{\tau^*} \leqslant \prod_{i=1}^{n}\left(\frac{2-\tau_i}{\tau_i}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} + 1 \leqslant \frac{2}{\tau_*} \quad \text{for all } i$$

$$\text{i.e., } \tau_* \leqslant \frac{2}{\prod_{i=1}^{n}\left(\frac{2-\tau_i}{\tau_i}\right)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} + 1} \leqslant \tau^* \quad \text{for all } i$$

$$\text{i.e., } \tau_* \leqslant \frac{2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\Sigma_{i=1}^n T_i}}}{\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\Sigma_{i=1}^n T_i}} + \prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\Sigma_{i=1}^n T_i}}} \leqslant \tau^* \quad \text{for all } i. \tag{13}$$

Then from inequalities (12) and (13), and using (6) we obtain

$$s\left(\widetilde{\alpha}^{-}\right) \leqslant s\left(DHFPEWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right)\right) \leqslant s\left(\widetilde{\alpha}^{+}\right).$$

Therefore from the comparative laws of DHFE, it is clear that

$$\widetilde{\alpha}^{-} \leqslant DHFPEWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) \leqslant \widetilde{\alpha}^{+}.$$

This completes the proof of the theorem.

Theorem 4 (Additivity) *Let $\widetilde{\alpha}_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collections of DHFEs, and if $\widetilde{\alpha} = (h, g)$ be another DHFE, then*

$$DHFPEWA(\widetilde{\alpha}_1 \oplus_{\varepsilon} \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_{\varepsilon} \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_{\varepsilon} \widetilde{\alpha}) = DHFPEWA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) \oplus_{\varepsilon} \widetilde{\alpha}.$$

Proof. Based on the operational laws of DHFEs, we have

$$\widetilde{\alpha}_i \oplus_{\varepsilon} \widetilde{\alpha} = \left(\bigcup_{\gamma_i \in h_i, \gamma \in h} \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}, \bigcup_{\tau_i \in g_i, \tau \in g} \frac{\tau_i \tau}{1 + (1-\tau_i)(1-\tau)} \right).$$

According to theorem 1, we have

$$\begin{aligned}
&DHFPEWA\left(\widetilde{\alpha}_1 \oplus_{\varepsilon} \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_{\varepsilon} \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_{\varepsilon} \widetilde{\alpha}\right) \\
&= \left(\bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \\ \ldots, \\ \gamma_n \in h_n, \gamma \in h}} \frac{\prod_{i=1}^{n} \left(1 + \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{j=1}^{n} \left(1 - \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{j=1}^{n} \left(1 + \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{j=1}^{n} \left(1 - \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right. \\
&\left. \bigcup_{\substack{\tau_1 \in g_1, \tau_2 \in g_2, \\ \ldots, \\ \tau_n \in g_n, \tau \in g}} \frac{2 \prod_{i=1}^{n} \left(\frac{\tau_i \tau}{1 + (1-\tau_i)(1-\tau)}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(2 - \frac{\tau_i \tau}{1 + (1-\tau_i)(1-\tau)}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{n} \left(\frac{\tau_i \tau}{1 + (1-\tau_i)(1-\tau)}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right) \\
&= \left(\bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \\ \ldots, \\ \gamma_n \in h_n, \gamma \in h}} \frac{\prod_{i=1}^{n} (1 + \gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} (1 + \gamma)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} (1 - \gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} (1 - \gamma)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} (1 + \gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} (1 + \gamma)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} (1 - \gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} (1 - \gamma)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.
\end{aligned}$$

$$\bigcup_{\substack{\tau_1\in g_1,\tau_2\in g_2,\\ \ldots,\\ \tau_n\in g_n,\tau\in g}} \frac{2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}(\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}(2-\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}(\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\Bigg)$$

$$=\Bigg(\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\\ \ldots,\gamma_n\in h_n,\gamma\in h}} \frac{(1+\gamma)\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-(1-\gamma)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{(1+\gamma)\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(1-\gamma)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},$$

$$\bigcup_{\substack{\tau_1\in g_1,\tau_2\in g_2,\\ \ldots,\\ \tau_n\in g_n,\tau\in g}} \frac{\tau.2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{(2-\tau)\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\tau\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\Bigg).$$

Again from the operational laws of DHFE

$$DHFPEWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)\oplus_{\varepsilon}\widetilde{\alpha}$$

$$=\Bigg(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n} \frac{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},$$

$$\bigcup_{\tau_1\in g_1,\tau_2\in g_2,\ldots,\tau_n\in g_n} \frac{2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\Bigg)\oplus_{\varepsilon}(h,g)$$

$$=\Bigg(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n,\gamma\in h} \frac{(1+\gamma)\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-(1-\gamma)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{(1+\gamma)\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(1-\gamma)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},$$

$$\bigcup_{\substack{\tau_1\in g_1,\tau_2\in g_2,\\ \ldots,\\ \tau_n\in g_n,\tau\in g}} \frac{\tau.2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{(2-\tau)\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\tau\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\Bigg).$$

Thus,

$$DHFPEWA(\widetilde{\alpha}_1 \oplus_\varepsilon \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_\varepsilon \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_\varepsilon \widetilde{\alpha}) = DHFPEWA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) \oplus_\varepsilon \widetilde{\alpha}.$$

This completes the proof.

Theorem 5 *Let $\widetilde{\alpha}_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collections of DHFEs, then the aggregated value by using DHFPEWG operator is also a DHFE and $DHFPEWG(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \otimes_{\varepsilon}{}_{i=1}^{n} (\widetilde{\alpha}_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}$ and*

$$\begin{aligned} DHFPEWG(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) &= \otimes_{\varepsilon}{}_{i=1}^{n} (\widetilde{\alpha}_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} \\ &= \left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \frac{2\prod_{i=1}^{n} (\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod_{i=1}^{n} (2-\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} + \prod_{i=1}^{n} (\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}, \right. \\ &\quad \left. \bigcup_{\tau_1 \in g_1, \tau_2 \in g_2, \ldots, \tau_n \in g_n} \frac{\prod_{i=1}^{n} (1+\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} - \prod_{i=1}^{n} (1-\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod_{i=1}^{n} (1+\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} + \prod_{i=1}^{n} (1-\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}} \right), \end{aligned}$$

where $T_i = \prod_{k=1}^{i-1} S(\widetilde{\alpha}_k)$ $(i = 2, 3, \ldots, n)$, $T_1 = 1$, and $S(\widetilde{\alpha}_k)$ is the score value of DHFE$\widetilde{\alpha}_k$.

Proof. The proof of this theorem is similar to the proof of Theorem 1.

5. An approach to solve MCDM problems with DHFEs

Let $X = \{x_1, x_2, \ldots, x_m\}$ be the set of alternatives and let $C = \{c_1, c_2, \ldots, c_n\}$ be a collection of criteria and there prioritization is given as $c_1 \succ c_2 \succ \ldots \succ c_n$ in such a manner that criteria c_j has a higher priority than c_i, if $j < i$. Now $E = \{e_1, e_2, \ldots, e_p\}$ represents a set of decision makers and the linear ordering $e_1 \succ e_2 \succ e_3 \succ \ldots \succ e_p$ represents prioritization between the decision makers in such a manner that decision maker e_η has a higher priority than decision maker e_ξ if $\eta < \xi$. Suppose that the decision matrix $R^{(q)} = \left(\widetilde{r}_{ij}^{(q)}\right)_{m\times n}$ $(q = 1, 2, \ldots, p)$ is in the form of dual hesitant fuzzy matrix. The elements of this matrix are represented by DHFEs as $\widetilde{r}_{ij}^{(q)} = \left(h_{ij}^{(q)},\ g_{ij}^{(q)}\right)$ which designates the value of the alternative $x_i \in X$ on the criteria $c_j \in C$ provided by the decision maker e_q, where $h_{ij}^{(q)}$ designates the membership degree of the alternative x_i satisfies the criteria C_j expressed by the decision maker e_q; where as $g_{ij}^{(q)}$ indicates the non-membership degree of the same alternative corresponding to the same criteria.

Now utilizing the DHFPEWA and DHFPEWG operators to develop an approach to multi-criteria group decision making under dual hesitant fuzzy environment, the main steps are described as follows:

Step 1. Calculate the value of $T_{ij}^{(q)}(q = 1, 2, \ldots, p)$with the following equations.

$$T_{ij}^{(q)} = \prod_{k=1}^{q-1} S\left(\widetilde{r}_{ij}^{(k)}\right) \qquad (q = 1, 2, \ldots, p), \tag{14}$$

$$T_{ij}^{(1)} = 1. \tag{15}$$

Step 2. To aggregate all the individual dual hesitant fuzzy decision matrix

$$R^{(q)} = \left(\widetilde{r}_{ij}^{(q)}\right)_{m\times n} \qquad (q = 1, 2, \ldots, p).$$

Thus using the DHFPEWA operator

$$\widetilde{r}_{ij} = \text{DHFPEWA}\left(\widetilde{r}_{ij}^{(1)}, \widetilde{r}_{ij}^{(2)}, \ldots, \widetilde{r}_{ij}^{(p)}\right)$$
$$= \left(\bigcup_{\gamma_{ij}^{(q)} \in h_{ij}^{(q)}} \frac{\prod_{q=1}^{p}\left(1+\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} - \prod_{q=1}^{p}\left(1-\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}{\prod_{q=1}^{p}\left(1+\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} + \prod_{q=1}^{p}\left(1-\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}, \right.$$
$$\left. \bigcup_{\tau_{ij}^{(q)} \in g_{ij}^{(q)}} \frac{2\prod_{q=1}^{p}\left(\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}{\prod_{q=1}^{p}\left(2-\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} + \prod_{q=1}^{p}\left(\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}} \right) \tag{16}$$

or using the DHFPEWG operator

$$\widetilde{r}_{ij} = \text{DHFPEWG}\left(\widetilde{r}_{ij}^{(1)}, \widetilde{r}_{ij}^{(2)}, \ldots, \widetilde{r}_{ij}^{(p)}\right)$$

$$= \left(\bigcup_{\gamma_{ij}^{(q)} \in h_{ij}^{(q)}} \frac{2\prod_{i=1}^{p}\left(\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}{\prod_{q=1}^{p}\left(2-\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} + \prod_{q=1}^{p}\left(\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}, \bigcup_{\tau_{ij}^{(q)} \in g_{ij}^{(q)}} \frac{\prod_{q=1}^{p}\left(1+\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} - \prod_{q=1}^{p}\left(1-\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}{\prod_{q=1}^{p}\left(1+\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} + \prod_{q=1}^{p}\left(1-\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}} \right). \tag{17}$$

Step 3. Calculate the values of T_{ij} as follows:

$$T_{ij} = \prod_{k=1}^{j-1} S\left(\widetilde{r}_{ik}\right), \qquad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n); \tag{18}$$

$$T_{i1} = 1, \qquad i = 1, 2, \ldots, m. \tag{19}$$

Step 4. Aggregate the DHFEs $\widetilde{r}_{ij}$ for each alternative x_i using the DHFPEWA (or DHFPEWG) operator as follows:

$$\widetilde{r}_{i} = \text{DHFPEWA}\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}\right)$$

$$= \left(\bigcup_{\gamma_{ij} \in h_{ij}} \frac{\prod_{j=1}^{n}\left(1+\gamma_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}} - \prod_{j=1}^{n}\left(1-\gamma_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n}\left(1+\gamma_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n}\left(1-\gamma_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}}}, \bigcup_{\tau_{ij} \in g_{ij}} \frac{2\prod_{j=1}^{n}\left(\tau_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n}\left(2-\tau_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n}\left(\tau_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}}} \right) \tag{20}$$

or

$$\widetilde{r}_i = \text{DHFPEWG}\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}\right)$$

$$= \left(\bigcup_{\gamma_{ij} \in h_{ij}} \frac{2\prod_{j=1}^{n} \left(\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n} \left(2-\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n} \left(\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}, \right.$$

$$\left. \bigcup_{\tau_{ij} \in g_{ij}} \frac{\prod_{j=1}^{n} \left(1+\tau_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} - \prod_{j=1}^{n} \left(1-\tau_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n} \left(1+\tau_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n} \left(1-\tau_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}} \right). \tag{21}$$

Step 5. Rank all the alternatives by the proposed score function $S(\widetilde{r}_i)$ described in the above, then the highest value of $S(\widetilde{r}_i)$, the larger the overall $\widetilde{r}_i$, and thus the best alternative x_i, is determined.

Based on the methodology developed in this paper, the following illustrative example is considered and solved.

6. An illustrative example

To illustrate the efficiency of the developed *DHFPEW* operators a practical example, studied earlier by Yu [27] in intuitionistic fuzzy context, is adopted in dual hesitant fuzzy environment. The problem is then solved using the ranking process developed in this article and is compared with the process developed by Yu [27] and Yu et al. [30].

The problem under consideration is presented in summarised form as follows:

For enriching academic environment of a Chinese university, three decision makers viz., e_1, e_2 and e_3 in order of priority levels $e_1 > e_2 > e_3$, wants to appoint outstanding teachers among five candidates, x_i $(i = 1, 2, \ldots, 5)$ based on four criteria C_1, C_2, C_3, C_4. The criteria possesses the prioritization relationship as $C_1 > C_2 > C_3 > C_4$. After evaluating the five candidates with respect to their criteria, the decision makers constructed the following three decision matrices $R^{(n)} = (r_{ij}^{(n)})_{5\times 4}$ $(n = 1, 2, 3)$ using DHFEs as follows:

$$R^{(1)} = \begin{bmatrix} \langle\{0.55,0.9\},\{0.0,0.1\}\rangle & \langle\{0.6\},\{0.3\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle \\ \langle\{0.7,0.9\},\{0.0,0.1\}\rangle & \langle\{0.75\},\{0.15,0.25\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.75\},\{0.15\}\rangle \\ \langle\{0.9\},\{0.0\}\rangle & \langle\{0.75,0.85\},\{0.0,0.15\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.45\},\{0.1,0.45\}\rangle \\ \langle\{0.5,0.75\},\{0.15\}\rangle & \langle\{0.6,0.75\},\{0.15,0.25\}\rangle & \langle\{0.7,0.9\},\{0.0,0.1\}\rangle & \langle\{0.3\},\{0.6\}\rangle \\ \langle\{0.4,0.75\},\{0.15,0.25\}\rangle & \langle\{0.4,0.6\},\{0.3,0.4\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.5,0.6\},\{0.3\}\rangle \end{bmatrix},$$

$$R^{(2)} = \begin{bmatrix} \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.3,0.4\},\{0.6\}\rangle \\ \langle\{0.85,0.75\},\{0.05,0.15\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.75,\},\{0.15\}\rangle \\ \langle\{0.7,0.9\},\{0.0,0.05\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.75\},\{0.05,0.15\}\rangle & \langle\{0.6,0.7\},\{0.1,0.3\}\rangle \\ \langle\{0.3,0.9\},\{0.0,0.1\}\rangle & \langle\{0.3,0.4\},\{0.6\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.6\},\{0.3\}\rangle \\ \langle\{0.45\},\{0.45,0.55\}\rangle & \langle\{0.6\},\{0.3\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.7,0.9\},\{0.0,0.1\}\rangle \end{bmatrix},$$

$$R^{(3)} = \begin{bmatrix} \langle\{0.75\},\{0.15,0.25\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.65,0.75\},\{0.15\}\rangle & \langle\{0.3\},\{0.4,0.6\}\rangle \\ \langle\{0.6\},\{0.1,0.3\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.6\},\{0.3\}\rangle \\ \langle\{0.9\},\{0.0\}\rangle & \langle\{0.6\},\{0.3\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.8,0.9\},\{0.0,0.1\}\rangle \\ \langle\{0.5,0.9\},\{0.0,0.1\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.75\},\{0.15,0.25\}\rangle \\ \langle\{0.75\},\{0.15\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.7,0.9\},\{0.0,0.1\}\rangle & \langle\{0.45\},\{0.45,0.55\}\rangle \end{bmatrix}.$$

To select the most preferable candidate the developed process is applied on the above matrices and the following steps are performed.

It is worthy to mention here that Step 1 is common for both the DHFPEWA and DHFPEWG operators.

Step 1. Calculate the value of $T_{ij}^{(i)}$ $(i = 1, 2, 3)$ using equations (16) and (17).

$$T_{ij}^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad T_{ij}^{(2)} = \begin{bmatrix} 0.8375 & 0.65 & 0.825 & 0.925 \\ 0.875 & 0.775 & 0.8 & 0.8 \\ 0.95 & 0.8625 & 0.825 & 0.5875 \\ 0.7375 & 0.7375 & 0.875 & 0.35 \\ 0.6875 & 0.575 & 0.8 & 0.625 \end{bmatrix},$$

$$T_{ij}^{(3)} = \begin{bmatrix} 0.6909 & 0.52 & 0.7631 & 0.3469 \\ 0.7438 & 0.7169 & 0.66 & 0.64 \\ 0.8431 & 0.7978 & 0.6806 & 0.4259 \\ 0.5716 & 0.2766 & 0.7219 & 0.2275 \\ 0.3266 & 0.3738 & 0.74 & 0.5469 \end{bmatrix}.$$

Step 2. Aggregate the three given decision matrices $R^{(k)}$ $(k = 1,2,3)$ by using DHFPEWA operator to aggregate the overall decision matrix R which is shown below:

$$R = \begin{bmatrix}
\left\langle \{0.6819, 0.7290, 0.8244, 0.8522\}, \{0.0, 0.1279, 0.0, 0.1478\} \right\rangle & \left\langle \{0.7459\}, \{0.0, 0.1894\} \right\rangle & \left\langle \{0.7909, 0.8117, 0.8285, 0.8459\}, \{0.0, 0.1320\} \right\rangle & \left\langle \{0.6758, 0.7002\}, \{0.0, 0.0, 0.2711, 0.2910\} \right\rangle \\
\left\langle \{0.6927, 0.7388, 0.7947, 0.8271\}, \{0, 0.0795, 0, 0.1103, 0, 0.1146, 0, 0.1579\} \right\rangle & \left\langle \{0.8104, 0.8366\}, \{0.0, 0.1324, 0.0, 0.1634\} \right\rangle & \left\langle \{0.8030, 0.8333\}, \{0.0, 0.1347\} \right\rangle & \left\langle \{0.7161\}, \{0.1809\} \right\rangle \\
\left\langle \{0.8528, 0.9000\}, \{0.0, 0.0\} \right\rangle & \left\langle \{0.7822, 0.8202\}, \{0, 0, 0, 0.1634\} \right\rangle & \left\langle \{0.7500, 0.7954\}, \{0.1052, 0.1500\} \right\rangle & \left\langle \{0.5885, 0.6378, 0.6207, 0.6670\}, \{0, 0.1000, 0, 0.1394, 0, 0.2189, 0, 0.2978\} \right\rangle \\
\left\langle \{0.4404, 0.6051, 0.6880, 0.7904, 0.5758, 0.7087, 0.7729, 0.8500\}, \{0, 0, 0, 0.1194\} \right\rangle & \left\langle \{0.5307, 0.5600, 0.6231, 0.6480\}, \{0.2596, 0.3300\} \right\rangle & \left\langle \{0.7317, 0.7730, 0.8227, 0.8512\}, \{0.0, 0.1285\} \right\rangle & \left\langle \{0.4545\}, \{0.4336, 0.4620\} \right\rangle \\
\left\langle \{0.4881, 0.6675\}, \{0.2227, 0.2414, 0.2848, 0.3078\} \right\rangle & \left\langle \{0.5431, 0.6332\}, \{0.2639, 0.3076\} \right\rangle & \left\langle \{0.8003, 0.8553\}, \{0, 0, 0, 0.1175\} \right\rangle & \left\langle \{0.5543, 0.6633, 0.5985, 0.6987\}, \{0, 0, 0.2475, 0.2625\} \right\rangle
\end{bmatrix}.$$

Step 3. To calculate the value of T_{ij} use the equation (20) and (21).

$$T_{ij} = \begin{bmatrix} 1 & 0.8515 & 0.7030 & 0.6162 \\ 1 & 0.8528 & 0.7460 & 0.6530 \\ 1 & 0.9382 & 0.8258 & 0.6793 \\ 1 & 0.8245 & 0.5341 & 0.4621 \\ 1 & 0.6568 & 0.4277 & 0.3846 \end{bmatrix}.$$

Step 4. Utilize DHFPEWA operator to aggregate all DHFEs $\widetilde{r}_{ij}$ ($i = 1, 2, 3, 4, 5$; $j = 1, 2, 3, 4$) for each alternative x_i to reduce it in DHFE $\widetilde{r}_i$ ($i = 1, 2, 3, 4, 5$).

Step 5. By the definition 3, calculate the score values $S(r_i)$ ($i = 1, 2, 3, 4, 5$) of the alternative x_i . The values are as follows:

$S(r_1) = 0.8770, S(r_2) = 0.8846, S(r_3) = 0.9223, S(r_4) = 0.8162, S(r_5) = 0.8091.$

Since $S_3 > S_2 > S_1 > S_4 > S_5$, the ordering of alternatives are found as

$$x_3 > x_2 > x_1 > x_4 > x_5\,.$$

Now, the given problem is solved using DHFPEWG operator, for finding the preference ordering of the candidates. The following steps are performed:

Step 1. Same as above step 1.

Step 2. Utilize the DHFPEWG operator to aggregate the given dual hesitant fuzzy decision matrix $R^{(q)} = \left(\widetilde{r}_{ij}^{(q)}\right)_{5\times 4}$ $(q = 1, 2, 3)$

$$R = \begin{bmatrix}
\left\langle \begin{Bmatrix} 0.6671, 0.6981, \\ 0.8083, 0.8418 \end{Bmatrix}, \begin{Bmatrix} 0.0911, 0.1193, \\ 0.1303, 0.1582 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7126 \end{Bmatrix}, \begin{Bmatrix} 0.1857, \\ 0.2088 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7660, 0.7969, \\ 0.8044, 0.8358 \end{Bmatrix}, \begin{Bmatrix} 0.1026, \\ 0.1341 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5148, \\ 0.5709 \end{Bmatrix}, \begin{Bmatrix} 0.3336, 0.3697, \\ 0.3723, 0.4072 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.6870, 0.7188, \\ 0.7607, 0.7939 \end{Bmatrix}, \begin{Bmatrix} 0.0452, 0.1043, 0.0788, \\ 0.1375, 0.0833, 0.1420, \\ 0.1168, 0.1749 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7957, \\ 0.8246 \end{Bmatrix}, \begin{Bmatrix} 0.1038, 0.1345, \\ 0.1450, 0.1754 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7894, \\ 0.8220 \end{Bmatrix}, \begin{Bmatrix} 0.1101, \\ 0.1366 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7091 \end{Bmatrix}, \begin{Bmatrix} 0.1903 \end{Bmatrix} \right\rangle \\
\left\langle \{0.8302, 0.9000\}, \{0, 0.0170\} \right\rangle & \left\langle \begin{Bmatrix} 0.7498, \\ 0.7869 \end{Bmatrix}, \begin{Bmatrix} 0.0926, 0.1247, \\ 0.1485, 0.1802 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7500, \\ 0.7894 \end{Bmatrix}, \begin{Bmatrix} 0.1173, \\ 0.1500 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5589, 0.5765, \\ 0.5861, 0.6042 \end{Bmatrix}, \begin{Bmatrix} .0789, .1, .1392, , .2636, \\ .16, .2833, .3195, .3384 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.4277, 0.5072, 0.615, \\ 0.7135, 0.5195, 0.6094, \\ 0.7284, 0.8341 \end{Bmatrix}, \begin{Bmatrix} 0.0654, 0.09, \\ 0.0972, 0.1217 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.4898, 0.5385, \\ 0.5542, 0.6067 \end{Bmatrix}, \begin{Bmatrix} 0.3360, \\ 0.3811 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7306, 0.7635, \\ 0.8068, 0.8408 \end{Bmatrix}, \begin{Bmatrix} 0.0927, \\ 0.1308 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.4081 \end{Bmatrix}, \begin{Bmatrix} 0.4853, \\ 0.4967 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.4653, \\ 0.6379 \end{Bmatrix}, \begin{Bmatrix} 0.2590, 0.3010, \\ 0.3066, 0.3473 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5149, \\ 0.6275 \end{Bmatrix}, \begin{Bmatrix} 0.2721, \\ 0.3254 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7813, \\ 0.8400 \end{Bmatrix}, \begin{Bmatrix} 0.0594, 0.0885, \\ 0.0909, 0.1198 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5397, 0.5884, \\ 0.5865, 0.6376 \end{Bmatrix}, \begin{Bmatrix} 0.2586, 0.2897, \\ 0.2853, 0.3159 \end{Bmatrix} \right\rangle
\end{bmatrix}.$$

Step 3. Calculate the value of T_{ij} $(i = 1, 2, 3, 4, 5)$, $(j = 1, 2, 3, 4)$

$$T_{ij} = \begin{bmatrix} 1 & 0.8146 & 0.6172 & 0.5192 \\ 1 & 0.8149 & 0.6806 & 0.5725 \\ 1 & 0.9283 & 0.7574 & 0.6196 \\ 1 & 0.7629 & 0.4534 & 0.3794 \\ 1 & 0.6241 & 0.3971 & 0.3417 \end{bmatrix}.$$

Step 4. Utilize the DHFPEWG operator to aggregate all DHFEs $\widetilde{r}_{ij}$ $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3, 4)$ for each alternative x_i to reduce in DHFE $\widetilde{r}_i$ $(i = 1, 2, 3, 4, 5)$.

Step 5. By Definition 3, calculate the score values $S(r_i)$ $(i = 1, 2, 3, 4, 5)$ of the alternative x_i. The score values are found as

$$S(r_1) = 0.7739, S(r_2) = 0.8154, S(r_3) = 0.8240, S(r_4) = 0.6742, S(r_5) = 0.6479.$$

Since $S_3 > S_2 > S_1 > S_4 > S_5$ the ordering is found as

$$x_3 > x_2 > x_1 > x_4 > x_5 .$$

It is evident that the ordering of the candidates are the same for both the operators.

Now, if the problem is considered in a hesitant fuzzy environment and is solved using hesitant fuzzy prioritized Einstein weighted averaging operator developed by Yu et al. [30] the score value of the candidates are found as

$$S(r_1) = 0.7673, S(r_2) = 0.7879, S(r_3) = 0.8009, S(r_4) = 0.6570, S(r_5) = 0.6367$$

with the ordering $x_3 > x_2 > x_1 > x_4 > x_5$.

But, if the problem is solved using hesitant fuzzy prioritized Einstein weighted geometric operator developed by Yu et al. [30] the score value of the candidates changed and are found as

$$S(r_1) = 0.7185, S(r_2) = 0.7681, S(r_3) = 0.7623, S(r_4) = 0.5896, S(r_5) = 0.5922$$

with the ordering $x_2 > x_3 > x_1 > x_5 > x_4$.

So, the methods developed by Yu et al. [30] are not found consistent in this context.

Further, if the problem under consideration is solved in intuitionistic fuzzy environment using the technique developed by Yu [27], the same inconsistencies are observed as in the case of Yu et al. [30]. In this context the solutions are found as

$$S(r_1) = 0.8901, S(r_2) = 0.8940, S(r_3) = 0.9003, S(r_4) = 0.8737, S(r_5) = 0.8574$$

using intuitionistic fuzzy prioritized averaging operator with the rank of the alternatives

$$x_3 > x_2 > x_1 > x_4 > x_5$$

and using intuitionistic fuzzy prioritized geometric operator

$$S(r_1) = 0.7586, S(r_2) = 0.8127, S(r_3) = 0.7983, S(r_4) = 0.6956, S(r_5) = 0.7097$$

with the rank

$$x_2 > x_3 > x_1 > x_5 > x_4 .$$

Thus the proposed method is consistent than the previous approaches and provides efficient solutions in the decision making context.

7. Conclusions

Most of the traditional hesitant fuzzy aggression operators are based on algebraic operations. However, algebraic sum and algebraic product are not only the operations for aggregation of HFS. Many aggression operators are available to solve group decision making problems in DHF environment. But those operators did not provide consistent satisfactory solution in the decision making environment. In this paper DHFPEWA and DHFPEWG are proposed which establishes their capabilities to provide efficient solution in the decision making process. A new score function for DHFEs is proposed to remove the drawback of earlier methods [28]. It is also to be noted here that this process evaluates the weights of the decision makers from the decision matrix not by assigning arbitrary weights to them. Thus the influence of outside values cannot affect the decision of the proposed model. The proposed method can be extended to solve MCDM problems in interval valued DHF as well as dual hesitant probabilistic fuzzy environment without any computational complexities. However, it is hoped that the developed method can add an extra dimension in the process of making decision in hesitant fuzzy contexts.

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Mixed generalized quasi-Einstein manifolds with applications to Relativity

Dipankar Hazra

Abstract. The present paper aims to study and analyse mixed generalized quasi-Einstein manifolds. Some geometric properties of $MG\,(QE)_n$ had been discussed. We had also outlined the behaviour of $MG\,(QE)_4$ space-time with space-matter tensor and discussed some of its related properties. Finally, we constructed examples of mixed generalized quasi-Einstein manifolds.

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Key words: Einstein manifolds; mixed generalized quasi-Einstein manifolds; quasi-conformal curvature tensor; energy momentum tensor; Einstein's field equation; space-matter tensor.

1 Introduction

An n-dimensional semi-Riemannian or Riemannian manifold (M^n, g) $(n > 2)$, is said to be an Einstein manifold if its Ricci tensor S satisfies the condition

$$S = \frac{r}{n} g, \tag{1.1}$$

where r denotes the scalar curvature of (M^n, g). In other words, an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [6]. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is a quasi-Einstein manifold if its Ricci tensor S satisfies the condition

$$S\,(X, Y) = ag\,(X, Y) + bA\,(X)\,A\,(Y) \tag{1.2}$$

and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g\,(X, U) = A\,(X)\,,$$

for all vector field X. U being a unit vector field.

Here a and b are called the associated scalars, A is called the associated 1-form and U is called the generator of the vector field of the manifold. Such an n-dimensional manifold is denoted by $(QE)_n$.

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The notion of a generalized quasi-Einstein manifold was introduced by U. C. De and G. C. Ghosh [8]. According to them, a non-flat Riemannian manifold is called a generalized quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y), \tag{1.3}$$

where a, b, c are certain non-zero scalars and A, B are two non-zero 1-forms such that for two unit vector fields U and V corresponding to the 1-forms A and B respectively, defined as

$$g(X,U) = A(X),\ g(X,V) = B(X) \text{ and } g(U,V) = 0.$$

In such a case a, b, c are called the associated scalars, A, B respectively are called the associated main and auxiliary 1-forms and U, V respectively are called the main and auxiliary generators of the vector fields of the manifold. This type of manifold is denoted by $G(QE)_n$.

In [4, 11], A. Bhattacharyya, T. De and S. Dey introduced the notion of mixed generalized quasi-Einstein manifold. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is called mixed generalized quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) = {} & ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ & + e[A(X)B(Y) + A(Y)B(X)], \end{aligned} \tag{1.4}$$

where a, b, c, e are scalars of which $b \neq 0$, $c \neq 0$, $e \neq 0$ and A, B are two non-zero 1-forms such that

$$g(X,U) = A(X),\ g(X,V) = B(X) \text{ and } g(U,V) = 0,$$

where U, V are unit vector fields. In this case a, b, c, e are called associated scalars. A, B are called the associated 1-forms and U, V are called the generators of the vector fields of the manifold. If $e = 0$, then the manifold becomes to $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

In [9], the authors introduce the notion of a manifold of generalized quasi-constant curvature.

A Riemannian manifold is said to be a manifold of generalized quasi-constant curvature if the curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the following condition

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = {} & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + q[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) \\ & + g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)] \\ & + s[g(X,W)B(Y)B(Z) - g(X,Z)B(Y)B(W) \\ & + g(Y,Z)B(X)B(W) - g(Y,W)B(X)B(Z)], \end{aligned} \tag{1.5}$$

where p, q, s are scalars, A and B are non-zero 1-forms. U and V are unit orthogonal vector fields such that

$$g(X,U) = A(X),\ g(X,V) = B(X) \text{ and } g(U,V) = 0.$$

A Riemannian manifold is said to be a manifold of mixed generalized quasi-constant curvature if the curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= p\left[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\right]\\ &+ q\left[g(X,W)A(Y)A(Z)-g(Y,W)A(X)A(Z)\right.\\ &\left.+g(Y,Z)A(X)A(W)-g(X,Z)A(Y)A(W)\right]\\ &+ s\left[g(X,W)B(Y)B(Z)-g(Y,W)B(X)B(Z)\right.\\ &\left.+g(Y,Z)B(X)B(W)-g(X,Z)B(Y)B(W)\right]\\ &+ t\left[\{A(Y)B(Z)+B(Y)A(Z)\}g(X,W)\right.\\ &-\{A(X)B(Z)+B(X)A(Z)\}g(Y,W)\\ &+\{A(X)B(W)+B(X)A(W)\}g(Y,Z)\\ &\left.-\{A(Y)B(W)+B(Y)A(W)\}g(X,Z)\right], \end{aligned} \tag{1.6}$$

where p, q, s, t are scalars. A, B are non-zero 1-forms. U and V are orthonormal unit vectors corresponding to A and B such that

$$g(X,U)=A(X),\ g(X,V)=B(X) \text{ and } g(U,V)=0.$$

The notion of quasi-conformal curvature tensor was introduced by Yano and Sawaki [18] and they defined it as:

$$\begin{aligned}C^*(X,Y)Z &= a_1R(X,Y)Z+b_1\left[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX\right.\\ &\left.-g(X,Z)QY\right]-\frac{r}{n}\left[\frac{a_1}{n-1}+2b_1\right]\left[g(Y,Z)X-g(X,Z)Y\right],\end{aligned} \tag{1.7}$$

where a_1 and b_1 are constants, R is the curvature tensor of type $(1,3)$, S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator and r is the scalar curvature of the manifold.
If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then (1.7) reduces to the conformal curvature tensor C. Thus the conformal curvature tensor C is a particular case of the tensor C^*. For this reason C^* is called the quasi-conformal curvature tensor. A Riemannian or a semi-Riemannian manifold is called quasi-conformally flat if $C^* = 0$ for $n > 3$.

In a smooth manifold (M^n, g) Petrov [17] introduced a tensor $\tilde{P}$ of the type $(0,4)$ and defined it by

$$\tilde{P} = \tilde{R} + \frac{\kappa}{2} g \wedge T - \sigma G, \tag{1.8}$$

where $\tilde{R}$ is the curvature tensor of type $(0,4)$, T is the energy momentum tensor of type $(0,2)$, κ is the gravitational constant, σ is the energy density, G is a tensor of type $(0,4)$ given by

$$G(X,Y,Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W), \tag{1.9}$$

for all $X,Y,Z,W \in \chi(M)$ and Kulkarni-Nomizu product $E \wedge F$ of two $(0,2)$ tensors E and F is defined by

$$\begin{aligned}(E\wedge F)(X,Y,Z,W) &= E(Y,Z)F(X,W)+E(X,W)F(Y,Z)\\ &- E(X,Z)F(Y,W)-E(Y,W)F(X,Z),\end{aligned} \tag{1.10}$$

where $X, Y, Z, W \in \chi(M)$. The tensor $\tilde{P}$ is called the space-matter tensor of type $(0,4)$ of the manifold M. The space-matter tensor have been studied by Ahsan, Ali and Siddiqui [1, 2, 3] and many others.

After studying and analyzing various papers [7, 10, 12, 13, 14], we got motivated to work in this area. We have tried to develop a new concept. This paper is organized as follows:

After introduction in Section 2, we have studied $MG(QE)_n$ with divergence free quasi-conformal curvature tensor. In Section 3, we have studied sectional curvatures at a point of a quasi-conformally flat $MG(QE)_n$. In the next two sections, we have studied $MG(QE)_4$ spacetime with vanishing space-matter tensor and divergence free space-matter tensor. In section 6, we have studied perfect fluid $MG(QE)_4$ spacetime. Finally, we have given two examples of $MG(QE)_n$.

2 $MG(QE)_n$ $(n > 3)$ with divergence free quasi-conformal curvature tensor

In this section we look for a sufficient condition in order that a $MG(QE)_n$ $(n > 3)$ may be quasi-conformally conservative.

Theorem 2.1. *If in a $MG(QE)_n$ the associated scalars are constants and the generators U and V of the vector fields of the manifold are parallel vector fields, then the manifold is quasi-conformally conservative.*

Proof. Quasi-conformal curvature tensor is said to be conservative if the divergence of C^* vanishes, i.e., $\operatorname{div}(C^*) = 0$.

In a $MG(QE)_n$ if the associated scalars a, b, c, e are constants, then contracting (1.4) we get

$$r = an + b + c,$$

which implies that the scalar curvature r is constant, i.e., $dr = 0$.
Using $dr = 0$ we obtain from (1.7) that

$$\begin{aligned}(\nabla_W C^*)(X,Y,Z) = a_1 (\nabla_W R)(X,Y)Z + b_1 [(\nabla_W S)(Y,Z)X \\ - (\nabla_W S)(X,Z)Y + g(Y,Z)(\nabla_W Q)(X) \\ -g(X,Z)(\nabla_W Q)(Y)]. \end{aligned} \tag{2.1}$$

We know $(\operatorname{div} R)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$ and from (1.4) we obtain

$$\begin{aligned}(\nabla_X S)(Y,Z) = b[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)] \\ + c[(\nabla_X B)(Y)B(Z) + B(Y)(\nabla_X B)(Z)] \\ + e[(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z) \\ + (\nabla_X A)(Z)B(Y) + A(Z)(\nabla_X B)(Y)], \end{aligned} \tag{2.2}$$

since a, b, c and e are constants.

Contracting (2.1) and using (2.2) we obtain

$$
\begin{aligned}
(\mathrm{div} C^*)(X,Y,Z) = (a_1+b_1)\,[b\,\{&(\nabla_X A)(Y)\,A(Z) + A(Y)\,(\nabla_X A)(Z)\\
&- (\nabla_Y A)(X)\,A(Z) - A(X)\,(\nabla_Y A)(Z)\}\\
&+ c\,\{(\nabla_X B)(Y)\,B(Z) + B(Y)\,(\nabla_X B)(Z)\\
&- (\nabla_Y B)(X)\,B(Z) - B(X)\,(\nabla_Y B)(Z)\}\\
&+ e\,\{(\nabla_X A)(Y)\,B(Z) + A(Y)\,(\nabla_X B)(Z)\\
&+ (\nabla_X A)(Z)\,B(Y) + A(Z)\,(\nabla_X B)(Y)\\
&- (\nabla_Y A)(X)\,B(Z) - A(X)\,(\nabla_Y B)(Z)\\
&- (\nabla_Y A)(Z)\,B(X) - A(Z)\,(\nabla_Y B)(X)\}]\,. \qquad (2.3)
\end{aligned}
$$

Using the condition that the generators U and V of the vector fields of the manifold are parallel vector fields which gives $\nabla_X U = 0$ and $\nabla_X V = 0$. Hence

$$g(\nabla_X U, Y) = 0, \text{ i.e., } (\nabla_X A)(Y) = 0$$

and

$$g(\nabla_X V, Y) = 0, \text{ i.e., } (\nabla_X B)(Y) = 0.$$

Therefore from (2.3) we get

$$(\mathrm{div} C^*)(X,Y,Z) = 0.$$

Thus the manifold is quasi-conformally conservative. □

3 Sectional curvatures at a point of a quasi-conformally flat $MG(QE)_n$

Let us consider $U^\perp$ and $V^\perp$ as $(n-1)$-dimensional distribution in a quasi-conformally flat $MG(QE)_n$ $(n>3)$ orthogonal to U and V respectively. Then for any $X \in U^\perp$ and $X \in V^\perp$, $g(X,U) = 0$ and $g(X,V) = 0$, i.e., $A(X) = 0$ and $B(X) = 0$. In this section we will determine sectional curvature K at the plane determined by the vectors $X, Y \in U^\perp$ and $X, Y \in V^\perp$ or by X, U and X, V.

Theorem 3.1. *In a quasi-conformally flat $MG(QE)_n$ $(n>3)$ the sectional curvature of the plane determined by two vectors $X, Y \in U^\perp$ and $X, Y \in V^\perp$ is*

$$\frac{a_1(an+b+c) + 2b_1(n-1)(b+c)}{n(n-1)\,a_1},$$

while the sectional curvature of the plane determined by two vectors X, U is

$$\frac{a_1(an+b+c) - b_1(n-1)\{b(n-2)-2c\}}{n(n-1)\,a_1}$$

and the sectional curvature of the plane determined by two vectors X, V is

$$\frac{a_1(an+b+c) - b_1(n-1)\{c(n-2)-2b\}}{n(n-1)\,a_1}.$$

Proof. In [5], A. Bhattacharyya, T. De and D. Debnath proved that every quasi-conformally flat $MG\left(QE\right)_n$ $(n>3)$ is a manifold of mixed generalized quasi-constant curvature, i.e.,

$$\begin{aligned}
\tilde{R}\left(X,Y,Z,W\right) = &\left\{\frac{a_1 r + 2b_1\left(n-1\right)\left(r-an\right)}{n\left(n-1\right)a_1}\right\}\left[g\left(Y,Z\right)g\left(X,W\right)\right.\\
&-g\left(X,Z\right)g\left(Y,W\right)\left.\right] + \left(-\frac{bb_1}{a_1}\right)\left[g\left(X,W\right)A\left(Y\right)A\left(Z\right)\right.\\
&-g\left(Y,W\right)A\left(X\right)A\left(Z\right) + g\left(Y,Z\right)A\left(X\right)A\left(W\right)\\
&-g\left(X,Z\right)A\left(Y\right)A\left(W\right)\left.\right] + \left(-\frac{cb_1}{a_1}\right)\left[g\left(X,W\right)B\left(Y\right)B\left(Z\right)\right.\\
&-g\left(Y,W\right)B\left(X\right)B\left(Z\right) + g\left(Y,Z\right)B\left(X\right)B\left(W\right)\\
&-g\left(X,Z\right)B\left(Y\right)B\left(W\right)\left.\right] + \left(-\frac{eb_1}{a_1}\right)\left[\left\{A\left(Y\right)B\left(Z\right)\right.\right.\\
&+B\left(Y\right)A\left(Z\right)\left.\right\}g\left(X,W\right) - \left\{A\left(X\right)B\left(Z\right)\right.\\
&+B\left(X\right)A\left(Z\right)\left.\right\}g\left(Y,W\right) + \left\{A\left(X\right)B\left(W\right)\right.\\
&+B\left(X\right)A\left(W\right)\left.\right\}g\left(Y,Z\right) - \left\{A\left(Y\right)B\left(W\right)\right.\\
&\left.+B\left(Y\right)A\left(W\right)\right\}g\left(X,Z\right)\left.\right].
\end{aligned} \tag{3.1}$$

Putting $Z=Y$ and $W=X$ in (3.1) we have

$$\tilde{R}\left(X,Y,Y,X\right) = \frac{a_1 r + 2b_1\left(n-1\right)\left(r-an\right)}{n\left(n-1\right)a_1}\left[g\left(X,X\right)g\left(Y,Y\right) - \left\{g\left(X,Y\right)\right\}^2\right]. \tag{3.2}$$

Putting $Y=Z=U$ and $W=X$ in (3.1) we have

$$\tilde{R}\left(X,U,U,X\right) = \left\{\frac{a_1 r + 2b_1\left(n-1\right)\left(r-an\right)}{n\left(n-1\right)a_1} - \frac{bb_1}{a_1}\right\}g\left(X,X\right). \tag{3.3}$$

Putting $Y=Z=V$ and $W=X$ in (3.1) we get

$$\tilde{R}\left(X,V,V,X\right) = \left\{\frac{a_1 r + 2b_1\left(n-1\right)\left(r-an\right)}{n\left(n-1\right)a_1} - \frac{cb_1}{a_1}\right\}g\left(X,X\right). \tag{3.4}$$

Now contracting (1.4) over X and Y we have

$$r = an + b + c. \tag{3.5}$$

Using (3.2), (3.5), (3.3) and (3.4) we obtain

$$K\left(X,Y\right) = \frac{\tilde{R}\left(X,Y,Y,X\right)}{g\left(X,X\right)g\left(Y,Y\right) - \left\{g\left(X,Y\right)\right\}^2} = \frac{a_1\left(an+b+c\right) + 2b_1\left(n-1\right)\left(b+c\right)}{n\left(n-1\right)a_1},$$

$$K\left(X,U\right) = \frac{\tilde{R}\left(X,U,U,X\right)}{g\left(X,X\right)g\left(U,U\right) - \left\{g\left(X,U\right)\right\}^2} = \frac{a_1\left(an+b+c\right) - b_1\left(n-1\right)\left\{b\left(n-2\right) - 2c\right\}}{n\left(n-1\right)a_1}$$

and

$$K\left(X,V\right) = \frac{\tilde{R}\left(X,V,V,X\right)}{g\left(X,X\right)g\left(V,V\right) - \left\{g\left(X,V\right)\right\}^2} = \frac{a_1\left(an+b+c\right) - b_1\left(n-1\right)\left\{c\left(n-2\right) - 2b\right\}}{n\left(n-1\right)a_1}.$$

Thus the proof of theorem is completed. □

4 $MG\left(QE\right)_4$ spacetime with vanishing space-matter tensor

In this section we study $MG\left(QE\right)_4$ spacetime with vanishing space-matter tensor.

Theorem 4.1. *A $MG\left(QE\right)_4$ spacetime satisfying Einstein's field equation and with vanishing space-matter tensor is a spacetime of mixed generalized quasi-constant curvature.*

Proof. The equation (1.8) can be written as

$$\begin{aligned}\tilde{P}\left(X,Y,Z,W\right) = \tilde{R}\left(X,Y,Z,W\right) + \frac{\kappa}{2}&\left[g\left(Y,Z\right)T\left(X,W\right) + g\left(X,W\right)T\left(Y,Z\right)\right.\\ &\left.-g\left(X,Z\right)T\left(Y,W\right) - g\left(Y,W\right)T\left(X,Z\right)\right]\\ &-\sigma\left[g\left(Y,Z\right)g\left(X,W\right) - g\left(X,Z\right)g\left(Y,W\right)\right].\end{aligned} \tag{4.1}$$

If $\tilde{P}=0$, then (4.1) becomes

$$\begin{aligned}\tilde{R}\left(X,Y,Z,W\right) = -\frac{\kappa}{2}&\left[g\left(Y,Z\right)T\left(X,W\right) + g\left(X,W\right)T\left(Y,Z\right)\right.\\ &\left.-g\left(X,Z\right)T\left(Y,W\right) - g\left(Y,W\right)T\left(X,Z\right)\right]\\ &+\sigma\left[g\left(Y,Z\right)g\left(X,W\right) - g\left(X,Z\right)g\left(Y,W\right)\right].\end{aligned} \tag{4.2}$$

The Einstein's field equation without cosmological constant is given by [15, 16]

$$S\left(X,Y\right) - \frac{r}{2}g\left(X,Y\right) = \kappa T\left(X,Y\right), \tag{4.3}$$

where κ is the gravitational constant and r is the scalar curvature of the spacetime. Using (1.4) and Einstein's field equation (4.3) in (4.2) we have

$$\begin{aligned}\tilde{R}\left(X,Y,Z,W\right) = \left(\sigma - a + \frac{r}{2}\right)&\left[g\left(Y,Z\right)g\left(X,W\right) - g\left(X,Z\right)g\left(Y,W\right)\right]\\ &-\frac{b}{2}\left[g\left(X,W\right)A\left(Y\right)A\left(Z\right) - g\left(Y,W\right)A\left(X\right)A\left(Z\right)\right.\\ &\left.+g\left(Y,Z\right)A\left(X\right)A\left(W\right) - g\left(X,Z\right)A\left(Y\right)A\left(W\right)\right]\\ &-\frac{c}{2}\left[g\left(X,W\right)B\left(Y\right)B\left(Z\right) - g\left(Y,W\right)B\left(X\right)B\left(Z\right)\right.\\ &\left.+g\left(Y,Z\right)B\left(X\right)B\left(W\right) - g\left(X,Z\right)B\left(Y\right)B\left(W\right)\right]\\ &-\frac{e}{2}\left[\left\{A\left(Y\right)B\left(Z\right) + B\left(Y\right)A\left(Z\right)\right\}g\left(X,W\right)\right.\\ &-\left\{A\left(X\right)B\left(Z\right) + B\left(X\right)A\left(Z\right)\right\}g\left(Y,W\right)\\ &+\left\{A\left(X\right)B\left(W\right) + B\left(X\right)A\left(W\right)\right\}g\left(Y,Z\right)\\ &\left.-\left\{A\left(Y\right)B\left(W\right) + B\left(Y\right)A\left(W\right)\right\}g\left(X,Z\right)\right].\end{aligned} \tag{4.4}$$

Comparing (1.6) and (4.4) we can say that the manifold under consideration is a manifold of mixed generalized quasi-constant curvature. □

5 $MG\,(QE)_4$ spacetime with divergence free space-matter tensor

In this section we look for a sufficient condition in order that a $MG\,(QE)_4$ may be of divergence free space-matter tensor.

Theorem 5.1. *In a $MG\,(QE)_4$ spacetime satisfying Einstein's field equation with divergence free space-matter tensor the energy density is constant.*

Proof. In a $MG\,(QE)_n$ if the associated scalars a, b, c and e are constants, then contracting (1.4) we get

$$r = an + b + c,$$

which implies that the scalar curvature r is constant, i.e., $dr = 0$.
Using (4.3), we obtain from (4.1) that

$$\begin{aligned}(\text{div}P)\,(X,Y,Z) &= \text{div}R\,(X,Y)\,Z + \frac{1}{2}\left[(\nabla_X S)\,(Y,Z) - (\nabla_Y S)\,(X,Z)\right] \\ &\quad - g\,(Y,Z)\left[d\sigma\,(X) + \frac{1}{4}dr\,(X)\right] + g\,(X,Z)\left[d\sigma\,(Y) + \frac{1}{4}dr\,(Y)\right]. \end{aligned} \tag{5.1}$$

We know that in a semi-Riemannian manifold

$$(\text{div}R)\,(X,Y,Z) = (\nabla_X S)\,(Y,Z) - (\nabla_Y S)\,(X,Z)\,. \tag{5.2}$$

From (5.1) and (5.2) we have

$$\begin{aligned}(\text{div}P)\,(X,Y,Z) = \frac{3}{2}\left[(\nabla_X S)\,(Y,Z) - (\nabla_Y S)\,(X,Z)\right] - g\,(Y,Z)\left[d\sigma\,(X) + \frac{1}{4}dr\,(X)\right] \\ + g\,(X,Z)\left[d\sigma\,(Y) + \frac{1}{4}dr\,(Y)\right]. \end{aligned} \tag{5.3}$$

By assuming $(\text{div}P)\,(X,Y,Z) = 0$ and then contracting (5.3) over Y and Z, we have

$$d\sigma\,(X) = 0.$$

Thus the energy density is constant. □

Theorem 5.2. *If in a $MG\,(QE)_4$ spacetime satisfying Einstein's field equation the associated scalars and the energy density σ are constants, then the divergence of the space-matter tensor vanishes.*

Proof. Using (1.4), equation (5.3) can be written as

$$\begin{aligned}(\mathrm{div}P)(X,Y,Z) = {} & \frac{3}{2}\left[da(X)g(Y,Z) - da(Y)g(X,Z)\right] + \frac{3}{2}\left[db(X)A(Y)A(Z)\right. \\ & \left.-db(Y)A(X)A(Z)\right] + \frac{3}{2}\left[dc(X)B(Y)B(Z)\right. \\ & \left.-dc(Y)B(X)B(Z)\right] + \frac{3}{2}\left[de(X)\{A(Y)B(Z)\right. \\ & \left.+A(Z)B(Y)\} - de(Y)\{A(X)B(Z) + A(Z)B(X)\}\right] \\ & + \frac{3b}{2}\left[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) - (\nabla_Y A)(X)A(Z)\right. \\ & \left.-A(X)(\nabla_Y A)(Z)\right] + \frac{3c}{2}\left[(\nabla_X B)(Y)B(Z) + B(Y)(\nabla_X B)(Z)\right. \\ & \left.-(\nabla_Y B)(X)B(Z) - B(X)(\nabla_Y B)(Z)\right] + \frac{3e}{2}\left[(\nabla_X A)(Y)B(Z)\right. \\ & + A(Y)(\nabla_X B)(Z) + (\nabla_X A)(Z)B(Y) + A(Z)(\nabla_X B)(Y) \\ & - (\nabla_Y A)(X)B(Z) - A(X)(\nabla_Y B)(Z) - (\nabla_Y A)(Z)B(X) \\ & \left.-A(Z)(\nabla_Y B)(X)\right] - g(Y,Z)\left[d\sigma(X) + \frac{1}{4}dr(X)\right] \\ & + g(X,Z)\left[d\sigma(Y) + \frac{1}{4}dr(Y)\right]. \end{aligned} \tag{5.4}$$

Using the conditions that the associated scalars and the energy density σ are constants and the generators U and V of the vector fields of the manifold are parallel vector fields which gives $\nabla_X U = 0$ and $\nabla_X V = 0$. Hence $dr(X) = 0$, $d\sigma(X) = 0$, for all X. Also

$$g(\nabla_X U, Y) = 0, \text{ i.e., } (\nabla_X A)(Y) = 0$$

and

$$g(\nabla_X V, Y) = 0, \text{ i.e., } (\nabla_X B)(Y) = 0.$$

Therefore from (5.4) we get

$$(\mathrm{div}P)(X,Y,Z) = 0.$$

Thus the divergence of the space-matter tensor vanishes. □

6 Perfect fluid $MG(QE)_4$ spacetime

Theorem 6.1. *If a perfect fluid $MG(QE)_4$ spacetime admits Einstein's field equation without cosmological constant, then in this case isotropic pressure is $\frac{-6a+b-c}{6\kappa}$ and energy density is $\frac{2a+3b+c}{2\kappa}$.*

Proof. In a perfect fluid spacetime, the energy momentum tensor T of type $(0,2)$ is of the form:

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y), \tag{6.1}$$

where σ and p are the energy density and the isotropic pressure, respectively. Then in the general relativistic spacetime whose matter content is perfect fluid satisfying the Einstein's field equation, the Ricci tensor holds the following equation

$$S(X,Y) - \frac{r}{2} g(X,Y) = \kappa T(X,Y). \tag{6.2}$$

From (6.1) and (6.2), we get

$$S(X,Y) - \frac{r}{2} g(X,Y) = \kappa \left[p g(X,Y) + (\sigma + p) A(X) A(Y) \right]. \tag{6.3}$$

Taking a frame field and contracting (6.3) over X and Y, we have

$$r = \kappa (\sigma - 3p). \tag{6.4}$$

Here, if we consider the general relativistic perfect fluid $MG(QE)_4$ spacetime with unit timelike velocity vector field U, then we have

$$g(U,U) = -1. \tag{6.5}$$

Now putting $X = Y = U$ in (6.3) and then using (6.4), we have

$$S(U,U) = \frac{\kappa}{2} (\sigma + 3p). \tag{6.6}$$

In the case of $MG(QE)_4$ spacetime, contracting (1.4) over X and Y, we have

$$r = 4a + b + c. \tag{6.7}$$

From (6.4) and (6.7) we have

$$4a + b + c = \kappa (\sigma - 3p). \tag{6.8}$$

Again from (1.4), we get

$$S(U,U) = -a + b. \tag{6.9}$$

From (6.6) and (6.9), we obtain

$$b - a = \frac{\kappa}{2} (\sigma + 3p). \tag{6.10}$$

Solving equations (6.8) and (6.10), we get

$$p = \frac{-6a + b - c}{6\kappa}, \quad \sigma = \frac{2a + 3b + c}{2\kappa}.$$

This completes the proof of the theorem. □

7 Examples of $MG\,(QE)_4$

In this section, we show the existence of $MG\,(QE)_4$ by constructing two non-trivial concrete examples.

Example 7.1. Let $\left(x^1, x^2, ..., x^n\right) \in \mathbb{R}^n$, where $\mathbb{R}^n$ an n-dimensional real number space. We consider a Riemannian metric g on $\mathbb{R}^4 = \left(x^1, x^2, x^3, x^4\right)$, by [10]

$$ds^2 = g_{ij} dx^i dx^j = \left(dx^1\right)^2 + \left(x^1\right)^2 \left(dx^2\right)^2 + \left(x^2\right)^2 \left(dx^3\right)^2 + \left(dx^4\right)^2, \tag{7.1}$$

where $i, j = 1, 2, 3, 4$. Using (7.1), we see the non-vanishing components of Riemannian metric are

$$g_{11} = 1,\ \ g_{22} = \left(x^1\right)^2,\ \ g_{33} = \left(x^2\right)^2,\ \ g_{44} = 1 \tag{7.2}$$

and its associated components are

$$g^{11} = 1,\ \ g^{22} = \frac{1}{\left(x^1\right)^2},\ \ g^{33} = \frac{1}{\left(x^2\right)^2},\ \ g^{44} = 1. \tag{7.3}$$

With the help of (7.2) and (7.3), it can be calculated that the non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are given by

$$\Gamma^1_{22} = -x^1,\ \ \Gamma^2_{33} = -\frac{x^2}{\left(x^1\right)^2},\ \ \Gamma^2_{12} = \frac{1}{x^1},\ \ \Gamma^3_{23} = \frac{1}{x^2},$$

$$R_{1332} = -\frac{x^2}{x^1},\ \ S_{12} = -\frac{1}{x^1 x^2}$$

and the other components are obtained by the symmetric properties. It can be easily shown that the scalar curvature r of the resulting manifold $\left(\mathbb{R}^4, g\right)$ is zero. We shall show that $\left(\mathbb{R}^4, g\right)$ is a $MG\,(QE)_4$.
Let us consider the associated scalars as follows:

$$a = \frac{1}{x^1 \left(x^2\right)^2},\ \ b = -\frac{1}{\left(x^2\right)^3},\ \ c = \frac{1}{\left(x^2\right)^4},\ \ e = -\frac{2}{\left(x^1\right)^2 x^2}. \tag{7.4}$$

We choose the 1-form as follows:

$$A_i\left(x\right) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{when } i = 1 \\ \frac{x^2}{\sqrt{2}}, & \text{when } i = 3 \\ 0, & \text{otherwise} \end{cases} \tag{7.5}$$

and

$$B_i\left(x\right) = \begin{cases} \frac{x^1}{\sqrt{2}}, & \text{when } i = 2 \\ \frac{1}{\sqrt{2}}, & \text{when } i = 4 \\ 0, & \text{otherwise} \end{cases} \tag{7.6}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equation

$$S_{12} = ag_{12} + bA_1A_2 + cB_1B_2 + e\left(A_1B_2 + A_2B_1\right), \tag{7.7}$$

since, for the other cases (1.4) holds trivially.
From the equations (7.4), (7.5), (7.6) and (7.7) we get

$$\begin{aligned}\text{Right hand side of (7.7)} &= ag_{12} + bA_1A_2 + cB_1B_2 + e\left(A_1B_2 + A_2B_1\right)\\ &= \frac{1}{x^1\left(x^2\right)^2}\cdot 0 + \left(-\frac{1}{\left(x^2\right)^3}\right)\cdot\frac{1}{\sqrt{2}}\cdot 0 + \frac{1}{\left(x^2\right)^4}\cdot 0\cdot\frac{x^1}{\sqrt{2}}\\ &\quad + \left(-\frac{2}{\left(x^1\right)^2x^2}\right)\left(\frac{1}{\sqrt{2}}\cdot\frac{x^1}{\sqrt{2}} + 0\right)\\ &= -\frac{1}{x^1x^2} = S_{12}.\end{aligned}$$

We shall now show that the associated vectors A_i and B_i are unit and also they are orthogonal.
Here,

$$g^{ij}A_iA_j = 1,\ \ g^{ij}B_iB_j = 1,\ \ g^{ij}A_iB_j = 0.$$

So, $\left(\mathbb{R}^4, g\right)$ is a $MG\left(QE\right)_4$.

Example 7.2. Let $\left(x^1, x^2, ..., x^n\right) \in \mathbb{R}^n$, where $\mathbb{R}^n$ denotes n-dimensional real number space. We consider a Lorentzian metric g on $\mathbb{R}^4 = \left(x^1, x^2, x^3, x^4; x^1 \neq \frac{(1+2p)\pi}{4}, p \in \mathbb{Z}\right)$, ($\mathbb{Z}$ is the set of positive integer) , by [7]

$$ds^2 = g_{ij}dx^idx^j = \left\{\sin\left(x^1\right) - \cos\left(x^1\right)\right\}\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right] - \left(dx^4\right)^2, \tag{7.8}$$

where $i, j = 1, 2, 3, 4$. Using (7.8), we see the non-vanishing components of the Lorentzian metric are

$$g_{11} = g_{22} = g_{33} = \sin\left(x^1\right) - \cos\left(x^1\right),\ \ g_{44} = -1 \tag{7.9}$$

and its associated components are

$$g^{11} = g^{22} = g^{33} = \frac{1}{\sin\left(x^1\right) - \cos\left(x^1\right)},\ \ g^{44} = -1. \tag{7.10}$$

With the help of (7.9) and (7.10), it can be found that the non-vanishing components of Christoffel symbols, curvature tensor, Ricci tensor and scalar curvature are given by

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \frac{\sin\left(x^1\right) + \cos\left(x^1\right)}{2\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)},\ \ \Gamma^1_{22} = \Gamma^1_{33} = \frac{\sin\left(x^1\right) + \cos\left(x^1\right)}{2\left(\cos\left(x^1\right) - \sin\left(x^1\right)\right)},$$

$$R_{1331} = \frac{1}{\cos\left(x^1\right) - \sin\left(x^1\right)},\ \ S_{33} = \frac{-3 + \sin 2\left(x^1\right)}{4\left(1 - \sin 2\left(x^1\right)\right)},$$

$$r = \frac{-3 + \sin 2\left(x^1\right)}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^3} \left(\neq 0\right) \tag{7.11}$$

and the other components are obtained by the symmetric properties. From (7.11), it is clear that the manifold $\left(\mathbb{R}^4, g\right)$ is a Lorentzian manifold. Now, we are to prove that $\left(\mathbb{R}^4, g\right)$ is a $MG\left(QE\right)_4$.
Let us consider the associated scalars as follows:

(7.12)

$$a = \frac{\sin 2\left(x^1\right)}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^3}, \quad b = \frac{1}{\sin\left(x^1\right) - \cos\left(x^1\right)}, \quad c = -\frac{3}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^3},$$
$$e = -\frac{1}{2\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^2}.$$

We choose the 1-form as follows:

$$A_i\left(x\right) = \begin{cases} \sqrt{\sin\left(x^1\right) - \cos\left(x^1\right)}, & \text{when } i = 1 \\ 0, & \text{otherwise} \end{cases} \tag{7.13}$$

and

$$B_i\left(x\right) = \begin{cases} \sqrt{\sin\left(x^1\right) - \cos\left(x^1\right)}, & \text{when } i = 3 \\ 0, & \text{otherwise} \end{cases} \tag{7.14}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equation

$$S_{33} = a g_{33} + b A_3 A_3 + c B_3 B_3 + 2e A_3 B_3, \tag{7.15}$$

since, for the other cases (1.4) holds trivially.
From the equations (7.12), (7.13), (7.14) and (7.15) we get

$$\begin{aligned}\text{Right hand side of (7.15)} &= a g_{33} + b A_3 A_3 + c B_3 B_3 + 2e A_3 B_3 \\ &= \frac{\sin 2\left(x^1\right)}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^2} + 0 - \frac{3}{4\left(\sin\left(x^1\right) - \cos\left(x^1\right)\right)^2} - 0 \\ &= \frac{-3 + \sin 2\left(x^1\right)}{4\left(1 - \sin 2\left(x^1\right)\right)} = S_{33}.\end{aligned}$$

We shall now show that the 1-forms are unit and orthogonal.
Here,

$$g^{ij} A_i A_j = 1, \quad g^{ij} B_i B_j = 1, \quad g^{ij} A_i B_j = 0.$$

So, $\left(\mathbb{R}^4, g\right)$ is a $MG\left(QE\right)_4$.

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Author's address:

Dipankar Hazra
Department of Mathematics,
Heramba Chandra College,
23/49, Gariahat Road, Kolkata - 700029,
West Bengal, India.
E-mail: dipankarsk524@gmail.com

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Impact of quasi-conformal curvature tensor in spacetimes and $f(\mathcal{R}, G)$-gravity

Uday Chand De[1,a], **Dipankar Hazra**[2,b]

[1] Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata 700019, West Bengal, India
[2] Department of Mathematics, University of Kalyani, Kalyani 741235, West Bengal, India

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Abstract At first, we show that a quasi-conformally flat perfect fluid spacetime is a de-Sitter spacetime as well as a Robertson-Walker spacetime. We study this spacetime as a solution of $f(\mathcal{R}, G)$-gravity theory, and obtain a relation among snap, jerk, and deceleration parameters using flat Friedmann-Robertson-Walker metric. Several energy conditions in terms of Ricci scalar are investigated with the models $f(\mathcal{R}, G) = \mu\mathcal{R}^{\beta_1}G^{\beta_2}$, $f(\mathcal{R}, G) = \alpha_1\mathcal{R} + \alpha_2\mathcal{R}^m G^n$ and $f(\mathcal{R}, G) = \mathcal{R} + a_1 G^{\alpha_3}$. For these models, the weak, null and dominant energy conditions are satisfied while the strong energy condition is violated, which is a good agreement with the recent observational studies which reveals that the current Universe is in accelerating phase.

1 Introduction

A spacetime is a Lorentzian manifold M^n with the Lorentzian metric g of signature $(-, +, +, \dots, +)$ which permits a globally timelike vector field. Different types of spacetimes have been studied in various ways, such as ([5, 14, 17, 20, 22, 23, 45]) and many others.

A Lorentzian manifold of dim. n $(n > 2)$ whose metric assume the appropriate local structure

$$ds^2 = -(d\zeta)^2 + q^2(\zeta) g^*_{u_1u_2} \mathrm{d}x^{u_1}\mathrm{d}x^{u_2} \tag{1}$$

is named a generalized Robertson-Walker (GRW) spacetime ([9, 10]), where $g^*_{u_1u_2} = g^*_{u_1u_2}(x^{u_3})$ are only functions of x^{u_3} $(u_1, u_2, u_3 = 2, 3, \dots, n)$ and q is a function dependent on ζ. So, $-\mathcal{I} \times q^2\bar{M}$ can be used to represent GRW spacetime in which $\mathcal{I} \subseteq \mathbb{R}$ is an open interval and $\bar{M}$ is an $(n-1)$-dimensional Riemannian manifold. If dim. of $\bar{M}$ is three and of constant sectional curvature, then the GRW spacetime terms into a Robertson-Walker (RW) spacetime.

M^n is referred to as perfect fluid spacetime (PFS) if the Ricci tensor $\mathcal{R}_{lk}$ takes the shape

$$\mathcal{R}_{lk} = c g_{lk} + d u_l u_k, \tag{2}$$

where c, d are scalars and u_k is a unit timelike vector, named velocity vector or flow vector.

The matter content of the spacetimes in general relativity (GR) theory is depicted by the energy-momentum tensor (EMT) $\mathcal{T}$ and the fluid is termed perfect fluid, since it does not have the heat conduction terms [24]. The form of the EMT [38] for a PFS is

$$\mathcal{T}_{lk} = \boldsymbol{p} g_{lk} + (\boldsymbol{p} + \boldsymbol{\sigma}) u_l u_k, \tag{3}$$

where $\boldsymbol{p}$ denotes isotropic pressure, $\boldsymbol{\sigma}$ denotes energy density of the ordinary matter. On a physical viewpoint, it is described the global structure of a PFS, in several relevant cases. The Einstein's field equations (EFE) are described by

$$\mathcal{R}_{lk} - \frac{1}{2} g_{lk}\mathcal{R} = \kappa\mathcal{T}_{lk}, \tag{4}$$

where $\mathcal{R}$ indicates the Ricci scalar and κ is the gravitational constant. Additionally, a state equation with the form $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{\sigma})$ connects $\boldsymbol{p}$ and $\boldsymbol{\sigma}$, and the PFS is known as isentropic. Furthermore, if $\boldsymbol{p} = \boldsymbol{\sigma}$, the PFS is referred to as stiff matter. The PFS is referred to as the dark matter era if $\boldsymbol{p} + \boldsymbol{\sigma} = 0$, the dust matter fluid if $\boldsymbol{p} = 0$, and the radiation era if $\boldsymbol{p} = \frac{\boldsymbol{\sigma}}{3}$ [8]. The Universe is represented as accelerating phase when $\frac{\boldsymbol{p}}{\boldsymbol{\sigma}} < -\frac{1}{3}$. It covers the quintessence phase if $-1 < \frac{\boldsymbol{p}}{\boldsymbol{\sigma}} < 0$ and phantom era if $\frac{\boldsymbol{p}}{\boldsymbol{\sigma}} < -1$.

It is widely accepted by the scientific community that our Universe is currently in an accelerated phase. GR in its standard form can not explain the accelerated expansion without extra terms or components, which have been gathered under the name of dark

[a] emails: ucde1950@gmail.com; uc_de@yahoo.com
[b] e-mail: dipankarsk524@gmail.com (corresponding author)

energy. As is widely known in GR, the energy conditions (ECs) are crucial tools for researching wormholes and black holes in various modified gravities ([4, 24]). The Raychaudhuri Eq. [41], which reflect the attractive nature of gravity through the positivity condition $\mathcal{R}_{lk}v^l v^k \geq 0$, where v^l is a null vector, are used to methodically create the ECs. In GR, through the EFE, this condition on geometry is equivalent to the null energy condition (NEC) $\mathcal{T}_{lk}v^l v^k \geq 0$ on matter. In specifically, the weak energy condition (WEC) asserts that $\mathcal{T}_{lk}u^l u^k \geq 0$, for any timelike vector u^l and assumes a positive local energy density. Numerous changes to EFE have been made and extensively researched (see [7, 36, 40] for examples of the modified gravity theories). One of these modified theories was the "$f(\mathcal{R}, G)$-gravity theory" [19], which was developed by substituting the earlier Ricci scalar $\mathcal{R}$ with a function of $\mathcal{R}$ and G, the Gauss-Bonnet scalar. In [12], the authors have studied the stability of de-Sitter and power-law solutions in $f(\mathcal{R}, G)$-gravity and shown that gravitational action plays an important role in the stability of the solutions both of which depending on the form of the $f(\mathcal{R}, G)$-theory and the parameters of the model. In [26], the authors develop the weak-field limit of $f(\mathcal{R}, G)$ considering the Parametrized Post-Newtonian formalism. The possibility to obtain inflation considering a generic $f(\mathcal{R}, G)$ theory discussed in [27]. Also in [3], the authors study ECs in terms of the Hubble, deceleration, jerk and snap parameters for $f(\mathcal{R}, G)$ modified theories of gravity and consider their realization for flat Friedmann cosmological models. Motivated by these studies, this paper is mainly organized to investigate ECs in terms of Ricci scalar in a quasi-conformally flat PFS obeying $f(\mathcal{R}, G)$-gravity. Some specific $f(\mathcal{R}, G)$-models, for instance, $f(\mathcal{R}, G) = \mu\mathcal{R}^{\beta_1}G^{\beta_2}$, $f(\mathcal{R}, G) = \alpha_1\mathcal{R}+\alpha_2\mathcal{R}^m G^n$ ($\alpha_1, \alpha_2, \beta_1, \beta_2, \mu, m$ and n are constants) suggested by de la Cruz-Dombriz et al. [12] and $f(\mathcal{R}, G) = \mathcal{R} + a_1 G^{\alpha_3}$ (a_1 is an arbitrary constant and α_3 is an even positive number) suggested by Nojiri et al. [37] were developed to explain late-time acceleration as a pure gravitational phenomenon without the usage of dark energy, and cosmic inflation without the use of scalar fields. Despite their limitations, these models managed to increase the popularity of $f(\mathcal{R}, G)$-models in general.

To study an infinitesimal nonhomothetic conformal transformation in compact orientable manifold of dim. $n > 2$ with constant Ricci scalar, Yano and Sawaki [44] created a new tensor denoted by $\mathcal{C}^*$ and defined by

$$\mathcal{C}^{*h}_{kji} = a\mathcal{Z}^h_{kji} + \frac{b}{n-2}\left\{\delta^h_k B_{ji} - \delta^h_j B_{ki} + g_{ji}B^h_k - g_{ki}B^h_j\right\},$$

where a, b being constants, $\mathcal{Z}^h_{kji} = \mathcal{R}^h_{kji} - \frac{\mathcal{R}}{n(n-1)}\left\{\delta^h_k g_{ji} - \delta^h_j g_{ki}\right\}$is the concircular curvature tensor and $B_{hk} = \mathcal{R}_{hk} - \frac{\mathcal{R}}{n}g_{hk}$. The authors [44] proved that if ££$\left(\mathcal{C}^*_{kjih}\mathcal{C}^{*kjih}\right) \leq 0$ and $a + b \neq 0$, then the manifold is isometric to a sphere, where £ stands for the Lie derivative.

The above expression of $\mathcal{C}^*$ for dim. 4 can be written explicitly as

$$\mathcal{C}^{*h}_{ijk} = \gamma\mathcal{R}^h_{ijk} + \delta\left\{g_{ij}\mathcal{R}^h_k - g_{ik}\mathcal{R}^h_j + \delta^h_k\mathcal{R}_{ij} - \delta^h_j\mathcal{R}_{ik}\right\} - \frac{\mathcal{R}}{4}\left(\frac{\gamma}{3} + 2\delta\right)\left\{\delta^h_k g_{ij} - \delta^h_j g_{ik}\right\}, \tag{5}$$

where γ and δ are constants, $\mathcal{R}^h_{ijk}$ denotes the curvature tensor. If $\gamma = 1$ and $\delta = -\frac{1}{2}$, then $\mathcal{C}^{*h}_{ijk}$ takes the form

$$\mathcal{C}^h_{ijk} = \mathcal{R}^h_{ijk} - \frac{1}{2}\left\{g_{ij}\mathcal{R}^h_k - g_{ik}\mathcal{R}^h_j + \delta^h_k\mathcal{R}_{ij} - \delta^h_j\mathcal{R}_{ik}\right\} + \frac{\mathcal{R}}{6}\left\{\delta^h_k g_{ij} - \delta^h_j g_{ik}\right\}, \tag{6}$$

where $\mathcal{C}^h_{ijk}$ is the conformal curvature tensor. Such a tensor $\mathcal{C}^{*h}_{ijk}$ is named quasi-conformal curvature tensor. A quasi-conformally flat manifold is known [2] to be either conformally flat for $\gamma \neq 0$ or Einstein for $\gamma = 0$ and $\delta \neq 0$. We must keep in mind the situation of $\gamma \neq 0$ or $\delta \neq 0$ because they do not impose any restrictions on manifolds if $\gamma = 0$ and $\delta = 0$. Throughout the paper we adopt that $\gamma + 2\delta \neq 0$. Some researchers investigated $\mathcal{C}^{*h}_{ijk}$ in Riemannian manifolds, contact manifolds and spacetimes, such as ([16, 32, 35, 39]) and many others.

Transvecting (5) with g_{lh}, we get

$$\begin{aligned}\mathcal{C}^*_{lijk} =& \gamma\mathcal{R}_{lijk} + \delta\left\{g_{ij}\mathcal{R}_{lk} - g_{ik}\mathcal{R}_{lj} + g_{lk}\mathcal{R}_{ij} - g_{lj}\mathcal{R}_{ik}\right\} \\ &- \frac{\mathcal{R}}{4}\left(\frac{\gamma}{3} + 2\delta\right)\left\{g_{ij}g_{lk} - g_{ik}g_{lj}\right\}.\end{aligned} \tag{7}$$

It is well-known that

$$\mathcal{C}^k_{lij,k} = \left(\frac{n-3}{n-2}\right)\left[\left\{\mathcal{R}_{li,j} - \mathcal{R}_{lj,i}\right\} - \frac{1}{2(n-1)}\left\{g_{li}\mathcal{R}_{,j} - g_{lj}\mathcal{R}_{,i}\right\}\right]. \tag{8}$$

Here comma (,) denotes the covariant differentiation.

Definition 1 [11] A generalized Ricci recurrent spacetime is characterized by

$$\mathcal{R}_{lk,h} = \omega_h\mathcal{R}_{lk} + \tau_h g_{lk}, \tag{9}$$

where ω_h and τ_h are nonzero vectors. Several researchers ([14, 15, 31]) and many others have investigated generalized Ricci recurrent spacetime.

After a brief introduction in Sect. 2, we characterize spacetimes with quasi-conformally flat curvature tensor. The physical properties of PFS admitting quasi-conformally flat curvature tensor are examined in the next Section. The analysis of quasi-conformally recurrent spacetimes is presented in Sect. 4. In the last Section, we explore quasi-conformally flat spacetimes in $f(\mathcal{R}, G)$-gravity theory.

2 Quasi-conformally flat spacetimes

Here we consider quasi-conformally flat $\left(\mathcal{C}^*_{lijk} = 0\right)$ spacetime of GR. Eq. (7) imply

$$\mathcal{R}_{lijk} = -\frac{\delta}{\gamma}\left\{g_{ij}\mathcal{R}_{lk} - g_{ik}\mathcal{R}_{lj} + g_{lk}\mathcal{R}_{ij} - g_{lj}\mathcal{R}_{ik}\right\} + \frac{\mathcal{R}}{4}\left(\frac{1}{3} + \frac{2\delta}{\gamma}\right)\left\{g_{ij}g_{lk} - g_{ik}g_{lj}\right\}. \tag{10}$$

Transvecting (10) with g^{ij} we obtain

$$\mathcal{R}_{lk} = \frac{\mathcal{R}}{4} g_{lk}. \tag{11}$$

Hence provide the result:

Theorem 1 *A spacetime which is $\mathcal{C}^*$-flat reveals an Einstein spacetime.*

Using (11) in (10) infers

$$\mathcal{R}_{lijk} = \frac{\mathcal{R}}{12}\left\{g_{ij}g_{lk} - g_{ik}g_{lj}\right\}. \tag{12}$$

Thus, we might conclude that:

Theorem 2 *A $\mathcal{C}^*$-flat spacetime represents a spacetime of constant curvature.*

M^n is referred to as a Yang pure space [21] if the metric fulfills the following

$$\mathcal{R}_{ij,l} = \mathcal{R}_{il,j}. \tag{13}$$

Since the Einstein spacetime satisfying this requirement (13). Thus we write:

Theorem 3 *A $\mathcal{C}^*$-flat spacetime is a Yang pure space.*

Definition 2 [18] A vector field ξ is known as conformal Ricci collineation (CRC), Ricci inheritance vector (RIV), conformal collineation (CC) if, for a scalar β, it satisfies

$$\pounds_\xi \mathcal{R}_{ij} = 2\beta g_{ij}, \tag{14}$$

$$\pounds_\xi \mathcal{R}_{ij} = 2\beta \mathcal{R}_{ij}, \tag{15}$$

$$\pounds_\xi g_{ij} = 2\beta g_{ij}, \tag{16}$$

respectively. In particular, if $\beta = 0$, then Eqs. (14) and (16) reduces to Ricci collineation and Killing equation, respectively.

Considering the Lie derivative on both sides of (11), we reach

$$\pounds_\xi \mathcal{R}_{ij} = \frac{\mathcal{R}}{4}\pounds_\xi g_{ij}. \tag{17}$$

If ξ is CRC, hence Eqs. (14) and (17) together imply

$$\pounds_\xi g_{ij} = 2\psi_1 g_{ij}, \quad \text{where} \quad \psi_1 = \frac{4\beta}{\mathcal{R}}. \tag{18}$$

In contrast, if ξ is CC, then Eqs. (16) and (17) gives us

$$\pounds_\xi \mathcal{R}_{ij} = 2\psi_2 g_{ij}, \quad \text{where} \quad \psi_2 = \frac{\beta\mathcal{R}}{4}. \tag{19}$$

This fact prompts us to make the conclusion:

Theorem 4 *A $\mathcal{C}^*$-flat spacetime admits CC with respect to ξ if and only if it permits CRC with respect to ξ.*

If ξ is RIV, then from Eqs. (11) and (15), it follows that

$$\pounds_\xi \mathcal{R}_{ij} = 2\psi_3 g_{ij}, \quad \text{where} \quad \psi_3 = \frac{\mathcal{R}\beta}{4}. \tag{20}$$

Consequently, we state:

Theorem 5 *In a $\mathcal{C}^*$-flat spacetime, RIV becomes CRC.*

3 Quasi-conformally flat perfect fluid spacetimes

In this part, we take a PFS with quasi-conformally flat curvature tensor obeying EFE. From (3), (4) and (11), it follows that

$$\left(\kappa \boldsymbol{p} + \frac{\mathcal{R}}{4}\right) g_{ij} + \kappa(\boldsymbol{p} + \boldsymbol{\sigma}) u_i u_j = 0. \tag{21}$$

Transvecting (21) with g^{ij} entails that

$$\kappa(\boldsymbol{\sigma} - 3\boldsymbol{p}) = \mathcal{R}. \tag{22}$$

Again, transvecting (21) with u^i we find

$$\mathcal{R} = 4\kappa\boldsymbol{\sigma}. \tag{23}$$

From (22) and (23), we get

$$\boldsymbol{p} + \boldsymbol{\sigma} = 0. \tag{24}$$

This represents a dark matter era [8]. Thus we conclude:

Theorem 6 *A $\mathcal{C}^*$-flat PFS represents a dark matter era.*

Since the quasi-conformally flat spacetime is an Einstein spacetime, the Ricci scalar $\mathcal{R}$ is constant and Eq. (8) infers $\mathcal{C}^i_{jkl,i} = 0$. Mantica et al. [33] proved that a PFS with $\mathcal{C}^i_{jkl,i} = 0$ and $\mathcal{R}_{,i} = 0$ is a GRW spacetime.

Thus, we assert the result:

Theorem 7 *A $\mathcal{C}^*$-flat PFS is a GRW spacetime.*

Mantica et al. [34] established that in a GRW spacetime, $u^i\mathcal{C}_{jkli} = 0$ (that is, the Weyl tensor is purely electric [25]) if and only if $\mathcal{C}^i_{jkl,i} = 0$.

Hence provide the result:

Theorem 8 *In a $\mathcal{C}^*$-flat PFS, the Weyl tensor is purely electric.*

It is known ([42], p. 73) that if the Weyl tensor is purely electric in a spacetime admitting a unit timelike vector, then it is of Petrov type I, D or O.

Consequently, we obtain:

Corollary 9 *A $\mathcal{C}^*$-flat PFS is of Petrov type I, D or O.*

For dimension 4, $u^i\mathcal{C}_{jkli} = 0$ is similar to $u_h\mathcal{C}_{lijk} + u_l\mathcal{C}_{ihjk} + u_i\mathcal{C}_{hljk} = 0$ ([30], p. 128). Transvecting with u^h, we have $\mathcal{C}_{lijk} = 0$. In [6], the authors proved that a GRW spacetime is conformally flat if and only if it is a RW spacetime.

Hence we write:

Theorem 10 *A $\mathcal{C}^*$-flat PFS is a RW spacetime.*

As the energy density cannot be negative, from (23) one gets

$$\mathcal{R} \geq 0. \tag{25}$$

that is, $\mathcal{R} = 0$ or, $\mathcal{R} > 0$.

Case 1 If $\mathcal{R} = 0$, then (12) infers $\mathcal{R}_{lijk} = 0$. This represents that the spacetime is of zero sectional curvature. Therefore the spacetime is locally isometric to Minkowski spacetime ([18], p. 67).

Case 2 If $\mathcal{R} > 0$, then Eq. (12) indicates that the constant curvature is positive. Note that the spacetime with constant positive curvature is a de-Sitter spacetime [18].

Therefore, we state:

Theorem 11 *A $\mathcal{C}^*$-flat PFS is either a de-Sitter spacetime or locally isometric to Minkowski spacetime.*

4 Quasi-conformally recurrent spacetimes

Definition 3 [43]M^n is said to be a recurrent spacetime if $\mathcal{R}^h_{ijk}$ fulfills the relation

$$\mathcal{R}^h_{ijk,l} = \theta_l \mathcal{R}^h_{ijk}, \tag{26}$$

where θ_l is a nonzero vector.

A 4-dimensional Lorentzian manifold is called a $\mathcal{C}^*$-recurrent spacetime if

$$\mathcal{C}^{*h}_{ijk,l} = \lambda_l \mathcal{C}^{*h}_{ijk}, \tag{27}$$

where λ_l is a unit timelike vector.

Inserting (5) in (27), we deduce

$$\begin{aligned}
&\gamma \mathcal{R}^h_{ijk,l} + \delta\left\{g_{ij}\mathcal{R}^h_{k,l} - g_{ik}\mathcal{R}^h_{j,l} + \delta^h_k \mathcal{R}_{ij,l} - \delta^h_j \mathcal{R}_{ik,l}\right\} \\
&\quad - \frac{\mathcal{R}_{,l}}{4}\left(\frac{\gamma}{3} + 2\delta\right)\left\{\delta^h_k g_{ij} - \delta^h_j g_{ik}\right\} \\
&= \lambda_l\Big[\gamma \mathcal{R}^h_{ijk} + \delta\left\{g_{ij}\mathcal{R}^h_k - g_{ik}\mathcal{R}^h_j + \delta^h_k \mathcal{R}_{ij} - \delta^h_j \mathcal{R}_{ik}\right\} \\
&\quad - \frac{\mathcal{R}}{4}\left(\frac{\gamma}{3} + 2\delta\right)\left\{\delta^h_k g_{ij} - \delta^h_j g_{ik}\right\}\Big].
\end{aligned} \tag{28}$$

Contracting h and k in (28), we infer

$$\begin{aligned}
&\gamma \mathcal{R}_{ij,l} + \delta\left\{g_{ij}\mathcal{R}_{,l} + 2\mathcal{R}_{ij,l}\right\} - \frac{\mathcal{R}_{,l}}{4}(\gamma + 6\delta) g_{ij} \\
&= \lambda_l\left[\gamma \mathcal{R}_{ij} + \delta\left\{g_{ij}\mathcal{R} + 2\mathcal{R}_{ij}\right\} - \frac{\mathcal{R}}{4}(\gamma + 6\delta) g_{ij}\right].
\end{aligned} \tag{29}$$

Therefore

$$\mathcal{R}_{ij,l} = \lambda_l \mathcal{R}_{ij} + \mu_l g_{ij}, \quad \text{where} \quad \mu_l = \frac{(\mathcal{R}_{,l} - \lambda_l \mathcal{R})}{4}. \tag{30}$$

This means that the spacetime represents a generalized Ricci recurrent spacetime.

Hence we write:

Theorem 12 *A $\mathcal{C}^*$-recurrent spacetime is a generalized Ricci recurrent spacetime.*

It is known [15] that a generalized Ricci recurrent GRW spacetime is an Einstein spacetime. Hence from (8), we find that $\mathcal{C}^k_{lij,k} = 0$. Also in [34], the authors demonstrated that a GRW spacetime with $\mathcal{C}^k_{lij,k} = 0$ becomes a PFS.

Thus we reach:

Corollary 13 *A $\mathcal{C}^*$-recurrent GRW spacetime is a PFS.*

5 Quasi-conformally flat spacetimes fulfilling $f(\mathcal{R}, G)$-gravity

Here, we focus on a few specific classes of $f(\mathcal{R}, G)$ modified gravity models. The term for gravitational action is

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g}\, f(\mathcal{R}, G)\mathrm{d}^4x + \mathcal{S}_{\text{mat}}, \tag{31}$$

$\mathcal{S}_{\text{mat}}$ being the matter action and Gauss-Bonnet invariant G is presented as

$$G = \mathcal{R}^2 + \mathcal{R}_{lijk}\mathcal{R}^{lijk} - 4\mathcal{R}_{lk}\mathcal{R}^{lk}. \tag{32}$$

The action term (31) provides the widely used gravitational field equations of $f(\mathcal{R}, G)$-gravity as

$$\mathcal{R}_{ij} - \frac{\mathcal{R}}{2} g_{ij} = \kappa \mathcal{T}_{ij} + \Omega_{ij} = \kappa \mathcal{T}^{\text{eff}}_{ij}, \tag{33}$$

where

$$\begin{aligned}
\Omega_{ij} &= \nabla_i \nabla_j f_{\mathcal{R}} - g_{ij}\Box f_{\mathcal{R}} + 2\mathcal{R}\nabla_i \nabla_j f_G - 2g_{ij}\mathcal{R}\Box f_G - 4\mathcal{R}^l_i \nabla_l \nabla_j f_G \\
&\quad - 4\mathcal{R}^l_j \nabla_l \nabla_i f_G + 4\mathcal{R}_{ij}\Box f_G + 4g_{ij}\mathcal{R}^{lk}\nabla_l \nabla_k f_G + 4\mathcal{R}_{ilkj}\nabla^l \nabla^k f_G
\end{aligned}$$

$$-\frac{1}{2}g_{ij}(\mathcal{R}f_{\mathcal{R}}+Gf_G-f)+(1-f_{\mathcal{R}})\left(\mathcal{R}_{ij}-\frac{1}{2}g_{ij}\mathcal{R}\right) \tag{34}$$

and $\mathcal{T}_{ij}^{\text{eff}}$ is the effective EMT. Observe that $f_{\mathcal{R}} \equiv \frac{\partial f}{\partial \mathcal{R}}$, $f_G \equiv \frac{\partial f}{\partial G}$ and $\Box$ represents the d'Alembert operator.

The modified gravitational field equations are used to derive the ECs in the frame of $f(\mathcal{R}, G)$ modified gravity, and the results are as follows

$$\text{NEC} \Longleftrightarrow \boldsymbol{\sigma}+\boldsymbol{p}\geq 0, \tag{35}$$

$$\text{WEC} \Longleftrightarrow \boldsymbol{\sigma}\geq 0 \quad \text{and} \quad \boldsymbol{\sigma}+\boldsymbol{p}\geq 0, \tag{36}$$

$$\text{DEC} \Longleftrightarrow \boldsymbol{\sigma}\geq 0 \quad \text{and} \quad \boldsymbol{\sigma}\pm\boldsymbol{p}\geq 0, \tag{37}$$

$$\text{SEC} \Longleftrightarrow \boldsymbol{\sigma}+3\boldsymbol{p}\geq 0 \quad \text{and} \quad \boldsymbol{\sigma}+\boldsymbol{p}\geq 0, \tag{38}$$

where DEC and SEC indicate the dominant and strong energy conditions, respectively.

From (11), it follows that

$$\mathcal{R}^{lk}=\frac{\mathcal{R}}{4}g^{lk}. \tag{39}$$

Equations (11) and (39) together imply

$$\mathcal{R}_{lk}\mathcal{R}^{lk}=\frac{\mathcal{R}^2}{4}. \tag{40}$$

From (12), it follows that

$$\mathcal{R}^{lijk}=\frac{\mathcal{R}}{12}\left\{g^{ij}g^{lk}-g^{ik}g^{lj}\right\}. \tag{41}$$

Multiplying (12) and (41), one infers

$$\mathcal{R}_{lijk}\mathcal{R}^{lijk}=\frac{\mathcal{R}^2}{6}. \tag{42}$$

Equations (32), (40) and (42) reflects that the Gauss-Bonnet invariant is

$$G=\frac{\mathcal{R}^2}{6}. \tag{43}$$

Since for a quasi-conformally flat spacetime $\mathcal{R}$ is constant, Eq. (34) becomes

$$\Omega_{ij}=\mathcal{R}_{ij}+\left(\frac{f}{2}-\frac{\mathcal{R}}{2}\right)g_{ij}. \tag{44}$$

For a PFS the EMT is given by

$$\mathcal{T}_{ij}=\boldsymbol{p}g_{ij}+(\boldsymbol{p}+\boldsymbol{\sigma})u_iu_j \tag{45}$$

and

$$\mathcal{T}_{ij}^{\text{eff}}=\boldsymbol{p}^{\text{eff}}g_{ij}+\left(\boldsymbol{p}^{\text{eff}}+\boldsymbol{\sigma}^{\text{eff}}\right)u_iu_j, \tag{46}$$

where $\boldsymbol{p}^{\text{eff}}$ and $\boldsymbol{\sigma}^{\text{eff}}$ are the effective isotropic pressure and the effective energy density of the effective matter.

Using (44) and (45) in (33), we obtain

$$\left(\kappa\boldsymbol{p}+\frac{f}{2}\right)g_{ij}+\kappa(\boldsymbol{p}+\boldsymbol{\sigma})u_iu_j=0. \tag{47}$$

Transvecting twice with u^i and g^{ij}, we have

$$\boldsymbol{\sigma}=\frac{f}{2\kappa} \tag{48}$$

and

$$\boldsymbol{p}=-\frac{f}{2\kappa}. \tag{49}$$

Theorem 14 *In a $\mathcal{C}^*$-flat spacetime obeying $f(\mathcal{R}, G)$-gravity, $\boldsymbol{\sigma}$ and $\boldsymbol{p}$ are given by (48) and (49), respectively.*

Equations (48) and (49) give $\boldsymbol{p} + \boldsymbol{\sigma} = 0$, that is, NEC is satisfied. Furthermore, NEC describes the null geodesic congruences, and it is convergent in sufficiently small neighbourhood of every point of spacetime. The physical interpretation of NEC is that particles following null geodesics will observe that gravity tends locally to be attractive (or at least not repulsive) when acting on nearby particles also following null geodesics [29].

From (33), (44)–(46), (48) and (49), we have

$$\left(\kappa \boldsymbol{p}^{\text{eff}} - \kappa \boldsymbol{p} - \frac{f}{2} + \frac{\mathcal{R}}{4}\right) g_{ij} + \kappa\left(\boldsymbol{p}^{\text{eff}} + \boldsymbol{\sigma}^{\text{eff}}\right) u_i u_j = 0. \tag{50}$$

Transvecting (50) with u^i and g^{ij}, respectively and using (49), we get

$$\boldsymbol{\sigma}^{\text{eff}} = \frac{\mathcal{R}}{4\kappa} \quad \text{and} \quad \boldsymbol{p}^{\text{eff}} = -\frac{\mathcal{R}}{4\kappa}. \tag{51}$$

Now, we consider the flat Friedmann-Robertson-Walker (FRW) metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{52}$$

where $a(t)$ is the scale factor of the Universe. In the FRW background, and taking into account a perfect fluid equation of state for ordinary matter, it follows that the field equations for $f(\mathcal{R}, G)$-gravity are given by

$$2\dot{H} f_{\mathcal{R}} + 8H\dot{H}\dot{f}_G = H\dot{f}_{\mathcal{R}} - \ddot{f}_{\mathcal{R}} + 4H^3\dot{f}_G - 4H^2\ddot{f}_G, \tag{53}$$

$$6H^2 f_{\mathcal{R}} + 24H^3\dot{f}_G = f_{\mathcal{R}}\mathcal{R} - f(\mathcal{R}, G) - 6H\dot{f}_{\mathcal{R}} + Gf_G, \tag{54}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and the overdot denotes a derivative with respect to the time coordinate, t. In addition, we have

$$\mathcal{R} = 6\left(2H^2 + \dot{H}\right) \tag{55}$$

and

$$G = 24H^2\left(H^2 + \dot{H}\right). \tag{56}$$

From (43), (55) and (56), we get

$$H^2 = \frac{\mathcal{R}}{12} \quad \text{and} \quad \dot{H} = 0. \tag{57}$$

As $H = \frac{\dot{a}}{a}$, $\frac{\dot{a}}{a} = \sqrt{\frac{\mathcal{R}}{12}}$. Thus

$$\ddot{a} = \frac{\dot{a}^2}{a}, \quad \dddot{a} = \frac{\dot{a}^3}{a^2} \quad \text{and} \quad \ddddot{a} = \frac{\dot{a}^4}{a^3}. \tag{58}$$

To continue, in analogy with the standard mechanics we introduce velocity, acceleration, jerk and snap in the cosmological context. It is appropriate to define the deceleration, jerk and snap parameters as

$$q = -\frac{1}{H^2}\frac{\ddot{a}}{a}, \quad j = \frac{1}{H^3}\frac{\dddot{a}}{a} \quad \text{and} \quad s = \frac{1}{H^4}\frac{\ddddot{a}}{a}, \tag{59}$$

respectively. Using (58) in (59), we obtain

$$s = j = -q. \tag{60}$$

Hence, in a quasi-conformally flat PFS satisfying $f(\mathcal{R}, G)$-gravity, the deceleration, jerk, and snap parameters are related by (60).

We now examine the ECs for three distinct $f(\mathcal{R}, G)$-gravity models in the following subsections.

5.1 $f(\mathcal{R}, G) = \mu\mathcal{R}^{\beta_1}G^{\beta_2}$

In this subsection, with the help of (43), the energy density and pressure are expressed as

$$\boldsymbol{\sigma} = \frac{\mu\mathcal{R}^{\beta_1+2\beta_2}}{2\kappa 6^{\beta_2}}, \tag{61}$$

$$\boldsymbol{p} = -\frac{\mu\mathcal{R}^{\beta_1+2\beta_2}}{2\kappa 6^{\beta_2}}. \tag{62}$$

The ECs for this setup can now be discussed using (61) and (62). Figures 1, 2 and 3, respectively, show the profiles of $\boldsymbol{\sigma}$, DEC and SEC. In this situation, $\boldsymbol{\sigma} + \boldsymbol{p}$ becomes zero. The energy density cannot be negative for $\mu > 0$ and $\mathcal{R} > 0$. One can see from Fig. 1 that

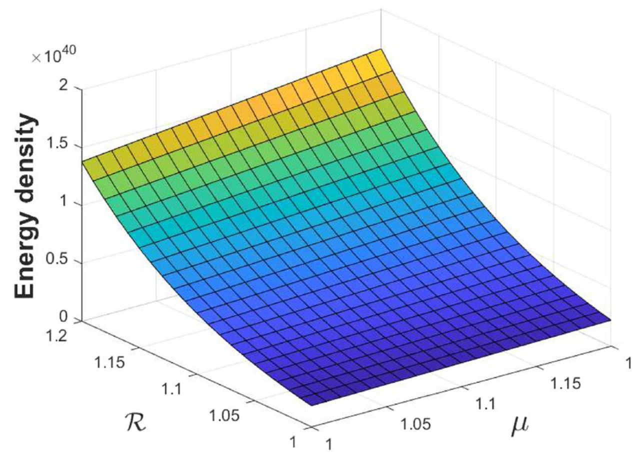

Fig. 1 Advancement of $\boldsymbol{\sigma}$ with respect to μ, $\mathcal{R}$ ($\beta_1 = 3, \beta_2 = 4$)

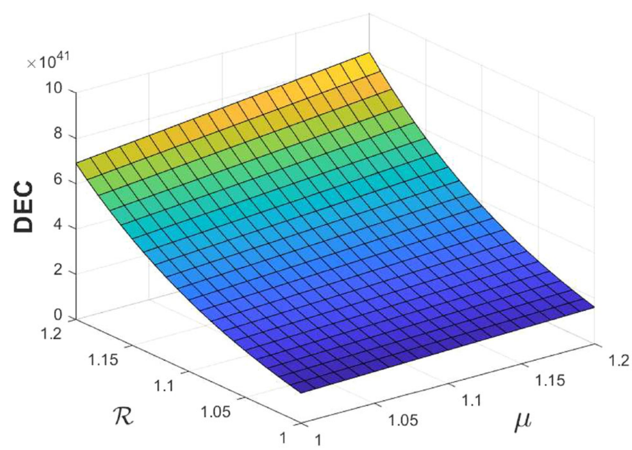

Fig. 2 Advancement of DEC with respect to μ, $\mathcal{R}$ ($\beta_1 = 5, \beta_2 = 2$)

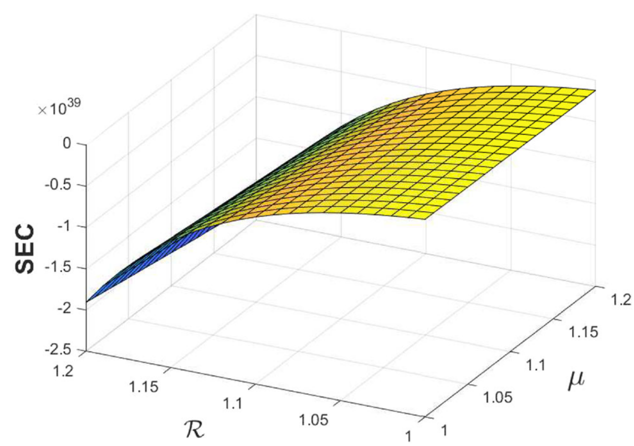

Fig. 3 Advancement of SEC with respect to μ, $\mathcal{R}$ ($\beta_1 = 4, \beta_2 = 6$)

the energy density is high for greater values of $\mathcal{R}$. As NEC is a part of WEC. Consequently, NEC and WEC are satisfied. Figure 2 shows the DEC profile, which has a positive range for its value. SEC is disobeyed, and this outcome shows that the Universe's late-time acceleration ([13, 28]). Moreover, all of the results are compatible with the ΛCDM model [1].

5.2 $f(\mathcal{R}, G) = \alpha_1\mathcal{R} + \alpha_2\mathcal{R}^m G^n$

Here, using (43), the energy density and pressure are represented as

$$\boldsymbol{\sigma} = \frac{1}{2\kappa}\left(\alpha_1\mathcal{R} + \frac{\alpha_2\mathcal{R}^{m+2n}}{6^n}\right), \tag{63}$$

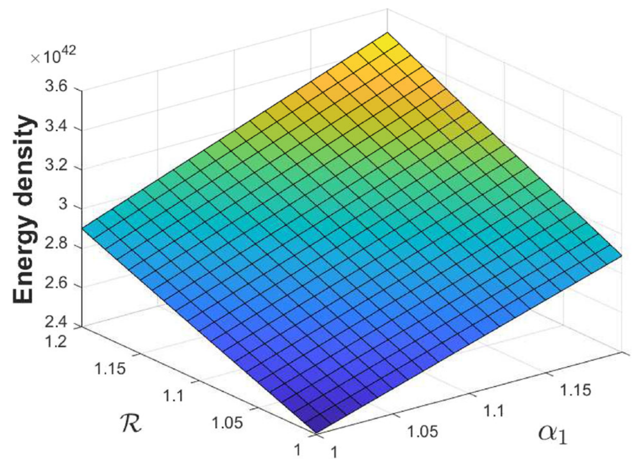

Fig. 4 Advancement of $\boldsymbol{\sigma}$ with respect to α_1, $\mathcal{R}$ ($\alpha_2 = 3, m = 4, n = 5$)

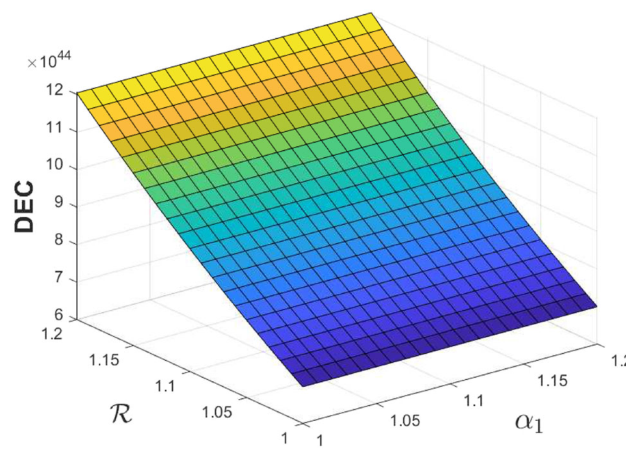

Fig. 5 Advancement of DEC with respect to α_1, $\mathcal{R}$ ($\alpha_2 = 4, m = 7, n = -2$)

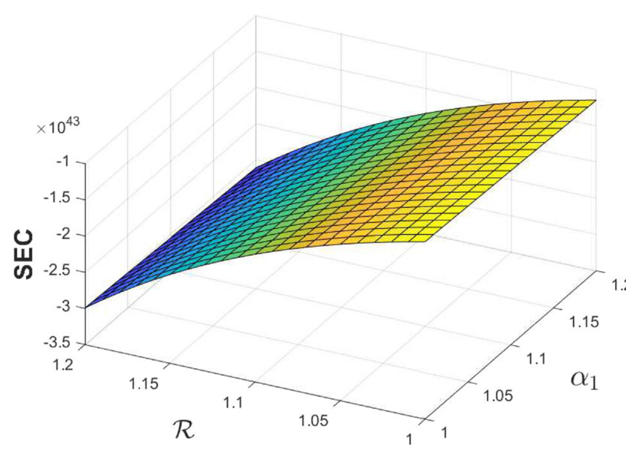

Fig. 6 Advancement of SEC with respect to α_1, $\mathcal{R}$ ($\alpha_2 = 7, m = 6, n = 1$)

$$\boldsymbol{p} = -\frac{1}{2\kappa}\left(\alpha_1\mathcal{R} + \frac{\alpha_2\mathcal{R}^{m+2n}}{6^n}\right). \tag{64}$$

Using (63) and (64), one can now talk about the ECs for this configuration. Figures 4, 5, and 6 indicate, in that order, the profiles of $\boldsymbol{\sigma}$, DEC, and SEC. From those figures, we notice that $\boldsymbol{\sigma}$ and DEC are satisfied but the SEC does not valid. Furthermore, as $\boldsymbol{\sigma} + \boldsymbol{p}$ goes to zero for this construction, WEC and NEC are also fulfilled.

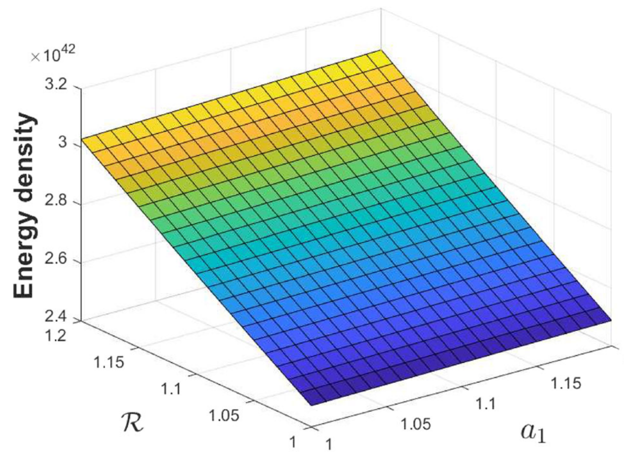

Fig. 7 Advancement of $\boldsymbol{\sigma}$ with respect to $\mathcal{R}$, a_1 ($\alpha_3 = 2$)

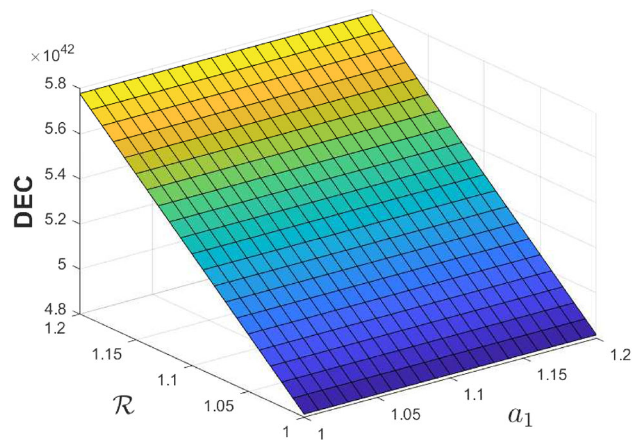

Fig. 8 Advancement of DEC with respect to $\mathcal{R}$, a_1 ($\alpha_3 = 8$)

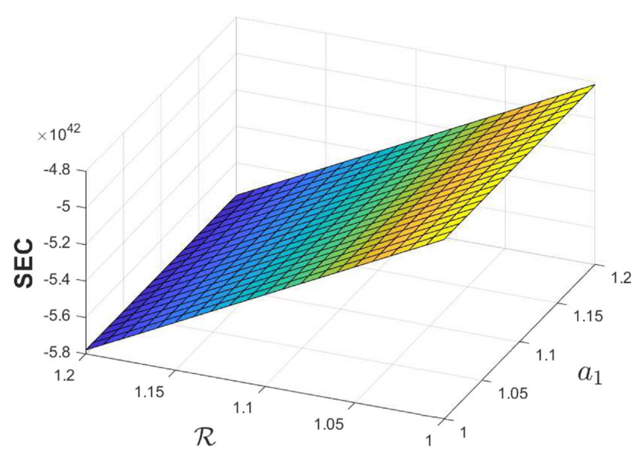

Fig. 9 Advancement of SEC with respect to $\mathcal{R}$, a_1 ($\alpha_3 = 6$)

5.3 $f(\mathcal{R}, G) = \mathcal{R} + a_1 G^{\alpha_3}$

For this model, the energy density and pressure are depicted as

$$\boldsymbol{\sigma} = \frac{1}{2\kappa}\left(\mathcal{R} + \frac{a_1 \mathcal{R}^{2\alpha_3}}{6^{\alpha_3}}\right), \tag{65}$$

$$\boldsymbol{p} = -\frac{1}{2\kappa}\left(\mathcal{R} + \frac{a_1 \mathcal{R}^{2\alpha_3}}{6^{\alpha_3}}\right). \tag{66}$$

We display $\boldsymbol{\sigma}$, the DEC and SEC profiles in Figs. 7, 8, and 9, respectively, using (65) and (66). We see that the figures satisfy the $\boldsymbol{\sigma}$ and DEC but not the SEC. Furthermore, NEC and WEC are also fulfilled as $\boldsymbol{\sigma} + \boldsymbol{p}$ goes to zero for this construction.

Remark 1 The equation of state is $\boldsymbol{p}+\boldsymbol{\sigma}=0$, that is, $|\boldsymbol{\sigma}|=|-\boldsymbol{p}|$, that is, $\boldsymbol{\sigma}=|\boldsymbol{p}|$ as the energy density cannot be negative. Hence in $f(\mathcal{R}, G)$-gravity, a quasi-conformally flat PFS satisfies the DEC. Thus in a quasi-conformally flat PFS satisfying $f(\mathcal{R}, G)$-gravity, the matter can not travel faster than the speed of light [18].

6 Conclusion

Spacetime, a torsion-free, time-oriented Lorentzian manifold, is the stage on which the physical world is now being modelled. According to GR theory, the Universe's matter content may be determined by choosing the appropriate EMT, and is accepted to act like a PFS in the cosmological models.

Here, we study a spacetime of quasi-conformally flat PFS, and we have demonstrated that this spacetime represents a dark matter era. Additionally, we show that a quasi-conformally flat PFS is either a de-Sitter spacetime or locally isometric to Minkowski spacetime. We further establish that this spacetime implies a RW spacetime and is of Petrov type I, D, or O. Also, we prove that a quasi-conformally recurrent GRW spacetime becomes a PFS and in a quasi-conformally flat PFS, the Weyl tensor is purely electric.

The investigation of quasi-conformally flat PFS within the context of $f(\mathcal{R}, G)$-gravity has been the foremost concern of this article. We have discussed the deceleration, jerk, and snap parameters and some physical interpretation of quasi-conformally flat PFS satisfying $f(\mathcal{R}, G)$-gravity. Here, both analytic and graphical analysis of our investigations have been done. For a better understanding, we have applied the analytical method to develop our formulation and graphical analysis has been done to assess the stability of cosmological models.

Additionally, we looked at the cosmological models' stability analysis using ECs. However, SEC broke the agreement, whereas DEC, NEC and WEC have been satisfied. The accelerated expansion of the cosmos is consistent with each of the aforementioned ECs profiles. These results indicate the accelerated expansion of the Universe and compatible with the ΛCDM model.

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Declarations

Conflict of interest The authors declare that they have no competing interest.

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SOME RESULTS ON MIXED SUPER QUASI-EINSTEIN MANIFOLDS SATISFYING CERTAIN VECTOR FIELDS

Dipankar Hazra

Department of Mathematics, Heramba Chandra College,
23/49, Gariahat Road, Kolkata - 700029, West Bengal, India

Abstract. The objective of this paper is to discuss various properties of mixed super quasi-Einstein manifolds admitting certain vector fields. We analyze the behaviour of $MS\left(QE\right)_n$ satisfying Codazzi type of Ricci tensor. We have also constructed a non-trivial example related to mixed super quasi-Einstein manifolds.
Keywords: Mixed super quasi-Einstein manifolds, pseudo quasi-Einstein manifold, Codazzi type of Ricci tensor, cyclic parallel Ricci tensor, Killing vector field, concurrent vector field.

1. Introduction

An n-dimensional semi-Riemannian or Riemannian manifold (M^n, g) $(n > 2)$, is called an Einstein manifold if its Ricci tensor S satisfies the criteria

$$S = \frac{r}{n}g, \tag{1.1}$$

where r denotes the scalar curvature of (M^n, g). We can also say an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi-Einstein manifold was introduced by M.C. Chaki and R.K. Maity [5]. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is a quasi-Einstein manifold if its Ricci tensor S satisfies the criteria

$$S\left(X, Y\right) = ag\left(X, Y\right) + bA\left(X\right)A\left(Y\right) \tag{1.2}$$

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Corresponding Author: Dipankar Hazra, Department of Mathematics, Heramba Chandra College, 23/49, Gariahat Road, Kolkata - 700029, West Bengal, India | E-mail: dipankarsk524@gmail.com
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and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X),$$

for all vector field X. U being a unit vector field.
Here a and b are called the associated scalars, A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold is denoted by $(QE)_n$. The quasi-Einstein manifolds have also been studied by De and Ghosh [7], Bejan [1], De and De [6], Han, De and Zhao [15] and many others. Quasi-Einstein manifolds have been generalized by many authors in several ways such as generalized quasi-Einstein manifolds [3, 9, 11, 23], $N(K)$-quasi Einstein manifolds [17, 24], super quasi-Einstein manifolds [4, 10, 19] etc.

Chaki [4] introduced the notion of a super quasi-Einstein manifold. His work suggested a non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a super quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) \\ &+ c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y), \end{aligned} \tag{1.3}$$

where a, b, c, d are scalars in which $b \neq 0$, $c \neq 0$ $d \neq 0$ and A, B are non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X),$$

where U, V are mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0,$$

for all X. In that case a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and auxiliary generators of the manifold and D is called the associated tensor of the manifold. Such an n-dimensional manifold is denoted by $S(QE)_n$.

In [2], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced the notion of mixed super quasi-Einstein manifolds. Their work suggested that a non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is said to be mixed super quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ &+ d[A(X)B(Y) + A(Y)B(X)] + eD(X,Y), \end{aligned} \tag{1.4}$$

where a, b, c, d, e are scalars on (M^n, g) of which $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$ and A, B are two non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X), \tag{1.5}$$

U, V being unit vector fields which are orthogonal, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \tag{1.6}$$

for all X. Here a, b, c, d, e are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and auxiliary generators of the manifold and D is called the associated tensor of the manifold. If $c = 0$, then the manifold becomes $S(QE)_n$. This type of manifold is denoted by the symbol $MS(QE)_n$. If $c = d = 0$, then the manifold is reduced to a pseudo quasi-Einstein manifold which was studied by Shaikh [22].

On the other hand, Gray [14] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class A consists of all Riemannian manifolds whose Ricci tensor S is a Codazzi type tensor, i.e.,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

The class B contains all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e.,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0.$$

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized Ricci recurrent manifold [8] if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = \gamma(X) S(Y,Z) + \delta(X) g(Y,Z),$$

where $\gamma(X)$ and $\delta(X)$ are non-zero 1-forms such that $\gamma(X) = g(X,\rho)$ and $\delta(X) = g(X,\mu)$; ρ and μ being associated vector fields of the 1-forms γ and δ, respectively. If $\delta = 0$, then the manifold reduces to a Ricci recurrent manifold [20].

After studying and analyzing various papers [12, 13, 18], we got motivation to work in this area. Recently in the paper [16], we have studied generalized Quasi-Einstein manifolds satisfying certain vector fields. In the present work we have tried to develop a new concept. This paper is organized as follows: After introduction in Section 2, we have studied that if the generators U and V of a $MS(QE)_n$ are Killing vector fields, then the manifold satisfies cyclic parallel Ricci tensor if and only if the associated tensor D is cyclic parallel. Section 3 is concerned with $MS(QE)_n$ satisfying Codazzi type of Ricci tensor. In the next two sections, we have studied $MS(QE)_n$ with generators U and V both as concurrent and recurrent vector fields. Finally the existence of $MS(QE)_n$ is shown by constructing non-trivial example.

2. The generators U and V as Killing vector fields

In this section we consider the generators U and V of the manifold are Killing vector fields.

Theorem 2.1. *If the generators of a $MS\left(QE\right)_n$ are Killing vector fields and the associated scalars are constants, then the manifold satisfies cyclic parallel Ricci tensor if and only if the associated tensor D is cyclic parallel.*

Proof. Let us assume that the generators U and V of the manifold are Killing vector fields. Then we have

$$\left(\pounds_U g\right)(X,Y)=0 \tag{2.1}$$

and

$$\left(\pounds_V g\right)(X,Y)=0, \tag{2.2}$$

where $\pounds$ denotes the Lie derivative.
From (2.1) and (2.2), we get

$$g\left(\nabla_X U,Y\right)+g\left(X,\nabla_Y U\right)=0 \tag{2.3}$$

and

$$g\left(\nabla_X V,Y\right)+g\left(X,\nabla_Y V\right)=0. \tag{2.4}$$

Since $g\left(\nabla_X U,Y\right)=\left(\nabla_X A\right)(Y)$ and $g\left(\nabla_X V,Y\right)=\left(\nabla_X B\right)(Y)$.
Thus from (2.3) and (2.4) we obtain

$$\left(\nabla_X A\right)(Y)+\left(\nabla_Y A\right)(X)=0 \tag{2.5}$$

and

$$\left(\nabla_X B\right)(Y)+\left(\nabla_Y B\right)(X)=0, \tag{2.6}$$

for all X, Y.
Similarly, we have

$$\left(\nabla_X A\right)(Z)+\left(\nabla_Z A\right)(X)=0, \tag{2.7}$$

$$\left(\nabla_Z A\right)(Y)+\left(\nabla_Y A\right)(Z)=0, \tag{2.8}$$

$$\left(\nabla_X B\right)(Z)+\left(\nabla_Z B\right)(X)=0, \tag{2.9}$$

$$\left(\nabla_Z B\right)(Y)+\left(\nabla_Y B\right)(Z)=0, \tag{2.10}$$

for all X, Y, Z.
We assume that the associated scalars are constants. Then from (1.4) we have

$$\begin{aligned}\left(\nabla_Z S\right)(X,Y) = {} & b\left[\left(\nabla_Z A\right)(X)\,A(Y)+A(X)\left(\nabla_Z A\right)(Y)\right]\\ & +c\left[\left(\nabla_Z B\right)(X)\,B(Y)+B(X)\left(\nabla_Z B\right)(Y)\right]\\ & +d\left[\left(\nabla_Z A\right)(X)\,B(Y)+A(X)\left(\nabla_Z B\right)(Y)\right.\\ & +\left.\left(\nabla_Z A\right)(Y)\,B(X)+A(Y)\left(\nabla_Z B\right)(X)\right]\\ & +e\left(\nabla_Z D\right)(X,Y).\end{aligned} \tag{2.11}$$

Using (2.11), we get

$$
\begin{aligned}
(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) &= b\,[\{(\nabla_X A)(Y) \\
&+ (\nabla_Y A)(X)\}\,A(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}\,A(Y) \\
&+ \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}\,A(X)] + c\,[\{(\nabla_X B)(Y) \\
&+ (\nabla_Y B)(X)\}\,B(Z) + \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}\,B(Y) \\
&+ \{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}\,B(X)] + d\,[\{(\nabla_X B)(Y) \\
&+ (\nabla_Y B)(X)\}\,A(Z) + \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}\,A(Y) \\
&+ \{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}\,A(X) + \{(\nabla_X A)(Y) \\
&+ (\nabla_Y A)(X)\}\,B(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}\,B(Y) \\
&+ \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}\,B(X)] + e\,[(\nabla_X D)(Y,Z) \\
&+ (\nabla_Y D)(Z,X) + (\nabla_Z D)(X,Y)]. \qquad (2.12)
\end{aligned}
$$

Using the equations (2.5) - (2.10) in (2.12), we get

$$
\begin{aligned}
(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) &= e\,[(\nabla_X D)(Y,Z) \\
&+ (\nabla_Y D)(Z,X) + (\nabla_Z D)(X,Y)].
\end{aligned}
$$

Thus the proof of theorem is completed. □

3. $MS\,(QE)_n$ admits Codazzi type of Ricci tensor

We know that a Riemannian or semi-Riemannian manifold satisfies Codazzi type of Ricci tensor if its Ricci tensor S satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)\,, \qquad (3.1)$$

for all X, Y, Z.

Theorem 3.1. *If a $MS\,(QE)_n$ admits the Codazzi type of Ricci tensor with the associated tensor D satisfying the relation $(\nabla_X D)(Y,V) = (\nabla_Y D)(V,X)$, then either $d = \pm\sqrt{bc}$ or the associated 1-forms A and B are closed.*

Proof. Using (2.11) and (3.1), we obtain

$$
\begin{aligned}
& b\,[(\nabla_X A)(Y)\,A(Z) + A(Y)\,(\nabla_X A)\,(Z)] + c\,[(\nabla_X B)(Y)\,B(Z) \\
& + B(Y)\,(\nabla_X B)(Z)] + d\,[(\nabla_X A)(Y)\,B(Z) + A(Y)\,(\nabla_X B)(Z) \\
& + (\nabla_X A)(Z)\,B(Y) + A(Z)\,(\nabla_X B)(Y)] + e\,(\nabla_X D)(Y,Z) \\
& - b\,[(\nabla_Y A)(Z)\,A(X) + A(Z)\,(\nabla_Y A)(X)] - c\,[(\nabla_Y B)(Z)\,B(X) \\
& + B(Z)\,(\nabla_Y B)(X)] - d\,[(\nabla_Y A)(Z)\,B(X) + A(Z)\,(\nabla_Y B)(X) \\
& + (\nabla_Y A)(X)\,B(Z) + A(X)\,(\nabla_Y B)(Z)] - e\,(\nabla_Y D)(Z,X) = 0. \qquad (3.2)
\end{aligned}
$$

Putting $Z = U$ in (3.2) and using $(\nabla_X A)(U) = 0$, we have

$$b\,[(\nabla_X A)(Y) - (\nabla_Y A)(X)] + d\,[(\nabla_X B)(Y) - (\nabla_Y B)(X)] = 0,$$

i.e.,

$$bdA(X,Y) = -ddB(X,Y). \tag{3.3}$$

Similarly, putting $Z = V$ in (3.2) and using $(\nabla_X B)(V) = 0$, we have

$$c[(\nabla_X B)(Y) - (\nabla_Y B)(X)] + d[(\nabla_X A)(Y) - (\nabla_Y A)(X)]$$
$$+e[(\nabla_X D)(Y,V) - (\nabla_Y D)(V,X)] = 0,$$

i.e.,

$$c\mathbf{d}B(X,Y) + d\mathbf{d}A(X,Y) + e[(\nabla_X D)(Y,V) - (\nabla_Y D)(V,X)] = 0. \tag{3.4}$$

If $(\nabla_X D)(Y,V) = (\nabla_Y D)(V,X)$, then from the equations (3.3) and (3.4) we get either

$$d = \pm\sqrt{bc}$$

or

$$\mathbf{d}A(X,Y) = 0$$

and

$$\mathbf{d}B(X,Y) = 0.$$

Thus, we complete the proof. □

Theorem 3.2. *If a $MS(QE)_n$ admits the Codazzi type of Ricci tensor with the associated tensor D satisfying the condition $(\nabla_V D)(Y,V) = (\nabla_Y D)(V,V)$, then the integral curves of the parallel vector fields U and V are geodesics.*

Proof. Putting $X = Z = U$ in (3.2), we get

$$b(\nabla_U A)(Y) + d(\nabla_U B)(Y) = 0,$$

which means that

$$bg(\nabla_U U, Y) + dg(\nabla_U V, Y) = 0. \tag{3.5}$$

Similarly, putting $X = Z = V$ in (3.2), we get

$$c(\nabla_V B)(Y) + d(\nabla_V A)(Y) + e[(\nabla_V D)(Y,V) - (\nabla_Y D)(V,V)] = 0,$$

i.e.,

$$cg(\nabla_V V, Y) + dg(\nabla_V U, Y) + e[(\nabla_V D)(Y,V) - (\nabla_Y D)(V,V)] = 0. \tag{3.6}$$

If U, V are parallel vector fields, then $\nabla_U V = 0 = \nabla_V U$.
We assume that $(\nabla_V D)(Y,V) = (\nabla_Y D)(V,V)$. So from (3.5) and (3.6), we obtain

$$g(\nabla_U U, Y) = 0, \text{ for all Y, i.e., } \nabla_{\mathrm{U}}\mathrm{U} = 0$$

and

$$g(\nabla_V V, Y) = 0, \text{ for all Y, i.e., } \nabla_{\mathrm{V}}\mathrm{V} = 0.$$

Thus the theorem is proved. □

4. The generators U and V as concurrent vector fields

A vector field ξ is called concurrent if [21]

$$\nabla_X \xi = \rho X, \tag{4.1}$$

where ρ is a non-zero constant. If $\rho = 0$, then the vector field reduces to a parallel vector field.

Theorem 4.1. *If the associated vector fields of a $MS\,(QE)_n$ are concurrent vector fields and the associated scalars are constants, then the manifold reduces to a pseudo quasi-Einstein manifold.*

Proof. We consider the vector fields U and V corresponding to the associated 1-forms A and B respectively are concurrent. Then

$$(\nabla_X A)(Y) = \alpha g(X, Y) \tag{4.2}$$

and

$$(\nabla_X B)(Y) = \beta g(X, Y), \tag{4.3}$$

where α and β are non-zero constants.
Using (4.2) and (4.3) in (2.11), we get

$$\begin{aligned}(\nabla_Z S)(X, Y) &= b\,[\alpha g(Z, X) A(Y) + \alpha g(Z, Y) A(X)] + c\,[\beta g(Z, X) B(Y) \\ &\quad + \beta g(Z, Y) B(X)] + d\,[\alpha g(Z, X) B(Y) + \beta g(Z, Y) A(X) \\ &\quad + \alpha g(Z, Y) B(X) + \beta g(Z, X) A(Y)] + e\,(\nabla_Z D)(X, Y).\end{aligned} \tag{4.4}$$

Contracting (4.4) over X and Y, we obtain

$$dr(Z) = 2\,[(b\alpha + d\beta) A(Z) + (c\beta + d\alpha) B(Z)], \tag{4.5}$$

where r is the scalar curvature of the manifold.
In a $MS\,(QE)_n$ if the associated scalars a, b, c, d and e are constants, then contracting (1.4) over X and Y we get

$$r = an + b + c,$$

which implies that the scalar curvature r is constant, i.e., $dr(X) = 0$, for all X.
Thus equation (4.5) gives

$$(b\alpha + d\beta) A(Z) + (c\beta + d\alpha) B(Z) = 0. \tag{4.6}$$

Since α and β are non-zero constants, using (4.6) in (1.4), we finally get

$$S(X, Y) = ag(X, Y) + \left[b + c\left(\frac{b\alpha + d\beta}{c\beta + d\alpha}\right)^2 - 2d\left(\frac{b\alpha + d\beta}{c\beta + d\alpha}\right)\right] A(X) A(Y) + eD(X, Y).$$

Thus the manifold reduces to a pseudo quasi-Einstein manifold. □

5. The generators U and V as recurrent vector fields

Definition 5.1. A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ will be called a pseudo generalized Ricci recurrent manifold if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = \beta(X) S(Y,Z) + \gamma(X) g(Y,Z) + \delta(X) D(Y,Z),$$

where $\beta(X)$, $\gamma(X)$ and $\delta(X)$ are non-zero 1-forms such that

$$\beta(X) = g(X,\xi_1), \quad \gamma(X) = g(X,\xi_2), \quad \delta(X) = g(X,\xi_3);$$

ξ_1, ξ_2 and ξ_3 are associated vector fields of the 1-forms β, γ and δ respectively, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,\xi_1) = 0,$$

for all X.

Theorem 5.1. *If the generators of a $MS(QE)_n$ corresponding to the associated 1-forms are recurrent with the same vector of recurrence and the associated scalars are constants with an additional condition that D is covariant constant, then the manifold is a pseudo generalized Ricci recurrent manifold.*

Proof. A vector field ξ corresponding to the associated 1-form η is said to be recurrent if [21]

$$(\nabla_X \eta)(Y) = \psi(X) \eta(Y), \tag{5.1}$$

where ψ is a non-zero 1-form.

Here, we consider the generators U and V corresponding to the associated 1-forms A and B as recurrent. Then we have

$$(\nabla_X A)(Y) = \lambda(X) A(Y) \tag{5.2}$$

and

$$(\nabla_X B)(Y) = \mu(X) B(Y), \tag{5.3}$$

where λ and μ are non-zero 1-forms.
Using (5.2) and (5.3) in (2.11), we obtain

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2b\lambda(Z) A(X) A(Y) + 2c\mu(Z) B(X) B(Y) \\ &+ d[\lambda(Z) + \mu(Z)][A(X) B(Y) + A(Y) B(X)] \\ &+ e(\nabla_Z D)(X,Y).\end{aligned} \tag{5.4}$$

We assume that the 1-forms λ and μ are equal, i.e.,

$$\lambda(Z) = \mu(Z), \tag{5.5}$$

for all Z. From the equations (5.4) and (5.5), we get

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2\lambda(Z)\left[bA(X)A(Y)+cB(X)B(Y)\right.\\ &\quad \left.+d\{A(X)B(Y)+A(Y)B(X)\}\right]\\ &\quad +e(\nabla_Z D)(X,Y).\end{aligned} \tag{5.6}$$

Using (1.4) and (5.6), we obtain

$$(\nabla_Z S)(X,Y)=\alpha_1(Z)S(X,Y)+\alpha_2(Z)g(X,Y)+\alpha_3(Z)D(X,Y)+e(\nabla_Z D)(X,Y),$$

where $\alpha_1(Z)=2\lambda(Z)$, $\alpha_2(Z)=-2a\lambda(Z)$ and $\alpha_3(Z)=-2e\lambda(Z)$.
So the proof is complete. □

6. Example of $MS(QE)_4$

In this section, we prove the existence of $MS(QE)_4$ by constructing a non-trivial concrete example.

Let $(x^1,x^2,\ldots,x^n)\in\mathbb{R}^n$, where $\mathbb{R}^n$ is an n-dimensional real number space. We consider a Riemannian metric g on $\mathbb{R}^4=(x^1,x^2,x^3,x^4)$, by

$$ds^2=g_{ij}dx^idx^j=(dx^1)^2+(x^1)^2(dx^2)^2+(x^2)^2(dx^3)^2+(dx^4)^2, \tag{6.1}$$

where $i,j=1,2,3,4$. Using (6.1), we see the non-vanishing components of Riemannian metric are

$$g_{11}=1,\ \ g_{22}=(x^1)^2,\ \ g_{33}=(x^2)^2,\ \ g_{44}=1 \tag{6.2}$$

and its associated components are

$$g^{11}=1,\ \ g^{22}=\frac{1}{(x^1)^2},\ \ g^{33}=\frac{1}{(x^2)^2},\ \ g^{44}=1. \tag{6.3}$$

Using (6.2) and (6.3), we can calculate that the non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are given by

$$\Gamma^1_{22}=-x^1,\ \ \Gamma^2_{33}=-\frac{x^2}{(x^1)^2},\ \ \Gamma^2_{12}=\frac{1}{x^1},\ \ \Gamma^3_{23}=\frac{1}{x^2},\ \ R_{1332}=-\frac{x^2}{x^1},\ \ S_{12}=-\frac{1}{x^1x^2}$$

and the other components are obtained by the symmetric properties. It can be easily shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4,g)$ is zero. We shall show that $(\mathbb{R}^4,g)$ is a $MS(QE)_4$.
Let us consider the associated scalars as follows:

$$a=\frac{1}{x^1(x^2)^2},\ \ b=\frac{1}{(x^2)^3},\ \ c=-\frac{1}{x^2},\ \ d=\frac{1}{x^1},\ \ e=-\frac{1}{(x^1)^2x^2}. \tag{6.4}$$

We choose the 1-form as follows:

$$A_i(x)=\begin{cases}x^1, & \text{when } i=2\\ 0, & \text{otherwise}\end{cases} \tag{6.5}$$

and

$$B_i(x) = \begin{cases} x^2, & \text{when } i = 3 \\ 0, & \text{otherwise} \end{cases} \tag{6.6}$$

at any point $x \in \mathbb{R}^4$.
We take the associated tensor as follows:

$$D_{ij}(x) = \begin{cases} 1, & \text{when } i = j = 1,3 \\ -2, & \text{when } i = j = 2 \\ x^1, & \text{when } i = 1, j = 2 \\ 0, & \text{otherwise} \end{cases} \tag{6.7}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equation

$$S_{12} = ag_{12} + bA_1A_2 + cB_1B_2 + d\left[A_1B_2 + A_2B_1\right] + eD_{12}, \tag{6.8}$$

since, for the other cases (1.4) holds trivially.
From the equations (6.4), (6.5), (6.6), (6.7) and (6.8) we get

$$\begin{aligned} \text{Right hand side of (6.8)} &= ag_{12} + bA_1A_2 + cB_1B_2 + d\left[A_1B_2 + A_2B_1\right] + eD_{12} \\ &= \frac{1}{x^1\left(x^2\right)^2} \cdot 0 + \frac{1}{\left(x^2\right)^3} \cdot 0 \cdot x^1 + \left(-\frac{1}{x^2}\right) \cdot 0 \cdot 0 \\ &+ \frac{1}{x^1}\left[0 + x^1 \cdot 0\right] + \left(-\frac{1}{\left(x^1\right)^2 x^2}\right) \cdot x^1 \\ &= -\frac{1}{x^1x^2} = S_{12}. \end{aligned}$$

Clearly, the trace of the $(0,2)$ tensor D is zero.
We shall now show that the 1-forms A_i and B_i are unit and also they are orthogonal.
Here,

$$g^{ij}A_iA_j = 1, \quad g^{ij}B_iB_j = 1, \quad g^{ij}A_iB_j = 0.$$

So, $\left(\mathbb{R}^4, g\right)$ is a $MS\left(QE\right)_4$.

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Some geometric and physical properties of pseudo $\mathcal{M}^*$-projective symmetric manifolds

Dipankar Hazra[a,*], Uday Chand De[b], Sameh Shenawy[c], Abdallah Abdelhameed Syied[d]

[a]*Department of Mathematics, Heramba Chandra College, 23/49, Gariahat Road, Kolkata 700029, West Bengal, India*
[b]*Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata 700019, West Bengal, India*
[c]*Basic Science Department, Modern Academy for Engineering and Technology, Maadi, Egypt*
[d]*Department of Mathematics, Faculty of Science, Zagazig University, P.O. Box 44519, Zagazig, Egypt*

Abstract. In this study we introduce a new tensor in a semi-Riemannian manifold, named the $\mathcal{M}^*$-projective curvature tensor which generalizes the m-projective curvature tensor. We start by deducing some fundamental geometric properties of the $\mathcal{M}^*$-projective curvature tensor. After that, we study pseudo $\mathcal{M}^*$-projective symmetric manifolds $(PM^*S)_n$. A non-trivial example has been used to show the existence of such a manifold. We introduce a series of interesting conclusions. We establish, among other things, that if the scalar curvature ρ is non-zero, the associated 1-form is closed for a $(PM^*S)_n$ with $\operatorname{div}M^* = 0$. We also deal with pseudo $\mathcal{M}^*$-projective symmetric spacetimes, $\mathcal{M}^*$-projectively flat perfect fluid spacetimes, and $\mathcal{M}^*$-projectively flat viscous fluid spacetimes. As a result, we establish some significant theorems.

1. Introduction

In Differential geometry, the investigation of curvature characteristics is the prime problem among others. In this context, S. S. Chern had uttered in [7] "A fundamental notion is curvature, in its different forms". Hence, the discovery of the Riemann curvature tensor creates an extremely significant subject matter. In this paper, due to the above sense, we have introduced a new curvature tensor, called $\mathcal{M}^*$-projective curvature tensor which generalizes some known curvature tensors.

According to Chaki [3], for a non-vanishing 1-form D, a non-flat Riemannian or a semi-Riemannian manifold (M^n, g), $(n > 2)$ is named pseudosymmetric if its curvature tensor obeys

$$\begin{aligned}(\nabla_Z\mathcal{R})(G,H,J,K) = 2D(Z)\mathcal{R}(G,H,J,K) + D(G)\mathcal{R}(Z,H,J,K) + D(H)\mathcal{R}(G,Z,J,K)\\ + D(J)\mathcal{R}(G,H,Z,K) + D(K)\mathcal{R}(G,H,J,Z),\end{aligned}$$

∇ is the Levi-Civita connection and $\mathcal{R}(G,H,J,K) = g(R(G,H)J,K)$, R being the curvature tensor of type $(1,3)$. Let π be the associated vector field corresponding to the 1-form D, i.e.,

$$g(H,\pi) = D(H),$$

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* Corresponding author: Dipankar Hazra
Email addresses: dipankarsk524@gmail.com (Dipankar Hazra), uc_de@yahoo.com (Uday Chand De), drshenawy@mail.com (Sameh Shenawy), a.a_syied@yahoo.com (Abdallah Abdelhameed Syied)

for all H. Pseudosymmetric manifolds have been investigated by several authors ([18], [20], [21], [32], [33]) and many others.

In a Riemannian or a semi-Riemannian manifold the Ricci tensor $\mathcal{S}$ is said to be of Codazzi type [10] if the covariant derivative of Ricci tensor satisfies

$$(\nabla_G \mathcal{S})(H,J) = (\nabla_H \mathcal{S})(G,J)$$

and the Ricci tensor is said to be cyclic parallel [10] if

$$(\nabla_G \mathcal{S})(H,J) + (\nabla_H \mathcal{S})(J,G) + \left(\nabla_J \mathcal{S}\right)(G,H) = 0.$$

A non-flat semi-Riemannian manifold obeying the condition

$$(\nabla_G \mathcal{S})(H,J) = \gamma(G)\,\mathcal{S}(H,J) + \delta(G)\,g(H,J),$$

where γ and δ are non-zero 1-forms, is named a generalized Ricci recurrent manifold [8]. The manifold becomes a Ricci recurrent manifold when $\delta = 0$.

In a Riemannian or a semi-Riemannian manifold (M^n, g) $(n \geq 2)$, the m-projective curvature tensor M is defined as [25]

$$M(G,H)J = R(G,H)J - \frac{1}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right],$$

where R is the curvature tensor of type $(1,3)$, $\mathcal{S}$ is the Ricci tensor of type $(0,2)$ and Q is the Ricci operator defined by $g(QG,H) = \mathcal{S}(G,H)$.

We define the M^*-projective curvature tensor of type $(1,3)$ as

$$M^*(G,H)J = R(G,H)J - \frac{\varphi}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right], \tag{1}$$

φ being scalar. The M^*-projective curvature tensor reduces to the m-projective curvature tensor when $\varphi = 1$. The M^*-projective curvature tensor and the curvature tensor are identical if $\varphi = 0$. Equation (1) can be expressed as

$$\begin{aligned}\mathcal{M}^*(G,H,J,K) = \mathcal{R}(G,H,J,K) - \frac{\varphi}{2(n-1)}\big[&\mathcal{S}(H,J)g(G,K) - \mathcal{S}(G,J)g(H,K)\\ &+g(H,J)\mathcal{S}(G,K) - g(G,J)\mathcal{S}(H,K)\big],\end{aligned} \tag{2}$$

where $\mathcal{M}^*(G,H,J,K) = g(M^*(G,H)J,K)$ and $\mathcal{R}(G,H,J,K) = g(R(G,H)J,K)$.

A non-flat Riemannian or a semi-Riemannian manifold $(M^n, g), (n > 2)$ is said to be a pseudo $\mathcal{M}^*$-projective symmetric manifold if the $\mathcal{M}^*$-projective curvature tensor of type $(0,4)$ satisfies the relation

$$\begin{aligned}(\nabla_Z \mathcal{M}^*)(G,H,J,K) = {}& 2D(Z)\,\mathcal{M}^*(G,H,J,K) + D(G)\,\mathcal{M}^*(Z,H,J,K) + D(H)\,\mathcal{M}^*(G,Z,J,K)\\ &+ D(J)\,\mathcal{M}^*(G,H,Z,K) + D(K)\,\mathcal{M}^*(G,H,J,Z),\end{aligned} \tag{3}$$

D being a non-vanishing 1-form and π is the associated vector field corresponding to the 1-form D, i.e.,

$$g(H,\pi) = D(H).$$

An n-dimensional pseudo $\mathcal{M}^*$-projective symmetric manifold is denoted by $(PM^*S)_n$, where P indicates pseudo, M^* is the $\mathcal{M}^*$-projective curvature tensor and S indicates symmetric. When $\varphi = 0$, the pseudo $\mathcal{M}^*$-projective symmetric manifold reduces to the pseudosymmetric manifold denoted by $(PS)_n$. Further, if $\varphi = 1$, pseudo $\mathcal{M}^*$-projective symmetric manifolds contain pseudo m-projective symmetric manifolds. Thus $(PM^*S)_n$ recovers some known geometric structures. We organized the paper as follows:

After preliminaries in section 3, we investigate the curvature property of $(PM^*S)_n$. The study of $(PM^*S)_n$ with Codazzi type of Ricci tensor is covered in section 4. In section 5, we analyze $(PM^*S)_n$ with $\operatorname{div} M^* = 0$. After that, in section 6, we construct an example of $(PM^*S)_4$. Pseudo $\mathcal{M}^*$-projective symmetric spacetimes are discussed in section 7. Section 8 is devoted to study $\mathcal{M}^*$-projectively flat spacetimes. We focus at $\mathcal{M}^*$-projectively flat perfect fluid and viscous fluid spacetimes in the last two sections.

2. Preliminaries

We can see from (1) the tensor M^* fulfills the following:

$$\begin{aligned} &(i) \quad M^*(G,H)J = -M^*(H,G)J, \\ &(ii) \quad M^*(G,H)J + M^*(H,J)G + M^*(J,G)H = 0. \end{aligned} \tag{4}$$

It is also clear from (2) that

$$\sum_{i=1}^{n} \varepsilon_i \mathcal{M}^*(G,H,e_i,e_i) = 0 = \sum_{i=1}^{n} \varepsilon_i \mathcal{M}^*(e_i,e_i,J,K) \tag{5}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \varepsilon_i \mathcal{M}^*(e_i,G,H,e_i) &= \left[1 - \frac{(n-2)\varphi}{2(n-1)}\right]\mathcal{S}(G,H) - \frac{\varphi\rho}{2(n-1)}g(G,H) \\ &= \sum_{i=1}^{n} \varepsilon_i \mathcal{M}^*(G,e_i,e_i,H) \\ &= \overline{\mathcal{M}^*}(G,H), \quad \text{(say)} \end{aligned} \tag{6}$$

where at each point of the manifold $\{e_k\}$, $k = 1,2,\ldots,n$ be an orthonormal basis of the tangent space, $\rho = \sum_{i=1}^{n} \varepsilon_i \mathcal{S}(e_i,e_i)$ is the scalar curvature and $\varepsilon_i = g(e_i,e_i) = \pm 1$.

The followings are derived from (1) and (4):

$$\begin{aligned} &(i) \quad \mathcal{M}^*(G,H,J,K) = -\mathcal{M}^*(H,G,J,K), \\ &(ii) \quad \mathcal{M}^*(G,H,J,K) = -\mathcal{M}^*(G,H,K,J), \\ &(iii) \quad \mathcal{M}^*(G,H,J,K) = \mathcal{M}^*(J,K,G,H), \\ &(iv) \quad \mathcal{M}^*(G,H,J,K) + \mathcal{M}^*(H,J,G,K) + \mathcal{M}^*(J,G,H,K) = 0, \end{aligned} \tag{7}$$

where $\mathcal{M}^*(G,H,J,K) = g(M^*(G,H)J,K)$.

Proposition 2.1. *A $\mathcal{M}^*$-projectively flat semi-Riemannnian manifold is an Einstein manifold.*

Proof. The $\mathcal{M}^*$-projective curvature tensor is given by

$$\begin{aligned} \mathcal{M}^*(G,H,J,K) = \mathcal{R}(G,H,J,K) - \frac{\varphi}{2(n-1)}[&\mathcal{S}(H,J)g(G,K) - \mathcal{S}(G,J)g(H,K) \\ &+ g(H,J)\mathcal{S}(G,K) - g(G,J)\mathcal{S}(H,K)], \end{aligned}$$

φ being an arbitrary scalar. If $\mathcal{M}^*$-projective curvature tensor vanishes, then

$$\mathcal{R}(G,H,J,K) = \frac{\varphi}{2(n-1)}[\mathcal{S}(H,J)g(G,K) - \mathcal{S}(G,J)g(H,K) + g(H,J)\mathcal{S}(G,K) - g(G,J)\mathcal{S}(H,K)]. \tag{8}$$

In (8), we obtain by contracting H and J

$$\mathcal{S}(G,K) = \frac{\varphi\rho}{2(n-1) - (n-2)\varphi}g(G,K). \tag{9}$$

Thus, we complete the proof.

In (9), contracting G and K we have

$$\rho(n-1)(1-\varphi) = 0,$$

which implies either $\rho = 0$ or $\varphi = 1$. If $\rho \neq 0$, then the $\mathcal{M}^*$-projective curvature tensor is the same as the m-projective curvature tensor M for $\varphi = 1$. As a result, $\mathcal{M}^*$-projectively flatness and m-projectively flatness are the same.

The following corollary was established by the author in [31]:

Corollary 2.2. *m-projectively flat Riemannian manifold is an Einstein manifold.*

Proposition 2.3. *If $\mathcal{M}^*$-projective curvature tensor is parallel, then the manifold reduces to a generalized Ricci recurrent manifold.*

Proof. The M^*-projective curvature tensor is given by

$$M^*(G,H)J = R(G,H)J - \frac{\varphi}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right], \tag{10}$$

φ being an arbitrary scalar. Taking the covariant derivative of (10) gives us

$$\begin{aligned}(\nabla_Z M^*)(G,H)J = (\nabla_Z R)(G,H)J &- \frac{d\varphi(Z)}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right]\\ &- \frac{\varphi}{2(n-1)}\left[(\nabla_Z\mathcal{S})(H,J)G - (\nabla_Z\mathcal{S})(G,J)H + g(H,J)(\nabla_Z Q)G - g(G,J)(\nabla_Z Q)H\right].\end{aligned}$$

By hypothesis, $\mathcal{M}^*$-projective curvature tensor is parallel. As a result of the previous equation,

$$\begin{aligned}&(\nabla_Z R)(G,H)J = \frac{d\varphi(Z)}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right]\\ &+ \frac{\varphi}{2(n-1)}\left[(\nabla_Z\mathcal{S})(H,J)G - (\nabla_Z\mathcal{S})(G,J)H + g(H,J)(\nabla_Z Q)G - g(G,J)(\nabla_Z Q)H\right].\end{aligned} \tag{11}$$

Contracting G in (11) we find

$$(\nabla_Z\mathcal{S})(H,J) = \frac{(n-2)}{2(n-1)-(n-2)\varphi}d\varphi(Z)\mathcal{S}(H,J) + \frac{1}{2(n-1)-(n-2)\varphi}\left[\rho d\varphi(Z) + \varphi d\rho(Z)\right]g(H,J). \tag{12}$$

Again, contracting H and J in (12) reveals that

$$(1-\varphi)\,d\rho(Z) = \rho d\varphi(Z). \tag{13}$$

This implies

$$d\varphi(Z) = (1-\varphi)(Z\log\rho). \tag{14}$$

From (12), (13) and (14) we obtain

$$(\nabla_Z\mathcal{S})(H,J) = \frac{(n-2)(1-\varphi)(Z\log\rho)}{2(n-1)-(n-2)\varphi}\mathcal{S}(H,J) + \frac{(Z\rho)}{2(n-1)-(n-2)\varphi}g(H,J).$$

So the proof is completed.

Proposition 2.4. *For a $\mathcal{M}^*$-projective curvature tensor with $divM^* = 0$, the curvature conditions $divM = 0$ and $divR = 0$ are equivalent, provided φ is constant.*

Proof. The M^*-projective curvature tensor is given by

$$M^*(G,H)J = R(G,H)J - \frac{\varphi}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right], \tag{15}$$

φ being an arbitrary scalar. Taking covariant derivative of the foregoing equation we find

$$(\nabla_Z M^*)(G,H)J = (\nabla_Z R)(G,H)J - \frac{d\varphi(Z)}{2(n-1)}\left[\mathcal{S}(H,J)G - \mathcal{S}(G,J)H + g(H,J)QG - g(G,J)QH\right]$$
$$- \frac{\varphi}{2(n-1)}\left[(\nabla_Z \mathcal{S})(H,J)G - (\nabla_Z \mathcal{S})(G,J)H + g(H,J)(\nabla_Z Q)G - g(G,J)(\nabla_Z Q)H\right]. \quad (16)$$

Contracting Z in (16), we have

$$(\mathrm{div} M^*)(G,H)J = (\mathrm{div} R)(G,H)J - \frac{1}{2(n-1)}\left[\mathcal{S}(H,J)(G\varphi) - \mathcal{S}(G,J)(H\varphi) + g(H,J)(QG\varphi)\right.$$
$$\left. - g(G,J)(QH\varphi)\right] - \frac{\varphi}{2(n-1)}\left[(\nabla_G \mathcal{S})(H,J) - (\nabla_H \mathcal{S})(G,J) + \frac{1}{2}g(H,J)\,d\rho(G) - \frac{1}{2}g(G,J)\,d\rho(H)\right].$$

Now if φ is constant, then the above equation reduces to

$$(\mathrm{div} M^*)(G,H)J = (1-\varphi)(\mathrm{div} R)(G,H)J + \varphi(\mathrm{div} M)(G,H)J.$$

Using $\mathrm{div} M^* = 0$, we have the following:

$$(\mathrm{div} M)(G,H)J = \left(\frac{\varphi - 1}{\varphi}\right)(\mathrm{div} R)(G,H)J.$$

This completes the proof.

3. Some curvature properties of $(PM^*S)_n$ $(n > 2)$

In this section, we show that the $\mathcal{M}^*$-projective curvature tensor satisfies Bianchi's second identity for a $(PM^*S)_n$ $(n > 2)$, i.e.,

$$(\nabla_Z \mathcal{M}^*)(G,H,J,K) + (\nabla_G \mathcal{M}^*)(H,Z,J,K) + (\nabla_H \mathcal{M}^*)(Z,G,J,K) = 0. \quad (17)$$

By virtue of (3) and (17) we acquire

$$2D(Z)\left[\mathcal{M}^*(G,H,J,K) + \mathcal{M}^*(H,G,J,K)\right] + 2D(G)\left[\mathcal{M}^*(Z,H,J,K) + \mathcal{M}^*(H,Z,J,K)\right]$$
$$+ 2D(H)\left[\mathcal{M}^*(G,Z,J,K) + \mathcal{M}^*(Z,G,J,K)\right] + D(J)\left[\mathcal{M}^*(G,H,Z,K) + \mathcal{M}^*(H,Z,G,K)\right.$$
$$\left. + \mathcal{M}^*(Z,G,H,K)\right] + D(K)\left[\mathcal{M}^*(G,H,J,Z) + \mathcal{M}^*(H,Z,J,G) + \mathcal{M}^*(Z,G,J,H)\right]$$
$$= (\nabla_Z \mathcal{M}^*)(G,H,J,K) + (\nabla_G \mathcal{M}^*)(H,Z,J,K) + (\nabla_H \mathcal{M}^*)(Z,G,J,K). \quad (18)$$

Using (7) in (18) we find

$$(\nabla_Z \mathcal{M}^*)(G,H,J,K) + (\nabla_G \mathcal{M}^*)(H,Z,J,K) + (\nabla_H \mathcal{M}^*)(Z,G,J,K) = 0. \quad (19)$$

This leads to the following theorem:

Theorem 3.1. *The $\mathcal{M}^*$-projective curvature tensor in $(PM^*S)_n$ $(n > 2)$ satisfies Bianchi's second identity.*

4. $(PM^*S)_n$ $(n > 2)$ with Codazzi type of Ricci tensor

Equation (2) provides us

$$\begin{aligned}
&-\frac{\varphi}{2(n-1)}\left[(\nabla_Z\mathcal{S})(H,J)\,g(G,K)-(\nabla_Z\mathcal{S})(G,J)\,g(H,K)+g(H,J)(\nabla_Z\mathcal{S})(G,K)-g(G,J)(\nabla_Z\mathcal{S})(H,K)\right.\\
&+(\nabla_G\mathcal{S})(Z,J)\,g(H,K)-(\nabla_G\mathcal{S})(H,J)\,g(Z,K)+g(Z,J)(\nabla_G\mathcal{S})(H,K)-g(H,J)(\nabla_G\mathcal{S})(Z,K)\\
&\left.+(\nabla_H\mathcal{S})(G,J)\,g(Z,K)-(\nabla_H\mathcal{S})(Z,J)\,g(G,K)+g(G,J)(\nabla_H\mathcal{S})(Z,K)-g(Z,J)(\nabla_H\mathcal{S})(G,K)\right]\\
&-\frac{(Z\varphi)}{2(n-1)}\left[\mathcal{S}(H,J)\,g(G,K)-\mathcal{S}(G,J)\,g(H,K)+g(H,J)\,\mathcal{S}(G,K)-g(G,J)\,\mathcal{S}(H,K)\right]\\
&-\frac{(G\varphi)}{2(n-1)}\left[\mathcal{S}(Z,J)\,g(H,K)-\mathcal{S}(H,J)\,g(Z,K)+g(Z,J)\,\mathcal{S}(H,K)-g(H,J)\,\mathcal{S}(Z,K)\right]\\
&-\frac{(H\varphi)}{2(n-1)}\left[\mathcal{S}(G,J)\,g(Z,K)-\mathcal{S}(Z,J)\,g(G,K)+g(G,J)\,\mathcal{S}(Z,K)-g(Z,J)\,\mathcal{S}(G,K)\right]\\
&=(\nabla_Z\mathcal{M}^*)(G,H,J,K)+(\nabla_G\mathcal{M}^*)(H,Z,J,K)+(\nabla_H\mathcal{M}^*)(Z,G,J,K). \qquad (20)
\end{aligned}$$

If $(PM^*S)_n$ admits the Codazzi type of Ricci tensor, then (20) becomes

$$\begin{aligned}
&-\frac{(Z\varphi)}{2(n-1)}\left[\mathcal{S}(H,J)\,g(G,K)-\mathcal{S}(G,J)\,g(H,K)+g(H,J)\,\mathcal{S}(G,K)-g(G,J)\,\mathcal{S}(H,K)\right]\\
&-\frac{(G\varphi)}{2(n-1)}\left[\mathcal{S}(Z,J)\,g(H,K)-\mathcal{S}(H,J)\,g(Z,K)+g(Z,J)\,\mathcal{S}(H,K)-g(H,J)\,\mathcal{S}(Z,K)\right]\\
&-\frac{(H\varphi)}{2(n-1)}\left[\mathcal{S}(G,J)\,g(Z,K)-\mathcal{S}(Z,J)\,g(G,K)+g(G,J)\,\mathcal{S}(Z,K)-g(Z,J)\,\mathcal{S}(G,K)\right]\\
&\qquad=(\nabla_Z\mathcal{M}^*)(G,H,J,K)+(\nabla_G\mathcal{M}^*)(H,Z,J,K)+(\nabla_H\mathcal{M}^*)(Z,G,J,K). \qquad (21)
\end{aligned}$$

Using (19) in (21), we obtain

$$\begin{aligned}
&\frac{(Z\varphi)}{2(n-1)}\left[\mathcal{S}(H,J)\,g(G,K)-\mathcal{S}(G,J)\,g(H,K)+g(H,J)\,\mathcal{S}(G,K)-g(G,J)\,\mathcal{S}(H,K)\right]\\
&+\frac{(G\varphi)}{2(n-1)}\left[\mathcal{S}(Z,J)\,g(H,K)-\mathcal{S}(H,J)\,g(Z,K)+g(Z,J)\,\mathcal{S}(H,K)-g(H,J)\,\mathcal{S}(Z,K)\right]\\
&+\frac{(H\varphi)}{2(n-1)}\left[\mathcal{S}(G,J)\,g(Z,K)-\mathcal{S}(Z,J)\,g(G,K)+g(G,J)\,\mathcal{S}(Z,K)-g(Z,J)\,\mathcal{S}(G,K)\right]=0. \qquad (22)
\end{aligned}$$

Contracting G and K in (22), we infer that

$$\begin{aligned}
&(Z\varphi)\left[(n-2)\,\mathcal{S}(H,J)+\rho g(H,J)\right]+\mathcal{S}(Z,J)(H\varphi)-\mathcal{S}(H,J)(Z\varphi)\\
&+g(Z,J)\,g(QH,\operatorname{grad}\varphi)-g(H,J)\,g(QZ,\operatorname{grad}\varphi)+(H\varphi)\left[(2-n)\,\mathcal{S}(J,Z)-\rho g(J,Z)\right]=0. \qquad (23)
\end{aligned}$$

Again contracting H and J in (23) yields

$$(Z\varphi)\,\rho=g(QZ,\operatorname{grad}\varphi)$$

which gives

$$\mathcal{S}(Z,\operatorname{grad}\varphi)=\rho g(Z,\operatorname{grad}\varphi).$$

Thus we conclude the following theorem:

Theorem 4.1. *For a $(PM^*S)_n$ admitting Codazzi type of Ricci tensor, ρ is an eigenvalue of the Ricci tensor $\mathcal{S}$ corresponding to the eigenvector gradφ.*

If φ is constant, we can deduce from (20) and Bianchi's second identity

$$\begin{aligned}
&(\nabla_Z \mathcal{S})(H,J)\, g(G,K) - (\nabla_Z \mathcal{S})(G,J)\, g(H,K) + g(H,J)(\nabla_Z \mathcal{S})(G,K) - g(G,J)(\nabla_Z \mathcal{S})(H,K) \\
&+ (\nabla_G \mathcal{S})(Z,J)\, g(H,K) - (\nabla_G \mathcal{S})(H,J)\, g(Z,K) + g(Z,J)(\nabla_G \mathcal{S})(H,K) - g(H,J)(\nabla_G \mathcal{S})(Z,K) \\
&+ (\nabla_H \mathcal{S})(G,J)\, g(Z,K) - (\nabla_H \mathcal{S})(Z,J)\, g(G,K) + g(G,J)(\nabla_H \mathcal{S})(Z,K) - g(Z,J)(\nabla_H \mathcal{S})(G,K) = 0. \qquad (24)
\end{aligned}$$

Contracting G and K in (24) reveals that

$$(n-3)(\nabla_Z \mathcal{S})(H,J) + \frac{1}{2} g(H,J)(Z\rho) - \frac{1}{2} g(Z,J)(H\rho) - (n-3)(\nabla_H \mathcal{S})(Z,J) = 0. \qquad (25)$$

Further, if ρ remains constant, (25) becomes

$$(\nabla_Z \mathcal{S})(H,J) = (\nabla_H \mathcal{S})(Z,J).$$

As a result, we can deduce the following corollary:

Corollary 4.2. *In a $(PM^*S)_n$ the Ricci tensor is of Codazzi type provided φ and the scalar curvature ρ are constants.*

Using (6) once again,

$$\overline{\mathcal{M}^*}(G,H) = \left[1 - \frac{(n-2)\varphi}{2(n-1)}\right] \mathcal{S}(G,H) - \frac{\varphi\rho}{2(n-1)} g(G,H).$$

Contracting G and H gives

$$\overline{m^*} = (1-\varphi)\rho. \qquad (26)$$

The $\mathcal{M}^*$-projective curvature tensor satisfies the relation for a $(PM^*S)_n$:

$$\begin{aligned}
(\nabla_Z \mathcal{M}^*)(G,H,J,K) = 2D(Z)\,\mathcal{M}^*(G,H,J,K) + D(G)\,\mathcal{M}^*(Z,H,J,K) + D(H)\,\mathcal{M}^*(G,Z,J,K) \\
+ D(J)\,\mathcal{M}^*(G,H,Z,K) + D(K)\,\mathcal{M}^*(G,H,J,Z), \qquad (27)
\end{aligned}$$

D being a non-vanishing 1-form and π is the associated vector field corresponding to the 1-form D, i.e.,

$$g(H,\pi) = D(H).$$

Contracting G and K in (27) we get

$$\left(\nabla_Z \overline{\mathcal{M}^*}\right)(H,J) = 2D(Z)\,\overline{\mathcal{M}^*}(H,J) + \mathcal{M}^*(Z,H,J,\pi) + D(H)\,\overline{\mathcal{M}^*}(Z,J) + D(J)\,\overline{\mathcal{M}^*}(H,Z) + \mathcal{M}^*(\pi,H,J,Z). \quad (28)$$

Again contracting H and J in (28) we find

$$\nabla_Z \overline{m^*} = 2D(Z)\,\overline{m^*} + 4\overline{\mathcal{M}^*}(Z,\pi). \qquad (29)$$

If we use (26) in (29), we obtain

$$(1-\varphi)\, d\rho(Z) - d\varphi(Z)\,\rho = 2D(Z)(1-\varphi)\rho + 4\overline{\mathcal{M}^*}(Z,\pi). \qquad (30)$$

Between (6) and (30), we have

$$(1-\varphi)\, d\rho(Z) - d\varphi(Z)\,\rho = \left[2(1-\varphi)\rho - \frac{2\varphi\rho}{(n-1)}\right] D(Z) + 4\left[1 - \frac{(n-2)\varphi}{2(n-1)}\right] D(\boldsymbol{Q}Z).$$

Thus we can state the following theorem:

Theorem 4.3. *For a* $(PM^*S)_n$ $(n > 2)$ *the following identity holds:*

$$(1-\varphi)\, d\rho(Z) - d\varphi(Z)\,\rho = \left[2(1-\varphi)\,\rho - \frac{2\varphi\rho}{(n-1)}\right] D(Z) + 4\left[1 - \frac{(n-2)\,\varphi}{2(n-1)}\right] D(QZ).$$

In particular, let us consider $\varphi = 0$, then from Theorem 4.2. we get

$$d\rho(Z) = 2D(Z)\,\rho + 4D(QZ).$$

Chaki established the following corollary in [3]:

Corollary 4.4. *For a* $(PS)_n$ *the following identity holds:*

$$d\rho(Z) = 2D(Z)\,\rho + 4D(QZ).$$

5. $(PM^*S)_n$ $(n > 2)$ with div$M^* = 0$

We know that for a $(PM^*S)_n$ $(n > 2)$,

$$(\nabla_Z M^*)(G,H)J = 2D(Z)\,M^*(G,H)J + D(G)\,M^*(Z,H)J + D(H)\,M^*(G,Z)J + D(J)\,M^*(G,H)Z + g(M^*(G,H)J, Z)\,\pi,$$

D being a non-vanishing 1-form and π is the associated vector field corresponding to the 1-form D, i.e.,

$$g(H,\pi) = D(H).$$

Hence,

$$\begin{aligned}(\mathrm{div}M^*)(G,H)J &= \sum_{i=1}^{n} \varepsilon_i g((\nabla_{e_i} M^*)(G,H)J, e_i)\\ &= \sum_{i=1}^{n} \varepsilon_i\,[2D(e_i)\,g(M^*(G,H)J,e_i) + D(G)\,g(M^*(e_i,H)J,e_i) + D(H)\,g(M^*(G,e_i)J,e_i)\\ &\qquad + D(J)\,g(M^*(G,H)e_i,e_i) + g(M^*(G,H)J,e_i)\,g(\pi,e_i)]\\ &= 3D(M^*(G,H)J) + D(G)\,\overline{\mathcal{M}^*}(H,J) - D(H)\,\overline{\mathcal{M}^*}(G,J).\end{aligned}$$

Now $(\mathrm{div}M^*)(G,H)J = 0$ implies

$$3D(M^*(G,H)J) + D(G)\,\overline{\mathcal{M}^*}(H,J) - D(H)\,\overline{\mathcal{M}^*}(G,J) = 0. \tag{31}$$

In (31), by contracting H and J,

$$2\overline{\mathcal{M}^*}(G,\pi) + \rho(1-\varphi)\,D(G) = 0. \tag{32}$$

Using (6) in (32) we deduce that

$$\mathcal{S}(G,\pi) = \frac{\rho(\varphi n - n + 1)}{[(n-1)(2-\varphi)+\varphi]}\,g(G,\pi).$$

This implies

$$\mathcal{S}(G,\pi) = \mu g(G,\pi), \tag{33}$$

where $\mu = \dfrac{\rho(\varphi n - n + 1)}{[(n-1)(2-\varphi)+\varphi]}$ is a scalar. Thus we can say that:

Theorem 5.1. *In a $(PM^*S)_n$ $(n > 2)$ with $divM^* = 0$, μ is an eigenvalue of the Ricci tensor $\mathcal{S}$ corresponding to the eigenvector π.*

Now if we take covariant derivative of (29), then we find

$$\nabla_W \nabla_Z \overline{m^*} = 2(\nabla_W D)(Z)\overline{m^*} + 2D(Z)\left(\nabla_W \overline{m^*}\right) + 4\left(\nabla_W \overline{\mathcal{M}^*}\right)(Z, \pi). \tag{34}$$

Using (28) and (29) in (34) we reach

$$\begin{aligned}\nabla_W \nabla_Z \overline{m^*} = {} & 2(\nabla_W D)(Z)\overline{m^*} + 4D(Z)D(W)\overline{m^*} + 8D(Z)\overline{\mathcal{M}^*}(W, \pi) + 8D(W)\overline{\mathcal{M}^*}(Z, \pi) \\ & + 4\mathcal{M}^*(W, Z, \pi, \pi) + 4D(Z)\overline{\mathcal{M}^*}(W, \pi) + 4D(\pi)\overline{\mathcal{M}^*}(Z, W) + 4\mathcal{M}^*(\pi, Z, \pi, W).\end{aligned} \tag{35}$$

Changing Z and W in (35) and subtracting these two equations, we obtain from (6) and (7) that

$$2\left[1 - \frac{(n-2)\varphi}{2(n-1)}\right][D(W)\mathcal{S}(Z, \pi) - D(Z)\mathcal{S}(W, \pi)] + (\nabla_Z D)(W)\overline{m^*} - (\nabla_W D)(Z)\overline{m^*} = 0. \tag{36}$$

Assume that the scalar curvature ρ is non-zero, then from (26), (33) and (36) we can derive

$$(\nabla_Z D)(W) = (\nabla_W D)(Z).$$

As a result, we may say the following:

Theorem 5.2. *The associated 1-form of a $(PM^*S)_n$ $(n > 2)$ with $divM^* = 0$ is closed provided the scalar curvature ρ is non-zero.*

6. Example of a $(PM^*S)_4$

Let us consider a Lorentzian metric g on $\mathbb{R}^4$ by [15]

$$ds^2 = g_{ij}dx^i dx^j = \left(dx^1\right)^2 + \left(x^1\right)^2\left(dx^2\right)^2 + \left(x^2\right)^2\left(dx^3\right)^2 - \left(dx^4\right)^2,$$

where $i, j = 1, 2, 3, 4$. We calculate the non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{22} = -x^1, \quad \Gamma^2_{33} = -\frac{x^2}{(x^1)^2}, \quad \Gamma^2_{12} = \frac{1}{x^1}, \quad \Gamma^3_{23} = \frac{1}{x^2}, \quad \mathcal{R}_{1332} = -\frac{x^2}{x^1}, \quad \mathcal{S}_{12} = -\frac{1}{x^1 x^2}$$

and using the symmetry properties, the other components are obtained.
Let us consider the scalar φ as follows:

$$\varphi = 6x^1. \tag{37}$$

The only non-vanishing $\mathcal{M}^*$-projective curvature tensor and its covariant derivatives are written by

$$\mathcal{M}^*_{1332} = -\frac{x^2}{x^1} + x^2, \quad \mathcal{M}^*_{1332,1} = \frac{x^2}{(x^1)^2}, \quad \mathcal{M}^*_{1332,2} = -\frac{1}{x^1} + 1. \tag{38}$$

The 1-form is chosen as follows:

$$D_i(x) = \begin{cases} \dfrac{1}{3x^1(x^1-1)}, & \text{when } i = 1 \\ \dfrac{1}{3x^2}, & \text{when } i = 2 \\ 0, & \text{otherwise} \end{cases} \tag{39}$$

Now, using these 1-forms, the equation (3) may be reduced to the following equations:

$$\mathcal{M}^*_{1332,1} = 3D_1\mathcal{M}^*_{1332} \tag{40}$$

and

$$\mathcal{M}^*_{1332,2} = 3D_2\mathcal{M}^*_{1332}. \tag{41}$$

From the equations (38), (39) and (41) we get

$$\begin{aligned}\text{Right hand side of (41)} &= 3D_2\mathcal{M}^*_{1332}\\ &= 3\cdot\frac{1}{3x^2}\cdot\left(-\frac{x^2}{x^1}+x^2\right)\\ &= -\frac{1}{x^1}+1 = \mathcal{M}^*_{1332,2}.\end{aligned}$$

It may be proved that (40) is also true using similar argument.
So, the manifold $\left(\mathbb{R}^4, g\right)$ under consideration is a $(PM^*S)_4$.

7. Pseudo $\mathcal{M}^*$-projective symmetric spacetimes

The spacetime of general relativity is a connected four-dimensional semi-Riemannian manifold $\left(M^4, g\right)$ with Lorentzian metric g whose signature $(-,+,+,+)$. The study of the casual character of vector of the manifold is the first step in the Lorentzian geometry. The Lorentzian manifold is a useful choice for studying general relativity because of this coincidence. Spacetimes have been studied by several authors in different ways such as ([2], [6], [9], [11], [12], [13], [16], [17], [19], [22], [26], [29], [34]).

Lorentzian manifolds with the Ricci tensor

$$\mathcal{S}(G,H) = \alpha g(G,H) + \beta D(G)D(H), \tag{42}$$

where α and β are scalars and π is a unit timelike vector field that corresponds to the 1-form D, are called perfect fluid spacetimes.

If the matter content of the spacetime is perfect fluid with velocity vector field π, the above form (42) of the Ricci tensor is derived from Einstein's equation. The energy momentum tensor T represents the matter content of spacetime, which is considered to be fluid. The energy momentum tensor for a perfect fluid spacetime has the form [24]

$$T(G,H) = (\sigma + p)D(G)D(H) + pg(G,H), \tag{43}$$

where σ represents the energy density and p represents the isotropic pressure. The velocity vector field π is a time-like vector that is metrically equal to the non-zero 1-form D. Because there are no heat conduction terms and stress factors corresponding to viscosity, the fluid is called perfect [14]. Furthermore, an equation of state governing the type of perfect fluid under consideration connects p and σ. In general, this is an equation of the form $p = p(\sigma, T_0)$, where T_0 denotes absolute temperature. We will just look at cases where T_0 is effectively constant and the state equation becomes $p = p(\sigma)$. The perfect fluid in this situation is known as isentropic [14]. In addition, if $p = \sigma$, the perfect fluid is referred to as stiff matter ([27], p. 66).

The Einstein's field equations (briefly, $\mathcal{EFE}$) without cosmological constant is as follows:

$$\mathcal{S}(G,H) - \frac{\rho}{2}g(G,H) = \kappa T(G,H), \tag{44}$$

where the Ricci tensor and scalar curvature are denoted by $\mathcal{S}$ and ρ, respectively. The gravitational constant is κ, whereas the energy momentum tensor is T. According to $\mathcal{EFE}$, matter controls the geometry of spacetime, while the motion of matter is dictated by the non-flat metric of space.

In this paper, we look at a special type of spacetime known as pseudo $\mathcal{M}^*$-projective symmetric spacetime. The results obtained for the pseudo $\mathcal{M}^*$-projective symmetric manifolds also holds for the Lorentzian setting. A 4-dimensional Lorentzian manifold (M, g) is said to be pseudo $\mathcal{M}^*$-projective symmetric spacetime if the $\mathcal{M}^*$-projective curvature tensor satisfies the relation (3). Here we consider the associated vector corresponding to the 1-form D is a unit timelike vector field, i.e., $g(\pi, \pi) = -1$. Thus equation (3) can be represented for a pseudo $\mathcal{M}^*$-projective symmetric spacetime as

$$\begin{aligned}
&(\nabla_Z\mathcal{R})(G,H,J,K) - \frac{\varphi}{6}\left[(\nabla_Z\mathcal{S})(H,J)\,g(G,K) - (\nabla_Z\mathcal{S})(G,J)\,g(H,K) + g(H,J)(\nabla_Z\mathcal{S})(G,K)\right.\\
&\left.-g(G,J)(\nabla_Z\mathcal{S})(H,K)\right] - \frac{d\varphi(Z)}{6}\left[\mathcal{S}(H,J)\,g(G,K) - \mathcal{S}(G,J)\,g(H,K) + g(H,J)\,\mathcal{S}(G,K) - g(G,J)\,\mathcal{S}(H,K)\right]\\
&= 2D(Z)\Big[\mathcal{R}(G,H,J,K) - \frac{\varphi}{6}\{\mathcal{S}(H,J)\,g(G,K) - \mathcal{S}(G,J)\,g(H,K) + g(H,J)\,\mathcal{S}(G,K)\\
&-g(G,J)\,\mathcal{S}(H,K)\}\Big] + D(G)\Big[\mathcal{R}(Z,H,J,K) - \frac{\varphi}{6}\{\mathcal{S}(H,J)\,g(Z,K) - \mathcal{S}(Z,J)\,g(H,K)\\
&+g(H,J)\,\mathcal{S}(Z,K) - g(Z,J)\,\mathcal{S}(H,K)\}\Big] + D(H)\Big[\mathcal{R}(G,Z,J,K) - \frac{\varphi}{6}\{\mathcal{S}(Z,J)\,g(G,K)\\
&-\mathcal{S}(G,J)\,g(Z,K) + g(Z,J)\,\mathcal{S}(G,K) - g(G,J)\,\mathcal{S}(Z,K)\}\Big] + D(J)\Big[\mathcal{R}(G,H,Z,K)\\
&-\frac{\varphi}{6}\{\mathcal{S}(H,Z)\,g(G,K) - \mathcal{S}(G,Z)\,g(H,K) + g(H,Z)\,\mathcal{S}(G,K) - g(G,Z)\,\mathcal{S}(H,K)\}\Big]\\
&+ D(K)\Big[\mathcal{R}(G,H,J,Z) - \frac{\varphi}{6}\{\mathcal{S}(H,J)\,g(G,Z) - \mathcal{S}(G,J)\,g(H,Z) + g(H,J)\,\mathcal{S}(G,Z) - g(G,J)\,\mathcal{S}(H,Z)\}\Big].
\end{aligned} \tag{45}$$

Taking a frame field and contracting G and K in (45) we get

$$\begin{aligned}
&2D(Z)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(H,J) - \frac{\varphi\rho}{6}g(H,J)\right] + D(H)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(Z,J) - \frac{\varphi\rho}{6}g(Z,J)\right]\\
&+ D(J)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(H,Z) - \frac{\varphi\rho}{6}g(H,Z)\right] + D(R(Z,H)J) - \frac{\varphi}{6}\left[D(Z)\,\mathcal{S}(H,J)\right.\\
&\left.-D(H)\,\mathcal{S}(Z,J) + D(QZ)\,g(H,J) - D(QH)\,g(Z,J)\right] + D(R(Z,J)H)\\
&-\frac{\varphi}{6}\left[D(Z)\,\mathcal{S}(H,J) - D(QJ)\,g(H,Z) + D(QZ)\,g(H,J) - D(J)\,\mathcal{S}(H,Z)\right]\\
&= \left(1-\frac{\varphi}{3}\right)(\nabla_Z\mathcal{S})(H,J) - \frac{\varphi}{6}d\rho(Z)\,g(H,J) - \frac{d\varphi(Z)}{6}\left[2\mathcal{S}(H,J) + \rho g(H,J)\right].
\end{aligned} \tag{46}$$

In $(PM^*S)_4$ spacetimes, we take the associated vector field π to be a parallel vector field. Then

$$\nabla_G\pi = 0, \tag{47}$$

for every vector field G.
Using Ricci identity, we can now deduce

$$R(G,H)\,\pi = 0. \tag{48}$$

It is clear from (48) that

$$\mathcal{R}(G,H,J,\pi) = 0, \tag{49}$$

where $\mathcal{R}(G,H,J,\pi) = g(R(G,H)J,\pi)$.
Hence,

$$D(R(G,H)J) = 0. \tag{50}$$

Contracting H in (48) we infer that

$$\mathcal{S}(G,\pi) = 0. \tag{51}$$

Now,

$$(\nabla_Z \mathcal{S})(G,\pi) = \nabla_Z \mathcal{S}(G,\pi) - \mathcal{S}(\nabla_Z G,\pi) - \mathcal{S}(G,\nabla_Z \pi).$$

As a result, if we use (47) and (51) in the previous equation, we find

$$(\nabla_Z \mathcal{S})(G,\pi) = 0. \tag{52}$$

Using (50), (51) and (52) in (46) we reach by putting $H = \pi$,

$$\left(1-\frac{\varphi}{6}\right)\mathcal{S}(Z,J) = \frac{\varphi\rho}{6}g(Z,J) - \frac{\varphi\rho}{2}D(Z)D(J) + \frac{1}{6}\left[\varphi d\rho(Z) + \rho d\varphi(Z)\right]D(J). \tag{53}$$

Using $J = \pi$, in the preceding equation once more, we get

$$\frac{1}{6}\left[\varphi d\rho(Z) + \rho d\varphi(Z)\right] = \frac{2\varphi\rho}{3}D(Z). \tag{54}$$

By combining the equations (53) and (54), we arrive to the following result:

$$\mathcal{S}(Z,J) = \alpha g(Z,J) + \beta D(Z)D(J),$$

where $\alpha = \beta = \dfrac{\varphi\rho}{6-\varphi}$.
As a result, we may say the following:

Theorem 7.1. *A pseudo $\mathcal{M}^*$-projective symmetric spacetime with associated vector field as a parallel vector field is a perfect fluid spacetime.*

According to $\mathcal{EFE}$ without cosmological constant, the Ricci tensor takes the form

$$\mathcal{S}(G,H) = \kappa\left(\frac{p-\sigma}{-2}\right)g(G,H) + \kappa(p+\sigma)D(G)D(H).$$

In contrast to equation (42) we notice $\alpha = \frac{\kappa}{2}(\sigma - p)$ and $\beta = \kappa(p+\sigma)$.
Now $\alpha = \beta$ gives $p = -\frac{1}{3}\sigma$.
In view of this observation, we can conclude:

Theorem 7.2. *A pseudo $\mathcal{M}^*$-projective symmetric spacetime with associated vector field as a parallel vector field represents the limiting case of dark energy and the limiting case of violating the strong energy condition.*

Now, we consider the pseudo $\mathcal{M}^*$-projective symmetric spacetime with cyclic parallel Ricci tensor. Then

$$(\nabla_Z \mathcal{S})(H,J) + (\nabla_H \mathcal{S})(J,Z) + \left(\nabla_J \mathcal{S}\right)(Z,H) = 0. \tag{55}$$

Since the scalar curvature ρ is constant in a spacetime with cyclic parallel Ricci tensor, $d\rho(G) = 0$, for all G.
If we consider that the scalar φ is constant, then $d\varphi(G) = 0$, for all G.
Using (46) in (55), we now have

$$\begin{aligned}
&4D(Z)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(H,J) - \frac{\varphi\rho}{6}g(H,J)\right] + 4D(H)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(Z,J) - \frac{\varphi\rho}{6}g(Z,J)\right] \\
&+ 4D(J)\left[\left(1-\frac{\varphi}{3}\right)\mathcal{S}(H,Z) - \frac{\varphi\rho}{6}g(H,Z)\right] + D(R(Z,H)J) + D(R(Z,J)H) \\
&+ D(R(H,J)Z) + D(R(H,Z)J) + D(R(J,Z)H) + D(R(J,H)Z) = 0.
\end{aligned}$$

Following some computations, we arrive at

$$D(Z)E(H,J) + D(H)E(Z,J) + D(J)E(H,Z) = 0,$$

where $E(G,H) = (6-2\varphi)\mathcal{S}(G,H) - \varphi\rho g(G,H)$. The above equation can be expressed in local coordinates as

$$D_iE_{jk} + D_jE_{ki} + D_kE_{ij} = 0.$$

Walker's Lemma [30] is now listed as follows:

Lemma 7.3. *If α_{ij}, β_i are numbers satisfying $\alpha_{ij} = \alpha_{ji}$, $\alpha_{ij}\beta_k + \alpha_{jk}\beta_i + \alpha_{ki}\beta_j = 0$ for $i, j, k = 1, 2, 3, \ldots, n$, then either all α_{ij} are zero or all β_i are zero.*

As $D(G) \neq 0$, according to Walker's Lemma, we have $E(H,J) = 0$, i.e.,

$$\mathcal{S}(H,J) = \left(\frac{\varphi\rho}{6-2\varphi}\right)g(H,J).$$

As a result, we arrive at the following theorem:

Theorem 7.4. *If the scalar φ is constant, then a pseudo $\mathcal{M}^*$-projective symmetric spacetime satisfying the cyclic parallel Ricci tensor is an Einstein spacetime.*

8. $\mathcal{M}^*$-projectively flat spacetimes

In this section we consider $\mathcal{M}^*$-projectively flat spacetimes, which are 4-dimensional Lorentzian manifolds with a timelike vector field. Hence from (2) we have

$$\mathcal{R}(G,H,J,K) = \frac{\varphi}{6}\left[\mathcal{S}(H,J)g(G,K) - \mathcal{S}(G,J)g(H,K) + g(H,J)\mathcal{S}(G,K) - g(G,J)\mathcal{S}(H,K)\right]. \tag{56}$$

Contracting H and J we can get

$$\mathcal{S}(G,K) = \frac{\varphi\rho}{2(3-\varphi)}g(G,K). \tag{57}$$

Again contracting G and K we reach

$$(1-\varphi)\rho = 0.$$

This means $\varphi = 1$, if $\rho \neq 0$.
Using this in (57) we obtain

$$\mathcal{S}(G,K) = \frac{\rho}{4}g(G,K). \tag{58}$$

As a consequence, $\mathcal{M}^*$-projectively flatness implies

$$R(G,H)J = \frac{\rho}{12}\left[g(H,J)G - g(G,J)H\right],$$

which reflects that the spacetime is of constant curvature. It is well known that space of constant curvature implies the spacetime is conformally flat and hence the spacetime is of Petrov type O.
Thus we get to the following conclusion:

Theorem 8.1. *A $\mathcal{M}^*$-projectively flat spacetime with non-zero scalar curvature is a space of constant curvature and of Petrov type O.*

Using (58) in (44) we find

$$T(G,H) = -\frac{\rho}{4\kappa} g(G,H). \tag{59}$$

Covariant differentiation of (59) yields

$$\left(\nabla_J T\right)(G,H) = -\frac{1}{4\kappa} d\rho(J)\, g(G,H). \tag{60}$$

The scalar curvature ρ is constant because the $\mathcal{M}^*$-projectively flat spacetime is Einstein. Thus

$$d\rho(J) = 0, \tag{61}$$

for all J.
Equations (60) and (61) implies

$$\left(\nabla_J T\right)(G,H) = 0.$$

Thus we can say that:

Theorem 8.2. *For a $\mathcal{M}^*$-projectively flat spacetime obeying $\mathcal{EFE}$ without cosmological constant the energy-momentum tensor is covariant constant.*

Remark 8.3. *It may be mentioned that Chaki and Ray [4] proved that a general relativistic spacetime with covariant constant energy-momentum tensor is Ricci symmetric.*

The matter collineation is defined by the energy momentum tensor T

$$\left(\pounds_\eta T\right)(G,H) = 0, \tag{62}$$

where η is the symmetry-generating vector field and $\pounds_\eta$ is the Lie derivative operator along the η vector field.
Let η be a Killing vector field with vanishing $\mathcal{M}^*$-projective curvature tensor on the spacetime. Then

$$\left(\pounds_\eta g\right)(G,H) = 0. \tag{63}$$

Taking the Lie derivatives on both sides of (59) with respect to η we reach

$$\left(\pounds_\eta T\right)(G,H) = -\frac{\rho}{4\kappa}\left(\pounds_\eta g\right)(G,H). \tag{64}$$

We have from (63) and (64)

$$\left(\pounds_\eta T\right)(G,H) = 0.$$

This means that matter collineation is possible in the spacetime.
If $\left(\pounds_\eta T\right)(G,H) = 0$, on the other hand, we get from (64) that

$$\left(\pounds_\eta g\right)(G,H) = 0.$$

As a consequence, the following theorem can be formulated:

Theorem 8.4. *For a $\mathcal{M}^*$-projectively flat spacetime obeying $\mathcal{EFE}$ without cosmological constant, the spacetime admits matter collineation with respect to a vector field η if and only if η is a Killing vector field.*

η is taken to be a conformal Killing vector field. Then we have

$$\left(\pounds_{\eta}g\right)(G,H) = 2\psi g(G,H), \tag{65}$$

where ψ being scalar.
Then from (64) and (65) we get

$$\left(\pounds_{\eta}T\right)(G,H) = -\frac{\rho}{4\kappa}2\psi g(G,H). \tag{66}$$

Using (59) in (66) we deduce that

$$\left(\pounds_{\eta}T\right)(G,H) = 2\psi T(G,H). \tag{67}$$

According to (67), the energy-momentum tensor possesses the Lie inheritance property along η. If (67) holds, then (65) holds as well, indicating that η is a conformal Killing vector field. As a result, we can say that:

Theorem 8.5. *If a $\mathcal{M}^*$-projectively flat spacetime obeying $\mathcal{EFE}$ without cosmological constant, then a vector field η on the spacetime is a conformal Killing vector field if and only if the energy-momentum tensor has the Lie inheritance property along η.*

9. $\mathcal{M}^*$-projectively flat perfect fluid spacetimes

Now imagine a matter distribution in a perfect fluid whose velocity vector field is the vector field π, which is identical with 1-form D of the spacetime. As a result, T is given by [24]:

$$T(G,H) = (\sigma + p)D(G)D(H) + pg(G,H). \tag{68}$$

The energy density and isotropic pressure are represented by σ and p, respectively.
As a result, we get from $\mathcal{EFE}$ without cosmological constant

$$\mathcal{S}(G,H) - \frac{\rho}{2}g(G,H) = \kappa\left[(\sigma + p)D(G)D(H) + pg(G,H)\right]. \tag{69}$$

Contracting the previous equation, we acquire

$$\rho = \kappa(\sigma - 3p). \tag{70}$$

Using (57) in (69) we infer that

$$\left[\frac{\varphi\rho}{2(3-\varphi)} - \frac{\rho}{2}\right]g(G,H) = \kappa\left[(\sigma + p)D(G)D(H) + pg(G,H)\right]. \tag{71}$$

Setting $H = \pi$ in (71) and using $D(G) \neq 0$, we acquire

$$\frac{\rho(3-2\varphi)}{(6-2\varphi)} = \kappa\sigma. \tag{72}$$

Equations (70) and (72) also produce

$$\sigma = (2\varphi - 3)p. \tag{73}$$

As a result, based on the above, we can conclude:

Theorem 9.1. *The energy density and the isotropic pressure are related by* (73) *for a $\mathcal{M}^*$-projectively flat perfect fluid spacetime obeying $\mathcal{EFE}$ without the cosmological constant.*

Remark 9.2. $p = \dfrac{\sigma}{(2\varphi - 3)}$, *i.e., $p = p(\sigma)$, in this situation. So we can conclude that the fluid is isentropic [14].*

From (73) we obtain $\dfrac{p}{\sigma} > -1$, when $\varphi < 1$. In most cases, the dark energy is characterised by an "equation-of-state" parameter $\omega \equiv \dfrac{p}{\sigma}$, the ratio of the spatially homogenous dark-energy pressure p and its energy density σ. Now $\omega > -1$ denotes that the model describes the evaluation in Quintessence region.
We can conclude the following from the preceding discussion:

Theorem 9.3. *A $\mathcal{M}^*$-projectively flat perfect fluid spacetime obeying $\mathcal{EFE}$ without the cosmological constant describes the model of the evaluation in Quintessence region, provided the scalar $\varphi < 1$.*

Now we consider the radiation era in $\mathcal{M}^*$-projectively flat perfect fluid spacetimes. In a perfect fluid spacetime, $\dfrac{p}{\sigma} = \dfrac{1}{3}$ defines the radiation era. In such instances, the energy–momentum tensor is of the form

$$T(G, H) = pg(G, H) + 4pD(G)D(H). \tag{74}$$

As a result, the $\mathcal{EFE}$ without the cosmological constant yields

$$\mathcal{S}(G, H) - \frac{\rho}{2}g(G, H) = \kappa\left[pg(G, H) + 4pD(G)D(H)\right]. \tag{75}$$

Using (57) in (75) reveals that

$$\left(\frac{\varphi\rho}{6 - 2\varphi} - \frac{\rho}{2}\right)g(G, H) = \kappa\left[pg(G, H) + 4pD(G)D(H)\right]. \tag{76}$$

Contracting the foregoing equation by taking a frame field, we obtain

$$4\left(\frac{\varphi\rho}{6 - 2\varphi} - \frac{\rho}{2}\right) = 0. \tag{77}$$

Setting $H = \pi$ in (76) we find

$$\left(\frac{\varphi\rho}{6 - 2\varphi} - \frac{\rho}{2}\right) = -3\kappa p. \tag{78}$$

By combining the equations (77) and (78), we arrive to the following result:

$$p = 0. \tag{79}$$

Thus, we can conclude from (74) and (79) that

$$T(G, H) = 0.$$

This indicates that the spacetime is devoid of matter. Thus we can state the following:

Theorem 9.4. *A radiation era in $\mathcal{M}^*$-projectively flat perfect fluid spacetime satisfying $\mathcal{EFE}$ without cosmological constant is vacuum.*

10. $\mathcal{M}^*$-projectively flat viscous fluid spacetimes

In a viscous fluid spacetime, the T is given by [23, 24]:

$$T(G, H) = pg(G, H) + (\sigma + p)D(G)D(H) + N(G, H), \tag{80}$$

where $N(G,H)$ is the fluid's anisotropic pressure. Also trace of $N = 0$ and $N(G,\pi) = 0$, where π is a velocity vector field.
Using (44) and (57) in (80) we get

$$\left(\frac{\varphi\rho}{6-2\varphi} - \frac{\rho}{2}\right) g(G,H) = \kappa\left[pg(G,H) + (\sigma+p)D(G)D(H) + N(G,H)\right]. \tag{81}$$

Putting $G = H = \pi$ in (81), yields

$$\sigma = -\frac{\rho(2\varphi-3)}{\kappa(6-2\varphi)}. \tag{82}$$

Again contracting (81) over G and H, we infer that

$$4\left(\frac{\varphi\rho}{6-2\varphi} - \frac{\rho}{2}\right) = \kappa\left[4p - (\sigma+p)\right],$$

implying

$$p = \frac{\rho(2\varphi-3)}{\kappa(6-2\varphi)}. \tag{83}$$

By combining the equations (82) and (83), we arrive to the following result:

$$\sigma + p = 0.$$

We know the scalar curvature ρ of a $\mathcal{M}^*$-projectively flat spacetime is constant. If the scalar φ is constant, then σ = constant follows from (82) and therefore $\sigma + p = 0$ gives p = constant. Now, $\sigma + p = 0$ indicates that the fluid behaves as a cosmological constant [28]. It is also known as a phantom barrier [5]. In cosmology, such a choice $\sigma = -p$ entails rapid expansion of spacetime, which is known as inflation [1].
In view of this observation, we can conclude:

Theorem 10.1. *If a $\mathcal{M}^*$-projectively flat viscous fluid spacetime obeying $\mathcal{EFE}$ without cosmological constant, then the spacetime has constant energy density and isotropic pressure and the spacetime represents inflation and also the fluid behaves as a cosmological constant provided the scalar φ is constant.*

We will now consider whether or not a $\mathcal{M}^*$-projectively flat viscous fluid spacetime can accept heat flux. Assume T has the following shape: [23, 24]:

$$T(G,H) = pg(G,H) + (\sigma+p)D(G)D(H) + D(G)B(H) + D(H)B(G), \tag{84}$$

where $B(G) = g(G,\nu)$ for all vector fields G, ν being the heat flux vector field. Thus we have $g(\pi,\nu) = 0$, i.e., $B(\pi) = 0$.
In virtue of (44) and (57), equation (84) takes the form

$$\left(\frac{\varphi\rho}{6-2\varphi} - \frac{\rho}{2}\right) g(G,H) = \kappa\left[pg(G,H) + (\sigma+p)D(G)D(H) + D(G)B(H) + D(H)B(G)\right]. \tag{85}$$

Setting $H = \pi$ in (85), we notice that

$$B(G) = -\frac{1}{\kappa}\left[\frac{\rho(2\varphi-3)}{(6-2\varphi)} + \kappa\sigma\right] D(G).$$

As a result, we arrive at the following theorem:

Theorem 10.2. *A $\mathcal{M}^*$-projectively flat viscous fluid spacetime obeying $\mathcal{EFE}$ without cosmological constant admits heat flux, provided* $\left[\frac{\rho(2\varphi-3)}{(6-2\varphi)} + \kappa\sigma\right] \neq 0.$

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Characterizations of mixed quasi-Einstein spacetimes under Gray's decomposition

Dipankar Hazra
Department of Mathematics, University of Kalyani
Kalyani 741235, West Bengal, India
dipankarsk524@gmail.com

Uday Chand De
Department of Pure Mathematics, University of Calcutta
35 Ballygunge Circular Road, Kolkata 700019, West Bengal, India
uc_de@yahoo.com

Avijit Sarkar*
Department of Mathematics, University of Kalyani
Kalyani 741235, West Bengal, India
avjaj@yahoo.co.in

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In this study, we analyze mixed quasi-Einstein spacetimes endowed with Gray's decomposition, as well as generalized Robertson–Walker spacetimes. For mixed quasi-Einstein spacetimes, the shape of the Ricci tensor in all $\mathcal{O}(n)$-invariant subspaces can be identified using Gray's decomposition of the gradient of the Ricci tensor. In case one, such a spacetime is found to be static; in three cases, the Ricci tensor is found to be in the form of a perfect fluid; and in the other three situations, the spacetime becomes a quasi-Einstein spacetime under certain restrictions on the associated vector fields. Finally, it is established that a mixed quasi-Einstein generalized Robertson–Walker spacetime is a perfect fluid spacetime.

Keywords: Mixed quasi-Einstein spacetime; Gray's decomposition; static spacetime; perfect fluid spacetime; generalized Robertson–Walker spacetime.

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1. Introduction

Lorentzian manifold is the subclass of a semi-Riemannian manifold. The index of the Lorentzian metric g is 1. A Lorentzian manifold $\mathbf{M}^n$ $(n \geq 4)$ admitting a globally

*Corresponding author.

timelike vector is physically known as spacetime. Several authors have explored spacetimes in various ways, such as Refs. 1–9 and also numerous others.

A mixed quasi-Einstein manifold [10] (briefly, $(\mathrm{MQE})_n$) is a non-flat semi-Riemannian manifold $(\mathbf{M}^n, g)$ $(n \geq 3)$ with its Ricci tensor (non-vanishing) $\mathcal{S}$ of type $(0, 2)$ meets the following requirements:

$$\mathcal{S}(Z_1, Z_2) = ag(Z_1, Z_2) + b[H(Z_1)F(Z_2) + H(Z_2)F(Z_1)], \tag{1}$$

a, b are scalars Â with b being nonzero and H, F are nonzero 1-forms in the sense that

$$g(Z_1, \mathcal{U}) = H(Z_1), \quad g(Z_1, \mathcal{V}) = F(Z_1), \quad g(\mathcal{U}, \mathcal{V}) = 0,$$

where unit vector fields are represented by $\mathcal{U}$ and $\mathcal{V}$. In this circumstance, H and F are referred to as associated 1-forms, while the generators of the manifold are referred to as $\mathcal{U}$ and $\mathcal{V}$. $(\mathrm{MQE})_n$ is transformed into an Einstein manifold for $b = 0$ and to a quasi-Einstein manifold if H equals F.

If the Ricci tensor fulfills the condition (1), a Lorentzian manifold is referred to as mixed quasi-Einstein spacetime. The existence of such a spacetime has been proved by Mallick *et al.* [11]. In this case, the vector field $\mathcal{V}$ related to the 1-form F is treated as a unit timelike vector field, that is, $F(\mathcal{V}) = g(\mathcal{V}, \mathcal{V}) = -1$.

The conformal curvature tensor of a Lorentzian manifold $(\mathbf{M}^n, g)$ $(n \geq 4)$ is stated as

$$\begin{aligned} \mathcal{C}(Z_1, Z_2)Z_3 = \mathcal{R}(Z_1, Z_2)Z_3 &- \frac{1}{(n-2)}[g(Z_2, Z_3)\mathcal{Q}Z_1 - g(Z_1, Z_3)\mathcal{Q}Z_2 \\ &+ \mathcal{S}(Z_2, Z_3)Z_1 - \mathcal{S}(Z_1, Z_3)Z_2] \\ &+ \frac{\rho}{(n-1)(n-2)}[g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2], \end{aligned} \tag{2}$$

$\mathcal{Q}$ is the Ricci operator satisfying the relation $\mathcal{S}(Z_1, Z_2) = g(\mathcal{Q}Z_1, Z_2)$ and ρ being the scalar curvature.

An n-dimensional $(n > 2)$ Lorentzian manifold is referred to be a generalized Robertson–Walker (briefly, GRW) spacetime if the metric adopts the following local structure:

$$ds^2 = -(d\varepsilon)^2 + q^2(\varepsilon)g^*_{u_1u_2}dx^{u_1}dx^{u_2}, \tag{3}$$

where q is a ε-dependent function and $g^*_{u_1u_2} = g^*_{u_1u_2}(x^{u_3})$ are only functions of x^{u_3} $(u_1, u_2, u_3 = 2, 3, \ldots, n)$. Thus a GRW spacetime can be represented as $-\mathcal{I} \times q^2\bar{\mathbf{M}}$, where $\bar{\mathbf{M}}$ is a Riemannian manifold of dimension $(n-1)$. If $\bar{\mathbf{M}}$ is of dim. 3 and possesses the constant sectional curvature, then the spacetime shrinks to a Robertson–Walker (briefly, RW) spacetime.

Lorentzian manifolds with the Ricci tensor

$$\mathcal{S}(Z_1, Z_2) = a_1g(Z_1, Z_2) + a_2F(Z_1)F(Z_2), \tag{4}$$

where $a_1 = \kappa(\frac{p-\sigma}{2-n})$ and $a_2 = \kappa(p+\sigma)$ are scalars and $\mathcal{V}$ is a unit timelike vector field corresponding to the 1-form F, are called perfect fluid spacetimes (briefly,

$\mathcal{PFS}$). If in particular a_1 and a_2 are constants, then the geometers called quasi-Einstein spacetimes.

The energy–momentum tensor (briefly, $\mathcal{EMT}$) $\mathfrak{T}$ represents the matter content of the spacetime, which is considered to be fluid. The $\mathcal{EMT}$ for a $\mathcal{PFS}$ resembles the shape [12]

$$\mathfrak{T}(Z_1, Z_2) = pg(Z_1, Z_2) + (p + \sigma)F(Z_1)F(Z_2), \tag{5}$$

where σ stands for energy density and p stands for isotropic pressure. The velocity vector field $\mathcal{V}$ is the metrically analogous unit timelike vector field to the nonzero 1-form F.

Einstein's field equations (briefly, $\mathcal{EFE}$) without cosmological constant are as follows:

$$\mathcal{S}(Z_1, Z_2) - \frac{\rho}{2}g(Z_1, Z_2) = \kappa\mathfrak{T}(Z_1, Z_2), \tag{6}$$

where $\mathcal{S}$ stands for the Ricci tensor and ρ stands for the scalar curvature, κ is the gravitational constant. According to $\mathcal{EFE}$, the geometry of spacetime is determined by matter, whereas the non-flat metric of spacetime governs matter motion. The above form (4) of the Ricci tensor is determined from Einstein's equation using (5).

Definition 1. If

$$\nabla_{Z_1}\zeta = \varrho Z_1, \tag{7}$$

ϱ is a constant ($\varrho \neq 0$), the vector field ζ is said to be concurrent.

If the vector fields $\mathcal{U}$ and $\mathcal{V}$ related to the 1-forms H and F are concurrent, then

$$(\nabla_{Z_1}H)(Z_2) = \gamma g(Z_1, Z_2) \tag{8}$$

and

$$(\nabla_{Z_1}F)(Z_2) = \delta g(Z_1, Z_2) \tag{9}$$

where γ and δ are nonzero constants.

We will look at mixed quasi-Einstein spacetime in this work, which is a unique kind of spacetime. The Lorentzian setting supports the results obtained for mixed quasi-Einstein manifolds. This paper is arranged as follows. After preliminaries in Sec. 3, we investigate each of the seven cases of Gray's decomposition of $(\mathrm{MQE})_n$. The analysis of $(\mathrm{MQE})_n$ with GRW spacetime is presented in Sec. 4.

2. Preliminaries

At any point on the manifold, consider an orthonormal frame field and contracting Z_1 and Z_2 in (1), yields

$$\rho = na. \tag{10}$$

Taking the covariant derivative of (1) gives us

$$\begin{aligned}(\nabla_{Z_3}\mathcal{S})(Z_1, Z_2) = da(Z_3)g(Z_1, Z_2) &+ db(Z_3)[H(Z_1)F(Z_2) + H(Z_2)F(Z_1)]\\ &+ b[(\nabla_{Z_3}H)(Z_1)F(Z_2) + H(Z_1)(\nabla_{Z_3}F)(Z_2)\\ &+ (\nabla_{Z_3}H)(Z_2)F(Z_1) + H(Z_2)(\nabla_{Z_3}F)(Z_1)].\end{aligned} \tag{11}$$

Setting $Z_2 = \mathcal{U}$ and $Z_2 = \mathcal{V}$ in (1) successively, we have the following:

$$\mathcal{S}(Z_1, \mathcal{U}) = aH(Z_1) + bF(Z_1) \tag{12}$$

and

$$\mathcal{S}(Z_1, \mathcal{V}) = aF(Z_1) - bH(Z_1). \tag{13}$$

3. Gray's Decompositions

According to Gray, [13] $\nabla\mathcal{S}$ can also be split into terms that are $\mathcal{O}(n)$-invariant (for additional information, read [14,15]. In $\mathcal{O}(n)$-invariant terms, $\nabla\mathcal{S}$ can be represented as follows [16]:

$$\begin{aligned}(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) = \hat{R}(Z_1, Z_2)Z_3 &+ \alpha_1(Z_1)g(Z_2, Z_3) + \alpha_2(Z_2)g(Z_1, Z_3)\\ &+ \alpha_2(Z_3)g(Z_1, Z_2),\end{aligned} \tag{14}$$

for all Z_1, Z_2, Z_3, where

$$\alpha_1(Z_1) = \frac{n}{(n-1)(n+2)}\nabla_{Z_1}\rho, \quad \alpha_2(Z_1) = \frac{(n-2)}{2(n-1)(n+2)}\nabla_{Z_1}\rho$$

with $\hat{R}(Z_1, Z_2)Z_3 = \hat{R}(Z_1, Z_3)Z_2$ is the tensor which has vanishing trace and represented as

$$\begin{aligned}\hat{R}(Z_1, Z_2)Z_3 = \frac{1}{3}[\hat{R}(Z_1, Z_2)Z_3 &+ \hat{R}(Z_2, Z_3)Z_1 + \hat{R}(Z_3, Z_1)Z_2]\\ &+ \frac{1}{3}[\hat{R}(Z_1, Z_2)Z_3 - \hat{R}(Z_2, Z_1)Z_3]\\ &+ \frac{1}{3}[\hat{R}(Z_1, Z_2)Z_3 - \hat{R}(Z_3, Z_1)Z_2].\end{aligned} \tag{15}$$

The decompositions (14) and (15) yield $\mathcal{O}(n)$-invariant subspace, which is characterized by linear invariant equations in $\nabla\mathcal{S}$.

The following equation can be used to determine the relationship between $\nabla\mathcal{S}$ and $\operatorname{div}\mathcal{C}$:

$$(\operatorname{div}\mathcal{C})(Z_1, Z_2)Z_3 = \left(\frac{n-3}{n-2}\right)[\hat{R}(Z_1, Z_2)Z_3 - \hat{R}(Z_2, Z_1)Z_3]. \tag{16}$$

The subspaces in Gray's decomposition are as described in the following:

(i) $\nabla\mathcal{S} = 0$ characterizes the subspaces that are **trivial**.

(ii) $\hat{R}(Z_1, Z_2)Z_3 = 0$ characterizes the **subspace** $\mathcal{J}$, i.e.

$$(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) = \alpha_1(Z_1)g(Z_2, Z_3) + \alpha_2(Z_2)g(Z_1, Z_3) + \alpha_2(Z_3)g(Z_1, Z_2), \tag{17}$$

where α_1, α_2 are 1-forms. Manifolds satisfying this requirement (17) are called *Sinyukov manifolds.* [17]

(iii) The **subspace** $\mathcal{J}'$ (referred to as the subspace $\mathcal{A}$) is defined as follows:

$$(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) + (\nabla_{Z_2}\mathcal{S})(Z_3, Z_1) + (\nabla_{Z_3}\mathcal{S})(Z_1, Z_2) = 0, \tag{18}$$

which yields that the scalar curvature ρ is constant. Also, the Ricci tensor is Killing [18] if Eq. (18) holds.

(iv) The Ricci tensor is of Codazzi type in **the subspaces** $\mathcal{B}$ **and** $\mathcal{B}'$ i.e.

$$(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) = (\nabla_{Z_2}\mathcal{S})(Z_1, Z_3). \tag{19}$$

(v) The cyclic condition for the Ricci tensor in **the subspace** $\mathcal{J} \oplus \mathcal{A}$ is

$$(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) + (\nabla_{Z_2}\mathcal{S})(Z_3, Z_1) + (\nabla_{Z_3}\mathcal{S})(Z_1, Z_2) = \frac{2\,g(Z_2, Z_3)}{(n+2)}\,d\rho(Z_1) + \frac{2\,g(Z_3, Z_1)}{(n+2)}\,d\rho(Z_2) + \frac{2\,g(Z_1, Z_2)}{(n+2)}\,d\rho(Z_3), \tag{20}$$

that is, $\mathcal{S}$ is conformal Killing. [19]

(vi) The Ricci tensor fulfills the following Codazzi condition in **the subspace** $\mathcal{J} \oplus \mathcal{B}$:

$$\nabla_{Z_1}\left[\mathcal{S}(Z_2, Z_3) - \frac{\rho}{2(n-1)}g(Z_2, Z_3)\right] = \nabla_{Z_2}\left[\mathcal{S}(Z_1, Z_3) - \frac{\rho}{2(n-1)}g(Z_1, Z_3)\right], \tag{21}$$

which gives $\operatorname{div}\mathcal{C} = 0$.

(vii) The scalar curvature is covariant constant in **the subspace** $\mathcal{A} \oplus \mathcal{B}$.

Consider each of these seven scenarios separately.

Case (i). $\nabla \mathcal{S} = 0$, therefore from (11) we find

$$\begin{aligned} & da(Z_3)g(Z_1, Z_2) + db(Z_3)[H(Z_1)F(Z_2) + H(Z_2)F(Z_1)] \\ & \quad + b[(\nabla_{Z_3} H)(Z_1)F(Z_2) + H(Z_1)(\nabla_{Z_3} F)(Z_2) \\ & \quad + (\nabla_{Z_3} H)(Z_2)F(Z_1) + H(Z_2)(\nabla_{Z_3} F)(Z_1)] = 0. \end{aligned} \tag{22}$$

Putting $Z_1 = \mathcal{U}$, $Z_2 = \mathcal{V}$ in (22), we infer that

$$db(Z_3) = 0. \tag{23}$$

Since $\nabla \mathcal{S} = 0$ implies $\rho =$ constant, therefore by Eq. (10) we conclude that

$$da(Z_3) = 0. \tag{24}$$

Using (23) and (24) in (22) infers

$$\begin{aligned} & (\nabla_{Z_3} H)(Z_1)F(Z_2) + H(Z_1)(\nabla_{Z_3} F)(Z_2) \\ & \quad + (\nabla_{Z_3} H)(Z_2)F(Z_1) + H(Z_2)(\nabla_{Z_3} F)(Z_1) = 0. \end{aligned} \tag{25}$$

Now setting $Z_1 = Z_2 = \mathcal{U}$ in (25) we deduce that

$$(\nabla_{Z_3} F)(\mathcal{U}) = 0. \tag{26}$$

Again setting $Z_2 = \mathcal{U}$ in (25) and using (26) we reach

$$g(\nabla_{Z_3} \mathcal{V}, Z_1) = 0, \tag{27}$$

for all Z_1. This implies that $\mathcal{V}$ is parallel.

A spacetime is called static (Ref. 20, p. 283) if it admits a unit timelike vector field $\mathcal{V}$ such that

$$(\nabla_{Z_1} F)(Z_2) = -F(Z_1)\dot{F}(Z_2) \quad \text{and} \quad \dot{F}(Z_1)\ddot{F}(Z_2) - \ddot{F}(Z_1)\dot{F}(Z_2) = 0,$$

where

$$\dot{F}(Z_1) = (\nabla_{\mathcal{V}} F)(Z_1), \quad \ddot{F}(Z_1) = (\nabla_{\mathcal{V}} \dot{F})(Z_1), \quad F(Z_1) = g(Z_1, \mathcal{V}),$$

for all Z_1. Since $\mathcal{V}$ is parallel, therefore $(\nabla_{Z_1} F)(Z_2) = 0$ and hence the above conditions hold for a static spacetime.

As a result, we are able to state the result.

Theorem 2. *If an $(MQE)_n$ spacetime is included in the trivial subspace, then the spacetime becomes a static spacetime.*

Case (ii). The Ricci tensor satisfies the relation $\hat{R}(Z_1, Z_2)Z_3 = 0$ in the subspace $\mathcal{J}$ and hence from the relation (16) we obtain $\operatorname{div} \mathcal{C} = 0$. So we arrive

$$(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) - (\nabla_{Z_3}\mathcal{S})(Z_1, Z_2) = \frac{1}{2(n-1)}[d\rho(Z_1)g(Z_2, Z_3) - d\rho(Z_3)g(Z_1, Z_2)]. \tag{28}$$

Equations (10), (11) and (28) together yield

$$\begin{aligned} & da(Z_1)g(Z_2, Z_3) + db(Z_1)[H(Z_2)F(Z_3) + H(Z_3)F(Z_2)] \\ & \quad + b[(\nabla_{Z_1}H)(Z_2)F(Z_3) + H(Z_2)(\nabla_{Z_1}F)(Z_3) + (\nabla_{Z_1}H)(Z_3)F(Z_2) \\ & \quad + H(Z_3)(\nabla_{Z_1}F)(Z_2)] - da(Z_3)g(Z_1, Z_2) \\ & \quad - db(Z_3)[H(Z_1)F(Z_2) + H(Z_2)F(Z_1)] - b[(\nabla_{Z_3}H)(Z_1)F(Z_2) \\ & \quad + H(Z_1)(\nabla_{Z_3}F)(Z_2) + (\nabla_{Z_3}H)(Z_2)F(Z_1) + H(Z_2)(\nabla_{Z_3}F)(Z_1)] \\ & = \frac{n}{2(n-1)}[da(Z_1)g(Z_2, Z_3) - da(Z_3)g(Z_1, Z_2)]. \end{aligned} \tag{29}$$

We now impose the condition that the scalars a and b are constants. After that, Eq. (29) turns into

$$\begin{aligned} & (\nabla_{Z_1}H)(Z_2)F(Z_3) + H(Z_2)[(\nabla_{Z_1}F)(Z_3) - (\nabla_{Z_3}F)(Z_1)] \\ & \quad + F(Z_2)[(\nabla_{Z_1}H)(Z_3) - (\nabla_{Z_3}H)(Z_1)] + H(Z_3)(\nabla_{Z_1}F)(Z_2) \\ & \quad - H(Z_1)(\nabla_{Z_3}F)(Z_2) - (\nabla_{Z_3}H)(Z_2)F(Z_1) = 0. \end{aligned} \tag{30}$$

Let us suppose that the vector fields $\mathcal{U}$ and $\mathcal{V}$ are concurrent. Then adopting (8) and (9) in (30) we obtain

$$[\gamma F(Z_3) + \delta H(Z_3)]g(Z_1, Z_2) - [\delta H(Z_1) + \gamma F(Z_1)]g(Z_3, Z_2) = 0. \tag{31}$$

Setting $Z_1 = \mathcal{U}$, $Z_3 = \mathcal{V}$ in (31), we obtain

$$H(Z_2) = -\frac{\delta}{\gamma}F(Z_2). \tag{32}$$

In view of (1) and (32) we get

$$\mathcal{S}(Z_1, Z_2) = ag(Z_1, Z_2) - \frac{2b\delta}{\gamma}F(Z_1)F(Z_2), \tag{33}$$

a quasi-Einstein spacetime. Thus, we might conclude the following.

Theorem 3. *If an* $(MQE)_n$ *spacetime belongs to the subspace* $\mathcal{J}$*, then the spacetime is a quasi-Einstein spacetime, provided the associated scalars are constants and the associated vector fields are concurrent.*

Case (iii). If $(\mathrm{MQE})_n$ is included in the subspace $\mathcal{A}$, then

$$(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) + (\nabla_{Z_2}\mathcal{S})(Z_3, Z_1) + (\nabla_{Z_3}\mathcal{S})(Z_1, Z_2) = 0.$$

The above equation reflects that ρ is constant.

In light of (10), (11) and (18) we have

$$\begin{aligned} & db(Z_1)[H(Z_2)F(Z_3) + H(Z_3)F(Z_2)] + db(Z_2)[H(Z_3)F(Z_1) + H(Z_1)F(Z_3)] \\ & \quad + db(Z_3)[H(Z_1)F(Z_2) + H(Z_2)F(Z_1)] + b[\{(\nabla_{Z_2}F)(Z_3) \\ & \quad + (\nabla_{Z_3}F)(Z_2)\}H(Z_1) + \{(\nabla_{Z_1}F)(Z_3) + (\nabla_{Z_3}F)(Z_1)\}H(Z_2) + \{(\nabla_{Z_1}F)(Z_2) \\ & \quad + (\nabla_{Z_2}F)(Z_1)\}H(Z_3) + \{(\nabla_{Z_2}H)(Z_3) + (\nabla_{Z_3}H)(Z_2)\}F(Z_1) + \{(\nabla_{Z_1}H)(Z_3) \\ & \quad + (\nabla_{Z_3}H)(Z_1)\}F(Z_2) + \{(\nabla_{Z_1}H)(Z_2) + (\nabla_{Z_2}H)(Z_1)\}F(Z_3)] = 0. \end{aligned} \tag{34}$$

Assume that the vector fields $\mathcal{U}$ and $\mathcal{V}$ are concurrent. From (8) and (9), we turn up

$$\begin{aligned} & db(Z_1)[H(Z_2)F(Z_3) + H(Z_3)F(Z_2)] + db(Z_2)[H(Z_3)F(Z_1) + H(Z_1)F(Z_3)] \\ & \quad + db(Z_3)[H(Z_1)F(Z_2) + H(Z_2)F(Z_1)] + 2b[\{\gamma F(Z_3) + \delta H(Z_3)\}g(Z_1, Z_2) \\ & \quad + \{\gamma F(Z_1) + \delta H(Z_1)\}g(Z_2, Z_3) + \{\gamma F(Z_2) + \delta H(Z_2)\}g(Z_1, Z_3)] = 0. \end{aligned} \tag{35}$$

Now, contraction of (35) gives

$$H(Z_1) = -\left[\frac{b\gamma(n+2) + (\mathcal{U}b)}{b\delta(n+2) + (\mathcal{V}b)}\right] F(Z_1). \tag{36}$$

From (1) and (36) it follows that

$$\mathcal{S}(Z_1, Z_2) = ag(Z_1, Z_2) - 2b\left[\frac{b\gamma(n+2) + (\mathcal{U}b)}{b\delta(n+2) + (\mathcal{V}b)}\right] F(Z_1)F(Z_2). \tag{37}$$

This leads the following results.

Theorem 4. *If an $(MQE)_n$ spacetime is included in the subspace $\mathcal{A}$, then the spacetime becomes a $\mathcal{PFS}$, provided the associated vector fields are concurrent.*

Also, Eqs. (18) and (6) reflect that the $\mathcal{EMT}$ is Killing, i.e.

$$(\nabla_{Z_1}\mathcal{T})(Z_2, Z_3) + (\nabla_{Z_2}\mathcal{T})(Z_3, Z_1) + (\nabla_{Z_3}\mathcal{T})(Z_1, Z_2) = 0,$$

for all $Z_1, Z_2, Z_3 \in \chi(\mathbf{M})$.

In Ref. 21, Sharma and Ghosh describe the following outcome.

Theorem A. *Let $(\boldsymbol{M}, g)$ be a $\mathcal{PFS}$ with Killing $\mathcal{EMT}$. Then*

(i) *the flow of spacetime is geodesic and the spacetime is expansion-free and shear-free, but not vorticity-free and*
(ii) *the spacetime admits constant energy density and pressure.*

Therefore, by Theorem A, we conclude the following.

Corollary 5. *Let $(MQE)_n$ spacetime belong to the subspace $\mathcal{A}$. Then*

(i) *the flow of spacetime is geodesic and the spacetime is expansion-free and shear-free, but not vorticity-free and*

(ii) *the spacetime admits constant energy density and pressure, provided the associated vector fields are concurrent.*

Case (iv). If $(\text{MQE})_n$ is included in $\mathcal{B}$ and $\mathcal{B}'$, then

$$(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) = (\nabla_{Z_2}\mathcal{S})(Z_1, Z_3),$$

from which it follows that ρ is constant, i.e. $d\rho(Z_1) = 0$, for all Z_1.

Hence, in view of (10) and (11), we acquire

$$\begin{aligned}(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) = {} & db(Z_1)[H(Z_2)F(Z_3) + H(Z_3)F(Z_2)] \\ & + b[(\nabla_{Z_1}H)(Z_2)F(Z_3) + H(Z_2)(\nabla_{Z_1}F)(Z_3) \\ & + (\nabla_{Z_1}H)(Z_3)F(Z_2) + H(Z_3)(\nabla_{Z_1}F)(Z_2)]. \end{aligned} \tag{38}$$

We now impose the condition that the associated vector fields $\mathcal{U}$ and $\mathcal{V}$ are concurrent. By virtue of (8), (9) and the foregoing equation, we provide

$$\begin{aligned}(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) = {} & db(Z_1)[H(Z_2)F(Z_3) + H(Z_3)F(Z_2)] \\ & + b[\{\gamma F(Z_3) + \delta H(Z_3)\}g(Z_1, Z_2) \\ & + \{\gamma F(Z_2) + \delta H(Z_2)\}g(Z_1, Z_3)]. \end{aligned} \tag{39}$$

Now, contraction of (39) gives

$$d\rho(Z_1) = 2b[\gamma F(Z_1) + \delta H(Z_1)]. \tag{40}$$

Since $d\rho(Z_1) = 0$, then from (40) we arrive at

$$H(Z_1) = -\frac{\gamma}{\delta}F(Z_1). \tag{41}$$

Adopting (41) in (1), we can derive

$$\mathcal{S}(Z_1, Z_2) = ag(Z_1, Z_2) - \frac{2b\gamma}{\delta}F(Z_1)F(Z_2). \tag{42}$$

Thus we write the following.

Theorem 6. *If an $(MQE)_n$ spacetime is included in the class $\mathcal{B}$ and $\mathcal{B}'$, then the spacetime becomes a $\mathcal{PFS}$, provided the associated vector fields are concurrent.*

A four-dimensional Lorentzian manifold is named a Yang pure space [22] whose metric satisfies Yang's equation:

$$(\nabla_{Z_1}\mathcal{S})(Z_2, Z_3) = (\nabla_{Z_2}\mathcal{S})(Z_1, Z_3).$$

They are identical to the condition $\operatorname{div}\mathcal{C} = 0$ in any dimension.

Mantica and Molinari [23] established the following result for $n \geq 4$.

Proposition 7. *A perfect fluid Yang pure space of dim. $n \geq 4$ with $p + \sigma \neq 0$ is a GRW spacetime.*

In contrast to Eq. (42) we notice $\kappa(\frac{p-\sigma}{2-n}) = a$ and $\kappa(p+\sigma) = -\frac{2b\gamma}{\delta}$, i.e. $p+\sigma \neq 0$.

In view of this observation, we can conclude the following.

Corollary 8. *An (MQE)$_n$ spacetime is belonging to the class $\mathcal{B}$ and $\mathcal{B}'$ is a GRW spacetime, provided the associated vector fields are concurrent.*

Also, Eqs. (19) and (6) reflect that $\mathfrak{T}$ satisfying

$$(\nabla_{Z_1}\mathfrak{T})(Z_2, Z_3) = (\nabla_{Z_2}\mathfrak{T})(Z_1, Z_3),$$

for all $Z_1, Z_2, Z_3 \in \chi(\mathbf{M})$.

However, it has been established [24] *that if the $\mathcal{EMT}$ is of Codazzi type in a $\mathcal{PFS}$, then the fluid is of vanishing shear and vorticity, and its velocity vector field becomes hypersurface orthogonal.*

Barnes [25] *observed that the probable local cosmological structures of the $\mathcal{PFS}$ are of Petrov types I, D, or O if the $\mathcal{PFS}$ is of vanishing shear and vorticity, the velocity vector field $\mathcal{V}$ is hypersurface orthogonal and the constant energy density over a hypersurface orthogonal to $\mathcal{V}$.*

As a result of the foregoing facts, we arrive at the following.

Corollary 9. *If an (MQE)$_n$ spacetime belongs to the class $\mathcal{B}$ and $\mathcal{B}'$, then the probable local cosmological structures of the spacetime are of Petrov types I, D, or O, provided the associated vector fields are concurrent.*

Case (v). In this subspace, $\mathcal{S}$ is conformal Killing (20). Mantica *et al.* [16] show that the subspaces $\mathcal{J} \oplus \mathcal{A}$ and $\mathcal{J}$ are equivalent. In this circumstance, we reach $\operatorname{div} \mathcal{C} = 0$. Consequently, the result is the same as in Theorem 3.

Case (vi). Let (MQE)$_n$ belong to $\mathcal{J} \oplus \mathcal{B}$. In this case, we obtain $\operatorname{div} \mathcal{C} = 0$. So in this case, also we get the same outcome of Theorem 3.

Case (vii). The scalar curvature is covariant constant in the subspace $\mathcal{A} \oplus \mathcal{B}$ and hence in this case, we have the same outcome of Theorem 6.

4. Mixed Quasi-Einstein GRW Spacetimes

We will assume that (MQE)$_n$ is a GRW spacetime during this whole section. In Ref. [23], the authors established that a Lorentzian manifold of dim. $n \geq 3$ admits a unit timelike torse forming vector field if and only if it is a GRW spacetime:

$$(\nabla_{Z_1}F)(Z_2) = \psi[g(Z_1, Z_2) + F(Z_1)F(Z_2)] \tag{43}$$

and

$$\mathcal{S}(Z_1, \mathcal{V}) = \mu g(Z_1, \mathcal{V}), \tag{44}$$

for some smooth functions $\psi\,(\neq 0)$ and μ on $\mathbf{M}$.

Now,

$$(\nabla_{Z_1}\mathcal{S})(Z_2,\mathcal{V}) = Z_1\mathcal{S}(Z_2,\mathcal{V}) - \mathcal{S}(\nabla_{Z_1}Z_2,\mathcal{V}) - \mathcal{S}(Z_2,\nabla_{Z_1}\mathcal{V}). \tag{45}$$

Using (43) and (44) in (45), we arrive

$$(\nabla_{Z_1}\mathcal{S})(Z_2,\mathcal{V}) = (Z_1\mu)F(Z_2) + \mu\psi g(Z_1,Z_2) - \psi\mathcal{S}(Z_1,Z_2), \tag{46}$$

where $(Z_1\mu) = g(Z_1, \operatorname{grad}\mu)$.

Differentiating (13) covariantly and applying (43) we reach

$$\begin{aligned}(\nabla_{Z_1}\mathcal{S})(Z_2,\mathcal{V}) &= da(Z_1)F(Z_2) + a\psi[g(Z_1,Z_2) + F(Z_1)F(Z_2)]\\ &\quad - db(Z_1)H(Z_2) - b(\nabla_{Z_1}H)(Z_2).\end{aligned} \tag{47}$$

Combining Eqs. (46) and (47), we reveal

$$\begin{aligned}&(Z_1\mu)F(Z_2) + \mu\psi g(Z_1,Z_2) - \psi\mathcal{S}(Z_1,Z_2)\\ &\quad = da(Z_1)F(Z_2) + a\psi[g(Z_1,Z_2) + F(Z_1)F(Z_2)]\\ &\quad - db(Z_1)H(Z_2) - b(\nabla_{Z_1}H)(Z_2).\end{aligned} \tag{48}$$

Setting $Z_2 = \mathcal{V}$ in (48) and using $F(\mathcal{V}) = -1$, we turn up

$$(Z_1\mu) = da(Z_1) + b(\nabla_{Z_1}H)(\mathcal{V}). \tag{49}$$

Equations (48) and (49) imply

$$\begin{aligned}\psi\mathcal{S}(Z_1,Z_2) &= \mu\psi g(Z_1,Z_2) + db(Z_1)H(Z_2) + b(\nabla_{Z_1}H)(\mathcal{V})F(Z_2)\\ &\quad + b(\nabla_{Z_1}H)(Z_2) - a\psi[g(Z_1,Z_2) + F(Z_1)F(Z_2)].\end{aligned} \tag{50}$$

Now from (13) and (44), we can derive

$$H(Z_2) = \left(\frac{a-\mu}{b}\right)F(Z_2). \tag{51}$$

Setting $Z_2 = \mathcal{U}$ in (50) and then applying (12), we reach

$$db(Z_1) = \psi[(2a-\mu)H(Z_1) + bF(Z_1)]. \tag{52}$$

Equations (51) and (52) together yield

$$db(Z_1) = \psi\left[\frac{(2a-\mu)(a-\mu)}{b} + b\right]F(Z_1). \tag{53}$$

In light of (50), (51) and (53) we infer that

$$\begin{aligned}\psi\mathcal{S}(Z_1,Z_2) &= \mu\psi g(Z_1,Z_2) + \psi\left[\frac{(2a-\mu)(a-\mu)}{b} + b\right]\\ &\quad \times \left(\frac{a-\mu}{b}\right)F(Z_1)F(Z_2) + b(\nabla_{Z_1}H)(\mathcal{V})F(Z_2)\\ &\quad + b(\nabla_{Z_1}H)(Z_2) - a\psi[g(Z_1,Z_2) + F(Z_1)F(Z_2)].\end{aligned} \tag{54}$$

Let us impose the condition that the associated vector field $\mathcal{U}$ is concurrent. Hence, in view of (8) and (54) we acquire

$$\mathcal{S}(Z_1, Z_2) = \left(\mu - a + \frac{b\gamma}{\psi}\right) g(Z_1, Z_2) + \left[\left(\frac{a-\mu}{b}\right)^2 (2a - \mu) + \frac{b\gamma}{\psi} - \mu\right] F(Z_1)F(Z_2).$$

Thus we arrive to the following result.

Theorem 10. *An $(MQE)_n$ GRW spacetime is a $\mathcal{PFS}$, provided the associated vector field $\mathcal{U}$ is concurrent.*

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Some physical properties of generalized quasi-Einstein spacetimes under Gray's decomposition

Dipankar Hazra*,‡, Uday Chand De†,¶ and Avijit Sarkar*,§

*Department of Mathematics, University of Kalyani
Kalyani 741235, West Bengal, India

†Department of Pure Mathematics, University of Calcutta
35 Ballygunge Circular Road, Kolkata 700019
West Bengal, India
‡dipankarsk524@gmail.com
§avjaj@yahoo.co.in
¶uc_de@yahoo.com

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In this study, we analyze generalized quasi-Einstein spacetimes endowed with Gray's decomposition, as well as generalized Robertson–Walker spacetimes. It is shown that the Ricci tensor of a generalized quasi-Einstein spacetime assumes the form of a perfect fluid in all Gray's subspaces under certain restrictions. Finally, it is established that a generalized quasi-Einstein generalized Robertson–Walker spacetime is a perfect fluid spacetime.

Keywords: Generalized quasi-Einstein spacetimes; Gray's decomposition; perfect fluid spacetime; generalized Robertson–Walker spacetime.

Mathematics Subject Classification 2020: 53C25, 53C50, 53Z05, 83C05

1. Introduction

A Lorentzian manifold is the subclass of a semi-Riemannian manifold. The index of the Lorentzian metric g is one. A spacetime is a Lorentzian manifold M^n $(n \geq 4)$ which admits a globally timelike vector field. Different types of spacetimes have been studied in various ways, such as $[4, 7, 9, 13, 15, 21, 25, 33, 34]$ and many others.

Lorentzian manifolds with the Ricci tensor

$$\mathcal{S} = \beta_1 g + \beta_2 A \otimes A, \tag{1.1}$$

where β_1, β_2 are scalars and $\boldsymbol{\rho}$ is a unit timelike vector field corresponding to the non-vanishing one-form A, that is, $A(\boldsymbol{\rho}) = g(\boldsymbol{\rho}, \boldsymbol{\rho}) = -1$, are called perfect fluid

‡Corresponding author.

spacetimes (briefly, $\mathcal{PFS}$). Throughout this paper, we consider $\boldsymbol{\rho}$ is a unit timelike vector field, which is called velocity vector field or flow vector field.

The energy–momentum tensor (briefly, $\mathcal{EMT}$) $\mathcal{T}$ represents the matter content of the spacetimes in general relativity theory. In general relativity theory, the fluid is termed perfect fluid, since it does not have the heat conduction terms [22]. The $\mathcal{EMT}$ for a $\mathcal{PFS}$ resembles the shape [26]

$$\mathcal{T} = \boldsymbol{p}g + (\boldsymbol{p} + \boldsymbol{\sigma})A \otimes A, \tag{1.2}$$

where $\boldsymbol{\sigma}$ stands for energy density and $\boldsymbol{p}$ stands for isotropic pressure. The velocity vector field $\boldsymbol{\rho}$ is metrically analogous unit timelike vector field to the non-vanishing one-form A.

Einstein's field equations (briefly, $\mathcal{EFE}$) are as follows:

$$\mathcal{S} - \frac{\boldsymbol{r}}{2} g = \boldsymbol{\kappa}\mathcal{T}, \tag{1.3}$$

$\mathcal{S}$ stands for the Ricci tensor and $\boldsymbol{r}$ stands for the scalar curvature, $\boldsymbol{\kappa}$ is the gravitational constant. According to $\mathcal{EFE}$, the geometry of spacetime is determined by matter, whereas the non-flat metric of spacetime governs the motion of matter. The above form (1.1) of the Ricci tensor is determined from Einstein's field equations using (1.2).

Using (1.1) and (1.2) from (1.3) we infer that

$$\beta_1 = \boldsymbol{\kappa}\left(\frac{\boldsymbol{p} - \boldsymbol{\sigma}}{2 - n}\right) \quad \text{and} \quad \beta_2 = \boldsymbol{\kappa}(\boldsymbol{p} + \boldsymbol{\sigma}). \tag{1.4}$$

An n-dimensional ($n > 2$) Lorentzian manifold is said to be a generalized Robertson–Walker (briefly, GRW) [1] spacetime if the metric adopts the following local structure:

$$ds^2 = -(d\zeta)^2 + q^2(\zeta) g^*_{u_1 u_2} dx^{u_1} dx^{u_2}, \tag{1.5}$$

where q is a ζ-dependent function and $g^*_{u_1u_2} = g^*_{u_1u_2}(x^{u_3})$ are only functions of x^{u_3} ($u_1, u_2, u_3 = 2, 3, \ldots, n$). Thus, a GRW spacetime can be represented as $-\mathcal{I} \times q^2 \bar{M}$, where $\bar{M}$ is a Riemannian manifold of dimension $(n-1)$. If the dimension of $\bar{M}$ is three and of constant sectional curvature, then the spacetime becomes a Robertson–Walker (briefly, RW) spacetime.

In 2001, Chaki [6] introduced the notion of generalized quasi-Einstein manifold. In [11], De and Ghosh introduced the same notion in another way. According to them, a generalized quasi-Einstein manifold (briefly, $(\mathbf{GQE})_\mathbf{n}$) is a non-flat semi-Riemannian manifold (M^n, g) ($n \geq 3$) with its Ricci tensor (non-vanishing) $\mathcal{S}$ of type $(0, 2)$ meeting the following requirements:

$$\mathcal{S}(\mathcal{U}_1, \mathcal{V}_1) = \gamma_1 g(\mathcal{U}_1, \mathcal{V}_1) + \gamma_2 A(\mathcal{U}_1)A(\mathcal{V}_1) + \gamma_3 B(\mathcal{U}_1)B(\mathcal{V}_1), \tag{1.6}$$

in which γ_1, γ_2, γ_3 are nontrivial functions and A, B are non-vanishing one-forms such that

$$A(\mathcal{U}_1) = g(\mathcal{U}_1, \boldsymbol{\rho}), \quad B(\mathcal{U}_1) = g(\mathcal{U}_1, \boldsymbol{\mu}) \quad \text{for all } \mathcal{U}_1 \text{ and } g(\boldsymbol{\rho}, \boldsymbol{\mu}) = 0,$$

where A, B are referred to as associated one-forms, while the generators of the manifold are referred to as $\boldsymbol{\rho}$, $\boldsymbol{\mu}$ and γ_1, γ_2, γ_3 are called associated scalars. In the same paper, the authors proved the existence of a $(\mathbf{GQE})_\mathbf{n}$ by showing that a two-quasi-umbilical hypersurface of an Euclidean space is a $(\mathbf{GQE})_\mathbf{n}$. Several authors continued to study generalized quasi-Einstein manifolds, especially [19, 20, 27, 32]. There is another notion of generalized quasi-Einstein manifolds in the literature. A complete Riemannian manifold is named a generalized quasi-Einstein manifold [5] if the Ricci tensor obeys

$$\mathcal{S}(\mathcal{U}_1, \mathcal{V}_1) + \nabla_{\mathcal{U}_1} \nabla_{\mathcal{V}_1} \phi - \nu(\mathcal{U}_1 \phi)(\mathcal{V}_1 \phi) - \theta g(\mathcal{U}_1, \mathcal{V}_1) = 0,$$

where ϕ being a smooth function and $\nu, \theta \in \mathbb{R}$.

If the Ricci tensor fulfills the condition (1.6), a Lorentzian manifold is referred to as generalized quasi-Einstein spacetime. The existence of such a spacetime has been proved by De and Mallick [12]. In this case, the vector field $\boldsymbol{\rho}$ related to the one-form A is treated as a unit timelike vector field, that is, $A(\boldsymbol{\rho}) = g(\boldsymbol{\rho}, \boldsymbol{\rho}) = -1$. We will look at generalized quasi-Einstein spacetime in this work, which is a unique kind of spacetime. The Lorentzian setting supports the results obtained for $(\mathbf{GQE})_\mathbf{n}$. $(\mathbf{GQE})_\mathbf{n}$ spacetime is transformed into a $\mathcal{PFS}$ for $\gamma_3 = 0$.

The conformal curvature tensor of a Lorentzian manifold $(M^n, g)(n \geq 4)$ is stated as

$$\begin{aligned} \mathcal{C}(\mathcal{U}_1, \mathcal{V}_1)\mathcal{W}_1 = \mathcal{R}(\mathcal{U}_1, \mathcal{V}_1)\mathcal{W}_1 &- \frac{1}{n-2}[g(\mathcal{V}_1, \mathcal{W}_1)\mathcal{Q}\mathcal{U}_1 - g(\mathcal{U}_1, \mathcal{W}_1)\mathcal{Q}\mathcal{V}_1 \\ &+ \mathcal{S}(\mathcal{V}_1, \mathcal{W}_1)\mathcal{U}_1 - \mathcal{S}(\mathcal{U}_1, \mathcal{W}_1)\mathcal{V}_1] \\ &+ \frac{\boldsymbol{r}}{(n-1)(n-2)}[g(\mathcal{V}_1, \mathcal{W}_1)\mathcal{U}_1 - g(\mathcal{U}_1, \mathcal{W}_1)\mathcal{V}_1], \end{aligned} \tag{1.7}$$

$\mathcal{Q}$ is the Ricci operator satisfying the relation $\mathcal{S}(\mathcal{U}_1, \mathcal{V}_1) = g(\mathcal{Q}\mathcal{U}_1, \mathcal{V}_1)$ and $\boldsymbol{r}$ being the scalar curvature.

From the above definition, it can be seen that

$$\begin{aligned} (\operatorname{div} \mathcal{C})(\mathcal{U}_1, \mathcal{V}_1)\mathcal{W}_1 = \left(\frac{n-3}{n-2}\right) \Bigg[&\{(\nabla_{\mathcal{U}_1} \mathcal{S})(\mathcal{V}_1, \mathcal{W}_1) - (\nabla_{\mathcal{V}_1} \mathcal{S})(\mathcal{U}_1, \mathcal{W}_1)\} \\ &- \frac{1}{2(n-1)}\{g(\mathcal{V}_1, \mathcal{W}_1) dr(\mathcal{U}_1) - g(\mathcal{U}_1, \mathcal{W}_1) dr(\mathcal{V}_1)\}\Bigg]. \end{aligned} \tag{1.8}$$

Definition 1.1 ([28]). A vector field $\boldsymbol{\rho}$ corresponding to the associated one-form A is said to be recurrent if

$$(\nabla_{\mathcal{U}_1} A)(\mathcal{V}_1) = \omega(\mathcal{U}_1) A(\mathcal{V}_1), \tag{1.9}$$

where ω is a non-vanishing one-form.

In a series of recent studies, many manifolds were widely studied in Gray's subspaces. In [24], the authors studied GRW spacetimes in Gray's subspaces. They proved that such spacetimes in all Gray's subspaces but one are perfect fluid or

Einstein. In [18], the authors studied $\mathcal{PFS}$ in Gray's decomposition. Weakly Ricci-symmetric spacetimes under Gray's decomposition were considered in [10]. Motivated by these studies and many others, this paper is mainly organized to investigate $(\mathbf{GQE})_\mathbf{n}$ spacetimes under Gray's decomposition subspaces, and to study $(\mathbf{GQE})_\mathbf{n}$ GRW spacetimes. The following is the outline of the paper.

After preliminaries in Sec. 3, we investigate each of the seven cases of Gray's decomposition of $(\mathbf{GQE})_\mathbf{n}$. The analysis of $(\mathbf{GQE})_\mathbf{n}$ with GRW spacetime is presented in Sec. 4.

2. Preliminaries

At any point on the manifold, consider an orthonormal frame field and contracting $\mathcal{U}_1$ and $\mathcal{V}_1$ in (1.6) yields

$$\boldsymbol{r} = n\gamma_1 - \gamma_2 + \gamma_3, \tag{2.1}$$

which implies

$$d\boldsymbol{r}(\mathcal{W}_1) = nd\gamma_1(\mathcal{W}_1) - d\gamma_2(\mathcal{W}_1) + d\gamma_3(\mathcal{W}_1). \tag{2.2}$$

Taking the covariant derivative of (1.6) gives us

$$\begin{aligned}(\nabla_{\mathcal{W}_1}\mathcal{S})(\mathcal{U}_1, \mathcal{V}_1) = {} & d\gamma_1(\mathcal{W}_1)g(\mathcal{U}_1, \mathcal{V}_1) + d\gamma_2(\mathcal{W}_1)A(\mathcal{U}_1)A(\mathcal{V}_1) \\ & + \gamma_2[(\nabla_{\mathcal{W}_1}A)(\mathcal{U}_1)A(\mathcal{V}_1) + A(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1)] \\ & + \gamma_3[(\nabla_{\mathcal{W}_1}B)(\mathcal{U}_1)B(\mathcal{V}_1) + B(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}B)(\mathcal{V}_1)] \\ & + d\gamma_3(\mathcal{W}_1)B(\mathcal{U}_1)B(\mathcal{V}_1). \end{aligned} \tag{2.3}$$

Setting $\mathcal{U}_1 = \boldsymbol{\rho}$ in (1.6), we have

$$\mathcal{S}(\mathcal{V}_1, \boldsymbol{\rho}) = (\gamma_1 - \gamma_2)A(\mathcal{V}_1). \tag{2.4}$$

Since $g(\boldsymbol{\rho}, \boldsymbol{\mu}) = 0$, $(\nabla_{\mathcal{W}_1}g)(\boldsymbol{\rho}, \boldsymbol{\mu}) = 0$ implies

$$(\nabla_{\mathcal{W}_1}A)(\boldsymbol{\mu}) + (\nabla_{\mathcal{W}_1}B)(\boldsymbol{\rho}) = 0. \tag{2.5}$$

3. $(\mathbf{GQE})_\mathbf{n}$ Spacetimes in Gray's Decomposition Subspaces

In [16], Gray showed that the gradient of the Ricci tensor $\nabla\mathcal{S}$ can be decomposed into $\mathcal{O}(n)$-invariant terms (for additional information, read [3, Chap. 16]). The decomposition of the gradient of the Ricci tensor gives $\mathcal{O}(n)$-invariant subspaces. These generated subspaces are called Gray's decomposition subspaces. Each subspace has a characteristic equation that is linear in $\nabla\mathcal{S}$. In Gray's trivial subspace, the characteristic equation is $\nabla\mathcal{S} = 0$. Manifolds lie in trivial subspace are called Ricci symmetric manifolds. The characteristic equation of Gray's subspace $\mathcal{J}$ is $(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1, \mathcal{W}_1) = \delta_1(\mathcal{U}_1)g(\mathcal{V}_1, \mathcal{W}_1) + \delta_2(\mathcal{V}_1)g(\mathcal{U}_1, \mathcal{W}_1) + \delta_2(\mathcal{W}_1)g(\mathcal{U}_1, \mathcal{V}_1)$, where $\delta_1(\mathcal{U}_1) = \frac{n}{(n-1)(n+2)}\nabla_{\mathcal{U}_1}\boldsymbol{r}$, $\delta_2(\mathcal{U}_1) = \frac{n-2}{2(n-1)(n+2)}\nabla_{\mathcal{U}_1}\boldsymbol{r}$. A manifold belonging to subspace $\mathcal{J}$ is called *Sinyukov manifolds* [31]. In Gray's subspace $\mathcal{A}$, the Ricci tensor is

Killing, that is, $(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1)+(\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1)+(\nabla_{\mathcal{W}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{V}_1)=0$ whereas in Gray's subspace $\mathcal{B}$ the Ricci tensor is of Codazzi type, that is, $(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1)=(\nabla_{\mathcal{W}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{V}_1)$. In Gray's subspace $\mathcal{J}\oplus\mathcal{A}$, the tensor $(\mathcal{S}-\frac{2\boldsymbol{r}g}{n+2})$ is Killing, while in Gray's subspace $\mathcal{J}\oplus\mathcal{B}$ the tensor $[\mathcal{S}-\frac{\boldsymbol{r}g}{2(n-1)}]$ is of Codazzi type. Gray's subspace $\mathcal{A}\oplus\mathcal{B}$ is distinguished by constant scalar curvature. In this part, we will investigate $(\mathbf{GQE})_\mathbf{n}$ spacetimes in all Gray's subspaces.

Case 1. The trivial subspace.

Gray's trivial class contains spacetimes whose Ricci tensor is symmetric, that is, $\nabla\mathcal{S}=0$. Thus, Eq. (2.3) becomes

$$\begin{aligned}
&d\gamma_1(\mathcal{W}_1)g(\mathcal{U}_1,\mathcal{V}_1)+d\gamma_2(\mathcal{W}_1)A(\mathcal{U}_1)A(\mathcal{V}_1)+d\gamma_3(\mathcal{W}_1)B(\mathcal{U}_1)B(\mathcal{V}_1)\\
&\quad+\gamma_2[(\nabla_{\mathcal{W}_1}A)(\mathcal{U}_1)A(\mathcal{V}_1)+A(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1)]\\
&\quad+\gamma_3[(\nabla_{\mathcal{W}_1}B)(\mathcal{U}_1)B(\mathcal{V}_1)+B(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}B)(\mathcal{V}_1)]=0. \qquad (3.1)
\end{aligned}$$

Putting $\mathcal{U}_1=\mathcal{V}_1=\boldsymbol{\rho}$ in (3.1), we infer that

$$-d\gamma_1(\mathcal{W}_1)+d\gamma_2(\mathcal{W}_1)=0. \qquad (3.2)$$

Again, putting $\mathcal{U}_1=\mathcal{V}_1=\boldsymbol{\mu}$ in (3.1), we find

$$d\gamma_1(\mathcal{W}_1)+d\gamma_3(\mathcal{W}_1)=0. \qquad (3.3)$$

Since $\nabla\mathcal{S}=0$, $\boldsymbol{r}$ is constant. Therefore, by Eqs. (2.2), (3.2) and (3.3) we conclude that

$$d\gamma_1(\mathcal{W}_1)=d\gamma_2(\mathcal{W}_1)=d\gamma_3(\mathcal{W}_1)=0. \qquad (3.4)$$

Using (3.4) in (3.1) infers

$$\begin{aligned}
&\gamma_2[(\nabla_{\mathcal{W}_1}A)(\mathcal{U}_1)A(\mathcal{V}_1)+A(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1)]\\
&\quad+\gamma_3[(\nabla_{\mathcal{W}_1}B)(\mathcal{U}_1)B(\mathcal{V}_1)+B(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}B)(\mathcal{V}_1)]=0. \qquad (3.5)
\end{aligned}$$

Setting $\mathcal{U}_1=\boldsymbol{\rho}$, $\mathcal{V}_1=\boldsymbol{\mu}$ in (3.5), we deduce that

$$-\gamma_2(\nabla_{\mathcal{W}_1}A)(\boldsymbol{\mu})+\gamma_3(\nabla_{\mathcal{W}_1}B)(\boldsymbol{\rho})=0. \qquad (3.6)$$

Utilizing (2.5) in (3.6) we reach

$$(\gamma_2+\gamma_3)(\nabla_{\mathcal{W}_1}B)(\boldsymbol{\rho})=0, \qquad (3.7)$$

which implies either $\gamma_2+\gamma_3=0$ or, $\gamma_2+\gamma_3\neq 0$.

If $\gamma_2+\gamma_3\neq 0$, then from (3.7) we get

$$(\nabla_{\mathcal{W}_1}B)(\boldsymbol{\rho})=0. \qquad (3.8)$$

Replacing $\mathcal{U}_1$ by $\boldsymbol{\rho}$ in (3.5) and using (3.8), we obtain

$$\gamma_2(\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1)=0. \qquad (3.9)$$

Since in a $(\mathbf{GQE})_\mathbf{n}$, $\gamma_2\neq 0$ and hence $(\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1)=0$, that is, $g(\mathcal{V}_1,\nabla_{\mathcal{W}_1}\boldsymbol{\rho})=0$, which demonstrates that $\boldsymbol{\rho}$ is parallel.

If $\gamma_2 + \gamma_3 = 0$, then (3.5) gives

$$\gamma_3[(\nabla_{\mathcal{W}_1} A)(\mathcal{U}_1)A(\mathcal{V}_1) + A(\mathcal{U}_1)(\nabla_{\mathcal{W}_1} A)(\mathcal{V}_1) \\ - (\nabla_{\mathcal{W}_1} B)(\mathcal{U}_1)B(\mathcal{V}_1) - B(\mathcal{U}_1)(\nabla_{\mathcal{W}_1} B)(\mathcal{V}_1)] = 0. \tag{3.10}$$

Since in a $(\mathbf{GQE})_{\mathbf{n}}$, $\gamma_3 \neq 0$ and hence the foregoing equation reduces to

$$[(\nabla_{\mathcal{W}_1} A)(\mathcal{U}_1)A(\mathcal{V}_1) + A(\mathcal{U}_1)(\nabla_{\mathcal{W}_1} A)(\mathcal{V}_1) \\ - (\nabla_{\mathcal{W}_1} B)(\mathcal{U}_1)B(\mathcal{V}_1) - B(\mathcal{U}_1)(\nabla_{\mathcal{W}_1} B)(\mathcal{V}_1)] = 0. \tag{3.11}$$

Contracting the previous Eq. (3.11) entails that

$$\mathrm{div}\boldsymbol{\rho} A(\mathcal{V}_1) + (\nabla_{\boldsymbol{\rho}} A)(\mathcal{V}_1) - \mathrm{div}\boldsymbol{\mu} B(\mathcal{V}_1) - (\nabla_{\boldsymbol{\mu}} B)(\mathcal{V}_1) = 0. \tag{3.12}$$

Setting $\mathcal{U}_1 = \mathcal{W}_1 = \boldsymbol{\rho}$ in (3.11), we infer

$$(\nabla_{\boldsymbol{\rho}} A)(\mathcal{V}_1) = -(\nabla_{\boldsymbol{\rho}} B)(\boldsymbol{\rho})B(\mathcal{V}_1). \tag{3.13}$$

Replacing $\mathcal{U}_1$ and $\mathcal{W}_1$ by $\boldsymbol{\mu}$ in (3.11) implies

$$(\nabla_{\boldsymbol{\mu}} B)(\mathcal{V}_1) = (\nabla_{\boldsymbol{\mu}} A)(\boldsymbol{\mu})A(\mathcal{V}_1). \tag{3.14}$$

In virtue of (3.12)–(3.14), we acquire that

$$B(\mathcal{V}_1) = \alpha_1 A(\mathcal{V}_1), \tag{3.15}$$

where $\alpha_1 = \frac{\mathrm{div}\boldsymbol{\rho} - (\nabla_{\boldsymbol{\mu}} A)(\boldsymbol{\mu})}{\mathrm{div}\boldsymbol{\mu} + (\nabla_{\boldsymbol{\rho}} B)(\boldsymbol{\rho})}$. Equations (1.6) and (3.15) give us

$$\mathcal{S}(\mathcal{U}_1, \mathcal{V}_1) = \gamma_1 g(\mathcal{U}_1, \mathcal{V}_1) + (\gamma_2 + \gamma_3 \alpha_1^2)A(\mathcal{U}_1)A(\mathcal{V}_1). \tag{3.16}$$

This represents a $\mathcal{PFS}$.

Also, from Eqs. (2.5) and (3.6) we obtain

$$(\gamma_2 + \gamma_3)(\nabla_{\mathcal{W}_1} A)(\boldsymbol{\mu}) = 0, \tag{3.17}$$

which implies either $\gamma_2 + \gamma_3 = 0$ or, $\gamma_2 + \gamma_3 \neq 0$.

If $\gamma_2 + \gamma_3 = 0$, then from (3.16) we conclude that the spacetime becomes a $\mathcal{PFS}$.

If $\gamma_2 + \gamma_3 \neq 0$, then from (3.17) we get

$$(\nabla_{\mathcal{W}_1} A)(\boldsymbol{\mu}) = 0. \tag{3.18}$$

Putting $\mathcal{U}_1 = \boldsymbol{\mu}$ in (3.5) and using (3.18), we have

$$\gamma_3 (\nabla_{\mathcal{W}_1} B)(\mathcal{V}_1) = 0. \tag{3.19}$$

Since in a $(\mathbf{GQE})_{\mathbf{n}}$, $\gamma_3 \neq 0$ and hence $(\nabla_{\mathcal{W}_1} B)(\mathcal{V}_1) = 0$, that is, $g(\mathcal{V}_1, \nabla_{\mathcal{W}_1} \boldsymbol{\mu}) = 0$, which implies that $\boldsymbol{\mu}$ is parallel. But, by the hypothesis $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ are orthogonal. Hence, $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ must be zero vector fields. But, by the hypothesis, $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ are nonzero. This case cannot occur. Therefore, $\gamma_2 + \gamma_3 = 0$.

As a result, we are able to state the result.

Theorem 3.1. *A* $(\mathbf{GQE})_{\mathbf{n}}$ *spacetime belonging to Gray's trivial subspace is a* $\mathcal{PFS}$.

Case 2. Gray's subspace $\mathcal{J}$.

The gradient of the Ricci tensor in Gray's subspace $\mathcal{J}$ has the form

$$\begin{aligned}&(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1)\\&\quad=\delta_1(\mathcal{U}_1)g(\mathcal{V}_1,\mathcal{W}_1)+\delta_2(\mathcal{V}_1)g(\mathcal{U}_1,\mathcal{W}_1)+\delta_2(\mathcal{W}_1)g(\mathcal{U}_1,\mathcal{V}_1).\end{aligned}\tag{3.20}$$

Using (2.3) in (3.20), we arrive at

$$\begin{aligned}&d\gamma_1(\mathcal{U}_1)g(\mathcal{V}_1,\mathcal{W}_1)+d\gamma_2(\mathcal{U}_1)A(\mathcal{V}_1)A(\mathcal{W}_1)+d\gamma_3(\mathcal{U}_1)B(\mathcal{V}_1)B(\mathcal{W}_1)\\&\quad+\gamma_2[(\nabla_{\mathcal{U}_1}A)(\mathcal{V}_1)A(\mathcal{W}_1)+A(\mathcal{V}_1)(\nabla_{\mathcal{U}_1}A)(\mathcal{W}_1)]\\&\quad+\gamma_3[(\nabla_{\mathcal{U}_1}B)(\mathcal{V}_1)B(\mathcal{W}_1)+B(\mathcal{V}_1)(\nabla_{\mathcal{U}_1}B)(\mathcal{W}_1)]\\&=\frac{n}{(n-1)(n+2)}g(\mathcal{V}_1,\mathcal{W}_1)\nabla_{\mathcal{U}_1}\boldsymbol{r}+\frac{n-2}{2(n-1)(n+2)}g(\mathcal{U}_1,\mathcal{W}_1)\nabla_{\mathcal{V}_1}\boldsymbol{r}\\&\quad+\frac{n-2}{2(n-1)(n+2)}g(\mathcal{U}_1,\mathcal{V}_1)\nabla_{\mathcal{W}_1}\boldsymbol{r}.\end{aligned}\tag{3.21}$$

Setting $\mathcal{V}_1=\boldsymbol{\rho}$, $\mathcal{W}_1=\boldsymbol{\mu}$ in (3.21), we obtain

$$\begin{aligned}&-\gamma_2(\nabla_{\mathcal{U}_1}A)(\boldsymbol{\mu})+\gamma_3(\nabla_{\mathcal{U}_1}B)(\boldsymbol{\rho})\\&\quad=\frac{(n-2)}{2(n-1)(n+2)}[d\boldsymbol{r}(\boldsymbol{\rho})B(\mathcal{U}_1)+d\boldsymbol{r}(\boldsymbol{\mu})A(\mathcal{U}_1)].\end{aligned}\tag{3.22}$$

Equations (2.5) and (3.22) together yield

$$-(\gamma_2+\gamma_3)(\nabla_{\mathcal{U}_1}A)(\boldsymbol{\mu})=\frac{(n-2)}{2(n-1)(n+2)}[d\boldsymbol{r}(\boldsymbol{\rho})B(\mathcal{U}_1)+d\boldsymbol{r}(\boldsymbol{\mu})A(\mathcal{U}_1)].\tag{3.23}$$

We now impose the condition that the velocity vector field is recurrent. Using this Eqs. (1.9) and (3.23) turn into

$$B(\mathcal{U}_1)=-\frac{d\boldsymbol{r}(\boldsymbol{\mu})}{d\boldsymbol{r}(\boldsymbol{\rho})}A(\mathcal{U}_1),\tag{3.24}$$

provided the scalar curvature is non-constant. Adopting (1.6) and (3.24), we obtain

$$\mathcal{S}(\mathcal{U}_1,\mathcal{V}_1)=\gamma_1g(\mathcal{U}_1,\mathcal{V}_1)+[\gamma_2+\gamma_3\left\{\frac{d\boldsymbol{r}(\boldsymbol{\mu})}{d\boldsymbol{r}(\boldsymbol{\rho})}\right\}^2]A(\mathcal{U}_1)A(\mathcal{V}_1).\tag{3.25}$$

Thus, we might conclude that

Theorem 3.2. *A* $(\mathbf{GQE})_\mathbf{n}$ *spacetime with non-constant scalar curvature belonging to Gray's subspace $\mathcal{J}$ represents a $\mathcal{PFS}$, provided the velocity vector field is recurrent.*

Case 3. Gray's subspace $\mathcal{A}$.

Spacetimes in Gray's subspace $\mathcal{A}$ are characterized by Killing Ricci tensor, that is,

$$(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1)+(\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1)+(\nabla_{\mathcal{W}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{V}_1)=0.\tag{3.26}$$

In light of (2.3) and (3.26) we have

$$\begin{aligned} d\gamma_1(\mathcal{U}_1)g(\mathcal{V}_1,\mathcal{W}_1) &+ d\gamma_2(\mathcal{U}_1)A(\mathcal{V}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{U}_1)B(\mathcal{V}_1)B(\mathcal{W}_1) \\ &+ d\gamma_1(\mathcal{V}_1)g(\mathcal{U}_1,\mathcal{W}_1) + d\gamma_2(\mathcal{V}_1)A(\mathcal{U}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{V}_1)B(\mathcal{U}_1)B(\mathcal{W}_1) \\ &+ d\gamma_1(\mathcal{W}_1)g(\mathcal{U}_1,\mathcal{V}_1) + d\gamma_2(\mathcal{W}_1)A(\mathcal{U}_1)A(\mathcal{V}_1) + d\gamma_3(\mathcal{W}_1)B(\mathcal{U}_1)B(\mathcal{V}_1) \\ &+ \gamma_2[\{(\nabla_{\mathcal{U}_1}A)(\mathcal{V}_1) + (\nabla_{\mathcal{V}_1}A)(\mathcal{U}_1)\}A(\mathcal{W}_1) + \{(\nabla_{\mathcal{U}_1}A)(\mathcal{W}_1) \\ &+ (\nabla_{\mathcal{W}_1}A)(\mathcal{U}_1)\}A(\mathcal{V}_1) + \{(\nabla_{\mathcal{V}_1}A)(\mathcal{W}_1) + (\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1)\}A(\mathcal{U}_1)] \\ &+ \gamma_3[\{(\nabla_{\mathcal{U}_1}B)(\mathcal{V}_1) + (\nabla_{\mathcal{V}_1}B)(\mathcal{U}_1)\}B(\mathcal{W}_1) + \{(\nabla_{\mathcal{U}_1}B)(\mathcal{W}_1) \\ &+ (\nabla_{\mathcal{W}_1}B)(\mathcal{U}_1)\}B(\mathcal{V}_1) + \{(\nabla_{\mathcal{V}_1}B)(\mathcal{W}_1) + (\nabla_{\mathcal{W}_1}B)(\mathcal{V}_1)\}B(\mathcal{U}_1)] \\ &= 0. \end{aligned} \tag{3.27}$$

Let us assume that the unit timelike vector field $\boldsymbol{\rho}$ and spacelike vector field $\boldsymbol{\mu}$ are Killing vector field. Then we have

$$(\pounds_{\boldsymbol{\rho}}g)(\mathcal{U}_1,\mathcal{V}_1) = 0 \tag{3.28}$$

and

$$(\pounds_{\boldsymbol{\mu}}g)(\mathcal{U}_1,\mathcal{V}_1) = 0, \tag{3.29}$$

where $\pounds$ stands for the Lie derivative.

In view of (3.28) and (3.29) we get

$$g(\nabla_{\mathcal{U}_1}\boldsymbol{\rho},\mathcal{V}_1) + g(\nabla_{\mathcal{V}_1}\boldsymbol{\rho},\mathcal{U}_1) = 0 \tag{3.30}$$

and

$$g(\nabla_{\mathcal{U}_1}\boldsymbol{\mu},\mathcal{V}_1) + g(\nabla_{\mathcal{V}_1}\boldsymbol{\mu},\mathcal{U}_1) = 0. \tag{3.31}$$

Since $g(\nabla_{\mathcal{U}_1}\boldsymbol{\rho},\mathcal{V}_1) = (\nabla_{\mathcal{U}_1}A)(\mathcal{V}_1)$ and $g(\nabla_{\mathcal{U}_1}\boldsymbol{\mu},\mathcal{V}_1) = (\nabla_{\mathcal{U}_1}B)(\mathcal{V}_1)$, Eqs. (3.30) and (3.31) turn into

$$(\nabla_{\mathcal{U}_1}A)(\mathcal{V}_1) + (\nabla_{\mathcal{V}_1}A)(\mathcal{U}_1) = 0 \tag{3.32}$$

and

$$(\nabla_{\mathcal{U}_1}B)(\mathcal{V}_1) + (\nabla_{\mathcal{V}_1}B)(\mathcal{U}_1) = 0. \tag{3.33}$$

Similarly, we infer that

$$(\nabla_{\mathcal{U}_1}A)(\mathcal{W}_1) + (\nabla_{\mathcal{W}_1}A)(\mathcal{U}_1) = 0, \tag{3.34}$$

$$(\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1) + (\nabla_{\mathcal{V}_1}A)(\mathcal{W}_1) = 0, \tag{3.35}$$

$$(\nabla_{\mathcal{U}_1}B)(\mathcal{W}_1) + (\nabla_{\mathcal{W}_1}B)(\mathcal{U}_1) = 0 \tag{3.36}$$

and

$$(\nabla_{\mathcal{W}_1}B)(\mathcal{V}_1) + (\nabla_{\mathcal{V}_1}B)(\mathcal{W}_1) = 0, \tag{3.37}$$

for all $\mathcal{U}_1$, $\mathcal{V}_1$, $\mathcal{W}_1$.

Utilizing Eqs. (3.32)–(3.37) in (3.27), we reach

$$\begin{aligned} &d\gamma_1(\mathcal{U}_1)g(\mathcal{V}_1,\mathcal{W}_1) + d\gamma_2(\mathcal{U}_1)A(\mathcal{V}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{U}_1)B(\mathcal{V}_1)B(\mathcal{W}_1) \\ &\quad + d\gamma_1(\mathcal{V}_1)g(\mathcal{U}_1,\mathcal{W}_1) + d\gamma_2(\mathcal{V}_1)A(\mathcal{U}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{V}_1)B(\mathcal{U}_1)B(\mathcal{W}_1) \\ &\quad + d\gamma_1(\mathcal{W}_1)g(\mathcal{U}_1,\mathcal{V}_1) + d\gamma_2(\mathcal{W}_1)A(\mathcal{U}_1)A(\mathcal{V}_1) \\ &\quad + d\gamma_3(\mathcal{W}_1)B(\mathcal{U}_1)B(\mathcal{V}_1) = 0. \end{aligned} \tag{3.38}$$

Setting $\mathcal{V}_1 = \boldsymbol{\rho}$, $\mathcal{W}_1 = \boldsymbol{\mu}$ in (3.38), we obtain

$$B(\mathcal{U}_1) = \alpha_2 A(\mathcal{U}_1), \tag{3.39}$$

where $\alpha_2 = \frac{d\gamma_2(\boldsymbol{\mu}) - d\gamma_1(\boldsymbol{\mu})}{d\gamma_1(\boldsymbol{\rho}) + d\gamma_3(\boldsymbol{\rho})}$, provided $\gamma_1 + \gamma_3 \neq$ constant.

From (1.6) and (3.39), it follows that

$$\mathcal{S}(\mathcal{U}_1,\mathcal{V}_1) = \gamma_1 g(\mathcal{U}_1,\mathcal{V}_1) + (\gamma_2 + \gamma_3\alpha_2^2)A(\mathcal{U}_1)A(\mathcal{V}_1). \tag{3.40}$$

This leads to the following results.

Theorem 3.3. *A* $(\mathbf{GQE})_\mathbf{n}$ *spacetime belonging to Gray's subspace* $\mathcal{A}$ *becomes a* $\mathcal{PFS}$, *provided the generators are Killing and the sum of the associated scalars* γ_1 *and* γ_3 *is non-constant.*

Also, Eqs. (1.3) and (3.26) reflect that the $\mathcal{EMT}$ is Killing, i.e.

$$(\nabla_{\mathcal{U}_1}\mathcal{T})(\mathcal{V}_1,\mathcal{W}_1) + (\nabla_{\mathcal{V}_1}\mathcal{T})(\mathcal{U}_1,\mathcal{W}_1) + (\nabla_{\mathcal{W}_1}\mathcal{T})(\mathcal{U}_1,\mathcal{V}_1) = 0,$$

for all $\mathcal{U}_1,\mathcal{V}_1,\mathcal{W}_1 \in \chi(M)$.

In [30], Sharma and Ghosh describe the following outcome.

Theorem A. *Let* (M,g) *be a* $\mathcal{PFS}$ *with Killing* $\mathcal{EMT}$. *Then*

(i) *the flow of spacetime is geodesic and the spacetime is expansion-free and shear-free, but not vorticity-free and*
(ii) *the spacetime admits constant energy density and pressure.*

Therefore, by Theorem A *we conclude the following.*

Corollary 3.1. *Let a* $(\mathbf{GQE})_\mathbf{n}$ *spacetime belong to the subspace* $\mathcal{A}$ *and the generators* $\boldsymbol{\rho}$ *and* $\boldsymbol{\mu}$ *are Killing vector fields. Then*

(i) *the flow of spacetime is geodesic and the spacetime is expansion-free and shear-free, but not vorticity-free and*
(ii) *the spacetime admits constant energy density and pressure, provided the sum of the associated scalars* γ_1 *and* γ_3 *is non-constant.*

Case 4. Gray's subspace $\mathcal{B}$.

The Ricci tensor of a spacetime belonging to Gray's subspace $\mathcal{B}$ obeys

$$(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1) = (\nabla_{\mathcal{W}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{V}_1). \tag{3.41}$$

In view of (2.3) and (3.41) we acquire

$$\begin{aligned} & d\gamma_1(\mathcal{U}_1)g(\mathcal{V}_1,\mathcal{W}_1) + d\gamma_2(\mathcal{U}_1)A(\mathcal{V}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{U}_1)B(\mathcal{V}_1)B(\mathcal{W}_1) \\ & \quad - d\gamma_1(\mathcal{W}_1)g(\mathcal{U}_1,\mathcal{V}_1) - d\gamma_2(\mathcal{W}_1)A(\mathcal{U}_1)A(\mathcal{V}_1) - d\gamma_3(\mathcal{W}_1)B(\mathcal{U}_1)B(\mathcal{V}_1) \\ & \quad + \gamma_2[(\nabla_{\mathcal{U}_1}A)(\mathcal{V}_1)A(\mathcal{W}_1) + A(\mathcal{V}_1)(\nabla_{\mathcal{U}_1}A)(\mathcal{W}_1) - (\nabla_{\mathcal{W}_1}A)(\mathcal{U}_1)A(\mathcal{V}_1). \\ & \quad - A(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1)] + \gamma_3[(\nabla_{\mathcal{U}_1}B)(\mathcal{V}_1)B(\mathcal{W}_1) + B(\mathcal{V}_1)(\nabla_{\mathcal{U}_1}B)(\mathcal{W}_1). \\ & \quad - (\nabla_{\mathcal{W}_1}B)(\mathcal{U}_1)B(\mathcal{V}_1) - B(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}B)(\mathcal{V}_1)] = 0. \end{aligned} \tag{3.42}$$

Setting $\mathcal{V}_1 = \boldsymbol{\rho}$, $\mathcal{W}_1 = \boldsymbol{\mu}$ in (3.42), we find

$$\begin{aligned} & [d\gamma_1(\boldsymbol{\mu}) - d\gamma_2(\boldsymbol{\mu})]A(\mathcal{U}_1) - \gamma_2(\nabla_{\boldsymbol{\mu}}A)(\mathcal{U}_1) + \gamma_3 B(\mathcal{U}_1)(\nabla_{\boldsymbol{\mu}}B)(\boldsymbol{\rho}) \\ & \quad = -\gamma_2(\nabla_{\mathcal{U}_1}A)(\boldsymbol{\mu}) + \gamma_3(\nabla_{\mathcal{U}_1}B)(\boldsymbol{\rho}). \end{aligned} \tag{3.43}$$

Again, setting $\mathcal{U}_1 = \boldsymbol{\rho}$, $\mathcal{V}_1 = \mathcal{W}_1 = \boldsymbol{\mu}$ in (3.42), we deduce that

$$d\gamma_1(\boldsymbol{\rho}) + d\gamma_3(\boldsymbol{\rho}) + \gamma_2(\nabla_{\boldsymbol{\mu}}A)(\boldsymbol{\mu}) = \gamma_3(\nabla_{\boldsymbol{\mu}}B)(\boldsymbol{\rho}). \tag{3.44}$$

By virtue of (2.5), (3.43) and (3.44) we arrive at

$$\begin{aligned} & [d\gamma_1(\boldsymbol{\mu}) - d\gamma_2(\boldsymbol{\mu})]A(\mathcal{U}_1) + [d\gamma_1(\boldsymbol{\rho}) + d\gamma_3(\boldsymbol{\rho}) + \gamma_2(\nabla_{\boldsymbol{\mu}}A)(\boldsymbol{\mu})]B(\mathcal{U}_1) \\ & \quad + (\gamma_2 + \gamma_3)(\nabla_{\mathcal{U}_1}A)(\boldsymbol{\mu}) - \gamma_2(\nabla_{\boldsymbol{\mu}}A)(\mathcal{U}_1) = 0. \end{aligned} \tag{3.45}$$

Adopting (1.9) in (3.45), we can derive

$$B(\mathcal{U}_1) = \alpha_3 A(\mathcal{U}_1), \tag{3.46}$$

where $\alpha_3 = \frac{d\gamma_2(\boldsymbol{\mu}) + \gamma_2\omega(\boldsymbol{\mu}) - d\gamma_1(\boldsymbol{\mu})}{d\gamma_1(\boldsymbol{\rho}) + d\gamma_3(\boldsymbol{\rho})}$, provided $\gamma_1 + \gamma_3 \neq$ constant.

Equations (1.6) and (3.46) give

$$\mathcal{S}(\mathcal{U}_1,\mathcal{V}_1) = \gamma_1 g(\mathcal{U}_1,\mathcal{V}_1) + (\gamma_2 + \gamma_3\alpha_3^2)A(\mathcal{U}_1)A(\mathcal{V}_1). \tag{3.47}$$

Thus, we write the following theorem.

Theorem 3.4. *A* $(\mathbf{GQE})_{\mathbf{n}}$ *spacetime belonging to Gray's subspace* $\mathcal{B}$ *reduces to a* $\mathcal{PFS}$, *provided the velocity vector field is recurrent and the sum of the associated scalars* γ_1 *and* γ_3 *is non-constant.*

A Lorentzian manifold is named a Yang pure space [17] whose metric satisfies Yang's equation:

$$(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1) = (\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1).$$

Mantica and Molinari [23] established the following result for $n \geq 4$.

Proposition 3.1. *A perfect fluid Yang pure space of dim.* $n \geq 4$ *with* $\boldsymbol{p} + \boldsymbol{\sigma} \neq 0$ *is a* GRW *spacetime.*

In contrast to Eq. (3.47) we notice $\boldsymbol{\kappa}(\frac{\boldsymbol{p}-\boldsymbol{\sigma}}{2-n}) = \gamma_1$ and $\boldsymbol{\kappa}(\boldsymbol{p}+\boldsymbol{\sigma}) = \gamma_2 + \gamma_3\alpha_3^2$, that is, $\boldsymbol{p} + \boldsymbol{\sigma} \neq 0$ for $\gamma_2 \neq -\gamma_3\alpha_3^2$.

In view of this observation, we can conclude as follows.

Corollary 3.2. *A* $(\mathbf{GQE})_\mathbf{n}$ *spacetime belonging to Gray's subspace* $\mathcal{B}$ *is a* GRW *spacetime, provided the velocity vector field is recurrent and the associated scalars satisfies the relations* $\gamma_2 + \gamma_3\alpha_3^2 \neq 0$ *and* $\gamma_1 + \gamma_3 \neq$ *constant.*

Also, Eqs. (1.3) and (3.41) reflect that $\mathcal{T}$ satisfies

$$(\nabla_{\mathcal{U}_1}\mathcal{T})(\mathcal{V}_1, \mathcal{W}_1) = (\nabla_{\mathcal{V}_1}\mathcal{T})(\mathcal{U}_1, \mathcal{W}_1),$$

for all $\mathcal{U}_1$, $\mathcal{V}_1$, $\mathcal{W}_1 \in \chi(M)$.

However, it has been established [14] *that if the* $\mathcal{EMT}$ *is of Codazzi type in a* $\mathcal{PFS}$, *then the fluid is of vanishing shear and vorticity, and its velocity vector field becomes hypersurface orthogonal.*

Barnes [2] *observed that the probable local cosmological structures of the* $\mathcal{PFS}$ *are of Petrov types I, D or O if the* $\mathcal{PFS}$ *is of vanishing shear and vorticity, the velocity vector field* $\boldsymbol{\rho}$ *is hypersurface orthogonal and the constant energy density over a hypersurface orthogonal to* $\boldsymbol{\rho}$.

As a result of the foregoing facts, we arrive at the following.

Corollary 3.3. *If a* $(\mathbf{GQE})_\mathbf{n}$ *spacetime belongs to Gray's subspace* $\mathcal{B}$, *then the probable local cosmological structures of the spacetime are of Petrov types I, D or O, provided the velocity vector field is recurrent and the sum of the associated scalars* γ_1 *and* γ_3 *is non-constant.*

Case 5. Gray's subspace $\mathcal{J} \oplus \mathcal{A}$.

Gray's subspace $\mathcal{J} \oplus \mathcal{A}$ possesses a spacetime whose Ricci tensor is conformal Killing, that is,

$$(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1, \mathcal{W}_1) + (\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1, \mathcal{W}_1) + (\nabla_{\mathcal{W}_1}\mathcal{S})(\mathcal{U}_1, \mathcal{V}_1)$$
$$= \frac{2d\boldsymbol{r}(\mathcal{U}_1)}{(n+2)} g(\mathcal{V}_1, \mathcal{W}_1) + \frac{2d\boldsymbol{r}(\mathcal{V}_1)}{(n+2)} g(\mathcal{U}_1, \mathcal{W}_1) + \frac{2d\boldsymbol{r}(\mathcal{W}_1)}{(n+2)} g(\mathcal{U}_1, \mathcal{V}_1). \quad (3.48)$$

The use of (2.3) and (3.48) implies

$$d\gamma_1(\mathcal{U}_1)g(\mathcal{V}_1, \mathcal{W}_1) + d\gamma_2(\mathcal{U}_1)A(\mathcal{V}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{U}_1)B(\mathcal{V}_1)B(\mathcal{W}_1)$$
$$+ d\gamma_1(\mathcal{V}_1)g(\mathcal{U}_1, \mathcal{W}_1) + d\gamma_2(\mathcal{V}_1)A(\mathcal{U}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{V}_1)B(\mathcal{U}_1)B(\mathcal{W}_1)$$
$$+ d\gamma_1(\mathcal{W}_1)g(\mathcal{U}_1, \mathcal{V}_1) + d\gamma_2(\mathcal{W}_1)A(\mathcal{U}_1)A(\mathcal{V}_1) + d\gamma_3(\mathcal{W}_1)B(\mathcal{U}_1)B(\mathcal{V}_1)$$
$$+ \gamma_2[\{(\nabla_{\mathcal{U}_1}A)(\mathcal{V}_1) + (\nabla_{\mathcal{V}_1}A)(\mathcal{U}_1)\}A(\mathcal{W}_1) + \{(\nabla_{\mathcal{U}_1}A)(\mathcal{W}_1)$$
$$+ (\nabla_{\mathcal{W}_1}A)(\mathcal{U}_1)\}A(\mathcal{V}_1) + \{(\nabla_{\mathcal{V}_1}A)(\mathcal{W}_1) + (\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1)\}A(\mathcal{U}_1)]$$
$$+ \gamma_3[\{(\nabla_{\mathcal{U}_1}B)(\mathcal{V}_1) + (\nabla_{\mathcal{V}_1}B)(\mathcal{U}_1)\}B(\mathcal{W}_1) + \{(\nabla_{\mathcal{U}_1}B)(\mathcal{W}_1)$$
$$+ (\nabla_{\mathcal{W}_1}B)(\mathcal{U}_1)\}B(\mathcal{V}_1) + \{(\nabla_{\mathcal{V}_1}B)(\mathcal{W}_1) + (\nabla_{\mathcal{W}_1}B)(\mathcal{V}_1)\}B(\mathcal{U}_1)]$$
$$= \frac{2d\boldsymbol{r}(\mathcal{U}_1)}{(n+2)} g(\mathcal{V}_1, \mathcal{W}_1) + \frac{2d\boldsymbol{r}(\mathcal{V}_1)}{(n+2)} g(\mathcal{U}_1, \mathcal{W}_1) + \frac{2d\boldsymbol{r}(\mathcal{W}_1)}{(n+2)} g(\mathcal{U}_1, \mathcal{V}_1). \quad (3.49)$$

Utilizing Eqs. (3.32)–(3.37) in (3.49), we reach

$$\begin{aligned} &d\gamma_1(\mathcal{U}_1)g(\mathcal{V}_1,\mathcal{W}_1) + d\gamma_2(\mathcal{U}_1)A(\mathcal{V}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{U}_1)B(\mathcal{V}_1)B(\mathcal{W}_1) \\ &\quad + d\gamma_1(\mathcal{V}_1)g(\mathcal{U}_1,\mathcal{W}_1) + d\gamma_2(\mathcal{V}_1)A(\mathcal{U}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{V}_1)B(\mathcal{U}_1)B(\mathcal{W}_1) \\ &\quad + d\gamma_1(\mathcal{W}_1)g(\mathcal{U}_1,\mathcal{V}_1) + d\gamma_2(\mathcal{W}_1)A(\mathcal{U}_1)A(\mathcal{V}_1) + d\gamma_3(\mathcal{W}_1)B(\mathcal{U}_1)B(\mathcal{V}_1) \\ &= \frac{2d\boldsymbol{r}(\mathcal{U}_1)}{(n+2)}g(\mathcal{V}_1,\mathcal{W}_1) + \frac{2d\boldsymbol{r}(\mathcal{V}_1)}{(n+2)}g(\mathcal{U}_1,\mathcal{W}_1) + \frac{2d\boldsymbol{r}(\mathcal{W}_1)}{(n+2)}g(\mathcal{U}_1,\mathcal{V}_1). \end{aligned} \tag{3.50}$$

Setting $\mathcal{V}_1 = \boldsymbol{\rho}$, $\mathcal{W}_1 = \boldsymbol{\mu}$ in (3.50), we get

$$B(\mathcal{U}_1) = \alpha_4 A(\mathcal{U}_1), \tag{3.51}$$

where $\alpha_4 = \frac{(n-2)d\gamma_1(\boldsymbol{\mu})+nd\gamma_2(\boldsymbol{\mu})+2d\gamma_3(\boldsymbol{\mu})}{-(n-2)d\gamma_1(\boldsymbol{\rho})+2d\gamma_2(\boldsymbol{\rho})+nd\gamma_3(\boldsymbol{\rho})}$, provided $(2-n)\gamma_1+2\gamma_2+n\gamma_3 \neq$ constant. From (1.6) and (3.51), it follows that

$$\mathcal{S}(\mathcal{U}_1,\mathcal{V}_1) = \gamma_1 g(\mathcal{U}_1,\mathcal{V}_1) + (\gamma_2+\gamma_3\alpha_4^2)A(\mathcal{U}_1)A(\mathcal{V}_1). \tag{3.52}$$

Thus, we can conclude the following result.

Theorem 3.5. *A* $(\mathbf{GQE})_\mathbf{n}$ *spacetime of class type* $\mathcal{J}\oplus\mathcal{A}$ *is a* $\mathcal{PFS}$, *provided the generators are Killing and the associated scalars satisfies the relation* $(2-n)\gamma_1 + 2\gamma_2 + n\gamma_3 \neq$ *constant.*

Case 6. Gray's subspace $\mathcal{J}\oplus\mathcal{B}$.

In Gray's subspace $\mathcal{J}\oplus\mathcal{B}$, the tensor $[\mathcal{S} - \frac{\boldsymbol{r}g}{2(n-1)}]$ is of Codazzi type, that is,

$$\begin{aligned} &\nabla_{\mathcal{U}_1}\left[\mathcal{S}(\mathcal{V}_1,\mathcal{W}_1) - \frac{\boldsymbol{r}}{2(n-1)}g(\mathcal{V}_1,\mathcal{W}_1)\right] \\ &\quad = \nabla_{\mathcal{V}_1}\left[\mathcal{S}(\mathcal{U}_1,\mathcal{W}_1) - \frac{\boldsymbol{r}}{2(n-1)}g(\mathcal{U}_1,\mathcal{W}_1)\right], \end{aligned} \tag{3.53}$$

which gives $\operatorname{div}\mathcal{C} = 0$. So, we arrive

$$\begin{aligned} &\frac{1}{2(n-1)}[d\boldsymbol{r}(\mathcal{U}_1)g(\mathcal{V}_1,\mathcal{W}_1) - d\boldsymbol{r}(\mathcal{W}_1)g(\mathcal{U}_1,\mathcal{V}_1)] \\ &\quad = (\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1) - (\nabla_{\mathcal{W}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{V}_1). \end{aligned} \tag{3.54}$$

Equations (2.3) and (3.54) together yield

$$\begin{aligned} &\frac{1}{2(n-1)}[d\boldsymbol{r}(\mathcal{U}_1)g(\mathcal{V}_1,\mathcal{W}_1) - d\boldsymbol{r}(\mathcal{W}_1)g(\mathcal{U}_1,\mathcal{V}_1)] \\ &\quad = d\gamma_1(\mathcal{U}_1)g(\mathcal{V}_1,\mathcal{W}_1) + d\gamma_2(\mathcal{U}_1)A(\mathcal{V}_1)A(\mathcal{W}_1) + d\gamma_3(\mathcal{U}_1)B(\mathcal{V}_1)B(\mathcal{W}_1) \\ &\qquad - d\gamma_1(\mathcal{W}_1)g(\mathcal{U}_1,\mathcal{V}_1) - d\gamma_2(\mathcal{W}_1)A(\mathcal{U}_1)A(\mathcal{V}_1) - d\gamma_3(\mathcal{W}_1)B(\mathcal{U}_1)B(\mathcal{V}_1) \\ &\qquad + \gamma_2[(\nabla_{\mathcal{U}_1}A)(\mathcal{V}_1)A(\mathcal{W}_1) + A(\mathcal{V}_1)(\nabla_{\mathcal{U}_1}A)(\mathcal{W}_1) - (\nabla_{\mathcal{W}_1}A)(\mathcal{U}_1)A(\mathcal{V}_1) \\ &\qquad - A(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}A)(\mathcal{V}_1)] + \gamma_3[(\nabla_{\mathcal{U}_1}B)(\mathcal{V}_1)B(\mathcal{W}_1) + B(\mathcal{V}_1)(\nabla_{\mathcal{U}_1}B)(\mathcal{W}_1) \\ &\qquad - (\nabla_{\mathcal{W}_1}B)(\mathcal{U}_1)B(\mathcal{V}_1) - B(\mathcal{U}_1)(\nabla_{\mathcal{W}_1}B)(\mathcal{V}_1)]. \end{aligned} \tag{3.55}$$

Substituting $\mathcal{V}_1$ by $\boldsymbol{\rho}$ and $\mathcal{W}_1$ by $\boldsymbol{\mu}$ in (3.55), we have

$$\gamma_2(\nabla_{\boldsymbol{\mu}} A)(\mathcal{U}_1) - \gamma_3 B(\mathcal{U}_1)(\nabla_{\boldsymbol{\mu}} B)(\boldsymbol{\rho}) + \frac{1}{2(n-1)}\, d\boldsymbol{r}(\boldsymbol{\mu}) A(\mathcal{U}_1)$$
$$- \gamma_2(\nabla_{\mathcal{U}_1} A)(\boldsymbol{\mu}) + \gamma_3(\nabla_{\mathcal{U}_1} B)(\boldsymbol{\rho}) + [d\gamma_2(\boldsymbol{\mu}) - d\gamma_1(\boldsymbol{\mu})]A(\mathcal{U}_1) = 0. \quad (3.56)$$

Putting $\mathcal{U}_1 = \boldsymbol{\rho}$, $\mathcal{V}_1 = \mathcal{W}_1 = \boldsymbol{\mu}$ in (3.55), we obtain

$$\gamma_3(\nabla_{\boldsymbol{\mu}} B)(\boldsymbol{\rho}) = d\gamma_1(\boldsymbol{\rho}) + d\gamma_3(\boldsymbol{\rho}) + \gamma_2(\nabla_{\boldsymbol{\mu}} A)(\boldsymbol{\mu}) - \frac{d\boldsymbol{r}(\boldsymbol{\rho})}{2(n-1)}. \quad (3.57)$$

Equations (2.5), (3.56) and (3.57) turn into

$$[d\gamma_2(\boldsymbol{\mu}) - d\gamma_1(\boldsymbol{\mu})]A(\mathcal{U}_1) - (\gamma_2 + \gamma_3)(\nabla_{\mathcal{U}_1} A)(\boldsymbol{\mu}) + \frac{d\boldsymbol{r}(\boldsymbol{\mu})}{2(n-1)} A(\mathcal{U}_1)$$
$$+ \gamma_2(\nabla_{\boldsymbol{\mu}} A)(\mathcal{U}_1) - \left[d\gamma_1(\boldsymbol{\rho}) + d\gamma_3(\boldsymbol{\rho}) + \gamma_2(\nabla_{\boldsymbol{\mu}} A)(\boldsymbol{\mu}) - \frac{d\boldsymbol{r}(\boldsymbol{\rho})}{2(n-1)}\right]$$
$$\times B(\mathcal{U}_1) = 0. \quad (3.58)$$

Adopting (1.9) in (3.58) we get

$$B(\mathcal{U}_1) = \alpha_5 A(\mathcal{U}_1), \quad (3.59)$$

where $\alpha_5 = \frac{(2-n)d\gamma_1(\boldsymbol{\mu}) + (2n-3)d\gamma_2(\boldsymbol{\mu}) + d\gamma_3(\boldsymbol{\mu}) + 2(n-1)\gamma_2\omega(\boldsymbol{\mu})}{(n-2)d\gamma_1(\boldsymbol{\rho}) + d\gamma_2(\boldsymbol{\rho}) + (2n-3)d\gamma_3(\boldsymbol{\rho})}$, provided $(n-2)\gamma_1 + \gamma_2 + (2n-3)\gamma_3 \neq$ constant.

Using (3.59) in (1.6) we infer

$$\mathcal{S}(\mathcal{U}_1, \mathcal{V}_1) = \gamma_1 g(\mathcal{U}_1, \mathcal{V}_1) + (\gamma_2 + \gamma_3 \alpha_5^2) A(\mathcal{U}_1) A(\mathcal{V}_1). \quad (3.60)$$

Hence, we write the following.

Theorem 3.6. *A* $\mathbf{(GQE)_n}$ *spacetime of class type* $\mathcal{J} \oplus \mathcal{B}$ *is a* $\mathcal{PFS}$, *provided the flow vector field is recurrent and the associated scalars satisfies the relation* $(n-2)\gamma_1 + \gamma_2 + (2n-3)\gamma_3 \neq$ *constant.*

In [29], Sharma proved the following.

Theorem B. *The conformal curvature tensor of a relativistic perfect fluid space-time M is divergence-free if and only if M is shear-free, irrotational, and its energy density is constant over the space-like hypersurface orthogonal to the four-velocity vector.*

Therefore, from Theorem 3.6 and Theorem B, we can state the following.

Corollary 3.4. *If a* $\mathbf{(GQE)_n}$ *spacetime belongs to Gray's subspace* $\mathcal{J} \oplus \mathcal{B}$, *then the spacetime is shear-free, irrotational, and its energy density is constant over the space-like hypersurface orthogonal to the four-velocity vector, provided the flow vector field is recurrent and the associated scalars satisfies the relation* $(n-2)\gamma_1 + \gamma_2 + (2n-3)\gamma_3 \neq$ *constant.*

Case 7. Gray's subspace $\mathcal{A} \oplus \mathcal{B}$.

Spacetimes belong to Gray's subspace $\mathcal{A} \oplus \mathcal{B}$ are characterized by having a constant scalar curvature, that is, $\nabla \boldsymbol{r} = 0$. Therefore, from (1.8), we reach

$$(\operatorname{div} \mathcal{C})(\mathcal{U}_1, \mathcal{V}_1)\mathcal{W}_1 = \left(\frac{n-3}{n-2}\right) \left[(\nabla_{\mathcal{U}_1} \mathcal{S})(\mathcal{V}_1, \mathcal{W}_1) - (\nabla_{\mathcal{V}_1} \mathcal{S})(\mathcal{U}_1, \mathcal{W}_1)\right], \quad (3.61)$$

which means that the divergence of the conformal curvature tensor vanishes if and only if the Ricci tensor is of Codazzi type.

Therefore, we conclude the following.

Theorem 3.7. *If a* $(\mathbf{GQE})_\mathbf{n}$ *spacetime belongs to Gray's subspace* $\mathcal{A} \oplus \mathcal{B}$, *then the subspaces* $\mathcal{B}$ *and* $\mathcal{J} \oplus \mathcal{B}$ *coincide.*

4. $(\mathbf{GQE})_\mathbf{n}$ GRW Spacetimes

In this part, we assume that $(\mathbf{GQE})_\mathbf{n}$ spacetime is a GRW spacetime. Mantica and Molinari [23] proved that a Lorentzian manifold of dimension $n \geq 3$ is a GRW spacetime if and only if it admits a unit timelike torse-forming vector field $\boldsymbol{\rho}$:

$$(\nabla_{\mathcal{U}_1} A)(\mathcal{V}_1) = \Psi[g(\mathcal{U}_1, \mathcal{V}_1) + A(\mathcal{U}_1)A(\mathcal{V}_1)] \quad (4.1)$$

and

$$\mathcal{S}(\mathcal{U}_1, \boldsymbol{\rho}) = \lambda g(\mathcal{U}_1, \boldsymbol{\rho}), \quad (4.2)$$

for some smooth functions $\Psi(\neq 0)$ and λ on M. Now,

$$(\nabla_{\mathcal{U}_1} \mathcal{S})(\mathcal{V}_1, \boldsymbol{\rho}) = \mathcal{U}_1 \mathcal{S}(\mathcal{V}_1, \boldsymbol{\rho}) - \mathcal{S}(\nabla_{\mathcal{U}_1} \mathcal{V}_1, \boldsymbol{\rho}) - \mathcal{S}(\mathcal{V}_1, \nabla_{\mathcal{U}_1} \boldsymbol{\rho}). \quad (4.3)$$

Using (4.1) and (4.2) in (4.3), we arrive at

$$(\nabla_{\mathcal{U}_1} \mathcal{S})(\mathcal{V}_1, \boldsymbol{\rho}) = (\mathcal{U}_1 \lambda)A(\mathcal{V}_1) + \lambda \Psi g(\mathcal{U}_1, \mathcal{V}_1) - \Psi \mathcal{S}(\mathcal{U}_1, \mathcal{V}_1), \quad (4.4)$$

where $(\mathcal{U}_1 \lambda) = g(\mathcal{U}_1, \operatorname{grad} \lambda)$.

Differentiating (2.4) covariantly and applying (4.1) we reach

$$\begin{aligned}(\nabla_{\mathcal{U}_1} \mathcal{S})(\mathcal{V}_1, \boldsymbol{\rho}) &= [d\gamma_1(\mathcal{U}_1) - d\gamma_2(\mathcal{U}_1)]A(\mathcal{V}_1) \\ &\quad + \Psi(\gamma_1 - \gamma_2)[g(\mathcal{U}_1, \mathcal{V}_1) + A(\mathcal{U}_1)A(\mathcal{V}_1)]. \quad (4.5)\end{aligned}$$

Combining Eqs. (4.4) and (4.5), we reveal

$$\begin{aligned}&[d\gamma_1(\mathcal{U}_1) - d\gamma_2(\mathcal{U}_1)]A(\mathcal{V}_1) + \Psi(\gamma_1 - \gamma_2)[g(\mathcal{U}_1, \mathcal{V}_1) + A(\mathcal{U}_1)A(\mathcal{V}_1)] \\ &\quad = (\mathcal{U}_1 \lambda)A(\mathcal{V}_1) + \lambda \Psi g(\mathcal{U}_1, \mathcal{V}_1) - \Psi \mathcal{S}(\mathcal{U}_1, \mathcal{V}_1). \quad (4.6)\end{aligned}$$

Setting $\mathcal{V}_1 = \boldsymbol{\rho}$ in (4.6) and using $A(\boldsymbol{\rho}) = -1$, we turn up

$$(\mathcal{U}_1 \lambda) = d\gamma_1(\mathcal{U}_1) - d\gamma_2(\mathcal{U}_1). \quad (4.7)$$

Contracting $\mathcal{U}_1$ and $\mathcal{V}_1$ in (4.6) reveals that

$$(\boldsymbol{\rho}\lambda) + n\lambda\Psi - \boldsymbol{r}\Psi = d\gamma_1(\boldsymbol{\rho}) - d\gamma_2(\boldsymbol{\rho}) + (n-1)\Psi(\gamma_1 - \gamma_2). \quad (4.8)$$

Equations (2.1), (4.7) and (4.8) imply

$$\lambda = \frac{(2n-1)\gamma_1 - n\gamma_2 + \gamma_3}{n}. \tag{4.9}$$

From (4.2) and (4.9), we can derive

$$\mathcal{S}(\mathcal{U}_1, \boldsymbol{\rho}) = \left[\frac{(2n-1)\gamma_1 - n\gamma_2 + \gamma_3}{n}\right] g(\mathcal{U}_1, \boldsymbol{\rho}). \tag{4.10}$$

This means that $\frac{(2n-1)\gamma_1 - n\gamma_2 + \gamma_3}{n}$ is an eigenvalue corresponding to the eigenvector $\boldsymbol{\rho}$. In light of (4.6), (4.7) and (4.9), we acquire

$$\mathcal{S}(\mathcal{U}_1, \mathcal{V}_1) = \left[\frac{(n-1)\gamma_1 + \gamma_3}{n}\right] g(\mathcal{U}_1, \mathcal{V}_1) - (\gamma_1 - \gamma_2)A(\mathcal{U}_1)A(\mathcal{V}_1). \tag{4.11}$$

This leads to the following result.

Theorem 4.1. *A* $(\mathbf{GQE})_\mathbf{n}$ GRW *spacetime represents a* $\mathcal{PFS}$.

According to $\mathcal{EFE}$ without cosmological constant, the Ricci tensor becomes

$$\mathcal{S}(\mathcal{U}_1, \mathcal{V}_1) = \boldsymbol{\kappa}\left(\frac{\boldsymbol{p} - \boldsymbol{\sigma}}{2-n}\right) g(\mathcal{U}_1, \mathcal{V}_1) + \boldsymbol{\kappa}(\boldsymbol{p} + \boldsymbol{\sigma})A(\mathcal{U}_1)A(\mathcal{V}_1). \tag{4.12}$$

From (4.11) and (4.12), it follows that

$$\boldsymbol{\kappa}\left(\frac{\boldsymbol{p} - \boldsymbol{\sigma}}{2-n}\right) = \frac{(n-1)\gamma_1 + \gamma_3}{n} \tag{4.13}$$

and

$$\boldsymbol{\kappa}(\boldsymbol{p} + \boldsymbol{\sigma}) = \gamma_2 - \gamma_1. \tag{4.14}$$

It is known that [23] a four-dimensional perfect fluid GRW spacetime is a RW spacetime. Guilfoyle and Nolan [17] proved that a four-dimensional perfect fluid spacetime with $\boldsymbol{p} + \boldsymbol{\sigma} \neq 0$ is a Yang pure space if and only if it is a RW spacetime.

Since $\gamma_1 \neq \gamma_2$ in general, so we obtain $\boldsymbol{p} + \boldsymbol{\sigma} \neq 0$.

This leads to the following corollary.

Theorem 4.2. *A* $(\mathbf{GQE})_\mathbf{4}$ GRW *spacetime is a Yang pure space.*

If $\gamma_1 = \gamma_2$, then we infer that $\boldsymbol{p} + \boldsymbol{\sigma} = 0$. This represents a dark matter era [8].

Hence, we conclude the result as follows.

Corollary 4.1. *A* $(\mathbf{GQE})_\mathbf{n}$ GRW *spacetime represents a dark matter era for* $\gamma_1 = \gamma_2$.

For $n = 4$, Eqs. (4.13) and (4.14) give us

$$\frac{\boldsymbol{p} - \boldsymbol{\sigma}}{\boldsymbol{p} + \boldsymbol{\sigma}} = \frac{3\gamma_1 + \gamma_3}{2(\gamma_1 - \gamma_2)}. \tag{4.15}$$

If $\gamma_2 = 4\gamma_1 + \gamma_3$, then (4.15) entails that $\boldsymbol{\sigma} = 3\boldsymbol{p}$. This represents a radiation era [8]. For $4\gamma_2 = 7\gamma_1 + \gamma_3$, the foregoing equation (4.15) assumes the form $\boldsymbol{\sigma} + 3\boldsymbol{p} = 0$. It is observed that the matter with $\frac{\boldsymbol{p}}{\boldsymbol{\sigma}} = -\frac{1}{3}$ represents quintessence phase.

Hence, we have the following two corollaries.

Corollary 4.2. *A* $(\mathbf{GQE})_4$ GRW *spacetime represents a radiation era for* $\gamma_2 = 4\gamma_1 + \gamma_3$.

Corollary 4.3. *A* $(\mathbf{GQE})_4$ GRW *spacetime represents quintessence phase for* $4\gamma_2 = 7\gamma_1 + \gamma_3$.

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A COMPARATIVE STUDY OF $N(k)$-QUASI EINSTEIN MANIFOLDS WITH CONHARMONIC CURVATURE TENSOR

DIPANKAR HAZRA

Abstract. The present paper deals with study of $N(k)$-quasi Einstein manifolds which satisfies the curvature conditions $\mathcal{L}(\xi, X) \cdot \mathcal{L} = 0$, $\mathcal{L}(\xi, X) \cdot \mathcal{V} = 0$, $\mathcal{L}(\xi, X) \cdot \mathcal{W}_2 = 0$ and $\mathcal{W}_2(\xi, X) \cdot \mathcal{L} = 0$, where $\mathcal{L}$, $\mathcal{V}$ and $\mathcal{W}_2$ denotes the conharmonic curvature tensor, the concircular curvature tensor and $\mathcal{W}_2$-curvature tensor respectively. Finally, an example has been constructed to verify various theorems related to the study of $N(k)$-quasi Einstein manifold.

1. INTRODUCTION

An n-dimensional semi-Riemannian or Riemannian manifold (M^n, g), $n > 2$, is called an *Einstein manifold* if its Ricci tensor S satisfies the criteria

$$S = \frac{r}{n} g,$$

where r denotes the scalar curvature of (M^n, g). We can also say an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi-Einstein manifold was introduced by M.C. Chaki and R.K. Maity [2]. A non-flat Riemannian manifold (M^n, g), $n \geq 3$, is a *quasi-Einstein manifold* if its Ricci tensor S satisfies the criteria

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{1.1}$$

and is not identically zero, where a and b are smooth functions of which $b \neq 0$ and η is a non-zero 1-form such that

$$g(X, \xi) = \eta(X), \quad g(\xi, \xi) = \eta(\xi) = 1, \tag{1.2}$$

for all vector field X.

We call η as associated 1-form and ξ as generator of the manifold, which is also an unit vector field. The study of quasi-Einstein manifolds was further continued by Guha [11], De and Ghosh [8], Bejan [1], De and De [6], Debnath and Konar [10], Jana and Shaikh [15] and many others.

Let R denotes the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ [22] of a Riemannian manifold M is defined by

$$N(k)\colon p \longrightarrow N_p(k) = \{Z \in T_pM\colon R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

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where k is a smooth function. M.M. Tripathi and Jeong Sik-Kim [24] introduced the notion of $N(k)$-quasi Einstein manifolds which is defined as follows: If the generator ξ belongs to the k-nullity distribution $N(k)$, then a quasi-Einstein manifold (M^n, g) is called $N(k)$*-quasi Einstein manifold.*

Lemma 1.1 ([19])**.** *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1}.$$

In [24], Tripathi and Kim proved that an n-dimensional conformally flat quasi-Einstein manifold is an $N\Big(\dfrac{a+b}{n-1}\Big)$-quasi Einstein manifold and in particular a 3-dimensional quasi-Einstein manifold is an $N\Big(\dfrac{a+b}{2}\Big)$-quasi Einstein manifold. Various geometrical properties of $N(k)$-quasi Einstein manifolds have been discussed by Taleshian and Hosseinzadeh [13,21], De, De and Gazi [7], Yang and Xu [25], Crâşmăreanu [5], Mallick and De [16], Chaubey [3], Chaubey, Bhaishya and Siddiqi [4] and many others. The above works inspired me to write up a study on this type of manifold.

In 1957, Ishii [14] defined the conharmonic curvature tensor $\mathcal{L}$ on a Riemannian manifold as

$$\mathcal{L}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right],$$

where Q is the Ricci operator, i.e., $S(X,Y) = g(QX,Y)$, for all X, Y.
The concircular curvature tensor $\mathcal{V}$ in a Riemannian manifold (M^n, g) is defined by ([26], [27])

$$\mathcal{V}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right],$$

where r is the scalar curvature.

In [20], Pokhariyal and Mishra defined the $\mathcal{W}_2$-curvature tensor by

$$\mathcal{W}_2(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\left[g(Y,Z)QX - g(X,Z)QY\right],$$

where Q is the Ricci operator.

The derivation conditions $R(\xi,X)\cdot R = 0$ and $R(\xi,X)\cdot S = 0$ have been discussed in [24], where R and S denotes the curvature tensor and Ricci tensor of the manifold respectively. In [19], Özgür and Tripathi studied the derivation conditions $\mathcal{V}(\xi,X)\cdot\mathcal{V} = 0$ and $\mathcal{V}(\xi,X)\cdot R = 0$ on $N(k)$-quasi Einstein manifolds, where $\mathcal{V}$ denotes the concircular curvature tensor. In [17], Özgür studied the conditions $R(\xi,X)\cdot\mathcal{P} = 0$, $\mathcal{P}(\xi,X)\cdot S = 0$ and $\mathcal{P}(\xi,X)\cdot\mathcal{P} = 0$, for an $N(k)$-quasi Einstein manifolds, where $\mathcal{P}$ denotes the projective curvature tensor and some physical examples of $N(k)$-quasi Einstein manifold are given. Again, in 2008, zgr and Sular [18] continued the study of $N(k)$-quasi Einstein manifolds satisfying the conditions $R(\xi,X)\cdot\mathcal{C} = 0$ and $R(\xi,X)\cdot\mathcal{C}_* = 0$, where $\mathcal{C}$ and $\mathcal{C}_*$ denotes the Weyl conformal and quasi-conformal curvature tensors, respectively. In 2011, Taleshian and Hosseinzadeh [21] continued the study of $N(k)$-quasi Einstein manifolds with conditions $R(\xi,X)\cdot\mathcal{L} = 0$, $\mathcal{L}(\xi,X)\cdot S = 0$, $\mathcal{P}(\xi,X)\cdot\mathcal{L} = 0$, $R(\xi,X)\cdot\mathcal{P}_* = 0$ and $\mathcal{P}_*(\xi,X)\cdot S = 0$, where $\mathcal{L}$, $\mathcal{P}$ and $\mathcal{P}_*$ denotes the conharmonic curvature tensor, the projective curvature tensor and pseudo projective curvature tensor, respectively.

After studying and analyzing the above mentioned papers and ([23], [9]), we got motivated to work in this area. Recently, in the paper [12] we have studied generalized Quasi-Einstein manifolds satisfying certain vector fields. In the present work we have tried to develop a new concept. This paper is organized as follows: After discussing preliminaries in Section 3, we have studied that an $N(k)$-quasi Einstein manifold satisfies $\mathcal{L}(\xi, X) \cdot \mathcal{L} = 0$. Section 4 is concerned with an $N(k)$-quasi Einstein manifold satisfies $\mathcal{L}(\xi, X) \cdot \mathcal{V} = 0$. In the next two sections, we study an $N(k)$-quasi Einstein manifold satisfies $\mathcal{L}(\xi, X) \cdot \mathcal{W}_2 = 0$ and $\mathcal{W}_2(\xi, X) \cdot \mathcal{L} = 0$ respectively. Finally, we construct an example of $N(k)$-quasi Einstein manifold to verify Theorem 3.1, Theorem 5.1 and Theorem 6.1.

2. PRELIMINARIES

From (1.1) and (1.2) it follows that

$$r = na + b, \quad QX = aX + b\eta(X)\xi \quad \text{and} \quad S(X,\xi) = k(n-1)\eta(X),$$

where r is the scalar curvature and Q is the Ricci operator.

In an n-dimensional $N(k)$-quasi Einstein manifold M, the conharmonic curvature tensor $\mathcal{L}$ takes the form

$$\mathcal{L}(X,Y)Z = \frac{nb-na-2b}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] - \frac{b}{n-2}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi].$$

Consequently, we have

$$\mathcal{L}(\xi,Y)Z = -\frac{na+b}{(n-1)(n-2)}[g(Y,Z)\xi - \eta(Z)Y], \tag{2.1}$$

$$\begin{aligned}\mathcal{L}(X,Y,Z,W) &= g(\mathcal{L}(X,Y)Z,W)\\ &= \frac{nb-na-2b}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &\quad -\frac{b}{n-2}[g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z)\\ &\quad +g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)],\end{aligned} \tag{2.2}$$

$$\eta(\mathcal{L}(X,Y)Z) = -\frac{na+b}{(n-1)(n-2)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.3}$$

$$\eta(\mathcal{L}(X,Y)\xi) = 0,$$

$$\eta(\mathcal{L}(X,\xi)Z) = -\frac{na+b}{(n-1)(n-2)}[\eta(X)\eta(Z) - g(X,Z)],$$

$$\eta(\mathcal{L}(\xi,Y)Z) = -\frac{na+b}{(n-1)(n-2)}[g(Y,Z) - \eta(Y)\eta(Z)], \tag{2.4}$$

for all vector fields X, Y, Z, W on M.

Also the concircular curvature tensor $\mathcal{V}$ in an n-dimensional $N(k)$-quasi Einstein manifold satisfies the following relations:

$$\mathcal{V}(X,Y)Z = \frac{b}{n}[g(Y,Z)X - g(X,Z)Y], \tag{2.5}$$

$$\mathcal{V}(X,Y)\xi = \frac{b}{n}[\eta(Y)X - \eta(X)Y],$$

$$\mathcal{V}(X,\xi)Z = \frac{b}{n}\left[\eta(Z)X - g(X,Z)\xi\right],$$
$$\mathcal{V}(\xi,Y)Z = \frac{b}{n}\left[g(Y,Z)\xi - \eta(Z)Y\right], \tag{2.6}$$
$$\mathcal{V}(X,Y,Z,W) = g(\mathcal{V}(X,Y)Z,W) = \frac{b}{n}\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right],$$
$$\eta(\mathcal{V}(X,Y)Z) = \frac{b}{n}\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right], \tag{2.7}$$

for all vector fields X, Y, Z, W on M.

Again in an n-dimensional $N(k)$-quasi Einstein manifold M, the $\mathcal{W}_2$-curvature tensor satisfies:

$$\mathcal{W}_2(X,Y)Z = \frac{b}{n-1}\left[g(Y,Z)X - g(X,Z)Y + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\right],$$
$$\mathcal{W}_2(\xi,Y)Z = \frac{b}{n-1}\left[\eta(Z)\eta(Y)\xi - \eta(Z)Y\right], \tag{2.8}$$
$$\begin{aligned}\mathcal{W}_2(X,Y,Z,U) &= g(\mathcal{W}_2(X,Y)Z,U)\\ &= \frac{b}{n-1}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U) + g(X,Z)\eta(Y)\eta(U)\\ &\quad -g(Y,Z)\eta(X)\eta(U)]\end{aligned} \tag{2.9}$$
$$\eta(\mathcal{W}_2(X,Y)Z) = 0, \tag{2.10}$$

for all vector fields X, Y, Z, U on M.

3. $N(K)$-QUASI EINSTEIN MANIFOLD SATISFYING $\mathcal{L}(\xi,X)\cdot\mathcal{L} = 0$

In this section we consider an n-dimensional $N(k)$-quasi Einstein manifold M satisfying the condition $(\mathcal{L}(\xi,X)\cdot\mathcal{L})(Y,Z)W = 0$.

Theorem 3.1. *An n-dimensional $N(k)$-quasi Einstein manifold satisfies the condition $\mathcal{L}(\xi,X)\cdot\mathcal{L} = 0$ if and only if the scalar curvature is zero.*

Proof. Let us assume that an n-dimensional $N(k)$-quasi Einstein manifold M satisfying the condition $(\mathcal{L}(\xi,X)\cdot\mathcal{L})(Y,Z)W = 0$. Then we have

$$\mathcal{L}(\xi,X)\mathcal{L}(Y,Z)W - \mathcal{L}(\mathcal{L}(\xi,X)Y,Z)W - \mathcal{L}(Y,\mathcal{L}(\xi,X)Z)W - \mathcal{L}(Y,Z)\mathcal{L}(\xi,X)W = 0. \tag{3.1}$$

Using (2.1) in (3.1) we have

$$\begin{aligned}(na+b)\,[\mathcal{L}(Y,Z,W,X)\xi - \eta(\mathcal{L}(Y,Z)W)X - g(X,Y)\mathcal{L}(\xi,Z)W + \eta(Y)\mathcal{L}(X,Z)W\\ -g(X,Z)\mathcal{L}(Y,\xi)W + \eta(Z)\mathcal{L}(Y,X)W - g(X,W)\mathcal{L}(Y,Z)\xi + \eta(W)\mathcal{L}(Y,Z)X] = 0.\end{aligned}$$

Then either $na+b = 0$ or,

$$\begin{aligned}\mathcal{L}(Y,Z,W,X)\xi - \eta(\mathcal{L}(Y,Z)W)X - g(X,Y)\mathcal{L}(\xi,Z)W + \eta(Y)\mathcal{L}(X,Z)W\\ - g(X,Z)\mathcal{L}(Y,\xi)W + \eta(Z)\mathcal{L}(Y,X)W - g(X,W)\mathcal{L}(Y,Z)\xi + \eta(W)\mathcal{L}(Y,Z)X = 0.\end{aligned} \tag{3.2}$$

Taking the inner product on both sides of (3.2) with ξ we get

$$\begin{aligned}\mathcal{L}(Y,Z,W,X) &- \eta(\mathcal{L}(Y,Z)W)\eta(X) - g(X,Y)\eta(\mathcal{L}(\xi,Z)W) + \eta(Y)\eta(\mathcal{L}(X,Z)W)\\ &- g(X,Z)\eta(\mathcal{L}(Y,\xi)W) + \eta(Z)\eta(\mathcal{L}(Y,X)W) - g(X,W)\eta(\mathcal{L}(Y,Z)\xi)\\ &+ \eta(W)\eta(\mathcal{L}(Y,Z)X) = 0. \end{aligned} \tag{3.3}$$

Now using the equations (2.2) to (2.4) in (3.3) we have

$$\begin{aligned} b\,[&g(Y,X)\eta(Z)\eta(W) - g(Z,X)\eta(Y)\eta(W) + g(Z,W)\eta(Y)\eta(X)\\ &- g(Y,W)\eta(Z)\eta(X) + g(X,Z)g(Y,W) - g(X,Y)g(Z,W)] = 0.\end{aligned}$$

In an $N(k)$-quasi Einstein manifold $b \neq 0$. So we obtain the following:

$$\begin{aligned} &g(Y,X)\eta(Z)\eta(W) - g(Z,X)\eta(Y)\eta(W) + g(Z,W)\eta(Y)\eta(X)\\ &\quad - g(Y,W)\eta(Z)\eta(X) + g(X,Z)g(Y,W) - g(X,Y)g(Z,W) = 0. \end{aligned} \tag{3.4}$$

Contracting X and Y in (3.4) we get $\eta(Z)\eta(W) - g(Z,W) = 0$. From (1.1) it follows that the manifold becomes an Einstein manifold. This is a cotradiction. Thus we have $na + b = 0$, that is, $r = 0$. Conversely, if $r = 0$, then in view of (2.1) the manifold satisfies $\mathcal{L}(\xi,X)\cdot\mathcal{L} = 0$.

Thus the proof of theorem is completed. □

4. $N(K)$-QUASI EINSTEIN MANIFOLD SATISFYING $\mathcal{L}(\xi,X)\cdot\mathcal{V} = 0$

In this section we show that an n-dimensional $N(k)$-quasi Einstein manifold M satisfies the condition $\mathcal{L}(\xi,X)\cdot\mathcal{V} = 0$.

Theorem 4.1. *An n-dimensional $N(k)$-quasi Einstein manifold M always satisfies the relation $\mathcal{L}(\xi,X)\cdot\mathcal{V} = 0$.*

Proof. Now,

$$\begin{aligned}(\mathcal{L}(\xi,X)\cdot\mathcal{V})(Y,Z)W =& \mathcal{L}(\xi,X)\mathcal{V}(Y,Z)W - \mathcal{V}(\mathcal{L}(\xi,X)Y,Z)W\\ &- \mathcal{V}(Y,\mathcal{L}(\xi,X)Z)W - \mathcal{V}(Y,Z)\mathcal{L}(\xi,X)W.\end{aligned} \tag{4.1}$$

From (2.1) and (4.1) we have

$$\begin{aligned}(\mathcal{L}(\xi,X)\cdot\mathcal{V})(Y,Z)W = \frac{na+b}{(n-1)(n-2)}&[-\mathcal{V}(Y,Z,W,X)\xi + \eta(\mathcal{V}(Y,Z)W)X\\ &+ g(X,Y)\mathcal{V}(\xi,Z)W - \eta(Y)\mathcal{V}(X,Z)W + g(X,Z)\mathcal{V}(Y,\xi)W\\ &- \eta(Z)\mathcal{V}(Y,X)W + g(X,W)\mathcal{V}(Y,Z)\xi - \eta(W)\mathcal{V}(Y,Z)X].\end{aligned} \tag{4.2}$$

Using (2.5) to (2.7) in (4.2) we get $(\mathcal{L}(\xi,X)\cdot\mathcal{V})(Y,Z)W = 0$.

Thus, we complete the proof. □

Theorem 4.2. *There is no n-dimensional $N(k)$-quasi Einstein manifold satisfying the condition $\mathcal{V}(\xi,X)\cdot\mathcal{L} = 0$.*

Proof. We consider an n-dimensional $N(k)$-quasi Einstein manifold satisfying the condition

$$(\mathcal{V}(\xi,X)\cdot\mathcal{L})(Y,Z)W = 0.$$

Then

$$\mathcal{V}(\xi, X)\mathcal{L}(Y,Z)W - \mathcal{L}(\mathcal{V}(\xi, X)Y, Z)W - \mathcal{L}(Y, \mathcal{V}(\xi, X)Z)W - \mathcal{L}(Y,Z)\mathcal{V}(\xi, X)W = 0. \tag{4.3}$$

Using (2.6) in (4.3) yields

$$b[\mathcal{L}(Y,Z,W,X)\xi - \eta(\mathcal{L}(Y,Z)W)X - g(X,Y)\mathcal{L}(\xi,Z)W + \eta(Y)\mathcal{L}(X,Z)W - g(X,Z)\mathcal{L}(Y,\xi)W + \eta(Z)\mathcal{L}(Y,X)W - g(X,W)\mathcal{L}(Y,Z)\xi + \eta(W)\mathcal{L}(Y,Z)X] = 0.$$

Since in an $N(k)$-quasi Einstein manifold $b \neq 0$, we have

$$\mathcal{L}(Y,Z,W,X)\xi - \eta(\mathcal{L}(Y,Z)W)X - g(X,Y)\mathcal{L}(\xi,Z)W + \eta(Y)\mathcal{L}(X,Z)W - g(X,Z)\mathcal{L}(Y,\xi)W + \eta(Z)\mathcal{L}(Y,X)W - g(X,W)\mathcal{L}(Y,Z)\xi + \eta(W)\mathcal{L}(Y,Z)X = 0. \tag{4.4}$$

Taking the inner product on both sides of (4.4) with ξ we get

$$\mathcal{L}(Y,Z,W,X) - \eta(\mathcal{L}(Y,Z)W)\eta(X) - g(X,Y)\eta(\mathcal{L}(\xi,Z)W) + \eta(Y)\eta(\mathcal{L}(X,Z)W) - g(X,Z)\eta(\mathcal{L}(Y,\xi)W) + \eta(Z)\eta(\mathcal{L}(Y,X)W) - g(X,W)\eta(\mathcal{L}(Y,Z)\xi) + \eta(W)\eta(\mathcal{L}(Y,Z)X) = 0. \tag{4.5}$$

By virtue of (2.2) to (2.4) we obtain from (4.5) that

$$b[g(X,Y)g(Z,W) - g(X,Z)g(Y,W) - g(Y,X)\eta(Z)\eta(W) + g(Z,X)\eta(Y)\eta(W) - g(Z,W)\eta(Y)\eta(X) + g(Y,W)\eta(Z)\eta(X)] = 0.$$

Since $b \neq 0$,

$$g(X,Y)g(Z,W) - g(X,Z)g(Y,W) - g(Y,X)\eta(Z)\eta(W) + g(Z,X)\eta(Y)\eta(W) - g(Z,W)\eta(Y)\eta(X) + g(Y,W)\eta(Z)\eta(X) = 0. \tag{4.6}$$

Putting $X = Y = e_i$ in (4.6), where $\{e_i\}$, $i = 1, 2, \ldots, n$, be an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq n$, we have

$$g(Z,W) - \eta(W)\eta(Z) = 0.$$

From (1.1) it follows that the manifold becomes an Einstein manifold, which is a contradiction. Thus the theorem is proved. □

5. $N(K)$-QUASI EINSTEIN MANIFOLD SATISFYING $\mathcal{L}(\xi, X) \cdot \mathcal{W}_2 = 0$

Theorem 5.1. *An $N(k)$-quasi Einstein manifold (M^n, g) satisfies the condition $\mathcal{L}(\xi, X) \cdot \mathcal{W}_2 = 0$ if and only if the scalar curvature is zero.*

Proof. Let us suppose that the manifold (M^n, g) be an $N(k)$-quasi Einstein manifold. Then the condition $\mathcal{L}(\xi, X) \cdot \mathcal{W}_2 = 0$ gives

$$\mathcal{L}(\xi, X)\mathcal{W}_2(Y,Z)U - \mathcal{W}_2(\mathcal{L}(\xi, X)Y, Z)U - \mathcal{W}_2(Y, \mathcal{L}(\xi, X)Z)U - \mathcal{W}_2(Y,Z)\mathcal{L}(\xi, X)U = 0. \tag{5.1}$$

In view of (2.1) and (5.1) we get

$$(na+b)[\mathcal{W}_2(Y,Z,U,X)\xi - \eta(\mathcal{W}_2(Y,Z)U)X - g(X,Y)\mathcal{W}_2(\xi,Z)U + \eta(Y)\mathcal{W}_2(X,Z)U - g(X,Z)\mathcal{W}_2(Y,\xi)U + \eta(Z)\mathcal{W}_2(Y,X)U - g(X,U)\mathcal{W}_2(Y,Z)\xi + \eta(U)\mathcal{W}_2(Y,Z)X] = 0,$$

which implies either $na + b = 0$ or

$$\begin{aligned} &\mathcal{W}_2(Y,Z,U,X)\,\xi - \eta(\mathcal{W}_2(Y,Z)U)X - g(X,Y)\mathcal{W}_2(\xi,Z)U + \eta(Y)\mathcal{W}_2(X,Z)U \\ &\quad - g(X,Z)\mathcal{W}_2(Y,\xi)U + \eta(Z)\mathcal{W}_2(Y,X)U - g(X,U)\mathcal{W}_2(Y,Z)\xi \\ &\quad + \eta(U)\mathcal{W}_2(Y,Z)X = 0. \end{aligned} \tag{5.2}$$

Taking the inner product on both sides of (5.2) with ξ we have

$$\begin{aligned} &\mathcal{W}_2(Y,Z,U,X) - \eta(\mathcal{W}_2(Y,Z)U)\eta(X) - g(X,Y)\eta(\mathcal{W}_2(\xi,Z)U) + \eta(Y)\eta(\mathcal{W}_2(X,Z)U) \\ &\quad - g(X,Z)\eta(\mathcal{W}_2(Y,\xi)U) + \eta(Z)\eta(\mathcal{W}_2(Y,X)U) - g(X,U)\eta(\mathcal{W}_2(Y,Z)\xi) \\ &\quad + \eta(U)\eta(\mathcal{W}_2(Y,Z)X) = 0. \end{aligned} \tag{5.3}$$

Using (2.9) and (2.10) in (5.3) we obtain

$$b\,[g(Z,U)g(Y,X) - g(Y,U)g(Z,X) + g(Y,U)\eta(Z)\eta(X) - g(Z,U)\eta(Y)\eta(X)] = 0.$$

In an $N(k)$-quasi Einstein manifold $b \neq 0$. So we have the following:

$$g(Z,U)g(Y,X) - g(Y,U)g(Z,X) + g(Y,U)\eta(Z)\eta(X) - g(Z,U)\eta(Y)\eta(X) = 0. \tag{5.4}$$

Contracting (5.4) over U and Z, we get $g(Y,X) - \eta(Y)\eta(X) = 0$, which contradicts the definition of an $N(k)$-quasi Einstein manifold. Then we have $na + b = 0$, i.e., $r = 0$. Conversely, let $r = 0$, then from (2.1), we have $\mathcal{L}(\xi,X)\cdot\mathcal{W}_2 = 0$.

So the proof is complete. □

6. $N(K)$-QUASI EINSTEIN MANIFOLD SATISFYING $\mathcal{W}_2(\xi,X)\cdot\mathcal{L} = 0$

In this section we assume that an n-dimensional $N(k)$-quasi Einstein manifold M satisfying the condition $(\mathcal{W}_2(\xi,X)\cdot\mathcal{L})(Y,Z)U = 0$.

Theorem 6.1. *An n-dimensional $N(k)$-quasi Einstein manifold (M^n, g) satisfies the condition $\mathcal{W}_2(\xi,X)\cdot\mathcal{L} = 0$ if and only if the scalar curvature is zero.*

Proof. We first consider an n-dimensional $N(k)$-quasi Einstein manifold M satisfying the condition $(\mathcal{W}_2(\xi,X)\cdot\mathcal{L})(Y,Z)U = 0$. Then we have

$$\begin{aligned} &\mathcal{W}_2(\xi,X)\mathcal{L}(Y,Z)U - \mathcal{L}(\mathcal{W}_2(\xi,X)Y,Z)U - \mathcal{L}(Y,\mathcal{W}_2(\xi,X)Z)U \\ &\quad - \mathcal{L}(Y,Z)\mathcal{W}_2(\xi,X)U = 0, \end{aligned} \tag{6.1}$$

for all vector fields X, Y, Z, U on M.

From (2.8) and (6.1) we get

$$\begin{aligned} &b\,[\eta(\mathcal{L}(Y,Z)U)\eta(X)\xi - \eta(\mathcal{L}(Y,Z)U)X - \eta(Y)\eta(X)\mathcal{L}(\xi,Z)U + \eta(Y)\mathcal{L}(X,Z)U \\ &\quad - \eta(Z)\eta(X)\mathcal{L}(Y,\xi)U + \eta(Z)\mathcal{L}(Y,X)U - \eta(U)\eta(X)\mathcal{L}(Y,Z)\xi \\ &\quad + \eta(U)\mathcal{L}(Y,Z)X] = 0. \end{aligned}$$

Since $b \neq 0$,

$$\begin{aligned} &\eta(\mathcal{L}(Y,Z)U)\eta(X)\xi - \eta(\mathcal{L}(Y,Z)U)X - \eta(Y)\eta(X)\mathcal{L}(\xi,Z)U + \eta(Y)\mathcal{L}(X,Z)U \\ &\quad - \eta(Z)\eta(X)\mathcal{L}(Y,\xi)U + \eta(Z)\mathcal{L}(Y,X)U - \eta(U)\eta(X)\mathcal{L}(Y,Z)\xi \\ &\quad + \eta(U)\mathcal{L}(Y,Z)X = 0. \end{aligned} \tag{6.2}$$

Taking the inner product on both sides of (6.2) with ξ we have

$$\eta(Y)\eta(X)\eta(\mathcal{L}(\xi,Z)U) - \eta(Y)\eta(\mathcal{L}(X,Z)U) + \eta(Z)\eta(X)\eta(\mathcal{L}(Y,\xi)U) - \eta(Z)\eta(\mathcal{L}(Y,X)U) + \eta(U)\eta(X)\eta(\mathcal{L}(Y,Z)\xi) - \eta(U)\eta(\mathcal{L}(Y,Z)X) = 0. \tag{6.3}$$

Using (2.3) to (2.4) in (6.3) we obtain

$$(na+b)\eta(U)[g(Y,X)\eta(Z) - g(Z,X)\eta(Y)] = 0. \tag{6.4}$$

Putting $Z = \xi$ in (6.4) we get

$$(na+b)\eta(U)[g(Y,X) - \eta(X)\eta(Y)] = 0. \tag{6.5}$$

Since in an $N(k)$-quasi Einstein manifold the 1-form η is non-zero and

$$g(Y,X) - \eta(X)\eta(Y) \neq 0,$$

from equation (6.5) it follows that $na + b = 0$, i.e., $r = 0$. Again, if we take $r = 0$, then the converse is trivial. This completes the proof. □

Therefore, by Theorem 3.1, Theorem 5.1 and Theorem 6.1 we can state the following corollary:

Corollary 6.1. *Let (M^n, g) be an n-dimensional $N(k)$-quasi Einstein manifold. Then the following statements are equivalent:*
(*i*) $\mathcal{L}(\xi,X)\cdot\mathcal{L} = 0$,
(*ii*) $\mathcal{L}(\xi,X)\cdot\mathcal{W}_2 = 0$,
(*iii*) $\mathcal{W}_2(\xi,X)\cdot\mathcal{L} = 0$,
(*iv*) *the scalar curvature is zero,*
for every vector field X on (M^n, g).

7. EXAMPLE OF $N(K)$-QUASI EINSTEIN MANIFOLDS

Let $(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$, where $\mathbb{R}^n$ is an n-dimensional real number space. We consider a Riemannian metric g on $\mathbb{R}^4 = (x^1, x^2, x^3, x^4)$, by [9]

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 + (dx^4)^2, \tag{7.1}$$

where $i, j = 1, 2, 3, 4$. Using (7.1), we see the non-vanishing components of Riemannian metric are

$$g_{11} = 1, \quad g_{22} = (x^1)^2, \quad g_{33} = (x^2)^2, \quad g_{44} = 1 \tag{7.2}$$

and its associated components are

$$g^{11} = 1, \quad g^{22} = \frac{1}{(x^1)^2}, \quad g^{33} = \frac{1}{(x^2)^2}, \quad g^{44} = 1. \tag{7.3}$$

Using (7.2) and (7.3), we can calculate that the non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are given by

$$\Gamma^1_{22} = -x^1, \quad \Gamma^2_{33} = -\frac{x^2}{(x^1)^2}, \quad \Gamma^2_{12} = \frac{1}{x^1}, \quad \Gamma^3_{23} = \frac{1}{x^2}, \quad R_{1332} = -\frac{x^2}{x^1}, \quad S_{12} = -\frac{1}{x^1x^2}$$

and the other components are obtained by the symmetric properties. It can be easily shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4, g)$ is zero. We shall now show that this $(\mathbb{R}^4, g)$ is an $N(k)$-quasi Einstein manifold.

Let us consider the associated scalars as follows:

$$a = \frac{3}{x^1 \left(x^2\right)^2}, \quad b = -\frac{3}{\left(x^1\right)^2 x^2}. \tag{7.4}$$

We choose the 1-form as follows:

$$\eta_i\left(x\right) = \begin{cases} \frac{1}{\sqrt{3}}, & \text{when } i = 1, \\ \frac{x^1}{\sqrt{3}}, & \text{when } i = 2, \\ \frac{x^2}{\sqrt{3}}, & \text{when } i = 3, \\ 0, & \text{when } i = 4 \end{cases} \tag{7.5}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.1) reduces to the equation

$$S_{12} = a g_{12} + b \eta_1 \eta_2, \tag{7.6}$$

since, for the other cases (1.1) holds trivially.

From the equations (7.4), (7.5) and (7.6) we get:

$$\begin{aligned} \text{Right hand side of (7.6)} = a g_{12} + b \eta_1 \eta_2 &= \frac{3}{x^1 (x^2)^2} \cdot 0 + \Big(-\frac{3}{(x^1)^2 x^2} \Big) \cdot \frac{1}{\sqrt{3}} \cdot \frac{x^1}{\sqrt{3}} \\ &= -\frac{1}{x^1 x^2} = S_{12}. \end{aligned}$$

By Lemma 1.1., here we see that $k = \frac{x^1 - x^2}{(x^1)^2 (x^2)^2}$. So, $\left(\mathbb{R}^4, g\right)$ is an $N\Big(\frac{x^1 - x^2}{(x^1)^2 (x^2)^2}\Big)$-quasi Einstein manifold.

Again, since the scalar curvature r is zero, therefore $\mathcal{L}\left(\xi, X\right) \cdot \mathcal{L} = 0$, $\mathcal{L}\left(\xi, X\right) \cdot \mathcal{W}_2 = 0$ and $\mathcal{W}_2\left(\xi, X\right) \cdot \mathcal{L} = 0$. Thus Theorem 3.1, Theorem 5.1 and Theorem 6.1 are verified.

CONCLUSIONS

Quasi Einstein manifold plays an important role in both mathematics and physics. In the present paper, an $N(k)$-quasi Einstein manifold has been considered, which is a special case of quasi-Einstein manifold. We have proven that in an $N(k)$-quasi Einstein manifold, the condition $\mathcal{L}\left(\xi, X\right) \cdot \mathcal{V} = 0$ holds, for all X. We noticed $N(k)$-quasi Einstein manifold satisfies the conditions $\mathcal{L}\left(\xi, X\right) \cdot \mathcal{L} = 0$, $\mathcal{L}\left(\xi, X\right) \cdot \mathcal{W}_2 = 0$, $\mathcal{W}_2\left(\xi, X\right) \cdot \mathcal{L} = 0$ if and only if the scalar curvature is zero. We have also shown that there is no $N(k)$-quasi Einstein manifold that satisfies $\mathcal{V}\left(\xi, X\right) \cdot \mathcal{L} = 0$. Finally, an example of $N(k)$-quasi Einstein manifold has been discussed to verify certain theorems.

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Department of Mathematics
Heramba Chandra College
23/49, Gariahat Road, Kolkata - 700029
West Bengal, India
E-mail address: dipankarsk524@gmail.com

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QUASI-CONFORMAL CURVATURE TENSOR ON $N(k)$-QUASI EINSTEIN MANIFOLDS

Dipankar Hazra and Avijit Sarkar*

Abstract. This paper deals with the study of $N(k)$-quasi Einstein manifolds that satisfies the certain curvature conditions $\mathcal{C}_*\cdot\mathcal{C}_* = 0$, $\mathcal{S}\cdot\mathcal{C}_* = 0$ and $\mathcal{R}\cdot\mathcal{C}_* = f\tilde{Q}(g, \mathcal{C}_*)$, where $\mathcal{C}_*$, $\mathcal{S}$ and $\mathcal{R}$ denotes the quasi-conformal curvature tensor, Ricci tensor and the curvature tensor respectively. Finally, we construct an example of $N(k)$-quasi Einstein manifold.

1. Introduction

An n-dimensional semi-Riemannian or Riemannian manifold (M^n, g) $(n > 2)$, is called an Einstein manifold if its Ricci tensor $\mathcal{S}$ satisfies the criteria

$$\mathcal{S} = \frac{\rho}{n} g,$$

where ρ denotes the scalar curvature of (M^n, g). We can also say an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [3]. A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$, is a quasi-Einstein manifold if its Ricci tensor $\mathcal{S}$ satisfies the criteria

$$\mathcal{S}(U, V) = ag(U, V) + b\eta(U)\eta(V) \tag{1}$$

and is not identically zero, where a and b are smooth functions of which $b \neq 0$ and η is a non-zero 1-form such that

$$g(U, \xi) = \eta(U), \qquad g(\xi, \xi) = \eta(\xi) = 1, \tag{2}$$

for all vector field U.
We call η as associated 1-form and ξ as generator of the manifold, which is also an unit vector field. The study of quasi-Einstein manifolds was further continued by Guha [11], De and Ghosh [8], Bejan [2], De and De [6], Debnath and Konar [9] and many others.

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* Corresponding author.

Let $\mathcal{R}$ denotes the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ [17] of a Riemannian manifold M is defined by

$$N(k) : p \longrightarrow N_p(k) = \{W \in T_pM : \mathcal{R}(U,V)W = k[g(V,W)U - g(U,W)V]\},$$

where k is a smooth function.
M. M. Tripathi and Jeong Sik-Kim [18] introduced the notion of $N(k)$-quasi Einstein manifolds which is defined as follows: If the generator ξ belongs to the k-nullity distribution $N(k)$, then a quasi-Einstein manifold (M^n, g) is called $N(k)$-quasi Einstein manifold. Here k is not arbitrary.

LEMMA 1.1. [15] *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1}. \tag{3}$$

So we note that in an $N(k)$-quasi Einstein manifold [15]

$$\mathcal{R}(U,V)\xi = \frac{a+b}{n-1}[\eta(V)U - \eta(U)V], \tag{4}$$

which is same as

$$\mathcal{R}(\xi,U)V = \frac{a+b}{n-1}[g(U,V)\xi - \eta(V)U]. \tag{5}$$

In [18], Tripathi and Kim proved that an n-dimensional conformally flat quasi-Einstein manifold is an $N\left(\frac{a+b}{n-1}\right)$-quasi Einstein manifold and in particular a 3-dimensional quasi-Einstein manifold is an $N\left(\frac{a+b}{2}\right)$-quasi Einstein manifold. Various geometrical properties of $N(k)$-quasi Einstein manifolds have been discussed by Taleshian and Hosseinzadeh [12, 16], De, De and Gazi [7], Crasmareanu [5], Yildiz, De and Cetinkaya [20], Mallick and De [13] and many others. The above works inspired me to write up a study on this type of manifold.

In 1968, Yano and Sawaki [19] defined the quasi-conformal curvature tensor $\mathcal{C}_*$ on a Riemannian manifold (M^n, g) as

$$\begin{aligned} \mathcal{C}_*(U,V)W = {} & a_0\mathcal{R}(U,V)W + a_1[\mathcal{S}(V,W)U \\ & -\mathcal{S}(U,W)V + g(V,W)QU - g(U,W)QV] \\ & -\frac{\rho}{n}\left(\frac{a_0}{n-1} + 2a_1\right)[g(V,W)U - g(U,W)V], \end{aligned} \tag{6}$$

where $\mathcal{S}(U,V) = g(QU,V)$, ρ is the scalar curvature, a_0 and a_1 are arbitrary constants, which are not simultaneously zero. If $a_0 = 1$ and $a_1 = -\frac{1}{n-2}$, then (6) reduces to the conformal curvature tensor. Thus the conformal curvature tensor is a particular case of the tensor $\mathcal{C}_*$. A Riemannian or a semi-Riemannian manifold is called quasi-conformally flat if $\mathcal{C}_* = 0$ for $n > 3$.

The derivation conditions $\mathcal{R}(\xi,U)\cdot\mathcal{R} = 0$ and $\mathcal{R}(\xi,U)\cdot\mathcal{S} = 0$ have been discussed in [18], where $\mathcal{R}$ and $\mathcal{S}$ denotes the curvature tensor and Ricci tensor of the manifold respectively. In 2008, Özgür and Sular [14] studied the derivation conditions $\mathcal{R}(\xi,U)\cdot\mathcal{C} = 0$ and $\mathcal{R}(\xi,U)\cdot\mathcal{C}_* = 0$ on $N(k)$-quasi Einstein manifolds, where $\mathcal{C}$ and $\mathcal{C}_*$ denotes the Weyl conformal and quasi-conformal curvature tensors, respectively.

After studying and analyzing the above papers, we got motivated to work in this area. In the present work we have tried to develop a new concept. This paper is organized as follows: Section 2 is preliminaries that covers various concepts and results of $N(k)$-quasi Einstein manifold and quasi-conformal curvature tensor. Section 3 deals with study of quasi-conformal curvature tensor of an $N(k)$-quasi Einstein manifold. Section 4 is concerned with an $N(k)$-quasi Einstein manifold satisfies $\mathcal{S}(U,\xi)\cdot\mathcal{C}_* = 0$. The properties of $\mathcal{C}_*$-pseudosymmetric $N(k)$-quasi Einstein manifolds had been analyzed in section 5. Finally, we give an example of $N(k)$-quasi Einstein manifold.

2. Preliminaries

From (1) and (2) it follows that

$$\rho = an + b \tag{7}$$

and

$$\mathcal{S}(U,\xi) = (a+b)\,\eta(U), \tag{8}$$

where ρ is the scalar curvature and Q is the Ricci operator.

In an n-dimensional $N(k)$-quasi Einstein manifold M, the quasi-conformal curvature tensor $\mathcal{C}_*$ takes the form

$$\begin{aligned}\mathcal{C}_*(U,V)W = {} & \frac{b}{n}(a_0 - 2a_1)\left[g(V,W)U - g(U,W)V\right] \\ & + ba_1\left[\eta(V)\eta(W)U - \eta(U)\eta(W)V\right. \\ & \left. + g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi\right].\end{aligned} \tag{9}$$

Consequently, we have

$$\mathcal{C}_*(\xi,U)V = \frac{b}{n}\left[a_0 + (n-2)a_1\right]\left[g(U,V)\xi - \eta(V)U\right], \tag{10}$$

$$\eta(\mathcal{C}_*(U,V)W) = \frac{b}{n}\left[a_0 + (n-2)a_1\right]\left[g(V,W)\eta(U) - g(U,W)\eta(V)\right], \tag{11}$$

$$\eta(\mathcal{C}_*(U,V)\xi) = 0 \tag{12}$$

and

$$\eta(\mathcal{C}_*(U,\xi)V) = \frac{b}{n}\left[a_0 + (n-2)a_1\right]\left[\eta(V)\eta(U) - g(U,V)\right] = -\eta(\mathcal{C}_*(\xi,U)V), \tag{13}$$

for all vector fields U, V, W on M.

3. The quasi-conformal curvature tensor of an $N(k)$-quasi Einstein manifold

In this section we consider an n-dimensional $N(k)$-quasi Einstein manifold M satisfying the condition $(\mathcal{C}_*(\xi,U)\cdot\mathcal{C}_*)(V,W)G = 0$. Then we have

$$\begin{aligned}&\mathcal{C}_*(\xi,U)\mathcal{C}_*(V,W)G - \mathcal{C}_*(\mathcal{C}_*(\xi,U)V,W)G \\ &-\mathcal{C}_*(V,\mathcal{C}_*(\xi,U)W)G - \mathcal{C}_*(V,W)\mathcal{C}_*(\xi,U)G = 0.\end{aligned} \tag{14}$$

Using (10) in (14) we have

$$\frac{b}{n}\left[a_0+(n-2)\,a_1\right]\left[g\left(U,\mathcal{C}_*\left(V,W\right)G\right)\xi-\eta\left(\mathcal{C}_*\left(V,W\right)G\right)U\right.$$
$$-g\left(U,V\right)\mathcal{C}_*\left(\xi,W\right)G+\eta\left(V\right)\mathcal{C}_*\left(U,W\right)G$$
$$-g\left(U,W\right)\mathcal{C}_*\left(V,\xi\right)G+\eta\left(W\right)\mathcal{C}_*\left(V,U\right)G$$
$$\left.-g\left(U,G\right)\mathcal{C}_*\left(V,W\right)\xi+\eta\left(G\right)\mathcal{C}_*\left(V,W\right)U\right]=0.$$

In an $N(k)$-quasi Einstein manifold $b\neq 0$. So we obtain the following:

$$\left[a_0+(n-2)\,a_1\right]\left[g\left(U,\mathcal{C}_*\left(V,W\right)G\right)\xi-\eta\left(\mathcal{C}_*\left(V,W\right)G\right)U\right.$$
$$-g\left(U,V\right)\mathcal{C}_*\left(\xi,W\right)G+\eta\left(V\right)\mathcal{C}_*\left(U,W\right)G$$
$$-g\left(U,W\right)\mathcal{C}_*\left(V,\xi\right)G+\eta\left(W\right)\mathcal{C}_*\left(V,U\right)G$$
$$\left.-g\left(U,G\right)\mathcal{C}_*\left(V,W\right)\xi+\eta\left(G\right)\mathcal{C}_*\left(V,W\right)U\right]=0.$$

Then either $a_0+(n-2)\,a_1=0$ or,

$$g\left(U,\mathcal{C}_*\left(V,W\right)G\right)\xi-\eta\left(\mathcal{C}_*\left(V,W\right)G\right)U$$
$$-\,g\left(U,V\right)\mathcal{C}_*\left(\xi,W\right)G+\eta\left(V\right)\mathcal{C}_*\left(U,W\right)G$$
$$-\,g\left(U,W\right)\mathcal{C}_*\left(V,\xi\right)G+\eta\left(W\right)\mathcal{C}_*\left(V,U\right)G$$
$$-\,g\left(U,G\right)\mathcal{C}_*\left(V,W\right)\xi+\eta\left(G\right)\mathcal{C}_*\left(V,W\right)U=0. \tag{15}$$

Assume that $a_0+(n-2)\,a_1\neq 0$. Taking the inner product on both sides of (15) with ξ we get

$$g\left(U,\mathcal{C}_*\left(V,W\right)G\right)-\eta\left(\mathcal{C}_*\left(V,W\right)G\right)\eta\left(U\right)$$
$$-\,g\left(U,V\right)\eta\left(\mathcal{C}_*\left(\xi,W\right)G\right)+\eta\left(V\right)\eta\left(\mathcal{C}_*\left(U,W\right)G\right)$$
$$-\,g\left(U,W\right)\eta\left(\mathcal{C}_*\left(V,\xi\right)G\right)+\eta\left(W\right)\eta\left(\mathcal{C}_*\left(V,U\right)G\right)$$
$$-\,g\left(U,G\right)\eta\left(\mathcal{C}_*\left(V,W\right)\xi\right)+\eta\left(G\right)\eta\left(\mathcal{C}_*\left(V,W\right)U\right)=0. \tag{16}$$

Now using the equations (11) - (13) in (16) we have

$$g\left(U,\mathcal{C}_*\left(V,W\right)G\right)=\frac{b}{n}\left[a_0+(n-2)\,a_1\right]\left[g\left(U,V\right)g\left(W,G\right)-g\left(U,W\right)g\left(V,G\right)\right].$$

Then using (6) and (7) we can write

$$a_0\mathcal{R}\left(V,W,G,U\right)+a_1\left[\mathcal{S}\left(W,G\right)g\left(V,U\right)\right.$$
$$\left.-\mathcal{S}\left(V,G\right)g\left(W,U\right)+g\left(W,G\right)\mathcal{S}\left(V,U\right)-g\left(V,G\right)\mathcal{S}\left(W,U\right)\right]$$
$$-\,\frac{an+b}{n}\left(\frac{a_0}{n-1}+2a_1\right)\left[g\left(W,G\right)g\left(V,U\right)-g\left(V,G\right)g\left(W,U\right)\right]$$
$$=\frac{b}{n}\left[a_0+(n-2)\,a_1\right]\left[g\left(U,V\right)g\left(W,G\right)-g\left(U,W\right)g\left(V,G\right)\right]. \tag{17}$$

Contracting (17) over U and V we obtain

$$\mathcal{S}\left(W,G\right)=(a+b)\,g\left(W,G\right).$$

This is a contradiction as M^n is not Einstein. Thus we have $a_0+(n-2)\,a_1=0$. Conversely, if $a_0+(n-2)\,a_1=0$, then in view of (10) the manifold satisfies $\mathcal{C}_*\left(\xi,U\right)\cdot\mathcal{C}_*=0$.
Thus we can state the following theorem:

THEOREM 3.1. *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Then M satisfies the condition $\mathcal{C}_*(\xi, U) \cdot \mathcal{C}_* = 0$ if and only if $a_0 + (n-2)a_1 = 0$.*

4. $N(k)$-quasi Einstein manifold satisfying $\mathcal{S}(U, \xi) \cdot \mathcal{C}_* = 0$

Let us suppose that an $N(k)$-quasi Einstein manifold (M^n, g) satisfying the condition

$$(\mathcal{S}(U, \xi) \cdot \mathcal{C}_*)(V, W)G = 0. \tag{18}$$

Now, $(\mathcal{S}(U, \xi) \cdot \mathcal{C}_*)(V, W)G = ((U \wedge_{\mathcal{S}} \xi).\mathcal{C}_*)(V, W)G$, where the endomorphism $(U \wedge_{\mathcal{S}} V)W$ is defined by

$$(U \wedge_{\mathcal{S}} V)W = \mathcal{S}(V, W)U - \mathcal{S}(U, W)V. \tag{19}$$

Then (18) takes the form,

$$\begin{aligned}(U \wedge_{\mathcal{S}} \xi)\mathcal{C}_*(V, W)G - \mathcal{C}_*((U \wedge_{\mathcal{S}} \xi)V, W)G \\ -\mathcal{C}_*(V, (U \wedge_{\mathcal{S}} \xi)W)G - \mathcal{C}_*(V, W)(U \wedge_{\mathcal{S}} \xi)G = 0.\end{aligned} \tag{20}$$

From (19) and (20), we get

$$\begin{aligned}&\mathcal{S}(\xi, \mathcal{C}_*(V, W)G)U - \mathcal{S}(U, \mathcal{C}_*(V, W)G)\xi \\ &- \mathcal{S}(\xi, V)\mathcal{C}_*(U, W)G + \mathcal{S}(U, V)\mathcal{C}_*(\xi, W)G \\ &- \mathcal{S}(\xi, W)\mathcal{C}_*(V, U)G + \mathcal{S}(U, W)\mathcal{C}_*(V, \xi)G \\ &- \mathcal{S}(\xi, G)\mathcal{C}_*(V, W)U + \mathcal{S}(U, G)\mathcal{C}_*(V, W)\xi = 0.\end{aligned} \tag{21}$$

Using (1) and (8) in (21), we have

$$\begin{aligned}(a+b)\eta(\mathcal{C}_*(V, W)G)U - ag(U, \mathcal{C}_*(V, W)G)\xi - b\eta(U)\eta(\mathcal{C}_*(V, W)G)\xi \\ - (a+b)\eta(V)\mathcal{C}_*(U, W)G + [ag(U, V) + b\eta(U)\eta(V)]\mathcal{C}_*(\xi, W)G \\ - (a+b)\eta(W)\mathcal{C}_*(V, U)G + [ag(U, W) + b\eta(U)\eta(W)]\mathcal{C}_*(V, \xi)G \\ - (a+b)\eta(G)\mathcal{C}_*(V, W)U + [ag(U, G) + b\eta(U)\eta(G)]\mathcal{C}_*(V, W)\xi = 0.\end{aligned} \tag{22}$$

Taking the inner product on both sides of (22) with ξ, we obtain

$$\begin{aligned}a\eta(\mathcal{C}_*(V, W)G)\eta(U) - ag(U, \mathcal{C}_*(V, W)G) - (a+b)\eta(V)\eta(\mathcal{C}_*(U, W)G) \\ + [ag(U, V) + b\eta(U)\eta(V)]\eta(\mathcal{C}_*(\xi, W)G) - (a+b)\eta(W)\eta(\mathcal{C}_*(V, U)G) \\ + [ag(U, W) + b\eta(U)\eta(W)]\eta(\mathcal{C}_*(V, \xi)G) - (a+b)\eta(G)\eta(\mathcal{C}_*(V, W)U) \\ + [ag(U, G) + b\eta(U)\eta(G)]\eta(\mathcal{C}_*(V, W)\xi) = 0.\end{aligned} \tag{23}$$

Using (9) and (11) - (13) in (23) we get

$$\begin{aligned}&aba_1[g(U, V)g(W, G) - g(U, W)g(V, G) - g(U, V)\eta(W)\eta(G) \\ &+g(W, U)\eta(V)\eta(G) - g(W, G)\eta(V)\eta(U) + g(V, G)\eta(W)\eta(U)] \\ &- \frac{b^2}{n}[a_0 + (n-2)a_1][g(W, U)\eta(V)\eta(G) - g(V, U)\eta(W)\eta(G)] = 0.\end{aligned} \tag{24}$$

Putting $W = \xi$ in (24), we obtain

$$\frac{b^2}{n}[a_0 + (n-2)a_1]\eta(G)[\eta(U)\eta(V) - g(U, V)] = 0. \tag{25}$$

Since in an $N(k)$-quasi Einstein manifold $b \neq 0$, the 1-form η is non-zero and $g(U,V) \neq \eta(U)\eta(V)$, from equation (25) it follows that $a_0 + (n-2)a_1 = 0$. Again, if we take $a_0 + (n-2)a_1 = 0$, then the converse is trivial.
This leads to the following theorem:

Theorem 4.1. *An n-dimensional $N(k)$-quasi Einstein manifold M satisfies $\mathcal{S}(U,\xi)\cdot \mathcal{C}_* = 0$ if and only if $a_0 + (n-2)a_1 = 0$.*

Therefore, by Theorem 3.1. and 4.1. we can state the following corollary:

Corollary 4.2. *Let (M^n, g) be an n-dimensional $N(k)$-quasi Einstein manifold. Then the following statements are equivalent:*

(i) $\mathcal{C}_*(\xi,U)\cdot \mathcal{C}_* = 0$,
(ii) $\mathcal{S}(U,\xi)\cdot \mathcal{C}_* = 0$,
(iii) $a_0 + (n-2)a_1 = 0$,

for every vector field U on (M^n, g).

5. $\mathcal{C}_*$-pseudosymmetric $N(k)$-quasi Einstein manifolds

In [14], Özgür and Sular studied the condition $\mathcal{R}(\xi,U)\cdot \mathcal{C}_* = 0$ for an $N(k)$-quasi Einstein manifolds, where $\mathcal{C}_*$ is the quasi-conformal curvature tensor and $\mathcal{R}$ is the curvature tensor of the manifold. In this section we generalize this condition.

An n-dimensional Riemannian or a semi-Riemannian manifold (M^n, g) is said to be $\mathcal{C}_*$-pseudosymmetric [10] if and only if the tensors $\mathcal{R}\cdot\mathcal{C}_*$ and $\tilde{Q}(g,\mathcal{C}_*)$ defined by

$$\begin{aligned}(\mathcal{R}(U,V)\cdot\mathcal{C}_*)(W,G)H &= \mathcal{R}(U,V)\mathcal{C}_*(W,G)H - \mathcal{C}_*(\mathcal{R}(U,V)W,G)H \\ &\quad - \mathcal{C}_*(W,\mathcal{R}(U,V)G)H - \mathcal{C}_*(W,G)\mathcal{R}(U,V)H \end{aligned} \tag{26}$$

and

$$\begin{aligned}\tilde{Q}(g,\mathcal{C}_*)(W,G,H;U,V) &= ((U\wedge V)\cdot\mathcal{C}_*)(W,G)H \\ &= (U\wedge V)\mathcal{C}_*(W,G)H - \mathcal{C}_*((U\wedge V)W,G)H \\ &\quad - \mathcal{C}_*(W,(U\wedge V)G)H - \mathcal{C}_*(W,G)(U\wedge V)H \end{aligned} \tag{27}$$

are linearly dependent, i.e.,

$$(\mathcal{R}(U,V)\cdot\mathcal{C}_*)(W,G)H = f\tilde{Q}(g,\mathcal{C}_*)(W,G,H;U,V), \tag{28}$$

for arbitrary vector fields U,V,W,G,H on M^n and the endomorphism $(U\wedge V)$ is defined by

$$(U\wedge V)W = g(V,W)U - g(U,W)V \tag{29}$$

and f is a smooth function on $\Omega_{\mathcal{C}_*} = \{x \in M^n : \mathcal{C}_* \neq 0 \text{ at } x\}$.
If $f = 0$, then the manifold (M^n, g) reduces to a quasi-conformally semisymmetric manifold (i.e. $\mathcal{R}\cdot\mathcal{C}_* = 0$).

From (26), (27) and (28) we have

$$\begin{aligned}&\mathcal{R}(U,V)\,\mathcal{C}_*(W,G)H-\mathcal{C}_*(\mathcal{R}(U,V)W,G)H\\&-\mathcal{C}_*(W,\mathcal{R}(U,V)G)H-\mathcal{C}_*(W,G)\mathcal{R}(U,V)H\\&=f\left[(U\wedge V)\,\mathcal{C}_*(W,G)H-\mathcal{C}_*((U\wedge V)W,G)H\right.\\&\left.-\mathcal{C}_*(W,(U\wedge V)G)H-\mathcal{C}_*(W,G)(U\wedge V)H\right].\end{aligned}\tag{30}$$

Putting $U=\xi$ in (30) and then using (2), (5) and (29), we obtain that

$$\begin{aligned}(k-f)\,&[g(V,\mathcal{C}_*(W,G)H)\xi-\eta(\mathcal{C}_*(W,G)H)V\\&-g(V,W)\,\mathcal{C}_*(\xi,G)H+\eta(W)\,\mathcal{C}_*(V,G)H\\&-g(V,G)\,\mathcal{C}_*(W,\xi)H+\eta(G)\,\mathcal{C}_*(W,V)H\\&-g(V,H)\,\mathcal{C}_*(W,G)\xi+\eta(H)\,\mathcal{C}_*(W,G)V]=0,\end{aligned}$$

which implies either $f=k$ or

$$\begin{aligned}&g(V,\mathcal{C}_*(W,G)H)\xi-\eta(\mathcal{C}_*(W,G)H)V\\&-g(V,W)\,\mathcal{C}_*(\xi,G)H+\eta(W)\,\mathcal{C}_*(V,G)H\\&-g(V,G)\,\mathcal{C}_*(W,\xi)H+\eta(G)\,\mathcal{C}_*(W,V)H\\&-g(V,H)\,\mathcal{C}_*(W,G)\xi+\eta(H)\,\mathcal{C}_*(W,G)V=0.\end{aligned}\tag{31}$$

Taking the inner product on both sides of (31) with ξ, we get

$$\begin{aligned}&g(V,\mathcal{C}_*(W,G)H)-\eta(\mathcal{C}_*(W,G)H)\,\eta(V)\\&-g(V,W)\,\eta(\mathcal{C}_*(\xi,G)H)+\eta(W)\,\eta(\mathcal{C}_*(V,G)H)\\&-g(V,G)\,\eta(\mathcal{C}_*(W,\xi)H)+\eta(G)\,\eta(\mathcal{C}_*(W,V)H)\\&-g(V,H)\,\eta(\mathcal{C}_*(W,G)\xi)+\eta(H)\,\eta(\mathcal{C}_*(W,G)V)=0.\end{aligned}\tag{32}$$

By virtue of (11) - (13) we obtain from (32) that

$$g(V,\mathcal{C}_*(W,G)H)=\frac{b}{n}\left[a_0+(n-2)\,a_1\right]\left[g(V,W)\,g(G,H)-g(V,G)\,g(W,H)\right].\tag{33}$$

Using (6) and (7), (33) can be written as

$$\begin{aligned}&a_0\mathcal{R}(W,G,H,V)+a_1\left[\mathcal{S}(G,H)\,g(W,V)\right.\\&\left.-\mathcal{S}(W,H)\,g(G,V)+g(G,H)\,\mathcal{S}(W,V)-g(W,H)\,\mathcal{S}(G,V)\right]\\&-\frac{an+b}{n}\left(\frac{a_0}{n-1}+2a_1\right)\left[g(G,H)\,g(W,V)-g(W,H)\,g(G,V)\right]\\&=\frac{b}{n}\left[a_0+(n-2)\,a_1\right]\left[g(V,W)\,g(G,H)-g(V,G)\,g(W,H)\right].\end{aligned}\tag{34}$$

Putting $V=W=e_i$ in (34), where $\{e_i\}$, $i=1,2,\ldots,n$ be an orthonormal basis of the tangent space at any point of the manifold (M^n,g) and taking summation over i, $1\leq i\leq n$, we have

$$\left[a_0+(n-2)\,a_1\right]\left[\mathcal{S}(G,H)-(a+b)\,g(G,H)\right]=0.$$

Since M^n is an $N(k)$-quasi Einstein manifold, $\mathcal{S}(G,H)\neq(a+b)\,g(G,H)$. So we obtain

$$a_0+(n-2)\,a_1=0.$$

Therefore, from (11)

$$\eta\left(\mathcal{C}_*\left(U,V\right)W\right)=0. \tag{35}$$

Using (35) in (32) yields

$$g\left(V,\mathcal{C}_*\left(W,G\right)H\right)=0.$$

This implies that the manifold is quasi-conformally flat. But, in this case $\mathcal{C}_*\neq 0$. Hence $f=k$, i.e., $f=\dfrac{a+b}{n-1}$.

Thus we conclude the following theorem:

THEOREM 5.1. *In a $\mathcal{C}_*$-pseudosymmetric $N\left(k\right)$-quasi Einstein manifold* $f=\dfrac{a+b}{n-1}$.

We know that [1] a quasi-conformally flat manifold is either conformally flat or Einstein.

In [14], authors proved the following corollary:

COROLLARY 5.2. *An $N\left(k\right)$-quasi Einstein manifold is quasi-conformally semisymmetric if and only if either $a+b=0$ or the manifold is conformally flat with $a_0=\left(2-n\right)a_1$.*

Now if we take $f\neq k$, then in view of (1) and (34) we have

$$\begin{aligned}\mathcal{R}\left(W,G,H,V\right)&=\lambda\left[g\left(G,H\right)g\left(W,V\right)-g\left(W,H\right)g\left(G,V\right)\right]\\&\quad+\mu\left[g\left(W,V\right)\eta\left(G\right)\eta\left(H\right)-g\left(G,V\right)\eta\left(W\right)\eta\left(H\right)\right.\\&\quad\left.+g\left(G,H\right)\eta\left(W\right)\eta\left(V\right)-g\left(W,H\right)\eta\left(G\right)\eta\left(V\right)\right],\end{aligned} \tag{36}$$

where $\lambda=\left(k+\dfrac{ba_1}{a_0}\right)$ and $\mu=-\dfrac{ba_1}{a_0}$.

A Riemannian or semi-Riemannian manifold is said to be a manifold of quasi-constant curvature [4] if the curvature tensor $\mathcal{R}$ of type $(0,4)$ satisfies the following condition

$$\begin{aligned}\mathcal{R}\left(U,V,W,G\right)&=p\left[g\left(V,W\right)g\left(U,G\right)-g\left(U,W\right)g\left(V,G\right)\right]\\&\quad+q\left[g\left(U,G\right)\eta\left(V\right)\eta\left(W\right)-g\left(U,W\right)\eta\left(V\right)\eta\left(G\right)\right.\\&\quad\left.+g\left(V,W\right)\eta\left(U\right)\eta\left(G\right)-g\left(V,G\right)\eta\left(U\right)\eta\left(W\right)\right],\end{aligned} \tag{37}$$

where p,q are scalar functions of which $q\neq 0$ and η is a non-zero 1-form defined by

$$g\left(U,\xi\right)=\eta\left(U\right),$$

for all U and ξ being a unit vector field.

From (36) and (37), we can state the following theorem:

THEOREM 5.3. *An n-dimensional $\mathcal{C}_*$-pseudosymmetric $N\left(k\right)$-quasi Einstein manifold $\left(M^n,g\right)$, $\left(n>2\right)$ with $f\neq k$ is a manifold of quasi-constant curvature.*

6. Example of $N\left(k\right)$-quasi Einstein manifolds

Let $\left(x^1,x^2,\ldots,x^n\right)\in\mathbb{R}^n$, where $\mathbb{R}^n$ is an n-dimensional real number space. We consider a Riemannian metric g on $\mathbb{R}^4=\left(x^1,x^2,x^3,x^4\right)$, by

$$ds^2=g_{ij}dx^idx^j=\left(dx^1\right)^2+\left(x^1\right)^2\left(dx^2\right)^2+\left(x^2\right)^2\left(dx^3\right)^2+\left(dx^4\right)^2, \tag{38}$$

where $i, j = 1, 2, 3, 4$. Using (38), we see the non-vanishing components of Riemannian metric are

$$g_{11} = 1, \;\; g_{22} = \left(x^1\right)^2, \;\; g_{33} = \left(x^2\right)^2, \;\; g_{44} = 1 \tag{39}$$

and its associated components are

$$g^{11} = 1, \;\; g^{22} = \frac{1}{\left(x^1\right)^2}, \;\; g^{33} = \frac{1}{\left(x^2\right)^2}, \;\; g^{44} = 1. \tag{40}$$

Using (39) and (40), we can calculate that the non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are given by

$$\Gamma^1_{22} = -x^1, \;\; \Gamma^2_{33} = -\frac{x^2}{\left(x^1\right)^2}, \;\; \Gamma^2_{12} = \frac{1}{x^1}, \;\; \Gamma^3_{23} = \frac{1}{x^2}, \;\; R_{1332} = -\frac{x^2}{x^1}, \;\; S_{12} = -\frac{1}{x^1 x^2}$$

and the other components are obtained by the symmetric properties. It can be easily shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4, g)$ is zero. We shall now show that this $(\mathbb{R}^4, g)$ is an $N(k)$-quasi Einstein manifold.
Let us consider the associated scalars as follows:

$$a = \frac{1}{x^1\left(x^2\right)^2}, \qquad b = -\frac{2}{\left(x^1\right)^2 x^2}. \tag{41}$$

We choose the 1-form as follows:

$$\eta_i(x) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{when } i = 1 \\ \frac{x^1}{\sqrt{2}}, & \text{when } i = 2 \\ 0, & \text{otherwise} \end{cases} \tag{42}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1) reduces to the equation

$$S_{12} = a g_{12} + b \eta_1 \eta_2, \tag{43}$$

since, for the other cases (1) holds trivially.
From the equations (41), (42) and (43) we get

$$\begin{aligned} \text{Right hand side of (43)} &= a g_{12} + b \eta_1 \eta_2 \\ &= \frac{1}{x^1\left(x^2\right)^2} \cdot 0 + \left(-\frac{2}{\left(x^1\right)^2 x^2}\right) \cdot \left(\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{x^1}{\sqrt{2}}\right) \\ &= -\frac{1}{x^1 x^2} = S_{12}. \end{aligned}$$

By Lemma 1.1., here we see that $k = \dfrac{x^1 - 2x^2}{3\left(x^1\right)^2\left(x^2\right)^2}$.
We shall now show that the 1-form η_i are unit.
Here,

$$g^{ij}\eta_i\eta_j = 1.$$

So, $(\mathbb{R}^4, g)$ is an $N\left(\dfrac{x^1 - 2x^2}{3\left(x^1\right)^2\left(x^2\right)^2}\right)$-quasi Einstein manifold.

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Dipankar Hazra
Department of Mathematics, Heramba Chandra College,
Kolkata - 700029, West Bengal, India
E-mail: `dipankarsk524@gmail.com`

Avijit Sarkar
Department of Mathematics, University of Kalyani,
Kalyani - 741235, West Bengal, India
E-mail: `avjaj@yahoo.co.in`

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PAPER

Characterizations of $\mathcal{H}$-flat curvature tensor on spacetimes and $f(r, \mathcal{T})$-gravity

Uday Chand De[1] **and Dipankar Hazra**[2,*]

[1] Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata 700019, West Bengal, India
[2] Department of Mathematics, University of Kalyani, Kalyani 741235, West Bengal, India
[*] Author to whom any correspondence should be addressed.

E-mail: uc_de@yahoo.com and dipankarsk524@gmail.com

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Abstract

The prime goal of this article is to characterize spacetimes endowed with $\mathcal{H}$-flat curvature tensor. It is demonstrated that a 4-dimensional $\mathcal{H}$-flat perfect fluid spacetime is either a de-Sitter spacetime or locally isometric to Minkowski spacetime under certain energy condition. Moreover, it is established that a Ricci simple $\mathcal{H}$-flat spacetime becomes a Robertson-Walker spacetime. In addition, we shown that a Ricci simple spacetime with harmonic $\mathcal{H}$-curvature tensor is a generalized Robertson-Walker spacetime. Also, we explain several conclusions based on $\mathcal{H}$-flat generalized Robertson-Walker spacetimes. Finally, we investigate $\mathcal{H}$-flat spacetimes that satisfy the $\mathbf{f}(\boldsymbol{r}, \mathcal{T})$-gravity and $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$-gravity.

1. Introduction

The subclass of a semi-Riemannian manifold is a Lorentzian manifold M^n. The signature of the metric g of a Lorentzian manifold is $(-,+,+,\ldots,+)$, that is, the index of g is 1. A spacetime is a Lorentzian manifold M^n $(n \geqslant 4)$ which admits a globally timelike vector field. Different types of spacetimes have been studied in various ways, such as [1–7] and many others.

Let the metric assume the appropriate local structure

$$ds^2 = -(d\zeta)^2 + q^2(\zeta) g^*_{u_1 u_2} dx^{u_1} dx^{u_2}, \tag{1}$$

then a Lorentzian manifold of dimension n $(n > 2)$ is named a generalized Robertson-Walker (GRW) [8] spacetime, where $g^*_{u_1 u_2} = g^*_{u_1 u_2}(x^{u_3})$ are only functions of x^{u_3} $(u_1, u_2, u_3 = 2, 3,\ldots,n)$ and q is a function dependent on ζ. So, $-\mathcal{I} \times q^2 \bar{M}$ can be used to represent GRW spacetime in which $\mathcal{I} \subseteq \mathbb{R}$ is an open interval and $\bar{M}$ is an $(n-1)$-dimensional Riemannian manifold. The GRW spacetime transforms to a Robertson-Walker spacetime (RW) if dimension of $\bar{M}$ is three and of constant sectional curvature.

A Lorentzian manifold M^n is referred to as a perfect fluid spacetime (PFS) if the Ricci tensor $\mathcal{S}$ takes the form

$$\mathcal{S} = \alpha_1 g + \alpha_2 A \otimes A, \tag{2}$$

where α_1, α_2 are scalars and $\boldsymbol{\rho}$ is a unit timelike vector corresponding to the non-vanishing one-form A, that is, $A(\boldsymbol{\rho}) = g(\boldsymbol{\rho}, \boldsymbol{\rho}) = -1$. Throughout this article, we consider $\boldsymbol{\rho}$ is a unit timelike vector, which is called velocity vector or flow vector.

In general relativity theory, the matter content of the spacetimes is represented by the energy-momentum tensor (EMT) $\mathcal{T}$ and the fluid is termed perfect fluid, since it does not have the heat conduction terms [9]. The form of the EMT [10] for a PFS is

$$\mathcal{T} = \boldsymbol{p} g + (\boldsymbol{p} + \boldsymbol{\sigma}) A \otimes A, \tag{3}$$

where $\boldsymbol{p}$ denotes isotropic pressure and $\boldsymbol{\sigma}$ denotes energy density. The non-vanishing one-form A is metrically equivalent to the velocity vector $\boldsymbol{\rho}$, which is a unit timelike vector. If $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{\sigma})$ holds for the equation of state (EoS), a PFS is termed to be isentropic [9].

The Einstein's field equations (EFE) are as follows:

$$\mathcal{S} - \frac{\boldsymbol{r}}{2} g = \boldsymbol{\kappa}\mathcal{T}, \tag{4}$$

$\mathcal{S}$ denotes the Ricci tensor, the gravitational constant is $\boldsymbol{\kappa}$ and $\boldsymbol{r}$ denotes the scalar curvature. In Einsteins general relativity theory, perfect fluids are of great importance for being special solutions of the Einsteins field equations having vanishing shear stresses, viscosity and heat conduction compatible with Bianchi identities. In cosmology, they represent the behaviour of the Hubble flow ranging from inflation to dark energy periods. Perfect fluids model the matter content of the interior of a star or an isotropic universe. For these reasons, the geometric studies of perfect fluids on modified gravity theories are very important areas of study that need to be explored. There are many different ways to modify the general relativity; Einstein-Hilbert gravitational action in some manner or another, specifically, $\mathbf{f}(\boldsymbol{r})$-gravity [11, 12] which is the most straightforward generalization of general relativity, Gauss-Bonnet gravity [13], $\mathbf{f}(\mathcal{T})$ theory [14] and $\mathbf{f}(\boldsymbol{r}, \mathcal{T})$-gravity [15, 16] and $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$ gravity [17–19]. In this paper, we consider PFS on the modified $\mathbf{f}(\boldsymbol{r}, \mathcal{T})$-gravity and $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$-gravity of general relativity. Equation (2) can be written as

$$\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_3) = \alpha_1 g(\mathcal{Z}_2, \mathcal{Z}_3) + \alpha_2 A(\mathcal{Z}_2)A(\mathcal{Z}_3). \tag{5}$$

Now putting $\mathcal{Z}_2 = \mathcal{Z}_3 = e_i$ in (5), where $\{e_i\}$ is an orthonormal basis of the tangent space at any point of the Lorentzian manifold and taking summation on i, $1 \leqslant i \leqslant n$, we obtain

$$\boldsymbol{r} = n\alpha_1 - \alpha_2. \tag{6}$$

Using the equations (3), (5) and (6) in the equation (4), we infer that

$$\begin{aligned}&\left[\frac{(2-n)\alpha_1 + \alpha_2}{2}\right] g(\mathcal{Z}_2, \mathcal{Z}_3) + \alpha_2 A(\mathcal{Z}_2)A(\mathcal{Z}_3)\\ &= \boldsymbol{\kappa p} g(\mathcal{Z}_2, \mathcal{Z}_3) + \boldsymbol{\kappa}(\boldsymbol{p} + \boldsymbol{\sigma})A(\mathcal{Z}_2)A(\mathcal{Z}_3).\end{aligned} \tag{7}$$

Putting $\mathcal{Z}_2 = \mathcal{Z}_3 = \boldsymbol{\rho}$ in the equation (7), we have

$$(n-2)\alpha_1 + \alpha_2 = 2\boldsymbol{\kappa\sigma}. \tag{8}$$

Contracting the equation (7), we get

$$-n(n-2)\alpha_1 + (n-2)\alpha_2 = 2(n-1)\boldsymbol{\kappa p} - 2\boldsymbol{\kappa\sigma}. \tag{9}$$

From the equations (8) and (9), we reach

$$\alpha_1 = \boldsymbol{\kappa}\left(\frac{\boldsymbol{p} - \boldsymbol{\sigma}}{2 - n}\right) \quad \text{and} \quad \alpha_2 = \boldsymbol{\kappa}(\boldsymbol{p} + \boldsymbol{\sigma}). \tag{10}$$

$\mathcal{T}$ must fulfill the following energy conditions to be a physically reasonable exact solution:

- A spacetime obeys the *weak energy condition* (WEC) if for every timelike vector field $\mathcal{Z}_2$, $\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_2) \geqslant 0$ holds.
- The WEC with the condition that the pressure does not exceed the energy density makes the *dominant energy condition* (DEC).
- If for any timelike vector field $\mathcal{Z}_2$, $\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_2) \geqslant 0$ holds, then a spacetime fulfills the *strong energy condition* (SEC).

The Weyl curvature tensor $\mathcal{C}$ and the projective curvature tensor $\mathcal{P}$ for a Lorentzian manifold M^n $(n \geqslant 4)$ are written by

$$\begin{aligned}\mathcal{C}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 = \mathcal{R}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 &- \frac{1}{n-2}[\mathcal{S}(\mathcal{Z}_3, \mathcal{W}_1)\mathcal{Z}_2 - \mathcal{S}(\mathcal{Z}_2, \mathcal{W}_1)\mathcal{Z}_3\\ &+ g(\mathcal{Z}_3, \mathcal{W}_1)\mathcal{Q}\mathcal{Z}_2 - g(\mathcal{Z}_2, \mathcal{W}_1)\mathcal{Q}\mathcal{Z}_3]\\ &+ \frac{\boldsymbol{r}}{(n-2)(n-1)}[g(\mathcal{Z}_3, \mathcal{W}_1)\mathcal{Z}_2 - g(\mathcal{Z}_2, \mathcal{W}_1)\mathcal{Z}_3]\end{aligned} \tag{11}$$

and

$$\mathcal{P}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 = \mathcal{R}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 - \frac{1}{n-1}[\mathcal{S}(\mathcal{Z}_3, \mathcal{W}_1)\mathcal{Z}_2 - \mathcal{S}(\mathcal{Z}_2, \mathcal{W}_1)\mathcal{Z}_3], \tag{12}$$

$\mathcal{R}$ denotes the curvature tensor, $\boldsymbol{r}$ denotes the scalar curvature and $\mathcal{Q}$, the Ricci operator fulfills the relation $\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_3) = g(\mathcal{Q}\mathcal{Z}_2, \mathcal{Z}_3)$.

De *et al* [20] introduced $\mathcal{H}$-curvature tensor of $(1, 3)$ type which is the linear combination of $\mathcal{C}$ and $\mathcal{P}$ defined by

$$\mathcal{H}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 = a_1\mathcal{C}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 + [a_1 + (n-2)b_1]\mathcal{P}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1, \tag{13}$$

where a_1 and b_1 are real numbers (not simultaneously zero). If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then $\mathcal{H} \equiv \mathcal{C}$, also if $a_1 = 0$ and $b_1 = \frac{1}{n-2}$, then $\mathcal{H} \equiv \mathcal{P}$. Since the conformal curvature tensor vanishes for $n = 3$, we consider the dimension of the manifold $n > 3$.

Definition 1.1. [21] A vector field $\boldsymbol{\rho}$ on a Lorentzian manifold (M^n, g) is named torse-forming if

$$(\nabla_{\mathcal{Z}_2}A)(\mathcal{Z}_3) = \Omega(\mathcal{Z}_2)A(\mathcal{Z}_3) + \varphi g(\mathcal{Z}_2, \mathcal{Z}_3),$$

being Ω a non-vanishing one-form, A is a non-vanishing one-form such that $g(\mathcal{Z}_2, \boldsymbol{\rho}) = A(\mathcal{Z}_2)$, for all $\mathcal{Z}_2$ and φ is a scalar function.

If $\boldsymbol{\rho}$ is a unit timelike, the above equation takes the form

$$(\nabla_{\mathcal{Z}_2}A)(\mathcal{Z}_3) = \varphi[g(\mathcal{Z}_2, \mathcal{Z}_3) + A(\mathcal{Z}_2)A(\mathcal{Z}_3)]. \tag{14}$$

Definition 1.2. [22] If the Ricci tensor has the following form

$$\mathcal{S} = -\boldsymbol{r}A \otimes A, \tag{15}$$

then a Lorentzian manifold is called Ricci simple, where A is a non-vanishing one-form.

In a series of recent studies, different curvature conditions were widely studied in spacetimes. In [23], the authors studied spacetimes with pseudo-projective curvature tensor. Spacetimes admitting quasi-conformal curvature tensor were considered in [24, 25]. Motivated by these studies and many others, this article is mainly organized to study spacetimes with $\mathcal{H}$-flat curvature tensor. The structure of the paper is as follows:

After a brief introduction in sect. 2, we characterize spacetimes with $\mathcal{H}$-flat curvature tensor. The physical properties of PFS admitting $\mathcal{H}$-flat curvature tensor are examined in the next section. The analysis of Ricci simple spacetimes with $\mathcal{H}$-curvature tensor is presented in the section 4. Section 5 examines GRW spacetimes with $\mathcal{H}$-flat curvature tensor. In the last section, we explore $\mathcal{H}$-flat spacetimes in $\mathbf{f}(\boldsymbol{r}, \mathcal{T})$-gravity and $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$-gravity theories.

2. $\mathcal{H}$-flat spacetimes

Here we consider $\mathcal{H}$-flat spacetime of general relativity. The equations (11), (12) and (13) implies

$$\begin{aligned} H(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) = {} & [2a_1 + (n-2)b_1]R(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) \\ & - \frac{(2n-3)a_1 + (n-2)^2 b_1}{(n-1)(n-2)}[\mathcal{S}(\mathcal{Z}_3, \mathcal{W}_1)g(\mathcal{Z}_2, \mathcal{Z}_1) - \mathcal{S}(\mathcal{Z}_2, \mathcal{W}_1)g(\mathcal{Z}_3, \mathcal{Z}_1)] \\ & - \frac{a_1}{n-2}[g(\mathcal{Z}_3, \mathcal{W}_1)\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_1) - g(\mathcal{Z}_2, \mathcal{W}_1)\mathcal{S}(\mathcal{Z}_3, \mathcal{Z}_1)] \\ & + \frac{a_1\boldsymbol{r}}{(n-2)(n-1)}[g(\mathcal{Z}_3, \mathcal{W}_1)g(\mathcal{Z}_2, \mathcal{Z}_1) - g(\mathcal{Z}_2, \mathcal{W}_1)g(\mathcal{Z}_3, \mathcal{Z}_1)], \end{aligned} \tag{16}$$

where $H(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) = g(\mathcal{H}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1, \mathcal{Z}_1)$ is the $\mathcal{H}$-curvature tensor of $(0, 4)$ type and $R(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) = g(\mathcal{R}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1, \mathcal{Z}_1)$.

For $\mathcal{H}$-flat curvature tensor, the equation (16) leads to

$$\begin{aligned} & [2a_1 + (n-2)b_1]R(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) \\ & - \frac{(2n-3)a_1 + (n-2)^2 b_1}{(n-2)(n-1)}[\mathcal{S}(\mathcal{Z}_3, \mathcal{W}_1)g(\mathcal{Z}_2, \mathcal{Z}_1) - \mathcal{S}(\mathcal{Z}_2, \mathcal{W}_1)g(\mathcal{Z}_3, \mathcal{Z}_1)] \\ & - \frac{a_1}{n-2}[g(\mathcal{Z}_3, \mathcal{W}_1)\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_1) - g(\mathcal{Z}_2, \mathcal{W}_1)\mathcal{S}(\mathcal{Z}_3, \mathcal{Z}_1)] \\ & + \frac{a_1\boldsymbol{r}}{(n-2)(n-1)}[g(\mathcal{Z}_3, \mathcal{W}_1)g(\mathcal{Z}_2, \mathcal{Z}_1) - g(\mathcal{Z}_2, \mathcal{W}_1)g(\mathcal{Z}_3, \mathcal{Z}_1)] = 0. \end{aligned} \tag{17}$$

Assume an orthonormal frame field at any point on the manifold and contracting $\mathcal{Z}_3$ and $\mathcal{W}_1$ in (17), yields

$$\mathcal{S}(\mathcal{Z}_1, \mathcal{Z}_2) = \frac{\boldsymbol{r}}{n}g(\mathcal{Z}_1, \mathcal{Z}_2). \tag{18}$$

Hence, we can assert the result:

Theorem 2.1. *A $\mathcal{H}$-flat spacetime is an Einstein spacetime.*

Using (18) in (17) infers

$$\begin{aligned}[2a_1 + (n-2)b_1]R(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) \\ = \frac{[2a_1 + (n-2)b_1]\boldsymbol{r}}{n(n-1)}[g(\mathcal{Z}_3, \mathcal{W}_1)g(\mathcal{Z}_2, \mathcal{Z}_1) - g(\mathcal{Z}_2, \mathcal{W}_1)g(\mathcal{Z}_3, \mathcal{Z}_1)].\end{aligned} \tag{19}$$

Thus, we might conclude that

Theorem 2.2. *A $\mathcal{H}$-flat spacetime is a spacetime of constant curvature, provided* $2a_1 + (n-2)b_1 \neq 0$.

3. Perfect fluid spacetimes with $\mathcal{H}$-flat curvature tensor

In this section, we take a PFS with $\mathcal{H}$-flat curvature tensor obeying EFE.

From (3), (4) and (18), it follows that

$$\left(\boldsymbol{\kappa p} + \frac{\boldsymbol{r}}{2} - \frac{\boldsymbol{r}}{n}\right)g(\mathcal{Z}_2, \mathcal{Z}_3) + \boldsymbol{\kappa}(\boldsymbol{p} + \boldsymbol{\sigma})A(\mathcal{Z}_2)A(\mathcal{Z}_3) = 0. \tag{20}$$

Contracting the equation (20) entails that

$$(n-2)\boldsymbol{r} + 2(n-1)\boldsymbol{\kappa p} - 2\boldsymbol{\kappa\sigma} = 0. \tag{21}$$

Setting $\mathcal{Z}_2 = \mathcal{Z}_3 = \boldsymbol{\rho}$ in (20), we find

$$(n-2)\boldsymbol{r} = 2n\boldsymbol{\kappa\sigma}. \tag{22}$$

Therefore by the equations (21) and (22) we conclude that

$$\boldsymbol{p} + \boldsymbol{\sigma} = 0. \tag{23}$$

This represents a dark matter era [26]. Thus we conclude:

Theorem 3.1. *A $\mathcal{H}$-flat perfect fluid spacetime satisfying Einstein's field equations represents a dark matter era.*

Now we suppose that the spacetime is of dimension 4. Equations (3) and (4) together yield

$$\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_3) = \left(\frac{\boldsymbol{r}}{2} + \boldsymbol{\kappa p}\right)g(\mathcal{Z}_2, \mathcal{Z}_3) + \boldsymbol{\kappa}(\boldsymbol{p} + \boldsymbol{\sigma})A(\mathcal{Z}_2)A(\mathcal{Z}_3). \tag{24}$$

Setting $\mathcal{Z}_2 = \mathcal{Z}_3 = \boldsymbol{\rho}$ in (24) and using (22), we deduce that

$$\mathcal{S}(\boldsymbol{\rho}, \boldsymbol{\rho}) = -\boldsymbol{\kappa\sigma}. \tag{25}$$

Suppose the spacetime under study meets the strong energy condition. Then we obtain

$$\boldsymbol{\kappa\sigma} \leqslant 0. \tag{26}$$

Since $\boldsymbol{\kappa} > 0$ and the energy density cannot be negative, the equations (22) and (26) gives us

$$\boldsymbol{r} = 0. \tag{27}$$

Then (19) infers $(a_1 + b_1)R(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) = 0$. This represents that the spacetime is of zero sectional curvature, provided $a_1 + b_1 \neq 0$. Therefore a 4-dimensional $\mathcal{H}$-flat spacetime is locally isometric to Minkowski spacetime ([27], p. 67), provided $a_1 + b_1 \neq 0$.

This observation leads us to the following conclusion:

Theorem 3.2. *A 4-dimensional $\mathcal{H}$-flat perfect fluid spacetime obeying the strong energy condition is locally isometric to Minkowski spacetime, provided* $a_1 + b_1 \neq 0$.

As the energy density cannot be negative, from (22) it follows that

$$\boldsymbol{r} \geqslant 0, \tag{28}$$

that is, $\boldsymbol{r} = 0$ or, $\boldsymbol{r} > 0$.

Case 1. If $\boldsymbol{r} = 0$, then (19) infers $(a_1 + b_1)R(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) = 0$, which means that the spacetime is locally isometric to Minkowski spacetime, provided $a_1 + b_1 \neq 0$.

Case 2. If $\boldsymbol{r} > 0$, then the equation (19) indicates that the constant curvature is positive, provided $a_1 + b_1 \neq 0$. Note that the spacetime with constant positive curvature is a de-Sitter spacetime [27].

Consequently, the following can be stated:

Theorem 3.3. *A 4-dimensional $\mathcal{H}$-flat perfect fluid spacetime is either a de-Sitter spacetime or locally isometric to Minkowski spacetime, provided* $a_1 + b_1 \neq 0$.

The de-Sitter spacetime is known to be conformally flat and, as a result, is of Petrov type O.

These facts lead us to the following result:

Corollary 3.1. *A 4-dimensional $\mathcal{H}$-flat perfect fluid spacetime is of Petrov type O, provided* $a_1 + b_1 \neq 0$.

4. Ricci simple spacetimes with $\mathcal{H}$-curvature tensor

In this section, we assume that Ricci simple spacetime with $\mathcal{H}$-flat curvature tensor and harmonic $\mathcal{H}$-curvature tensor.

Firstly, we consider Ricci simple $\mathcal{H}$-flat spacetime. Utilizing (15) in (17), we reach

$$\begin{aligned}&[2a_1 + (n-2)b_1]R(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) - \frac{(2n-3)a_1\boldsymbol{r} + (n-2)^2 b_1\boldsymbol{r}}{(n-1)(n-2)}\\ &\times[A(\mathcal{Z}_2)A(\mathcal{W}_1)g(\mathcal{Z}_3, \mathcal{Z}_1) - A(\mathcal{Z}_3)A(\mathcal{W}_1)g(\mathcal{Z}_2, \mathcal{Z}_1)]\\ &-\frac{a_1\boldsymbol{r}}{n-2}[A(\mathcal{Z}_3)A(\mathcal{Z}_1)g(\mathcal{Z}_2, \mathcal{W}_1) - A(\mathcal{Z}_2)A(\mathcal{Z}_1)g(\mathcal{Z}_3, \mathcal{W}_1)]\\ &+\frac{a_1\boldsymbol{r}}{(n-2)(n-1)}[g(\mathcal{Z}_3, \mathcal{W}_1)g(\mathcal{Z}_2, \mathcal{Z}_1) - g(\mathcal{Z}_2, \mathcal{W}_1)g(\mathcal{Z}_3, \mathcal{Z}_1)] = 0.\end{aligned} \tag{29}$$

Contracting $\mathcal{Z}_3$ and $\mathcal{W}_1$ in (29) reveals that

$$\begin{aligned}\mathcal{S}(\mathcal{Z}_1, \mathcal{Z}_2) = &\frac{[a_1 + (n-2)b_1]\boldsymbol{r}}{(n-1)[2a_1 + (n-2)b_1]}g(\mathcal{Z}_1, \mathcal{Z}_2)\\ &-\frac{(a_1 - b_1)(n-2)\boldsymbol{r}}{(n-1)[2a_1 + (n-2)b_1]}A(\mathcal{Z}_1)A(\mathcal{Z}_2),\end{aligned} \tag{30}$$

provided $2a_1 + (n-2)b_1 \neq 0$, which is the form of a PFS.

Combining the equations (15) and (30), we acquire

$$\frac{[(n-2)b_1 + a_1]\boldsymbol{r}}{(n-1)[2a_1 + (n-2)b_1]} = 0 \quad \text{and} \quad \frac{(a_1 - b_1)(n-2)\boldsymbol{r}}{(n-1)[2a_1 + (n-2)b_1]} = \boldsymbol{r}, \tag{31}$$

both the equation gives the same result

$$a_1 + (n-2)b_1 = 0. \tag{32}$$

Equations (32) and (13) turns into

$$\mathcal{H}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 = a_1\mathcal{C}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1. \tag{33}$$

Since by hypothesis the spacetime is $\mathcal{H}$-flat, therefore (33) infers the spacetime is conformally flat.

In [28], the authors established that a Ricci simple conformally flat PFS of dimension n ($n \geqslant 4$) is a RW spacetime.

Hence from (30) and the foregoing observation, we get to the following conclusion:

Theorem 4.1. *A Ricci simple $\mathcal{H}$-flat spacetime becomes a* RW *spacetime, provided* $2a_1 + (n-2)b_1 \neq 0$.

For a 4-dimensional Ricci simple $\mathcal{H}$-flat spacetime, the equation (30) becomes

$$\mathcal{S}(\mathcal{Z}_1, \mathcal{Z}_2) = \frac{(a_1 + 2b_1)\boldsymbol{r}}{6(a_1 + b_1)}g(\mathcal{Z}_1, \mathcal{Z}_2) - \frac{(a_1 - b_1)\boldsymbol{r}}{3(a_1 + b_1)}A(\mathcal{Z}_1)A(\mathcal{Z}_2). \tag{34}$$

For $n = 4$, from (5) and (10) we can derive

$$\mathcal{S}(\mathcal{Z}_1, \mathcal{Z}_2) = \frac{\boldsymbol{\kappa}}{2}(\boldsymbol{\sigma} - \boldsymbol{p})g(\mathcal{Z}_1, \mathcal{Z}_2) + \boldsymbol{\kappa}(\boldsymbol{p} + \boldsymbol{\sigma})A(\mathcal{Z}_1)A(\mathcal{Z}_2). \tag{35}$$

From (34) and (35), we infer that

$$\left\{\frac{(a_1 + 2b_1)\boldsymbol{r}}{6(a_1 + b_1)} - \frac{\boldsymbol{\kappa}}{2}(\boldsymbol{\sigma} - \boldsymbol{p})\right\}g(\mathcal{Z}_1, \mathcal{Z}_2) - \left\{\frac{(a_1 - b_1)\boldsymbol{r}}{3(a_1 + b_1)} + \boldsymbol{\kappa}(\boldsymbol{p} + \boldsymbol{\sigma})\right\}A(\mathcal{Z}_1)A(\mathcal{Z}_2) = 0. \tag{36}$$

Contracting the equation (36), we get

$$\boldsymbol{\kappa\sigma} - 3\boldsymbol{\kappa p} = \boldsymbol{r}. \tag{37}$$

Setting $\mathcal{Z}_1 = \mathcal{Z}_2 = \boldsymbol{\rho}$ in the equation (36), we have

$$\kappa\boldsymbol{\sigma} + 3\kappa\boldsymbol{p} = -\frac{a_1\boldsymbol{r}}{(a_1 + b_1)}. \tag{38}$$

Adding (37) and (38), we find

$$\kappa\boldsymbol{\sigma} = \frac{b_1\boldsymbol{r}}{2(a_1 + b_1)}. \tag{39}$$

Subtracting (37) from (38), we acquire

$$\kappa\boldsymbol{p} = -\frac{(2a_1 + b_1)\boldsymbol{r}}{6(a_1 + b_1)}. \tag{40}$$

From the equations (39) and (40), we reach

$$\kappa(\boldsymbol{\sigma} - \boldsymbol{p}) = \frac{(a_1 + 2b_1)\boldsymbol{r}}{3(a_1 + b_1)} \quad \text{and} \quad \kappa(\boldsymbol{p} + \boldsymbol{\sigma}) = \frac{(b_1 - a_1)\boldsymbol{r}}{3(a_1 + b_1)}. \tag{41}$$

From the above equation, we notice that the equation of state is of the form $\boldsymbol{p} = -\left(\frac{2a_1 + b_1}{3b_1}\right)\boldsymbol{\sigma}$, that is, the fluid is isentropic.

Hence, we conclude the result as:

Corollary 4.1. *A 4-dimensional Ricci simple $\mathcal{H}$-flat spacetime represents an isentropic perfect fluid spacetime with the equation of state is given by* $\boldsymbol{p} = -\left(\frac{2a_1+b_1}{3b_1}\right)\boldsymbol{\sigma}$.

A Lorentzian manifold is referred to as a Yang pure space [29] if the metric fulfills the Yang's equation:

$$(\nabla_{\mathcal{Z}_2}\mathcal{S})(\mathcal{Z}_3, \mathcal{W}_1) = (\nabla_{\mathcal{Z}_3}\mathcal{S})(\mathcal{Z}_2, \mathcal{W}_1).$$

From (41), we obtain that $\boldsymbol{p} + \boldsymbol{\sigma} \neq 0$ for $a_1 + b_1 \neq 0$. Guilfoyle and Nolan [29] established that a PFS with $\boldsymbol{\sigma} + \boldsymbol{p} \neq 0$ of dimension 4 is a Yang pure space if and only if it is a RW spacetime.

Therefore, we can conclude as follows:

Corollary 4.2. *A 4-dimensional Ricci simple $\mathcal{H}$-flat spacetime is a Yang pure space for* $b_1 + a_1 \neq 0$.

If $b_1 = a_1$, then from (41) we infer that $\boldsymbol{p} + \boldsymbol{\sigma} = 0$. This represents a dark matter era. Consequently, we get the following conclusion:

Corollary 4.3. *A 4-dimensional Ricci simple $\mathcal{H}$-flat spacetime represents a dark matter era for* $a_1 = b_1$.

Mantica and Molinari [30] proved that the subsequent theorem:

Theorem A. *A Lorentzian manifold of dimension* n $(n \geqslant 3)$ *is a GRW spacetime if and only if it admits a unit timelike torse-forming vector field* $\boldsymbol{\rho}$: $(\nabla_{\mathcal{Z}_2}A)(\mathcal{Z}_3) = \varphi[g(\mathcal{Z}_2, \mathcal{Z}_3) + A(\mathcal{Z}_2)A(\mathcal{Z}_3)]$ *and* $\boldsymbol{\rho}$ *is an eigenvector of the Ricci tensor.*

Using the definition of $\mathcal{H}$-curvature tensor, we acquire that

$$\begin{aligned}(\operatorname{div}\mathcal{H})(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 &= \frac{(3-n)a_1}{2(n-1)(n-2)}[g(\mathcal{Z}_3, \mathcal{W}_1)d\boldsymbol{r}(\mathcal{Z}_2) - g(\mathcal{Z}_2, \mathcal{W}_1)d\boldsymbol{r}(\mathcal{Z}_3)]\\ &+ \frac{(2n^2 - 8n + 7)a_1 + (n-2)^3 b_1}{(n-1)(n-2)}[(\nabla_{\mathcal{Z}_2}\mathcal{S})(\mathcal{Z}_3, \mathcal{W}_1) - (\nabla_{\mathcal{Z}_3}\mathcal{S})(\mathcal{Z}_2, \mathcal{W}_1)].\end{aligned} \tag{42}$$

If the $\mathcal{H}$-curvature tensor is harmonic, i.e., $(\operatorname{div}\mathcal{H})(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 = 0$, then (42) reduces to

$$\begin{aligned}&[(2n^2 - 8n + 7)a_1 + (n-2)^3 b_1][(\nabla_{\mathcal{Z}_2}\mathcal{S})(\mathcal{Z}_3, \mathcal{W}_1) - (\nabla_{\mathcal{Z}_3}\mathcal{S})(\mathcal{Z}_2, \mathcal{W}_1)]\\ &-\frac{(n-3)a_1}{2}[g(\mathcal{Z}_3, \mathcal{W}_1)d\boldsymbol{r}(\mathcal{Z}_2) - g(\mathcal{Z}_2, \mathcal{W}_1)d\boldsymbol{r}(\mathcal{Z}_3)] = 0.\end{aligned} \tag{43}$$

Differentiating (15) covariantly and applying in (43) we reach

$$\begin{aligned}&[(2n^2-8n+7)a_1+(n-2)^3b_1][-d\boldsymbol{r}(\mathcal{Z}_2)A(\mathcal{Z}_3)A(\mathcal{W}_1)\\&-\boldsymbol{r}(\nabla_{\mathcal{Z}_2}A)(\mathcal{Z}_3)A(\mathcal{W}_1)-\boldsymbol{r}A(\mathcal{Z}_3)(\nabla_{\mathcal{Z}_2}A)(\mathcal{W}_1)\\&+d\boldsymbol{r}(\mathcal{Z}_3)A(\mathcal{Z}_2)A(\mathcal{W}_1)+\boldsymbol{r}(\nabla_{\mathcal{Z}_3}A)(\mathcal{Z}_2)A(\mathcal{W}_1)+\boldsymbol{r}A(\mathcal{Z}_2)(\nabla_{\mathcal{Z}_3}A)(\mathcal{W}_1)]\\&-\frac{(n-3)a_1}{2}[g(\mathcal{Z}_3,\mathcal{W}_1)d\boldsymbol{r}(\mathcal{Z}_2)-g(\mathcal{Z}_2,\mathcal{W}_1)d\boldsymbol{r}(\mathcal{Z}_3)]=0.\end{aligned}\tag{44}$$

Contracting $\mathcal{Z}_3$ and $\mathcal{W}_1$ in (44) reveals that

$$\begin{aligned}&[(2n^2-8n+7)a_1+(n-2)^3b_1][d\boldsymbol{r}(\boldsymbol{\rho})A(\mathcal{Z}_2)+\boldsymbol{r}(\nabla_{\boldsymbol{\rho}}A)(\mathcal{Z}_2)+\boldsymbol{r}A(\mathcal{Z}_2)\operatorname{div}\boldsymbol{\rho}]\\&+\left[\left(\frac{3n^2-12n+11}{2}\right)a_1+(n-2)^3b_1\right]d\boldsymbol{r}(\mathcal{Z}_2)=0.\end{aligned}\tag{45}$$

Setting $\mathcal{W}_1=\boldsymbol{\rho}$ in (44), we find

$$\begin{aligned}&[(2n^2-8n+7)a_1+(n-2)^3b_1][d\boldsymbol{r}(\mathcal{Z}_2)A(\mathcal{Z}_3)+\boldsymbol{r}(\nabla_{\mathcal{Z}_2}A)(\mathcal{Z}_3)\\&-d\boldsymbol{r}(\mathcal{Z}_3)A(\mathcal{Z}_2)-\boldsymbol{r}(\nabla_{\mathcal{Z}_3}A)(\mathcal{Z}_2)]\\&-\frac{(n-3)a_1}{2}[d\boldsymbol{r}(\mathcal{Z}_2)A(\mathcal{Z}_3)-d\boldsymbol{r}(\mathcal{Z}_3)A(\mathcal{Z}_2)]=0.\end{aligned}\tag{46}$$

Setting $\mathcal{Z}_3=\boldsymbol{\rho}$ in (46), we deduce that

$$\begin{aligned}&[(2n^2-8n+7)a_1+(n-2)^3b_1]\boldsymbol{r}(\nabla_{\boldsymbol{\rho}}A)(\mathcal{Z}_2)\\&=\left[\frac{(17n-4n^2-17)a_1-2(n-2)^3b_1}{2}\right][d\boldsymbol{r}(\mathcal{Z}_2)+d\boldsymbol{r}(\boldsymbol{\rho})A(\mathcal{Z}_2)].\end{aligned}\tag{47}$$

In light of (45) and (47) we have

$$\begin{aligned}&\frac{a_1(n-3)}{2}d\boldsymbol{r}(\boldsymbol{\rho})A(\mathcal{Z}_2)-\frac{a_1(n-2)(n-3)}{2}d\boldsymbol{r}(\mathcal{Z}_2)\\&+[(2n^2-8n+7)a_1+(n-2)^3b_1]\boldsymbol{r}A(\mathcal{Z}_2)\operatorname{div}\boldsymbol{\rho}=0.\end{aligned}\tag{48}$$

Putting $\mathcal{Z}_2=\boldsymbol{\rho}$ in (48), we obtain

$$[(n-2)^3b_1+(2n^2-8n+7)a_1]\boldsymbol{r}\operatorname{div}\boldsymbol{\rho}=-\frac{a_1(n-1)(n-3)}{2}d\boldsymbol{r}(\boldsymbol{\rho}).\tag{49}$$

The equations (48) and (49) reflects that

$$d\boldsymbol{r}(\mathcal{Z}_2)=-d\boldsymbol{r}(\boldsymbol{\rho})A(\mathcal{Z}_2).\tag{50}$$

Adopting (46) and (50), we can derive

$$[(2n^2-8n+7)a_1+(n-2)^3b_1][(\nabla_{\mathcal{Z}_2}A)(\mathcal{Z}_3)-(\nabla_{\mathcal{Z}_3}A)(\mathcal{Z}_2)]\boldsymbol{r}=0.\tag{51}$$

By virtue of (44), (50) and (51) we arrive at

$$\begin{aligned}&[(2n^2-8n+7)a_1+(n-2)^3b_1][A(\mathcal{Z}_2)(\nabla_{\mathcal{Z}_3}A)(\mathcal{W}_1)-A(\mathcal{Z}_3)(\nabla_{\mathcal{Z}_2}A)(\mathcal{W}_1)]\boldsymbol{r}\\&-\frac{(n-3)a_1}{2}[d\boldsymbol{r}(\boldsymbol{\rho})A(\mathcal{Z}_3)g(\mathcal{Z}_2,\mathcal{W}_1)-d\boldsymbol{r}(\boldsymbol{\rho})A(\mathcal{Z}_2)g(\mathcal{Z}_3,\mathcal{W}_1)]=0.\end{aligned}\tag{52}$$

Substituting $\mathcal{Z}_3$ by $\boldsymbol{\rho}$ in (52), we acquire that

$$\begin{aligned}&[(2n^2-8n+7)a_1+(n-2)^3b_1][(\nabla_{\mathcal{Z}_2}A)(\mathcal{W}_1)+A(\mathcal{Z}_2)(\nabla_{\boldsymbol{\rho}}A)(\mathcal{W}_1)]\boldsymbol{r}\\&+\frac{a_1(n-3)d\boldsymbol{r}(\boldsymbol{\rho})}{2}[A(\mathcal{Z}_2)A(\mathcal{W}_1)+g(\mathcal{Z}_2,\mathcal{W}_1)]=0.\end{aligned}\tag{53}$$

In virtue of (47) and (53), we get

$$\begin{aligned}&\left[\frac{(17n-4n^2-17)a_1-2(n-2)^3b_1}{2}\right][d\boldsymbol{r}(\mathcal{W}_1)+d\boldsymbol{r}(\boldsymbol{\rho})A(\mathcal{W}_1)]A(\mathcal{Z}_2)\\&+[(2n^2-8n+7)a_1+(n-2)^3b_1]\boldsymbol{r}(\nabla_{\mathcal{Z}_2}A)(\mathcal{W}_1)\\&+\frac{a_1(n-3)d\boldsymbol{r}(\boldsymbol{\rho})}{2}[A(\mathcal{Z}_2)A(\mathcal{W}_1)+g(\mathcal{Z}_2,\mathcal{W}_1)]=0.\end{aligned}\tag{54}$$

The equations (50) and (54) entails that

$$(\nabla_{\mathcal{Z}_2}A)(\mathcal{W}_1)=-\frac{a_1(n-3)d\boldsymbol{r}(\boldsymbol{\rho})}{2[(2n^2-8n+7)a_1+(n-2)^3b_1]\boldsymbol{r}}[g(\mathcal{Z}_2,\mathcal{W}_1)+A(\mathcal{Z}_2)A(\mathcal{W}_1)],\tag{55}$$

provided $(2n^2-8n+7)a_1+(n-2)^3b_1\neq 0$. This means that $\boldsymbol{\rho}$ is torse-forming.

Putting $\mathcal{Z}_3 = \boldsymbol{\rho}$ in (15), we infer that

$$\mathcal{S}(\mathcal{Z}_2, \boldsymbol{\rho}) = \boldsymbol{r}g(\mathcal{Z}_2, \boldsymbol{\rho}), \tag{56}$$

which reflects that $\boldsymbol{r}$ is an eigenvalue corresponding to the eigenvector $\boldsymbol{\rho}$.

In view of this observation and theorem A, we conclude the following:

Theorem 4.2. *A Ricci simple spacetime with harmonic $\mathcal{H}$-curvature tensor is a* GRW *spacetime, provided* $(2n^2 - 8n + 7)a_1 + (n - 2)^3 b_1 \neq 0$.

In a Lorentzian manifold, the Raychaudhuri equation [31] for the fluid can be expressed as

$$(\nabla_{\mathcal{Z}_2} A)(\mathcal{Z}_3) = \omega(\mathcal{Z}_2, \mathcal{Z}_3) + \tau(\mathcal{Z}_2, \mathcal{Z}_3) + \varphi[g(\mathcal{Z}_2, \mathcal{Z}_3) + A(\mathcal{Z}_2)A(\mathcal{Z}_3)], \tag{57}$$

ω stands for the vorticity tensor, τ stands for the shear tensor and φ is a scalar function. The equation (51) provides us the one-form A is closed, provided $(2n^2 - 8n + 7)a_1 + (n - 2)^3 b_1 \neq 0$. Hence $\boldsymbol{\rho}$ is irrotational. Consequently, the fluid has zero vorticity, that is, $\omega(\mathcal{Z}_2, \mathcal{Z}_3) = 0$. Therefore the foregoing equation reduces to

$$(\nabla_{\mathcal{Z}_2} A)(\mathcal{Z}_3) = \tau(\mathcal{Z}_2, \mathcal{Z}_3) + \varphi[g(\mathcal{Z}_2, \mathcal{Z}_3) + A(\mathcal{Z}_2)A(\mathcal{Z}_3)]. \tag{58}$$

Since $\boldsymbol{\rho}$ is a torse-forming vector field, the equations (14) and (58) turns into

$$\tau(\mathcal{Z}_2, \mathcal{Z}_3) = 0. \tag{59}$$

Thus we can state:

Corollary 4.4. *A Ricci simple spacetime with harmonic $\mathcal{H}$-curvature tensor is vorticity-free and shear-free, provided* $(2n^2 - 8n + 7)a_1 + (n - 2)^3 b_1 \neq 0$.

5. GRW spacetimes with $\mathcal{H}$-flat curvature tensor

For this section, we need the following result which is important for the subsequent results.

Theorem B. *[32] A Lorentzian manifold of dimension* n $(n \geqslant 3)$ *is a GRW spacetime if and only if it admits a timelike vector field $\mathcal{W}_1$ such that $\nabla_{\mathcal{Z}_2} \mathcal{W}_1 = f\,\mathcal{Z}_2$, f being a scalar.*

From the above theorem B we acquire that

$$\mathcal{R}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 = (\mathcal{Z}_2 f)\mathcal{Z}_3 - (\mathcal{Z}_3 f)\mathcal{Z}_2. \tag{60}$$

Contracting (60) with $\mathcal{Z}_2$ provides

$$\mathcal{S}(\mathcal{Z}_3, \mathcal{W}_1) = (1 - n)(\mathcal{Z}_3 f). \tag{61}$$

Replacing $\mathcal{W}_1$ by $\boldsymbol{\rho}$ in (61) we infer

$$\mathcal{S}(\mathcal{Z}_3, \boldsymbol{\rho}) = (1 - n)(\mathcal{Z}_3 f). \tag{62}$$

Since by hypothesis the GRW spacetime is $\mathcal{H}$-flat, from (18) we deduce that

$$\mathcal{S}(\mathcal{Z}_2, \boldsymbol{\rho}) = \frac{\boldsymbol{r}}{n} A(\mathcal{Z}_2). \tag{63}$$

In view of (62) and (63) we reach

$$\mathcal{Z}_3 f = \frac{\boldsymbol{r}}{n(1 - n)} A(\mathcal{Z}_3). \tag{64}$$

For the flow vector field $\boldsymbol{\rho}$, equation (60) gives us

$$\mathcal{R}(\mathcal{Z}_2, \mathcal{Z}_3)\boldsymbol{\rho} = (\mathcal{Z}_2 f)\mathcal{Z}_3 - (\mathcal{Z}_3 f)\mathcal{Z}_2. \tag{65}$$

Equations (64) and (65) turns into

$$\mathcal{R}(\mathcal{Z}_2, \mathcal{Z}_3)\boldsymbol{\rho} = \frac{\boldsymbol{r}}{n(1 - n)}[A(\mathcal{Z}_2)\mathcal{Z}_3 - A(\mathcal{Z}_3)\mathcal{Z}_2]. \tag{66}$$

Now the equation (11) implies

$$C(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \boldsymbol{\rho}) = R(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \boldsymbol{\rho}) - \frac{1}{n-2}[\mathcal{S}(\mathcal{Z}_3, \mathcal{W}_1)A(\mathcal{Z}_2) - \mathcal{S}(\mathcal{Z}_2, \mathcal{W}_1)A(\mathcal{Z}_3) + g(\mathcal{Z}_3, \mathcal{W}_1)\mathcal{S}(\mathcal{Z}_2, \boldsymbol{\rho}) - g(\mathcal{Z}_2, \mathcal{W}_1)\mathcal{S}(\mathcal{Z}_3, \boldsymbol{\rho})] + \frac{\boldsymbol{r}}{(n-1)(n-2)}[g(\mathcal{Z}_3, \mathcal{W}_1)A(\mathcal{Z}_2) - g(\mathcal{Z}_2, \mathcal{W}_1)A(\mathcal{Z}_3)]. \tag{67}$$

Adopting (18), (63), (66) and (67) we arrive at $C(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \boldsymbol{\rho}) = 0$, that is, $\mathcal{C}(\mathcal{Z}_2, \mathcal{Z}_3)\boldsymbol{\rho} = 0$, which means that the Weyl curvature tensor is purely electric [33]. It is noted that in a GRW spacetime, $\mathcal{C}(\mathcal{Z}_2, \mathcal{Z}_3)\boldsymbol{\rho} = 0$ if and only if div $\mathcal{C} = 0$ [30]. Moreover, it is well known

$$(\mathrm{div}\, \mathcal{C})(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{W}_1 = \left(\frac{n-3}{n-2}\right)\left[\{(\nabla_{\mathcal{Z}_2}\mathcal{S})(\mathcal{Z}_3, \mathcal{W}_1) - (\nabla_{\mathcal{Z}_3}\mathcal{S})(\mathcal{Z}_2, \mathcal{W}_1)\} - \frac{1}{2(n-1)}\{g(\mathcal{Z}_3, \mathcal{W}_1)dr(\mathcal{Z}_2) - g(\mathcal{Z}_2, \mathcal{W}_1)dr(\mathcal{Z}_3)\}\right]. \tag{68}$$

Since the $\mathcal{H}$-flat spacetime is Einstein, therefore obviously div $\mathcal{C} = 0$.

Thus we can state:

Theorem 5.1. *In a $\mathcal{H}$-flat* GRW *spacetime the Weyl curvature tensor is purely electric and harmonic.*

For $n = 4$, $\mathcal{C}(\mathcal{Z}_2, \mathcal{Z}_3)\boldsymbol{\rho} = 0$ is similar to

$$A(\mathcal{Z}_4)C(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) + A(\mathcal{Z}_2)C(\mathcal{Z}_3, \mathcal{Z}_4, \mathcal{W}_1, \mathcal{Z}_1) + A(\mathcal{Z}_3)C(\mathcal{Z}_4, \mathcal{Z}_2, \mathcal{W}_1, \mathcal{Z}_1) = 0, \tag{69}$$

$A(\mathcal{Z}_4) = g(\mathcal{Z}_4, \boldsymbol{\rho})$. Substituting $\mathcal{Z}_4$ by $\boldsymbol{\rho}$ in (69) we have $C(\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{W}_1, \mathcal{Z}_1) = 0$ [34]. In [35], the authors proved that a GRW spacetime is a RW spacetime if and only if it is conformally flat.

Hence we write:

Corollary 5.1. *A 4-dimensional $\mathcal{H}$-flat* GRW *spacetime is a* RW *spacetime.*

It is known ([36], p. 73) that if a spacetime admits a unit timelike vector field in which the Weyl curvature tensor is purely electric, then it is of Petrov type *I*, *D* or *O*.

Consequently, the following is our conclusion:

Corollary 5.2. *A $\mathcal{H}$-flat* GRW *spacetime is of Petrov type I*, D *or O*.

6. $\mathcal{H}$-flat spacetimes obeying f(r, $\mathcal{T}$)-gravity and f(r, $\mathcal{L}_m$)-gravity

In this section, we characterize $\mathcal{H}$-flat spacetimes satisfying $\mathbf{f}(\boldsymbol{r}, \mathcal{T})$-gravity and $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$-gravity.

Firstly, we consider $\mathcal{H}$-flat spacetimes obeying $\mathbf{f}(\mathbf{r}, \mathcal{T})$-gravity. The generalization of $\mathbf{f}(\boldsymbol{r})$-gravity, which provides the physical aspect of the matter distribution in order to formulate a theoretical model, is termed $\mathbf{f}(\boldsymbol{r}, \mathcal{T})$-gravity. This modified gravity theory was first presented by Harko *et al* [16]. For a number of unique cases of $\mathbf{f}(\boldsymbol{r}, \mathcal{T})$-gravity, the related field equations have been studied in metric formalism. Here, we select [16]

$$\mathbf{f}(\boldsymbol{r}, \mathcal{T}) = 2\mathbf{f}(\mathcal{T}) + \boldsymbol{r}, \tag{70}$$

$\mathbf{f}(\mathcal{T})$ be a function of trace $\mathcal{T}$ of the EMT and $\mathbf{f}(\boldsymbol{r}, \mathcal{T})$ be a function of the trace $\mathcal{T}$ of the EMT and the scalar curvature $\boldsymbol{r}$.

According to the modified Einstein-Hilbert action term,

$$E = \int \left[\frac{16\pi\mathcal{L}_m + \mathbf{f}(\boldsymbol{r}, \mathcal{T})}{16\pi}\right]\sqrt{-g}\ d^4x, \tag{71}$$

where $\mathcal{L}_m$ denotes the matter Lagrangian of the scalar field. The stress energy tensor of the matter is presented as

$$\mathcal{T}_{ij} = \frac{-2\delta(\sqrt{-g})\mathcal{L}_m}{\sqrt{-g}\ \delta^{ij}}. \tag{72}$$

Consider that $\mathcal{L}_m$ depends solely on g and not its derivatives.

The following field equations of $\mathbf{f}(\boldsymbol{r}, \mathcal{T})$-gravity are obtained from the variation of action (71) with respect to the metric tensor g:

$$\mathbf{f}_r(r, \mathcal{T})\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_3) - \frac{1}{2}\mathbf{f}(r, \mathcal{T})g(\mathcal{Z}_2, \mathcal{Z}_3) + [g(\mathcal{Z}_2, \mathcal{Z}_3)\Box - \nabla_{\mathcal{Z}_2}\nabla_{\mathcal{Z}_3}]\mathbf{f}_r(r, \mathcal{T})$$
$$= 8\pi\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3) - [\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3) + \Theta(\mathcal{Z}_2, \mathcal{Z}_3)]\mathbf{f}_{\mathcal{T}}(r, \mathcal{T}), \tag{73}$$

$\mathbf{f}_r(r, \mathcal{T}) = \frac{\partial\{\mathbf{f}(r, \mathcal{T})\}}{\partial r}$, $\mathbf{f}_{\mathcal{T}}(r, \mathcal{T}) = \frac{\partial\{\mathbf{f}(r, \mathcal{T})\}}{\partial \mathcal{T}}$, $\Box$ indicates the d'Alembert operator and

$$\Theta(\mathcal{Z}_2, \mathcal{Z}_3) = -2\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3) + g(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{L}_m - 2g^{lk}\frac{\partial^2\mathcal{L}_m}{\partial g^{ab}\partial g^{lk}}. \tag{74}$$

If $\mathbf{f}(r, \mathcal{T})$ equals $\mathbf{f}(r)$, then (73) provides the field equations of $\mathbf{f}(r)$-gravity.

Since the Lagrangian has no single value, we suppose that $\mathcal{L}_m$ is equal to $-p$ and utilizing (3), we infer that

$$\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3) = -pg(\mathcal{Z}_2, \mathcal{Z}_3) + (p + \sigma)A(\mathcal{Z}_2)A(\mathcal{Z}_3), \tag{75}$$

where $A(\mathcal{Z}_2) = g(\mathcal{Z}_2, \rho)$ and $g(\rho, \rho) = -1$. In view of (75), the variation of stress energy can be simply obtained in the following form:

$$\Theta(\mathcal{Z}_2, \mathcal{Z}_3) = -2\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3) - pg(\mathcal{Z}_2, \mathcal{Z}_3). \tag{76}$$

Equations (70) and (73) together yield

$$\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_3) = \frac{r}{2}g(\mathcal{Z}_2, \mathcal{Z}_3) + 8\pi\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3) + \mathbf{f}(\mathcal{T})g(\mathcal{Z}_2, \mathcal{Z}_3)$$
$$- 2[\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3) + \Theta(\mathcal{Z}_2, \mathcal{Z}_3)]\mathbf{f}'(\mathcal{T}). \tag{77}$$

Harko *et al* [16] did not take the conservation of the EMT in the derivation of the field equations. But in [15], the author assumed the conservation of the EMT. This section examines the PFS solution to the $\mathbf{f}(r, \mathcal{T})$-gravity equation under the assumption that the EMT is conserved.

Adopting (75), (76) and (77), we reach

$$\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_3) = \left[\frac{r}{2} - 8p\pi + \mathbf{f}(\mathcal{T})\right]g(\mathcal{Z}_2, \mathcal{Z}_3)$$
$$+ (p + \sigma)\{2\mathbf{f}'(\mathcal{T}) + 8\pi\}A(\mathcal{Z}_2)A(\mathcal{Z}_3). \tag{78}$$

It indicates that the Ricci tensor of the PFS in $\mathbf{f}(r, \mathcal{T})$-gravity theory takes the form (78). Equations (18) and (78) reflects that

$$\left[\frac{r}{4} - 8p\pi + \mathbf{f}(\mathcal{T})\right]g(\mathcal{Z}_2, \mathcal{Z}_3) + (p + \sigma)\{2\mathbf{f}'(\mathcal{T}) + 8\pi\}A(\mathcal{Z}_2)A(\mathcal{Z}_3) = 0. \tag{79}$$

Contracting (79), reveals that

$$r + 4\mathbf{f}(\mathcal{T}) - 32p\pi - (p + \sigma)\{8\pi + 2\mathbf{f}'(\mathcal{T})\} = 0. \tag{80}$$

Replacing $\mathcal{Z}_2$ and $\mathcal{Z}_3$ by ρ in (79), we acquire that

$$-r - 4\mathbf{f}(\mathcal{T}) + 32p\pi + 4(p + \sigma)\{8\pi + 2\mathbf{f}'(\mathcal{T})\} = 0. \tag{81}$$

Equations (80) and (81) turns into

$$\{4\pi + \mathbf{f}'(\mathcal{T})\}(p + \sigma) = 0, \tag{82}$$

which means that either $p + \sigma = 0$ or, $p + \sigma \neq 0$.

Case 1. If $p + \sigma = 0$, then a dark matter era is represented in spacetime.

Case 2. If $p + \sigma \neq 0$, then $4\pi + \mathbf{f}'(\mathcal{T}) = 0$. Thus the equation (78) reflects that it is an Einstein spacetime. The result is as follows:

Theorem 6.1. *A $\mathcal{H}$-flat spacetime obeying $\mathbf{f}(r, \mathcal{T})$-gravity for the model $\mathbf{f}(r, \mathcal{T}) = r + 2\mathbf{f}(\mathcal{T})$ represents either a dark matter era or an Einstein spacetime.*

Equations (80) and (81) infers

$$p = \frac{r + 4\mathbf{f}(\mathcal{T})}{32\pi}. \tag{83}$$

For dust matter era, that is, p is equal to zero, the foregoing equation reduces to $\mathbf{f}(\mathcal{T}) = -\frac{r}{4}$. Consequently, we can say that:

Corollary 6.1. *A $\mathcal{H}$-flat spacetime is unable to illustrate dust matter era for any viable $\mathbf{f}(r, \mathcal{T})$.*

Remark 6.1. $\mathbf{f}(r, \mathcal{T})$-gravity reduces to $\mathbf{f}(r)$-gravity for $\mathbf{f}(\mathcal{T})$ equals zero. Therefore by theorem 6.1, for $\mathbf{f}(r)$-gravity a dark matter era is represented in $\mathcal{H}$-flat spacetime. The equation of state is $p + \sigma = 0$, that is,

$|\boldsymbol{\sigma}| = |-\boldsymbol{p}|$, that is, $\boldsymbol{\sigma} = |\boldsymbol{p}|$ as the energy density cannot be negative. Hence in $\mathbf{f}(\boldsymbol{r})$-gravity a $\mathcal{H}$-flat spacetime satisfies the DEC. Thus in a $\mathcal{H}$-flat spacetime satisfying $\mathbf{f}(\boldsymbol{r})$-gravity the speed of light is the faster than the speed of matter [27].

Now we consider $\mathcal{H}$-flat spacetimes in the context of $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$-gravity, which is the another extension of $\mathbf{f}(\boldsymbol{r})$-gravity, proposed by Harko and Lobo [18]. As a result of the coupling the motion of the massive particles is non-geodesic, and an extra force, orthogonal to the four-velocity, arises. The connections with MOND and the Pioneer anomaly were also explored. This model was extended to the case of the arbitrary couplings in both geometry and matter in [17]. We assume that the action term for the modified theories of gravity takes the following form:

$$S = \int \mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)\sqrt{-g}\; d^4x, \tag{84}$$

where $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$ is an arbitrary function of the scalar curvature $\boldsymbol{r}$, and of the Lagrangian density corresponding to matter, $\mathcal{L}_m$. We define the energy-momentum tensor of the matter as

$$\mathcal{T}_{ij} = \frac{-2\delta(\sqrt{-g}\,\mathcal{L}_m)}{\sqrt{-g}\;\delta g^{ij}}. \tag{85}$$

Assuming that $\mathcal{L}_m$ depends on the metric tensor g and not on its derivatives.

The following field equations of $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$-gravity are obtained from the variation of action (84) with respect to the metric tensor g:

$$\begin{aligned}&\mathbf{f}_{\boldsymbol{r}}(\boldsymbol{r}, \mathcal{L}_m)\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_3) + [g(\mathcal{Z}_2, \mathcal{Z}_3)\,\Box - \nabla_{\mathcal{Z}_2}\nabla_{\mathcal{Z}_3}]\mathbf{f}_{\boldsymbol{r}}(\boldsymbol{r}, \mathcal{L}_m)\\ &-\frac{1}{2}[\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m) - \mathbf{f}_{\mathcal{L}_m}(\boldsymbol{r}, \mathcal{L}_m)\mathcal{L}_m]g(\mathcal{Z}_2, \mathcal{Z}_3) = \frac{1}{2}\mathbf{f}_{\mathcal{L}_m}(\boldsymbol{r}, \mathcal{L}_m)\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3).\end{aligned} \tag{86}$$

Contracting the field equation (86), we get

$$\mathbf{f}_{\boldsymbol{r}}(\boldsymbol{r}, \mathcal{L}_m)\boldsymbol{r} + 3\,\Box\,\mathbf{f}_{\boldsymbol{r}}(\boldsymbol{r}, \mathcal{L}_m) - 2[\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m) - \mathbf{f}_{\mathcal{L}_m}(\boldsymbol{r}, \mathcal{L}_m)\mathcal{L}_m] = \frac{1}{2}\mathbf{f}_{\mathcal{L}_m}(\boldsymbol{r}, \mathcal{L}_m)t, \tag{87}$$

where t is the trace of the energy-momentum tensor $\mathcal{T}$. Eliminating the term $\Box\mathbf{f}_{\boldsymbol{r}}(\boldsymbol{r}, \mathcal{L}_m)$ from (86) and (87), we obtain the modified form of the gravitational field equations as [18]

$$\begin{aligned}&\left[\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_3) - \frac{\boldsymbol{r}}{3}g(\mathcal{Z}_2, \mathcal{Z}_3)\right]\mathbf{f}_{\boldsymbol{r}}(\boldsymbol{r}, \mathcal{L}_m) + \frac{1}{6}[\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m) - \mathbf{f}_{\mathcal{L}_m}(\boldsymbol{r}, \mathcal{L}_m)\mathcal{L}_m]g(\mathcal{Z}_2, \mathcal{Z}_3)\\ &=\frac{1}{2}\left[\mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3) - \frac{t}{3}g(\mathcal{Z}_2, \mathcal{Z}_3)\right]\mathbf{f}_{\mathcal{L}_m}(\boldsymbol{r}, \mathcal{L}_m) + \nabla_{\mathcal{Z}_2}\nabla_{\mathcal{Z}_3}\mathbf{f}_{\boldsymbol{r}}(\boldsymbol{r}, \mathcal{L}_m).\end{aligned} \tag{88}$$

We consider the following model [18] for our investigations:

$$\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m) = \frac{\boldsymbol{r}}{2} + \mathcal{L}_m. \tag{89}$$

Then for this particular $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$ model with $\mathcal{L}_m = \boldsymbol{\sigma}$ [19], the equation (88) becomes

$$\mathcal{S}(\mathcal{Z}_2, \mathcal{Z}_3) + \left(\frac{t}{3} - \frac{\boldsymbol{r}}{6}\right)g(\mathcal{Z}_2, \mathcal{Z}_3) = \mathcal{T}(\mathcal{Z}_2, \mathcal{Z}_3). \tag{90}$$

In this case, we consider PFS solution of $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$-gravity equation assuming the EMT is of the form (3).

Contracting the equation (3), we have

$$t = 3\boldsymbol{p} - \boldsymbol{\sigma}. \tag{91}$$

Equations (3), (18), (90) and (91) gives us

$$\left(\frac{\boldsymbol{\sigma}}{3} - \frac{\boldsymbol{r}}{12}\right)g(\mathcal{Z}_2, \mathcal{Z}_3) + (\boldsymbol{p} + \boldsymbol{\sigma})A(\mathcal{Z}_2)A(\mathcal{Z}_3) = 0. \tag{92}$$

Contracting the equation (92), we obtain

$$\boldsymbol{r} = \boldsymbol{\sigma} - 3\boldsymbol{p}. \tag{93}$$

Putting $\mathcal{Z}_2 = \mathcal{Z}_3 = \boldsymbol{\rho}$ in the equation (92), we find

$$-\boldsymbol{r} = 12\boldsymbol{p} + 8\boldsymbol{\sigma}. \tag{94}$$

From the equations (93) and (94), we reach

$$\boldsymbol{p} + \boldsymbol{\sigma} = 0. \tag{95}$$

Thus we conclude:

Theorem 6.2. *A $\mathcal{H}$-flat perfect fluid spacetime satisfying* $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m)$*-gravity for the model* $\mathbf{f}(\boldsymbol{r}, \mathcal{L}_m) = \frac{\boldsymbol{r}}{2} + \boldsymbol{\sigma}$ *represents a dark matter era.*

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Data availability statement

No new data were created or analysed in this study.

Conflict of interest

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ORCID iDs

Dipankar Hazra https://orcid.org/0000-0001-8004-9039

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CHARACTERIZATIONS OF ALMOST PSEUDO-RICCI SYMMETRIC SPACETIMES UNDER GRAY'S DECOMPOSITION

Dipankar Hazra

Department of Mathematics, Heramba Chandra College,
23/49 Gariahat Road, Kolkata 700029,
West Bengal, India
(e-mail: dipankarsk524@gmail.com)

and

Uday Chand De*

Department of Pure Mathematics, University of Calcutta,
35 Ballygunge Circular Road, Kolkata 700019,
West Bengal, India
(e-mail: uc_de@yahoo.com)

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In this study, we analyze almost pseudo-Ricci symmetric spacetimes endowed with Gray's decomposition, as well as generalized Robertson–Walker spacetimes. For almost pseudo-Ricci symmetric spacetimes, we determine the form of the Ricci tensor in all the $O(n)$-invariant subspaces provided by Gray's decomposition of the gradient of the Ricci tensor. In three cases we obtain that the Ricci tensor is in the form of perfect fluid and in one case the spacetime becomes a generalized Robertson–Walker spacetime. In other cases we obtain some algebraic results. Finally, it is shown that an almost pseudo-Ricci symmetric generalized Robertson–Walker spacetime is a perfect fluid spacetime.

Keywords: almost pseudo-Ricci symmetric spacetime, Gray's decomposition, perfect fluid spacetime, generalized Robertson–Walker spacetime.

1. Introduction

Lorentzian geometry is the mathematical framework that supports some of the most important theories in modern physics, general relativity and string theory. From a purely mathematical point of view, a Lorentzian manifold M is a smooth manifold endowed with a symmetric nondegenerate bilinear form g, called the metric of signature $(-,+,+,+,\dots,+)$, that is, index of g is 1. In general, a Lorentzian manifold (M,g) may not have a globally timelike vector field. If (M,g) admits a globally timelike vector field, it is called a time-oriented Lorentzian manifold,

*Corresponding author.

physically known as spacetime. Several authors have explored spacetimes in various ways, such as [1–3] and also numerous others.

A nonflat semi-Riemannian manifold is named *pseudo-Ricci symmetric* [4] and indicated by $(\mathrm{PRS})_n$ if the Ricci tensor $\mathcal{S}$ of type $(0,2)$ of the manifold is nonzero and satisfies the relation

$$(\nabla_X \mathcal{S})(Y,Z) = 2\omega(X)\mathcal{S}(Y,Z) + \omega(Y)\mathcal{S}(X,Z) + \omega(Z)\mathcal{S}(X,Y),$$

where ∇ signifies the covariant differentiation with respect to the metric g, ω is a nonvanishing one-form and $\tilde{\rho}$ is a vector field metrically equivalent to the one-form ω, that is,

$$\omega(X) = g(X,\tilde{\rho})$$

for all X. The manifold reduces to a Ricci symmetric manifold if $\omega = 0$. Several authors [4, 5] have investigated pseudo-Ricci symmetric manifolds as well as pseudo-Ricci symmetric spacetimes.

Thus, in general theory of relativity, the pseudo-Ricci symmetric manifolds have some applications. In view of this, Chaki and Kawaguchi [6] were inspired to generalize pseudo-Ricci symmetric manifolds and introduced the notion of a new manifold called almost pseudo-Ricci symmetric.

A semi-Riemannian manifold (M^n, g) is called *an almost pseudo-Ricci symmetric* (denoted by $\mathrm{A}(\mathrm{PRS})_n$) if the covariant derivative of the Ricci tensor satisfies

$$(\nabla_X \mathcal{S})(Y,Z) = [\omega(X) + \eta(X)]\mathcal{S}(Y,Z) + \omega(Y)\mathcal{S}(X,Z) + \omega(Z)\mathcal{S}(X,Y), \tag{1}$$

where $\mathcal{S}$ is the nonvanishing Ricci tensor, ω and η are nonvanishing one-forms such that $g(X,\tilde{\rho}) = \omega(X)$ and $g(X,\rho) = \eta(X)$, for all X and $\tilde{\rho}$, ρ are called the basic vector fields of the manifold corresponding to the associated one-forms ω and η, respectively.

If $\omega = \eta$, then $\mathrm{A}(\mathrm{PRS})_n$ yields $(\mathrm{PRS})_n$. Almost pseudo-Ricci symmetric manifolds have been investigated by several authors [7, 8] and many others.

Changing X and Y in (1) and subtracting these two equations, we obtain

$$(\nabla_X \mathcal{S})(Y,Z) - (\nabla_Y \mathcal{S})(X,Z) = \eta(X)\mathcal{S}(Y,Z) - \eta(Y)\mathcal{S}(X,Z). \tag{2}$$

Considering a frame field and contracting Y and Z in (2), yields

$$dr(X) = 2r\eta(X) - 2\mathcal{S}(X,\rho), \tag{3}$$

where $r = \sum_{i=1}^{n} \varepsilon_i \mathcal{S}(e_i,e_i)$ is the scalar curvature and $\varepsilon_i = g(e_i,e_i) = \pm 1$.

The conformal curvature tensor in a Lorentzian manifold (M^n, g) $(n > 3)$ is given by

$$C(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}[g(Y,Z)QX - g(X,Z)QY + \mathcal{S}(Y,Z)X - \mathcal{S}(X,Z)Y]$$
$$+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \tag{4}$$

Q is the Ricci operator defined as $\mathcal{S}(X,Y) = g(QX,Y)$ and r is the scalar curvature. De, Özgür and De [9] proved that a conformally flat $\mathrm{A}(\mathrm{PRS})_4$ spacetime is the Robertson–Walker spacetime.

In cosmology, the observation that the space is isotropic and homogeneous in the large scale one chooses the Robertson–Walker (briefly, RW) metric. In 1995, Alías, Romero and Sánchez [10] generalized the notion of RW metric to the generalized Robertson–Walker (briefly, GRW) metric. A Lorentzian manifold M of dimension $n \geq 3$ endowed with the Lorentzian metric g defined by

$$ds^2 = -(dt)^2 + \varphi^2(t)\, g^*_{lm}(x)\, dx^l dx^m, \tag{5}$$

where t is the time and $g^*_{lm}(x)$ is the metric tensor of a Riemannian manifold M^*, is a GRW spacetime. In other words, a GRW spacetime is the warped product $-I \times \varphi^2 M^*$, where I is an open interval of the real line, φ is a smooth warping function or scale factor such that $\varphi > 0$ and M^* is an $(n-1)$-dimensional Riemannian manifold. In particular, if M^* is a 3-dimensional Riemannian manifold of constant sectional curvature, then the warped product $-I \times \varphi^2 M^*$ is said to be a RW spacetime. A Robertson–Walker spacetime compiles the cosmological principle, that is, the spacetime is locally spatially isotropic and locally spatially homogeneous, although the GRW spacetime is not necessarily spatially homogeneous [3].

Lorentzian manifolds with the Ricci tensor

$$\mathcal{S}(X,Y) = \alpha g(X,Y) + \beta\eta(X)\,\eta(Y), \tag{6}$$

where α and β are scalars and ρ is a unit timelike vector field corresponding to the one-form η, are called perfect fluid spacetimes.

The energy momentum tensor $\mathcal{T}$ represents the matter content of the spacetime, which is considered to be fluid. The energy momentum tensor for a perfect fluid spacetime has the form [11]

$$\mathcal{T}(X,Y) = pg(X,Y) + (p+\sigma)\,\eta(X)\,\eta(Y), \tag{7}$$

where σ represents the energy density and p represents the isotropic pressure. The velocity vector field ρ is a unit timelike vector field that is metrically equivalent to the nonzero one-form η. The perfect fluid is known as isentropic for $p = p(\sigma)$ and for $p = \sigma$, the fluid is called stiff matter fluid ([12], p. 66).

Einstein's field equation (briefly, $\mathcal{EFE}$) without cosmological constant is as follows:

$$\mathcal{S}(X,Y) - \frac{r}{2} g(X,Y) = \kappa\mathcal{T}(X,Y), \tag{8}$$

$\mathcal{S}$ and r being the Ricci tensor and scalar curvature, respectively, κ is the gravitational constant. According to $\mathcal{EFE}$, the geometry of spacetime is determined by matter, whereas the motion of matter is dictated by the nonflat metric of the spacetime. The above form (6) of the Ricci tensor is derived from Einstein's equation using Eq. (7).

In this paper, we consider almost pseudo-Ricci symmetric spacetimes. The Lorentzian setting supports the results obtained for almost pseudo-Ricci symmetric manifolds. An n-dimensional Lorentzian manifold (M, g) is said to be an almost pseudo-Ricci symmetric spacetime if the Ricci tensor satisfies (1). Almost pseudo-Ricci symmetric spacetimes have been investigated by several authors [9, 13] and many others. The existence of such a spacetime has been proved by De et al. [9]. In this case, the associated vector field corresponding to the one-form η is treated as a unit timelike velocity vector field or flow vector field, i.e. $g(\rho, \rho) = -1$.

The present paper is structured as follows. In Section 2, we investigate all the seven cases of Gray's decomposition of $\mathrm{A(PRS)}_n$. The study of $\mathrm{A(PRS)}_n$ with GRW spacetime is presented in Section 3.

2. Gray's decomposition and almost pseudo-Ricci symmetric spacetimes

Considering the action of the orthogonal group on the space of tensors with the symmetries of the covariant derivative of the Ricci curvature, Gray decomposed such space into irreducible components [14]. Gray proposed that the covariant derivative of the Ricci tensor, that is $\nabla\mathcal{S}$, can be decomposed into $O(n)$-invariant terms (for additional information, see [15]). According to [14], the covariant derivative of the Ricci tensor $(\nabla_X\mathcal{S})(Y,Z)$ can be converted into $O(n)$-invariant term as follows [16]:

$$(\nabla_X\mathcal{S})(Y,Z) = \tilde{\mathcal{R}}(X,Y)Z + \gamma(X)g(Y,Z) + \delta(Y)g(X,Z) + \delta(Z)g(X,Y), \tag{9}$$

for all vector fields X, Y, Z, where

$$\gamma(X) = \frac{n}{(n-1)(n+2)}\nabla_X r, \qquad \delta(X) = \frac{n-2}{2(n-1)(n+2)}\nabla_X r,$$

and $\tilde{\mathcal{R}}(X,Y)Z = \tilde{\mathcal{R}}(X,Z)Y$ is a tensor with zero trace that can be written as a sum of its orthogonal components

$$\tilde{\mathcal{R}}(X,Y)Z = \frac{1}{3}\left[\tilde{\mathcal{R}}(X,Y)Z + \tilde{\mathcal{R}}(Y,Z)X + \tilde{\mathcal{R}}(Z,X)Y\right] + \frac{1}{3}\left[\tilde{\mathcal{R}}(X,Y)Z - \tilde{\mathcal{R}}(Y,X)Z\right] + \frac{1}{3}\left[\tilde{\mathcal{R}}(X,Y)Z - \tilde{\mathcal{R}}(Z,X)Y\right]. \tag{10}$$

The decompositions (9) and (10) yield $O(n)$-invariant subspace, which is characterized by linear invariant equations in $(\nabla_X\mathcal{S})(Y,Z)$.

Therefore, the relation between $\nabla\mathcal{S}$ and the divergence of the Weyl conformal curvature tensor C can be given by the equation [16]

$$(\operatorname{div} C)(X,Y)Z = \frac{n-3}{n-2}\left[\tilde{\mathcal{R}}(X,Y)Z - \tilde{\mathcal{R}}(Y,X)Z\right]. \tag{11}$$

The subspaces in Gray's decomposition are as follows:

(i) The *trivial subspace* is characterized by $\nabla\mathcal{S} = 0$.

(ii) The *subspace* $\mathcal{J}$ is characterized by $\tilde{\mathcal{R}}(X,Y)Z = 0$, i.e.

$$(\nabla_X\mathcal{S})(Y,Z) = \gamma(X)g(Y,Z) + \delta(Y)g(X,Z) + \delta(Z)g(X,Y), \tag{12}$$

where γ, δ are one-forms. Manifolds satisfying this requirement (12) are called *Sinyukov manifolds* [17, 18].

(iii) The *orthogonal complements* $\mathcal{J}'$ (also referred to as the subspace $\mathcal{A}$) are characterized by

$$(\nabla_X \mathcal{S})(Y, Z) + (\nabla_Y \mathcal{S})(Z, X) + (\nabla_Z \mathcal{S})(X, Y) = 0. \tag{13}$$

which yields that the scalar curvature r is constant. Also, the Ricci tensor is Killing [19] if Eq. (13) holds.

(iv) In *the subspaces* $\mathcal{B}$ *and* $\mathcal{B}'$ the Ricci tensor is of Codazzi type, i.e.

$$(\nabla_X \mathcal{S})(Y, Z) = (\nabla_Y \mathcal{S})(X, Z). \tag{14}$$

(v) The Ricci tensor fulfills the following cyclic condition in *the subspace* $\mathcal{J} \oplus \mathcal{A}$,

$$\begin{aligned} &(\nabla_X \mathcal{S})(Y, Z) + (\nabla_Y \mathcal{S})(Z, X) + (\nabla_Z \mathcal{S})(X, Y) \\ &= 2\frac{dr(X)}{(n+2)} g(Y, Z) + 2\frac{dr(Y)}{(n+2)} g(Z, X) + 2\frac{dr(Z)}{(n+2)} g(X, Y), \end{aligned} \tag{15}$$

that is, the Ricci tensor is conformal Killing [19].

(vi) The Ricci tensor fulfills the following Codazzi condition in *the subspace* $\mathcal{J} \oplus \mathcal{B}$,

$$\nabla_X \left[\mathcal{S}(Y, Z) - \frac{r}{2(n-1)} g(Y, Z) \right] = \nabla_Y \left[\mathcal{S}(X, Z) - \frac{r}{2(n-1)} g(X, Z) \right], \tag{16}$$

which gives $\operatorname{div} C = 0$.

(vii) In *the subspace* $\mathcal{A} \oplus \mathcal{B}$, the scalar curvature is covariant constant.

Let us consider each of these seven cases separately.

Case (i): The *trivial subspace* $\nabla \mathcal{S} = 0$.

Theorem 1. *An* $\mathrm{A(PRS)}_n$ *spacetime does not belong to the trivial subspace.*

Proof: Since $\nabla \mathcal{S} = 0$, then from the definition of $\mathrm{A(PRS)}_n$ the one-forms ω and η must vanish at any point of the manifold, which contradicts the definition of $\mathrm{A(PRS)}_n$. □

Case (ii): The subspace $\mathcal{J}$ where $\tilde{\mathcal{R}}(X, Y) Z = 0$.

Theorem 2. *If an* $\mathrm{A(PRS)}_n$ *spacetime belongs to the subspace* $\mathcal{J}$*, then the spacetime is a perfect fluid spacetime.*

Proof: The Ricci tensor satisfies the relation $\tilde{\mathcal{R}}(X, Y) Z = 0$ in the subspace $\mathcal{J}$ and hence from the relation (11) we obtain $\operatorname{div} C = 0$. So we have

$$(\nabla_X \mathcal{S})(Y, Z) - (\nabla_Z \mathcal{S})(X, Y) = \frac{1}{2(n-1)} \left[dr(X) g(Y, Z) - dr(Z) g(X, Y) \right]. \tag{17}$$

Using (1) and (3) in (17) we get

$$\eta(X)\mathcal{S}(Y,Z) - \eta(Z)\mathcal{S}(X,Y) = \frac{r}{(n-1)}\left[g(Y,Z)\eta(X) - g(X,Y)\eta(Z)\right] - \frac{1}{(n-1)}\left[g(Y,Z)\mathcal{S}(X,\rho) - g(X,Y)\mathcal{S}(Z,\rho)\right]. \tag{18}$$

Now, putting $X = Y = \rho$ in (18) and using $\eta(\rho) = -1$, we observe that

$$\mathcal{S}(Z,\rho) = -t\eta(Z), \tag{19}$$

where $t = \mathcal{S}(\rho,\rho)$.

Again setting $X = \rho$ in (18) and using (19) we arrive at

$$\mathcal{S}(Y,Z) = \frac{(r+t)}{(n-1)}g(Y,Z) + \frac{(r+nt)}{(n-1)}\eta(Y)\eta(Z). \tag{20}$$

This implies that an $\mathrm{A(PRS)}_n$ spacetime is a perfect fluid spacetime. □

COROLLARY 1. *If an* $\mathrm{A(PRS)}_4$ *spacetime belongs to the subspace* $\mathcal{J}$, *then the spacetime represents a dark matter era for* $r = -4t$.

Proof: According to $\mathcal{EFE}$ without cosmological constant, the Ricci tensor becomes

$$\mathcal{S}(Y,Z) = \kappa\left(\frac{p-\sigma}{2-n}\right)g(Y,Z) + \kappa(p+\sigma)\eta(Y)\eta(Z).$$

In contrast to Eq. (20) we notice that

$$\kappa\left(\frac{p-\sigma}{2-n}\right) = \frac{r+t}{n-1}$$

and

$$\kappa(p+\sigma) = \frac{r+nt}{n-1}.$$

Now, for $n = 4$, $p+\sigma = 0$, provided $r = -4t$. Hence the spacetime represents a dark matter era [20]. □

Case (iii): The subspace $\mathcal{A}$ is characterized by the condition (13).

THEOREM 3. *If an* $\mathrm{A(PRS)}_n$ *spacetime belongs to the subspace* $\mathcal{A}$, *then the associated one-forms are related by* $3\omega(X) + \eta(X) = 0$.

Proof: From (1) and (13) we obtain

$$[3\omega(X) + \eta(X)]\mathcal{S}(Y,Z) + [3\omega(Y) + \eta(Y)]\mathcal{S}(Z,X) + [3\omega(Z) + \eta(Z)]\mathcal{S}(X,Y) = 0. \tag{21}$$

Walker's Lemma [21] is now listed as follows.

Lemma 1. *If α_{ij}, β_i are numbers satisfying $\alpha_{ij} = \alpha_{ji}$, $\alpha_{ij}\beta_k + \alpha_{jk}\beta_i + \alpha_{ki}\beta_j = 0$ for $i, j, k = 1, 2, 3, \ldots, n$, then either all α_{ij} are zero or all β_i are zero.*

As $\mathcal{S} \neq 0$, according to Walker's Lemma, from (21) we have $3\omega(X) + \eta(X) = 0$. □

Case (iv): In this subspace, the Ricci tensor is of Codazzi type.

Proposition 1. *If an $\mathrm{A}(\mathrm{PRS})_n$ spacetime belongs to the subspaces $\mathcal{B}$ and $\mathcal{B}'$, then the spacetime is a Ricci simple spacetime.*

Proof: If an $\mathrm{A}(\mathrm{PRS})_n$ belongs to $\mathcal{B}$ and $\mathcal{B}'$, then

$$(\nabla_X \mathcal{S})(Y, Z) = (\nabla_Y \mathcal{S})(X, Z). \tag{22}$$

Using (1) in (22) reveals that

$$\eta(X)\mathcal{S}(Y, Z) = \eta(Y)\mathcal{S}(X, Z). \tag{23}$$

Putting $X = \rho$ in (23), we infer that

$$\mathcal{S}(Y, Z) = -\eta(Y)\mathcal{S}(\rho, Z). \tag{24}$$

Contracting Y and Z in (23) reveals that

$$\eta(X)\, r = \mathcal{S}(X, \rho). \tag{25}$$

Using (25) in (24) we deduce that

$$\mathcal{S}(Y, Z) = -r\eta(Y)\eta(Z),$$

which implies the spacetime is Ricci simple [22]. □

Remark 1. The physical interpretation of a Ricci simple spacetime is explored in [22]. The authors proved that a Ricci simple spacetime becomes a stiff matter fluid [12]. Thus we conclude that if an $\mathrm{A}(\mathrm{PRS})_n$ spacetime belongs to the subspaces $\mathcal{B}$ and $\mathcal{B}'$, then the spacetime becomes a stiff matter fluid.

It is known that

$$(\operatorname{div} C)(X, Y)Z = \left(\frac{n-3}{n-2}\right)\Big[\{(\nabla_X \mathcal{S})(Y, Z) - (\nabla_Y \mathcal{S})(X, Z)\} - \frac{1}{2(n-1)}\{g(Y, Z)\,dr(X) - g(X, Z)\,dr(Y)\}\Big]. \tag{26}$$

Since in our case $\mathcal{S}$ satisfies (22), this implies that the scalar curvature r is constant. Hence, from (26) we get $(\operatorname{div} C)(X, Y)Z = 0$.

Mantica, Suh and De [22] proved the following theorem.

Theorem A. *If an n-dimensional $(n > 3)$ Lorentzian manifold (M^n, g) with the Ricci tensor of the form $\mathcal{S}(X, Y) = -r\eta(X)\eta(Y)$ satisfies the curvature condition* $\operatorname{div} C = 0$, *where "div" denotes the divergence, then (M^n, g) is a GRW spacetime.*

Theorem 4. *If an* $\mathrm{A}\,(\mathrm{PRS})_n$ *spacetime belongs to the class* $\mathcal{B}$ *and* $\mathcal{B}'$*, then the spacetime becomes a* GRW *spacetime.*

Proof: Since the Ricci tensor is of Codazzi type, $\operatorname{div} C = 0$. Hence, from Proposition 1, and Theorem A, we conclude that the spacetime is a GRW spacetime. □

Case (v): In this subspace, the Ricci tensor satisfies Eq. (15). Mantica et al. [16] showed that the subspaces $\mathcal{J} \oplus \mathcal{A}$ and $\mathcal{J}$ are equivalent. In this circumstances, we reach $\operatorname{div} C = 0$. Consequently, the result is the same as in Theorem 2.

Case (vi): Let the $\mathrm{A}\,(\mathrm{PRS})_n$ belong to $\mathcal{J} \oplus \mathcal{B}$. In this case, we get $\operatorname{div} C = 0$. So we can state the same result as in Theorem 2.

Case (vii): In the subspace $\mathcal{A} \oplus \mathcal{B}$, the scalar curvature is covariant constant.

Theorem 5. *If an* $\mathrm{A}\,(\mathrm{PRS})_n$ *spacetime belongs to the subspace* $\mathcal{A} \oplus \mathcal{B}$*, then the velocity vector field* ρ *is an eigenvector corresponding to the eigenvalue* r*.*

Proof: Since the scalar curvature r is covariant constant, then from Eq. (3) we get

$$\mathcal{S}\,(X, \rho) = r g\,(X, \rho)\,.$$

This completes the proof. □

3. $\mathrm{A}\,(\mathrm{PRS})_n$ GRW spacetimes

In this section, we characterize almost pseudo-Ricci symmetric GRW spacetimes. Mantica and Molinari [23] proved that a Lorentzian manifold of dimension $n \geq 3$ is a GRW spacetime if and only if it admits a unit timelike torse forming vector field: $\nabla_k u_j = \phi\left(g_{kj} + u_k u_j\right)$, that is also an eigenvector of the Ricci tensor.

Theorem 6. *An* $\mathrm{A}\,(\mathrm{PRS})_n$ GRW *spacetime is a perfect fluid spacetime.*

Proof: We assume that the $\mathrm{A}\,(\mathrm{PRS})_n$ spacetime be a GRW spacetime. Then we have

$$(\nabla_X \eta)\,(Y) = \psi\,[g\,(X, Y) + \eta\,(X)\,\eta\,(Y)] \quad \text{and} \quad \mathcal{S}\,(X, \rho) = \mu g\,(X, \rho)\,, \tag{27}$$

for some smooth functions $\psi\,(\neq 0)$ and μ on M.
Now,

$$(\nabla_X \mathcal{S})\,(Y, \rho) = X\mathcal{S}\,(Y, \rho) - \mathcal{S}\,(\nabla_X Y, \rho) - \mathcal{S}\,(Y, \nabla_X \rho)\,. \tag{28}$$

Using (27) in (28), we arrive at

$$(\nabla_X \mathcal{S})\,(Y, \rho) = X\,(\mu)\,\eta\,(Y) + \mu\psi g\,(X, Y) - \psi\mathcal{S}\,(Y, X)\,, \tag{29}$$

where $X\,(\mu) = g\,(X, \mathrm{grad}\mu)$.

Combining Eqs. (1) and (27), we reveal

$$(\nabla_X \mathcal{S})\,(Y, \rho) = \mu\,[\omega\,(X)\,\eta\,(Y) + \eta\,(X)\,\eta\,(Y) + \omega\,(Y)\,\eta\,(X)] + \omega\,(\rho)\,\mathcal{S}\,(X, Y)\,. \tag{30}$$

Comparing Eqs. (29) and (30), we obtain

$$\mu\left[\omega(X)\eta(Y)+\eta(X)\eta(Y)+\omega(Y)\eta(X)\right]+\omega(\rho)\mathcal{S}(X,Y) = X(\mu)\eta(Y)+\mu\psi g(X,Y)-\psi\mathcal{S}(Y,X). \quad (31)$$

Setting $Y=\rho$ in (31) and using $\eta(\rho)=-1$, we infer that

$$X(\mu)=\mu\left[\omega(X)+\eta(X)\right]-2\mu\omega(\rho)\eta(X). \quad (32)$$

Contracting X and Y in (31) reveals that

$$\rho(\mu)+n\mu\psi-r\psi=\mu\left[2\omega(\rho)-1\right]+r\omega(\rho). \quad (33)$$

Eqs. (32) and (33) imply

$$\mu=\frac{r\left[\omega(\rho)+\psi\right]}{\omega(\rho)+n\psi}. \quad (34)$$

From (27) and (34), we can derive

$$\mathcal{S}(X,\rho)=\frac{r\left[\omega(\rho)+\psi\right]}{\omega(\rho)+n\psi}g(X,\rho). \quad (35)$$

This means that ρ is an eigenvector corresponding to the eigenvalue $\frac{r\left[\omega(\rho)+\psi\right]}{\omega(\rho)+n\psi}$.

In view of Eqs. (31) and (32), we obtain

$$\mathcal{S}(X,Y)=\frac{\mu}{\omega(\rho)+\psi}\left[\psi g(X,Y)-2\omega(\rho)\eta(X)\eta(Y)-\omega(Y)\eta(X)\right]. \quad (36)$$

Using (34) in (36), we reach

$$\mathcal{S}(X,Y)=\frac{r}{\omega(\rho)+n\psi}\left[\psi g(X,Y)-2\omega(\rho)\eta(X)\eta(Y)-\eta(X)\omega(Y)\right]. \quad (37)$$

Switching X and Y in (37) and subtracting these two equations, we obtain either $r=0$, or

$$\omega(Y)\eta(X)=\omega(X)\eta(Y). \quad (38)$$

Substituting $X=\rho$ in (38) gives

$$\omega(Y)=-\omega(\rho)\eta(Y). \quad (39)$$

Using (39) in (37) we infer that

$$\mathcal{S}(X,Y)=\frac{r}{\omega(\rho)+n\psi}\left[\psi g(X,Y)-\omega(\rho)\eta(X)\eta(Y)\right].$$

If $r=0$, then (34) gives us $\mu=0$.
Since μ cannot be zero, $r\neq 0$. Hence an $\mathrm{A(PRS)}_n$ GRW spacetime is a perfect fluid spacetime. □

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A Study of Public Key Cryptography Based Signature Algorithms

***Malabika Das**
Heramba Chandra College, Kolkata, India.
malabika.mail@gamil.com

Rajdeep Chakraborty
Netaji Subhas Engineering College, Kolkata, India.
rajdeep.chak@gmail.com

*****_Corresponding author_

Abstract:

Security in the world of the internet has become very important in all aspects of social life. One of the methods of securing the information on the internet is public-key cryptography or asymmetric cryptography. Public key cryptography is not only a process of encrypting information, it also provides confidentiality, data integrity and authentication. RSA and ElGamal are two very important public-key cryptosystems. These two cryptosystems are also used for digital signature scheme because of their high level of security.

In this paper, we discussed RSA algorithm in details with example. RSA is one of the most widely used public-key cryptography in various application. Then we discussed the ElGamal algorithm in details with example. ElGamal public-key cryptography is also used in many applications nowadays. In the next section, we discussed digital signature using RSA algorithm and ElGamal algorithm as digital signature is one of the important application of public-key cryptography. Therefore, Reader will have a good understanding of public-key cryptography. In this paper we have also included a relative study between RSA and ElGamal.

Keywords: Public-key cryptography, Digital signature, RSA, ElGamal

Introduction:

With the increased use of computers and communication systems, information security has become the biggest concern. Cryptography provides for secure communication in the presence of adversaries through encryption. Two forms of encryption are common in use symmetric and asymmetric or public-key encryption. In public-key cryptography [1][2], two different keys are used for encryption and decryption, one is the public key and the other is the private key. Each receiver has its own set of public and private keys. Public keys are kept public and any person can encrypt a message using the intended receiver's public key, but only by the receiver's private key, the message can be decrypted.

Digital signature [1] is a mathematical scheme to verify the authenticity of digital message or document. In this process, the sender attaches a code with the message that acts as a signature. Digital signature is based on asymmetric cryptography or public cryptography. It ensures the source and authenticity of the message. The signer creates the digital signature using a private key to encrypt signature-related data, while the only way to decrypt that data is the signer's public key. Now how it works, when a signer digitally signs a document, a cryptographic hash is generated for the document. The cryptographic hash is then encrypted using the sender's private key which creates the digital signature. Then It is appended to the document and sent to the receiver side along with the signer's public key. The recipient can decrypt the Digital signature with the signer's public key. A cryptographic hash is again generated on the receiver's side from the document. Then both cryptographic hashes are compared to check its authenticity. If they match, the document considered valid. Fig 1 gives the block diagram of Public-key cryptography.

Section 2 describes the RSA cryptosystem with example and used as digital signature, Section 3 describes the ElGamal cryptosystem with example and used as digital signature, Section 4 gives the study in details as literature review. Section 5 draws the conclusion and references are given at last.

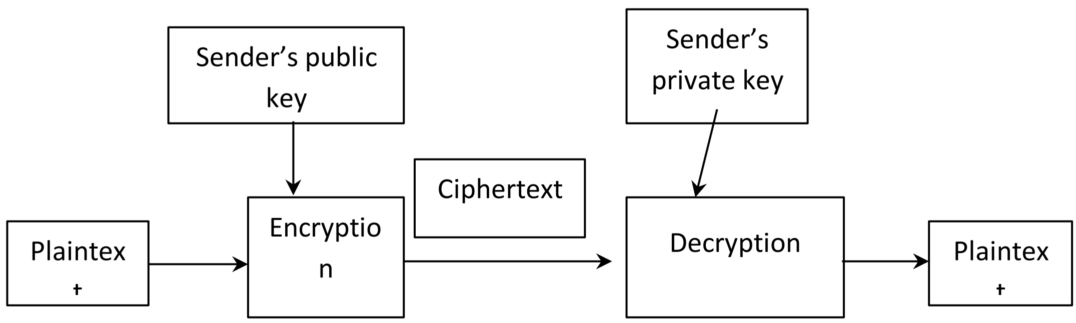

Fig.1: Block diagram of public key cryptography

RSA cryptosystem

RSA cryptosystem [1] is a public key cryptosystem algorithm based on exponentiation in modular arithmetic. The system was invented by Ron Rivest, Adi Shamir, and Len Adleman in 1978 and hence, it is termed as RSA Cryptosystem. RSA Cryptosystem involves three steps: Key generation, Encryption, and Decryption.

Section 2.1 gives key generation, section 2.2 gives encryption method, section 2.3 gives decryption method, section 2.4 gives an example and section 2.5 illustrates digital signature algorithm using RSA.

Key generation:

Each participant needs to generate public and private keys.

- Select two large prime numbers, p, and q.
- Calculate N=p×q, For strong encryption N must be large, a minimum of 1024 bits.
- Calculate totient function $\phi(N)=(p-1)(q-1)$.
- Select an integer e such that e is co-prime to $\phi(N)$, and $1< e< \phi(N)$.
- The pair of (N,e) is RSA public key and made public.
- Calculate d such that ed $\equiv 1 \bmod \phi(N)$.
- The pair (N,d) makes the private key.

Encryption:

Suppose the sender wish to send some text message to someone whose public key is (N, e). The sender then represents the plaintext as a numbers less than N. To encrypt the plaintext P, which is a number modulo N. The Ciphertext C Calculated as-

$$C \equiv P^e \mod N.$$

Decryption:

Using private key (N,d), Plaintext can be found-

$$P \equiv C^d \mod N.$$

Example using small numbers:

Let the plaintext be P=9.

- First, Select two large prime numbers p = 7 and q = 11.
- Calculate N = p × q =7 ×11=77.
- Calculate φ (N) = (p - 1) x (q-1) = (7 - 1) x (11 - 1)=6×10=60.
- Let us now choose relative prime e Such that e is co-prime to φ (N) = 60.Say, e=7.
- Thus the public key is (N, e) = (77, 7).
- A plaintext message P=9 is encrypted using public key (N, e). To find ciphertext from the plain text following formula is used to get ciphertext C. $C \equiv P^e \mod N = 9^7 \mod 77 = 37$.
- The private key is (N,d). To determine the private key, we use the following formula d such that: ed mod φ (N)=1, 7d mod 60 = 1, which gives d = 43.
- The private key is (N,d)= (77,43).
- A ciphertext message C is decrypted using private key (N,d). To calculate plain text P from the ciphertext C following formula is used. $P = c^d \mod N = 37^{43} \mod 77 = 9$

In this example, Plain text P = 9 and the ciphertext C= 37.

Digital siganature with RSA Algorithm:

The RSA public-key cryptosystem is also used to sign and verify messages. Since it is based on the math of the modular exponentiations and discrete logarithms and it's computational difficulty provides a strong security. RSA is an asymmetric digital signature [1] such that one key is used for signing a message and only by the other key the message can be verified.
Where section 2.51 gives key generation, 2.5.2 illustrates signing method and section 2.5.3 illustrates verifying method of digital signature using RSA.

Key generation:
The RSA key-pair consists of:

- public key (N *e*)
- private key (*N*,*d*)

RSA Sign
To Sign a message '*m'* with the private key exponent *d*:

- Compute the message hash: h = hash(m)
- Encrypt h to calculate the signature: s = h^d(mod N).
- The hash h should be in the range [0...N-1). The signature s is in the range [0...N-1).

RSA Verify Signature

Signature s for the message 'm' is verified with the public key exponent e.

- Compute the message hash: h = hash(m)
- Decrypt the signature**:** h′=s^e(mod N)
- Compare h with h' to find whether the signature is valid or not.

If h′=s^e(mod N) = (h^$d)^e$ (mod N) =h, then signature is correct.

Elgamal Cryptosystem:

Elgamal encryption [2][3] is a public key cryptosystem. It is based on the difficulty of finding discrete logarithm in a cyclic group.

Section 3.1 gives key generation, section 3.2 gives encryption method, section 3.3 gives decryption method, section 3.4 gives an example and section 3.5 illustrates digital signature algorithm using RSA.

Key generation:
Participant generates the public and private key pair.

- Select a large number p and a generator g of the multiplicative group F_p of integers modulo p.
- Select a random integer b, 1 ≤ b ≤ p-2, and compute g^bmod p.
- Now public key is (p, g, g^b) and private key is b.

Encryption:
To encrypts a message M, the sender represent M as integers in the range {1,.., p-1}.

- Then sender obtain receiver's public key (p,g,g^b).
- Select a random integer k, 1≤ k ≤ p-2.
- Compute c1= g^k mod p, c2=M × $(g$^$b)^k$.
- Send ciphertext C = (c1,c2).

Decryption:

- Use private key b to compute ($c1^{p-1-b}$) mod p. Note $c1^{p-1-b}$ = $c1^{-b}$.
- Recover M by computing $c1^{-b} \times c2 \ modp$

Example Using Small Numbers:
This is a simple example of ElGamal cryptosystem.
Let Participant A wants to send a message to Participant B. First A needs B's public keys.
Now, B choses a number p=17 (In practical this number is very large).
Then B Selects a random number b=5 and a generator g =6. Then B calculates g^b=7 and makes (17,6,7) to public. And keep b=5 as private key.

A encrypts a message M=13 (in the range of {1,2,…16}). Then A chooses a random number k=10 (1≤ k ≤ 15). He calculates c1=6^{10} (mod17) =15. He encrypts c2 =13× 7^{10}=9 and sends (15,9) to B.

B receives (15 9) from A.
B's public key is (17,6,7) and private key is b=5,
B now decrypts the message by using the private key.
Decryption : M= 9$\times$ 15^{11} (mod 17)=13 [15^{17-1-5} =15^{11}]

B now decrypted the message and received 13, which is the original message.

Digital Signature with ElGamal Algoirthm:

The ElGamal signature [6] scheme was described by Taher ElGamal in 1985.It involves following steps-

Where section 3.5.1 illustrates system parameters, section 3.5.2 illustrates key generation, section 3.5.3 illustrates signature generation and section 3.5.4 illustrates verification.

Sytem parameters

- Let H be a collision resistant hash function.
- Let p be a large prime such that computing discrete logarithm modulo p is difficult.
- Let g<p be a randomly chosen generator of the multiplicative group of integers modulo p.
- The algorithm parameters are (p,g).these system parameters may be shared between users.

Key generation:

- Choose randomly a select key x with 1<x<p-1.
- Compute y=g^x mod p.
- The public key is (p,g,y).
- The secret key is x.
- These steps are performed by the signer.

Signature generation:

- To sign a message the signers performs the following steps.
- Choose a random k such that 0<k<p-1 and gcd (k,p-1)=1.
- Compute r = g^kmod p.
- Compute s = (H(m)-xr) k^{-1}(mod p-1)
- If s=0, start over again.

Then the pair (r,s) is the digital signature of m. The signer repeats these steps for every signature.

Verification:

- A signature (r,s) of a message m is verified as follows.
- 0<r<p and 0<s<p-1.
- $g^{H(m)}$=y^r r^s(mod p).

The verifier accepts a signature if all conditions are satisfied and rejects it otherwise.

Literature Review and Relative Study

Section 1 gives analysis of RSA cryptosystem, section 4.2 gives disadvantages of RSA algorithm, and section 4.3 shows attack on RSA cryptosystem.

Section 4 gives analysis of ElGamal cryptosystem, section 4.5 gives disadvantages of ELGamal algorithm, and section 4.6 shows comparison between RSA and Elgamal.

Analysis Of RSA Cryptosystem:

RSA algorithm is the first algorithm which can be used both for data encryption and digital signatures. It's security depends on the difficulty of decomposition of large prime numbers. It uses Public-key Cryptosystem that have two different keys, called the key pair for the encryption and decryption. It is estimated that the difficulty of guessing the plaintext from single key and the ciphertext equals to the decomposition of product of two large prime numbers.

Disadvantages of RSA algorithm:

In [4] this paper The author mentions some of the limitations associated with RSA. If any of p,q,e and d is known, then the other values are can be calculated and therefore secrecy is needed. Also, In RSA the message length should be less than bit length, otherwise the algorithm may fail. Another limitation of RSA is that it is much slower than other symmetric cryptosystems since it uses the public key. Also, in RSA the length of plain text that can be encrypted is required within the size of N=p*q.

Attacks on RSA Cryptosystem:

In [13] this paper author discussed some of the attacks on RSA

1. Factoring RSA Modulus: Factoring the public modulus is the most evident way to attack RSA cryptosystem. It is assumed that By 2020 1024 bits number will be factored and will not be secured. As a result 2024 bit key should be more secured.
2. Timing Attacks: It has been observed that the RSA algorithm takes different amount of time to perform its crypto operations according to the key's value, so based on the time required to apply the private key to some information, some estimate can be made of the private key.
3. Chosen Cipher text Attack: In this attacker is able to find out plain text based on cipher text using to extended Euclidean Algorithm.

Analysis of ElGamal Algorithm

ElGamal encryption scheme is based on the difficulty of finding discrete logarithm in a cyclic group. One of the strength of ElGamal is its non-deterministic encrypting the same plaintext multiple times will result in different ciphertext, since a random k is chosen each time.

In [15] this paper points out that ElGamal is used in the free GNU privacy Guard Soft-Ware, and other cryptosystems.

Disadvantages of Elgamal

In [15] this paper the author point out that the main disadvantages of El-Gamal is its need for randomness and its slower speed especially for signing. Another disadvantage is that the message expansion by a factor of two takes place during encryption that means the ciphertext is twice as long as the plain text.

Comparison Between RSA and ElGamal cryptosystem

RSA and ElGamal Both are implementation of Public-key cryptosystem. The strength of this algorithm lies in the bit length used. The difficulty level in RSA lies in the factorization of large primes whereas in ElGamal lies in the calculation of discrete logarithms. RSA is a deterministic algorithm while ElGamal is a probabilistic algorithm.
The [6] author in this paper compared the two algorithms RSA and ElGamal. After testing it is concluded that RSA and ElGamal took the same time in the key generation. Though it takes longer time to generate 2048 bit keys as the calculation result have modular expression. It's also concluded that the encryption and decryption time of RSA algorithm is better than ElGamal algorithm. RSA algorithm is faster than ElGamal algorithm. Security wise, The ElGamal algorithm will be more difficult to solve than the RSA algorithm because ElGamal has a complicated calculation to solve discrete logarithms.

Table 1 illustrates the summary between RSA and ElGamal.

Factors	RSA	El-Gamal
Developed	1978	1985
Key-length	>1024 bits	1024 bits
Type of Algorithm	Asymmetric	Asymmetric
Power Consumption	High	low
Key used	Different key for encryption and decryption	Different key for encryption and decryption
Hardware and software implementation	Not very efficient	Faster and efficient

Table 1.Summery Table of RSA and ElGamal:[16]

Conclusion:

In this paper, we have discussed public-key cryptography, which is based on a pair of keys, a public key and a private key. The use of public-key cryptography in digital signature ensures the authenticity and integrity of a message. RSA and ElGamal cryptosystems are implementation of public-key cryptosystem. Both cryptosystems can be used for encryption and signing a message without direct interaction. Also, we have discussed the strengths and weaknesses of RSA and ElGamal algorithms. Finally, through a better understanding of their strengths and weaknesses, further research can be conducted effectively.

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